Past Paper Questions: Rational Functions

1 The curve C has equation $y = \dfrac{x^2 - 4}{x - 3}$.

(i) Find the equations of the asymptotes of C. [3]

(ii) Draw a sketch of C and its asymptotes. Give the coordinates of the points of intersection of C with the coordinate axes. [4]

[You are not required to find the coordinates of any turning points.]

2 The curve C has equation

$$y = \frac{x - ax^2}{x - 1},$$

where a is a constant and $a > 1$.

(i) Find the equations of the asymptotes of C. [3]

(ii) Show that the x-coordinates of both the turning points of C are positive. [4]

3 The curve Γ, which has equation

$$y = \frac{ax^2 + bx + c}{x^2 + px + q},$$

has asymptotes $x = 1$, $x = 4$ and $y = 2$. Find the values of a, p and q. [4]

It is given that Γ has a stationary point at $x = 2$.

(i) Find the value of c. [3]

(ii) Show that if $b \neq -10$ then Γ has exactly 2 stationary points. [2]

(iii) Draw a sketch of Γ for the case where $b = -6$. [4]

4 The curve C has equation

$$y = 2x + \frac{3(x-1)}{x+1}.$$

(i) Write down the equations of the asymptotes of C. [2]

(ii) Find the set of values of x for which C is above its oblique asymptote and the set of values of x for which C is below its oblique asymptote. [3]

(iii) Draw a sketch of C, stating the coordinates of the points of intersection of C with the coordinate axes. [4]

5 The curve C has equation

$$y = \lambda x + \frac{x}{x+2},$$

where λ is a non-zero constant.

(i) Find the asymptotes of C. [3]

(ii) Show that if $\lambda > 0$ then $\frac{dy}{dx} > 0$ at all points of C. [2]

(iii) Show that, for $\lambda < -\frac{1}{2}$, C has two distinct stationary points, both to the left of the y-axis. [3]

(iv) In separate diagrams draw sketches of C for each of the cases $\lambda > 0$ and $\lambda < -\frac{1}{2}$. [6]

6 The curve C has equation

$$y = \frac{x^2 - 2x + \lambda}{x + 1},$$

where λ is a constant. Show that the equations of the asymptotes of C are independent of λ. [3]

Find the value of λ for which the x-axis is a tangent to C, and sketch C in this case. [4]

Sketch C in the case $\lambda = -4$, giving the exact coordinates of the points of intersection of C with the x-axis. [3]

7 The curve C has equation

$$y = \frac{x^2}{x + \lambda},$$

where λ is a non-zero constant. Obtain the equation of each of the asymptotes of C. [3]

In separate diagrams, sketch C for the cases $\lambda > 0$ and $\lambda < 0$. In both cases the coordinates of the turning points must be indicated. [8]

8 The curve C has equation

$$y = \frac{x(x+1)}{(x-1)^2}.$$

(i) Obtain the equations of the asymptotes of C. [3]

(ii) Show that there is exactly one point of intersection of C with the asymptotes and find its coordinates. [2]

(iii) Find $\frac{dy}{dx}$ and hence

(a) find the coordinates of any stationary points of C,

(b) state the set of values of x for which the gradient of C is negative.

[6]

(iv) Draw a sketch of C. [3]

9 The curve C has equation

$$y = \frac{x^2 - 3x - 7}{x + 1}.$$

(i) Obtain the equations of the asymptotes of C. [3]

(ii) Show that $\frac{dy}{dx} > 1$ at all points of C. [2]

(iii) Draw a sketch of C. [3]

10 The curve C has equation

$$y = \frac{x^2 + \lambda x - 6\lambda^2}{x + 3},$$

where λ is a constant such that $\lambda \neq 1$ and $\lambda \neq -\frac{3}{2}$.

(i) Find $\frac{dy}{dx}$ and deduce that if C has two stationary points then $-\frac{3}{2} < \lambda < 1$. [5]

(ii) Find the equations of the asymptotes of C. [3]

(iii) Draw a sketch of C for the case $0 < \lambda < 1$. [3]

(iv) Draw a sketch of C for the case $\lambda > 3$. [3]

11 The curve C with equation

$$y = \frac{ax^2 + bx + c}{x - 1},$$

where a, b and c are constants, has two asymptotes. It is given that $y = 2x - 5$ is one of these asymptotes.

