Past Paper Questions: Roots of Polynomial Equations

1 The equation

$$8x^3 + 12x^2 + 4x - 1 = 0$$

has roots α, β, γ. Show that the equation with roots $2\alpha + 1, 2\beta + 1, 2\gamma + 1$ is

$$y^3 - y - 1 = 0.$$ [3]

The sum $(2\alpha + 1)^n + (2\beta + 1)^n + (2\gamma + 1)^n$ is denoted by S_n. Find the values of S_3 and S_{-2}. [5]

2 The roots of the equation

$$x^3 - x - 1 = 0$$

are α, β, γ, and

$$S_n = \alpha^n + \beta^n + \gamma^n.$$

(i) Use the relation $y = x^2$ to show that α^2, β^2, γ^2 are the roots of the equation

$$y^3 - 2y^2 + y - 1 = 0.$$ [3]

(ii) Hence, or otherwise, find the value of S_4. [2]

(iii) Find the values of S_8, S_{12} and S_{16}. [9]

3 Show that the sum of the cubes of the roots of the equation

$$x^3 + \lambda x + 1 = 0$$

is -3. [3]

Show also that there is no real value of λ for which the sum of the fourth powers of the roots is negative. [3]

4 Obtain the sum of the squares of the roots of the equation

$$x^4 + 3x^3 + 5x^2 + 12x + 4 = 0.$$ [2]

Deduce that this equation does not have more than 2 real roots. [3]

Show that, in fact, the equation has exactly 2 real roots in the interval $-3 < x < 0$. [5]

Denoting these roots by α and β, and the other 2 roots by γ and δ, show that $|\gamma| = |\delta| = \frac{2}{\sqrt{(\alpha\beta)}}$. [4]

5 The equation

$$x^3 + 3x - 1 = 0$$

has roots α, β, γ. Use the substitution $y = x^3$ to show that the equation whose roots are α^3, β^3, γ^3 is

$$y^3 - 3y^2 + 30y - 1 = 0.$$

[3]

Find the value of $\alpha^9 + \beta^9 + \gamma^9$. [5]

6 The equation

$$x^3 + x - 1 = 0$$

has roots α, β, γ. Show that the equation with roots $\alpha^3, \beta^3, \gamma^3$ is

$$y^3 - 3y^2 + 4y - 1 = 0.$$ [4]

Hence find the value of $\alpha^6 + \beta^6 + \gamma^6$. [3]

7 The equation

$$x^4 - x^3 - 1 = 0$$

has roots $\alpha, \beta, \gamma, \delta$. By using the substitution $y = x^3$, or by any other method, find the exact value of $\alpha^6 + \beta^6 + \gamma^6 + \delta^6$. [5]

8 The equation

$$x^3 + x - 1 = 0$$

has roots α, β, γ. Use the relation $x = \sqrt{y}$ to show that the equation

$$y^3 + 2y^2 + y - 1 = 0$$

has roots α^2, β^2, γ^2. [2]

Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(i) Write down the value of S_2 and show that $S_4 = 2$. [3]

(ii) Find the values of S_6 and S_8. [4]

9 The equation

$$x^4 + x^3 + cx^2 + 4x - 2 = 0,$$

where c is a constant, has roots $\alpha, \beta, \gamma, \delta$.

(i) Use the substitution $y = \frac{1}{x}$ to find an equation which has roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$. [2]

(ii) Find, in terms of c, the values of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ and $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$. [3]

(iii) Hence find

$$\left(\alpha - \frac{1}{\alpha}\right)^2 + \left(\beta - \frac{1}{\beta}\right)^2 + \left(\gamma - \frac{1}{\gamma}\right)^2 + \left(\delta - \frac{1}{\delta}\right)^2$$

in terms of c. [2]

(iv) Deduce that when $c = -3$ the roots of the given equation are not all real. [3]

10 The roots of the equation

$$x^3 + px^2 + qx + r = 0$$

are $\frac{\beta}{k}$, β, $k\beta$, where p, q, r, k and β are non-zero real constants. Show that $\beta = -\frac{q}{p}$. [4]

Deduce that $rp^3 = q^3$. [2]

11 Find a cubic equation with roots α, β and γ, given that

$$\alpha + \beta + \gamma = -6, \qquad \alpha^2 + \beta^2 + \gamma^2 = 38, \qquad \alpha\beta\gamma = 30.$$ [3]