(i) State the equation of the other asymptote. [1]

(ii) Find the value of a and show that $b = -7$. [3]

(iii) Given also that C has a turning point when $x = 2$, find the value of c. [3]

(iv) Find the set of values of k for which the line $y = k$ does not intersect C. [4]

12 The curve C has equation

$$y = \frac{2x^2 + 2x + 3}{x^2 + 2}.$$

Show that, for all x, $1 \leqslant y \leqslant \frac{5}{2}$. [4]

Find the coordinates of the turning points on C. [3]

Find the equation of the asymptote of C. [2]

Sketch the graph of C, stating the coordinates of any intersections with the y-axis and the asymptote. [2]

13 The curve C has equation $y = \dfrac{x^2}{x-2}$. Find the equations of the asymptotes of C. [3]

Find the coordinates of the turning points on C. [3]

Draw a sketch of C. [3]

14 The curve C has equation $y = \dfrac{2x^2 - 3x - 2}{x^2 - 2x + 1}$. State the equations of the asymptotes of C. [2]

Show that $y \leqslant \frac{25}{12}$ at all points of C. [4]

Find the coordinates of any stationary points of C. [3]

Sketch C, stating the coordinates of any intersections of C with the coordinate axes and the asymptotes. [4]

15 The curve C has equation

$$y = \frac{ax^2 + bx + c}{x + d},$$

where a, b, c and d are constants. The curve cuts the y-axis at $(0, -2)$ and has asymptotes $x = 2$ and $y = x + 1$.

(i) Write down the value of d. [1]

(ii) Determine the values of a, b and c. [6]

(iii) Show that, at all points on C, either $y \leqslant 3 - 2\sqrt{6}$ or $y \geqslant 3 + 2\sqrt{6}$. [7]

16 Express $\dfrac{2x^2 - x + 5}{x^2 - 1}$ in the form $2 + \dfrac{A}{x-1} + \dfrac{B}{x+1}$, where A and B are integers to be found. [3]

The curve C has equation $y = \dfrac{2x^2 - x + 5}{x^2 - 1}$. Show that there are two distinct values of x for which $\dfrac{dy}{dx} = 0$. [4]

Sketch C, stating the equations of the asymptotes and giving the coordinates of any points of intersection with the coordinate axes and with the asymptotes. You do not need to find the coordinates of the turning points. [7]

17 The curve C has equation $y = \frac{4x^2 - 3x}{x^2 + 1}$. Verify that the equation of C may be written in the form $y = -\frac{1}{2} + \frac{(3x - 1)^2}{2(x^2 + 1)}$ and also in the form $y = \frac{9}{2} - \frac{(x + 3)^2}{2(x^2 + 1)}$. [3]

Hence show that $-\frac{1}{2} \leqslant y \leqslant \frac{9}{2}$. [2]

Without differentiating, write down the coordinates of the turning points of C. [2]

State the equation of the asymptote of C. [1]

Sketch the graph of C, stating the coordinates of the intersections with the coordinate axes and the asymptote. [3]

18 A curve C has equation $y = \dfrac{x^2}{x-2}$. Find the equations of the asymptotes of C. [3]

Show that there are no points on C for which $0 < y < 8$. [4]

Sketch C, giving the coordinates of the turning points. [3]

19 The curve C has equation $y = \dfrac{x+2}{x^2-9}$. Show that $\dfrac{dy}{dx} < 0$ at all points on C. [3]

State the equations of the asymptotes of C. [2]

Sketch C, showing the coordinates of any points of intersection with the coordinate axes. [3]

20 The curve C has equation $y = \dfrac{x^2 - 3x + 6}{1 - x}$.

(i) Find the equations of the asymptotes of C. [3]

(ii) Find the coordinates of the turning points of C. [3]

(iii) Find the coordinates of any intersections with the coordinate axes. [2]

(iv) Sketch C. [3]

21 The curve C has equation

$$y = \frac{x^2 + b}{x + b},$$

where b is a positive constant.

(i) Find the equations of the asymptotes of C. [3]

(ii) Show that C does not intersect the x-axis. [1]

(iii) Justifying your answer, find the number of stationary points on C. [2]

(iv) Sketch C. Your sketch should indicate the coordinates of any points of intersection with the y-axis. You do not need to find the coordinates of any stationary points. [3]

22 The curve C has equation

$$y = \frac{x^2 + 7x + 6}{x - 2}.$$

(i) Find the coordinates of the points of intersection of C with the axes. [2]

(ii) Find the equation of each of the asymptotes of C. [3]

(iii) Sketch C. [3]

23 The curves C_1 and C_2 have equations

$$y = \frac{ax}{x+5} \quad \text{and} \quad y = \frac{x^2 + (a+10)x + 5a + 26}{x+5}$$

respectively, where a is a constant and $a > 2$.