Hence find the numerical values of the roots. [3]

12 The roots of the cubic equation $x^3 - 7x^2 + 2x - 3 = 0$ are α, β, γ. Find the values of

(i) $\alpha^2 + \beta^2 + \gamma^2$, [2]

(ii) $\alpha^3 + \beta^3 + \gamma^3$. [3]

13 The cubic equation $x^3 - x^2 - 3x - 10 = 0$ has roots α, β, γ.

(i) Let $u = -\alpha + \beta + \gamma$. Show that $u + 2\alpha = 1$, and hence find a cubic equation having roots $-\alpha + \beta + \gamma$, $\alpha - \beta + \gamma$, $\alpha + \beta - \gamma$. [5]

(ii) State the value of $\alpha\beta\gamma$ and hence find a cubic equation having roots $\frac{1}{\beta\gamma}, \frac{1}{\gamma\alpha}, \frac{1}{\alpha\beta}$. [5]

14 The cubic equation $x^3 - 2x^2 - 3x + 4 = 0$ has roots α, β, γ. Given that $c = \alpha + \beta + \gamma$, state the value of c. [1]

Use the substitution $y = c - x$ to find a cubic equation whose roots are $\alpha + \beta$, $\beta + \gamma$, $\gamma + \alpha$. [3]

Find a cubic equation whose roots are $\frac{1}{\alpha + \beta}$, $\frac{1}{\beta + \gamma}$, $\frac{1}{\gamma + \alpha}$. [2]

Hence evaluate $\frac{1}{(\alpha + \beta)^2} + \frac{1}{(\beta + \gamma)^2} + \frac{1}{(\gamma + \alpha)^2}$. [2]

15 The roots of the equation $x^4 - 4x^2 + 3x - 2 = 0$ are α, β, γ and δ; the sum $\alpha^n + \beta^n + \gamma^n + \delta^n$ is denoted by S_n. By using the relation $y = x^2$, or otherwise, show that α^2, β^2, γ^2 and δ^2 are the roots of the equation

$$y^4 - 8y^3 + 12y^2 + 7y + 4 = 0.$$ [3]

State the value of S_2 and hence show that

$$S_8 = 8S_6 - 12S_4 - 72.$$ [3]

16 The equation $x^3 + px + q = 0$, where p and q are constants, with $q \neq 0$, has one root which is the reciprocal of another root. Prove that $p + q^2 = 1$. [5]

17 The roots of the cubic equation $x^3 - 7x^2 + 2x - 3 = 0$ are α, β and γ. Find the values of [6]

(i) $\dfrac{1}{(\alpha\beta)(\beta\gamma)(\gamma\alpha)}$,

(ii) $\dfrac{1}{\alpha\beta} + \dfrac{1}{\beta\gamma} + \dfrac{1}{\gamma\alpha}$,

(iii) $\dfrac{1}{\alpha^2\beta\gamma} + \dfrac{1}{\alpha\beta^2\gamma} + \dfrac{1}{\alpha\beta\gamma^2}$.

Deduce a cubic equation, with integer coefficients, having roots $\dfrac{1}{\alpha\beta}$, $\dfrac{1}{\beta\gamma}$ and $\dfrac{1}{\gamma\alpha}$. [2]

18 The quartic equation $x^4 - px^2 + qx - r = 0$, where p, q and r are real constants, has two pairs of equal roots. Show that $p^2 + 4r = 0$ and state the value of q. [6]

19 The roots of the cubic equation $2x^3 + x^2 - 7 = 0$ are α, β and γ. Using the substitution $y = 1 + \frac{1}{x}$, or otherwise, find the cubic equation whose roots are $1 + \frac{1}{\alpha}$, $1 + \frac{1}{\beta}$ and $1 + \frac{1}{\gamma}$, giving your answer in the form $ay^3 + by^2 + cy + d = 0$, where a, b, c and d are constants to be found. [4]

20 The cubic equation

$$z^3 - z^2 - z - 5 = 0$$

has roots α, β and γ. Show that the value of $\alpha^3 + \beta^3 + \gamma^3$ is 19. [4]

Find the value of $\alpha^4 + \beta^4 + \gamma^4$. [2]

Show that the cubic equation with roots $\frac{\alpha - 1}{\alpha}$, $\frac{\beta - 1}{\beta}$ and $\frac{\gamma - 1}{\gamma}$ may be found using the substitution $z = \frac{1}{1 - x}$, and find this equation, giving your answer in the form $px^3 + qx^2 + rx + s = 0$, where p, q, r and s are constants to be determined. [4]

21 By finding a cubic equation whose roots are α, β and γ, solve the set of simultaneous equations

$$\alpha + \beta + \gamma = -1,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 29,$$
$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = -1.$$

[8]

22 The roots of the cubic equation $x^3 + 2x^2 - 3 = 0$ are α, β and γ.

(i) By using the substitution $y = \frac{1}{x^2}$, find the cubic equation with roots $\frac{1}{\alpha^2}$, $\frac{1}{\beta^2}$ and $\frac{1}{\gamma^2}$. [3]

(ii) Hence find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$. [1]

(iii) Find also the value of $\frac{1}{\alpha^2\beta^2} + \frac{1}{\beta^2\gamma^2} + \frac{1}{\gamma^2\alpha^2}$. [1]

23 It is given that the equation

$$x^3 - 21x^2 + kx - 216 = 0,$$

where k is a constant, has real roots a, ar and ar^{-1}.

(i) Find the numerical values of the roots. [6]

(ii) Deduce the value of k. [2]

24 The equation

$$9x^3 - 9x^2 + x - 2 = 0$$

has roots α, β, γ.

(i) Use the substitution $y = 3x - 1$ to show that $3\alpha - 1$, $3\beta - 1$, $3\gamma - 1$ are the roots of the equation

$$y^3 - 2y - 7 = 0.$$ [2]

The sum $(3\alpha - 1)^n + (3\beta - 1)^n + (3\gamma - 1)^n$ is denoted by S_n.

(ii) Find the value of S_3. [2]

(iii) Find the value of S_{-2}. [4]

25 The equation

$$x^3 - x + 1 = 0$$

has roots α, β, γ.

(i) Use the relation $x = y^{\frac{1}{3}}$ to show that the equation

$$y^3 + 3y^2 + 2y + 1 = 0$$

has roots α^3, β^3, γ^3. Hence write down the value of $\alpha^3 + \beta^3 + \gamma^3$. [3]

Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(ii) Find the value of S_{-3}. [2]

(iii) Show that $S_6 = 5$ and find the value of S_9. [4]

26 A cubic equation $x^3 + bx^2 + cx + d = 0$ has real roots α, β and γ such that

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = -\frac{5}{12},$$
$$\alpha\beta\gamma = -12,$$
$$\alpha^3 + \beta^3 + \gamma^3 = 90.$$

(i) Find the values of c and d. [3]

(ii) Express $\alpha^2 + \beta^2 + \gamma^2$ in terms of b. [2]

(iii) Show that $b^3 - 15b + 126 = 0$. [4]

(iv) Given that $3 + i\sqrt{(12)}$ is a root of $y^3 - 15y + 126 = 0$, deduce the value of b. [2]

27 Find the sum of the squares of the roots of the equation

$$x^3 + x + 12 = 0,$$

and deduce that only one of the roots is real. [4]

The real root of the equation is denoted by α. Prove that $-3 < \alpha < -2$, and hence prove that the modulus of each of the other roots lies between 2 and $\sqrt{6}$. [5]

28 Given that

$$\alpha + \beta + \gamma = 0, \quad \alpha^2 + \beta^2 + \gamma^2 = 14, \quad \alpha^3 + \beta^3 + \gamma^3 = -18,$$

find a cubic equation whose roots are α , β , γ. [4]

Hence find possible values for α , β , γ. [2]

29 In the equation

$$x^3 + ax^2 + bx + c = 0,$$

the coefficients a, b and c are real. It is given that all the roots are real and greater than 1.

(i) Prove that $a < -3$. [1]

(ii) By considering the sum of the squares of the roots, prove that $a^2 > 2b + 3$. [2]

(iii) By considering the sum of the cubes of the roots, prove that $a^3 < -9b - 3c - 3$. [4]

30 The roots of the equation

$$x^3 + x + 1 = 0$$

are α, β, γ. Show that the equation whose roots are

$$\frac{4\alpha + 1}{\alpha + 1}, \quad \frac{4\beta + 1}{\beta + 1}, \quad \frac{4\gamma + 1}{\gamma + 1}$$

is of the form

$$y^3 + py + q = 0,$$

where the numbers p and q are to be determined. [5]

Hence find the value of

$$\left(\frac{4\alpha + 1}{\alpha + 1}\right)^n + \left(\frac{4\beta + 1}{\beta + 1}\right)^n + \left(\frac{4\gamma + 1}{\gamma + 1}\right)^n,$$

for $n = 2$ and for $n = 3$. [4]

31 The roots of the equation

$$x^3 - 8x^2 + 5 = 0$$

are α, β, γ. Show that

$$\alpha^2 = \frac{5}{\beta + \gamma}.$$ [4]

It is given that the roots are all real. Without reference to a graph, show that one of the roots is negative and the other two roots are positive. [3]

32 The roots of the equation

$$x^4 - 5x^2 + 2x - 1 = 0$$

are $\alpha, \beta, \gamma, \delta$. Let $S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$.

(i) Show that

$$S_{n+4} - 5S_{n+2} + 2S_{n+1} - S_n = 0.$$ [2]

(ii) Find the values of S_2 and S_4. [3]

(iii) Find the value of S_3 and hence find the value of S_6. [6]

(iv) Hence find the value of [3]

$$\alpha^2(\beta^4 + \gamma^4 + \delta^4) + \beta^2(\gamma^4 + \delta^4 + \alpha^4) + \gamma^2(\delta^4 + \alpha^4 + \beta^4) + \delta^2(\alpha^4 + \beta^4 + \gamma^4).$$

33 The equation

$$x^3 + 5x + 3 = 0$$

has roots α, β, γ. Use the substitution $x = -\frac{3}{y}$ to find a cubic equation in y and show that the roots of this equation are $\beta\gamma, \gamma\alpha, \alpha\beta$. [4]

Find the exact values of $\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2$ and $\beta^3\gamma^3 + \gamma^3\alpha^3 + \alpha^3\beta^3$. [5]

34 The roots of the equation $x^3 + 4x - 1 = 0$ are α, β and γ. Use the substitution $y = \frac{1}{1 + x}$ to show that the equation $6y^3 - 7y^2 + 3y - 1 = 0$ has roots $\frac{1}{\alpha + 1}$, $\frac{1}{\beta + 1}$ and $\frac{1}{\gamma + 1}$. [2]

For the cases $n = 1$ and $n = 2$, find the value of

$$\frac{1}{(\alpha + 1)^n} + \frac{1}{(\beta + 1)^n} + \frac{1}{(\gamma + 1)^n}.$$ [2]

Deduce the value of $\dfrac{1}{(\alpha+1)^3} + \dfrac{1}{(\beta+1)^3} + \dfrac{1}{(\gamma+1)^3}$. [2]

Hence show that $\dfrac{(\beta+1)(\gamma+1)}{(\alpha+1)^2} + \dfrac{(\gamma+1)(\alpha+1)}{(\beta+1)^2} + \dfrac{(\alpha+1)(\beta+1)}{(\gamma+1)^2} = \dfrac{73}{36}$. [3]

35 The equation $x^3 + px + q = 0$ has a repeated root. Prove that $4p^3 + 27q^2 = 0$. [5]

36 The equation

$$x^3 + 5x^2 - 3x - 15 = 0$$

has roots α, β, γ. Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [3]

Hence show that the matrix $\begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{pmatrix}$ is singular. [4]

37 The roots of the equation $x^4 - 3x^2 + 5x - 2 = 0$ are α, β, γ, δ, and $\alpha^n + \beta^n + \gamma^n + \delta^n$ is denoted by S_n. Show that

$$S_{n+4} - 3S_{n+2} + 5S_{n+1} - 2S_n = 0.$$ [2]

Find the values of

(i) S_2 and S_4, [3]

(ii) S_3 and S_5. [6]

Hence find the value of

$$\alpha^2(\beta^3 + \gamma^3 + \delta^3) + \beta^2(\gamma^3 + \delta^3 + \alpha^3) + \gamma^2(\delta^3 + \alpha^3 + \beta^3) + \delta^2(\alpha^3 + \beta^3 + \gamma^3).$$ [3]

38 A cubic equation has roots α , β and γ such that

$$\alpha + \beta + \gamma = 4,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 14,$$
$$\alpha^3 + \beta^3 + \gamma^3 = 34.$$

Find the value of $\alpha\beta + \beta\gamma + \gamma\alpha$. [2]

Show that the cubic equation is

$$x^3 - 4x^2 + x + 6 = 0,$$

and solve this equation. [6]

39 The cubic equation $x^3 - px - q = 0$, where p and q are constants, has roots α, β, γ. Show that

(i) $\alpha^2 + \beta^2 + \gamma^2 = 2p$, [1]

(ii) $\alpha^3 + \beta^3 + \gamma^3 = 3q$, [2]

(iii) $6(\alpha^5 + \beta^5 + \gamma^5) = 5(\alpha^3 + \beta^3 + \gamma^3)(\alpha^2 + \beta^2 + \gamma^2)$. [3]

40 The equation

$$8x^3 + 36x^2 + kx - 21 = 0,$$

where k is a constant, has roots $a - d$, a, $a + d$. Find the numerical values of the roots and determine the value of k. [8]

41 The roots of the quartic equation $x^4 + 4x^3 + 2x^2 - 4x + 1 = 0$ are α, β, γ and δ. Find the values of

(i) $\alpha + \beta + \gamma + \delta$, [1]

(ii) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$, [2]

(iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} + \dfrac{1}{\delta}$, [2]

(iv) $\dfrac{\alpha}{\beta\gamma\delta} + \dfrac{\beta}{\alpha\gamma\delta} + \dfrac{\gamma}{\alpha\beta\delta} + \dfrac{\delta}{\alpha\beta\gamma}$. [2]

Using the substitution $y = x + 1$, find a quartic equation in y. Solve this quartic equation and hence find the roots of the equation $x^4 + 4x^3 + 2x^2 - 4x + 1 = 0$. [7]

42 The cubic equation $x^3 + px^2 + qx + r = 0$, where p, q and r are integers, has roots α, β and γ, such that

$$\alpha + \beta + \gamma = 15,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 83.$$

Write down the value of p and find the value of q. [3]

Given that α, β and γ are all real and that $\alpha\beta + \alpha\gamma = 36$, find α and hence find the value of r. [5]

43 Find the cubic equation with roots α, β and γ such that

$$\alpha + \beta + \gamma = 3,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 1,$$
$$\alpha^3 + \beta^3 + \gamma^3 = -30,$$

giving your answer in the form $x^3 + px^2 + qx + r = 0$, where p, q and r are integers to be found. [6]

44 The cubic equation $2x^3 - 3x^2 + 4x - 10 = 0$ has roots α, β and γ.

(i) Find the value of $(\alpha + 1)(\beta + 1)(\gamma + 1)$. [4]

(ii) Find the value of $(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta)$. [4]

45 The roots of the equation

$$x^3 + px^2 + qx + r = 0$$

are α, 2α, 4α, where p, q, r and α are non-zero real constants.

(i) Show that

$$2p\alpha + q = 0.$$ [4]

(ii) Show that

$$p^3 r - q^3 = 0.$$ [2]

46 The roots of the cubic equation

$$x^3 - 5x^2 + 13x - 4 = 0$$

are α, β, γ.

(i) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [3]

(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. [2]

47 The equation $x^3 + 2x^2 + x + 7 = 0$ has roots α, β, γ.

(i) Use the relation $x^2 = -7y$ to show that the equation

$$49y^3 + 14y^2 - 27y + 7 = 0$$

has roots $\dfrac{\alpha}{\beta\gamma}$, $\dfrac{\beta}{\gamma\alpha}$, $\dfrac{\gamma}{\alpha\beta}$. [4]

(ii) Show that $\dfrac{\alpha^2}{\beta^2\gamma^2} + \dfrac{\beta^2}{\gamma^2\alpha^2} + \dfrac{\gamma^2}{\alpha^2\beta^2} = \dfrac{58}{49}$. [3]

(iii) Find the exact value of $\dfrac{\alpha^3}{\beta^3\gamma^3} + \dfrac{\beta^3}{\gamma^3\alpha^3} + \dfrac{\gamma^3}{\alpha^3\beta^3}$. [2]

48 The cubic equation $x^3 + bx^2 + cx + d = 0$, where b, c and d are constants, has roots α, β, γ. It is given that $\alpha\beta\gamma = -1$.

(a) State the value of d. [1]

(b) Find a cubic equation, with coefficients in terms of b and c, whose roots are $\alpha + 1$, $\beta + 1$, $\gamma + 1$. [3]

(c) Given also that $\gamma + 1 = -\alpha - 1$, deduce that $(c - 2b + 3)(b - 3) = b - c$. [4]

49 The cubic equation $7x^3+3x^2+5x+1=0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$. [3]

(b) Find the value of $\alpha^{-2}+\beta^{-2}+\gamma^{-2}$. [2]

(c) Find the value of $\alpha^{-3}+\beta^{-3}+\gamma^{-3}$. [2]

50 The cubic equation $6x^3+px^2-3x-5=0$, where p is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are α^2, β^2, γ^2. [3]

(b) It is given that $\alpha^2+\beta^2+\gamma^2=2(\alpha+\beta+\gamma)$.

(i) Find the value of p. [3]

(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. [2]

51 The cubic equation $x^3 + cx + 1 = 0$, where c is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are α^3, β^3, γ^3. [3]

(b) Show that $\alpha^6 + \beta^6 + \gamma^6 = 3 - 2c^3$. [3]

(c) Find the real value of c for which the matrix $\begin{pmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{pmatrix}$ is singular. [5]

52 The equation $x^4 - 2x^3 - 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\alpha^3, \beta^3, \gamma^3, \delta^3$ and state the value of $\alpha^3 + \beta^3 + \gamma^3 + \delta^3$. [4]

(b) Find the value of $\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\gamma^3}+\frac{1}{\delta^3}$. [3]

(c) Find the value of $\alpha^4+\beta^4+\gamma^4+\delta^4$. [2]

53 The cubic equation $2x^3 - 4x^2 + 3 = 0$ has roots α, β, γ. Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(a) State the value of S_1 and find the value of S_2. [3]

(b) **(i)** Express S_{n+3} in terms of S_{n+2} and S_n. [1]

(ii) Hence, or otherwise, find the value of S_4. [2]

(c) Use the substitution $y = S_1 - x$, where S_1 is the numerical value found in part **(a)**, to find and simplify an equation whose roots are $\alpha+\beta$, $\beta+\gamma$, $\gamma+\alpha$. [3]

(d) Find the value of $\frac{1}{\alpha+\beta}+\frac{1}{\beta+\gamma}+\frac{1}{\gamma+\alpha}$. [2]

54 It is given that

$$\alpha+\beta+\gamma = 3, \quad \alpha^2+\beta^2+\gamma^2 = 5, \quad \alpha^3+\beta^3+\gamma^3 = 6.$$

The cubic equation $x^3+bx^2+cx+d=0$ has roots α, β, γ.

Find the values of b, c and d. [6]

55 The cubic equation $x^3 + 2x^2 + 3x + 3 = 0$ has roots α, β, γ.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [2]

(b) Show that $\alpha^3 + \beta^3 + \gamma^3 = 1$. [2]

56 The cubic equation $2x^3 + 5x^2 - 6 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\frac{1}{\alpha^3}, \frac{1}{\beta^3}, \frac{1}{\gamma^3}$. [3]

(b) Find the value of $\frac{1}{\alpha^6} + \frac{1}{\beta^6} + \frac{1}{\gamma^6}$. [3]

(c) Find also the value of $\frac{1}{\alpha^9} + \frac{1}{\beta^9} + \frac{1}{\gamma^9}$. [2]

57 The cubic equation $x^3 + 5x^2 + 10x - 2 = 0$ has roots α, β, γ.

Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [3]

58 The cubic equation $x^3 + bx^2 + d = 0$ has roots α, β, γ, where $\alpha = \beta$ and $d \neq 0$.

(a) Show that $4b^3 + 27d = 0$. [5]

(b) Given that $2\alpha^2 + \gamma^2 = 3b$, find the values of b and d. [3]

59 The equation $x^4 + 3x^2 + 2x + 6 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}, \frac{1}{\delta^2}$ and state the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$.

[4]

(b) Find the value of $\beta^2\gamma^2\delta^2 + \alpha^2\gamma^2\delta^2 + \alpha^2\beta^2\delta^2 + \alpha^2\beta^2\gamma^2$. [3]

(c) Find the value of $\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} + \frac{1}{\delta^4}$. [2]

60 The equation $x^4 - x^2 + 2x + 5 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\alpha^2, \beta^2, \gamma^2, \delta^2$ and state the value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$. [4]

(b) Find the value of $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}$. [3]

(c) Find the value of $\alpha^4+\beta^4+\gamma^4+\delta^4$. [2]