(i) Find the equations of the asymptotes of C_1. [2]

(ii) Find the equation of the oblique asymptote of C_2. [2]

(iii) Show that C_1 and C_2 do not intersect. [2]

(iv) Find the coordinates of the stationary points of C_2. [3]

(v) Sketch C_1 and C_2 on a single diagram. [You do not need to calculate the coordinates of any points where C_2 crosses the axes.] [3]

24 The curve C has equation

$$y = \frac{x^2}{kx - 1},$$

where k is a positive constant.

(i) Obtain the equations of the asymptotes of C. [3]

(ii) Find the coordinates of the stationary points of C. [3]

(iii) Sketch C. [3]

25 The curve C has equation $y = \dfrac{5(x-1)(x+2)}{(x-2)(x+3)}$.

(i) Express y in the form $P + \dfrac{Q}{x-2} + \dfrac{R}{x+3}$. [3]

(ii) Show that $\dfrac{dy}{dx} = 0$ for exactly one value of x and find the corresponding value of y. [4]

(iii) Write down the equations of all the asymptotes of C. [3]

(iv) Find the set of values of k for which the line $y = k$ does not intersect C. [4]

26 The curve C has equation

$$y = \frac{x^2 + 2x - 3}{(\lambda x + 1)(x + 4)},$$

where λ is a constant.

(i) Find the equations of the asymptotes of C for the case where $\lambda = 0$. [4]

(ii) Find the equations of the asymptotes of C for the case where λ is not equal to any of -1, 0, $\frac{1}{4}$, $\frac{1}{3}$. [3]

(iii) Sketch C for the case where $\lambda = -1$. Show, on your diagram, the equations of the asymptotes and the coordinates of the points of intersection of C with the coordinate axes. [4]

27 The curve C has equation

$$y = \frac{x^2}{x + \lambda},$$

where λ is a non-zero constant. Obtain the equations of the asymptotes of C. [3]

In separate diagrams, sketch C for the cases where

(i) $\lambda > 0$,

(ii) $\lambda < 0$.

[4]

28 The curve C has equation

$$y = \frac{x^2 + qx + 1}{2x + 3},$$

where q is a positive constant.

(i) Obtain the equations of the asymptotes of C. [3]

(ii) Find the value of q for which the x-axis is a tangent to C, and sketch C in this case. [4]

(iii) Sketch C for the case $q = 3$, giving the exact coordinates of the points of intersection of C with the x-axis. [3]

(iv) It is given that, for all values of the constant λ, the line

$$y = \lambda x + \tfrac{3}{2}\lambda + \tfrac{1}{2}(q - 3)$$

passes through the point of intersection of the asymptotes of C. Use this result, with the diagrams you have drawn, to show that if $\lambda < \frac{1}{2}$ then the equation

$$\frac{x^2 + qx + 1}{2x + 3} = \lambda x + \tfrac{3}{2}\lambda + \tfrac{1}{2}(q - 3)$$

has no real solution if q has the value found in part **(ii)**, but has 2 real distinct solutions if $q = 3$. [4]

29 The curve C has equation

$$y = \frac{ax^2 + bx + c}{x + 4},$$

where a, b and c are constants. It is given that $y = 2x - 5$ is an asymptote of C.

(i) Find the values of a and b. [3]

(ii) Given also that C has a turning point at $x = -1$, find the value of c. [3]

(iii) Find the set of values of y for which there are no points on C. [4]

(iv) Draw a sketch of the curve with equation

$$y = \frac{2(x-7)^2 + 3(x-7) - 2}{x - 3}.$$ [3]

[You should state the equations of the asymptotes and the coordinates of the turning points.]

30 The curve C has equation

$$y = \frac{(x-2)(x-a)}{(x-1)(x-3)},$$

where a is a constant not equal to 1, 2 or 3.

(i) Write down the equations of the asymptotes of C. [2]

(ii) Show that C meets the asymptote parallel to the x-axis at the point where $x = \frac{2a-3}{a-2}$. [2]

(iii) Show that the x-coordinates of any stationary points on C satisfy

$$(a-2)x^2 + (6-4a)x + (5a-6) = 0,$$

and hence find the set of values of a for which C has stationary points. [6]

(iv) Sketch the graph of C for

(a) $a > 3$,

(b) $2 < a < 3$.

[4]

31 The curve C has equation

$$y = \frac{x^2 - 5x + 4}{x + 1}.$$

(i) Obtain the coordinates of the points of intersection of C with the axes. [2]

(ii) Obtain the equation of each of the asymptotes of C. [3]

(iii) Draw a sketch of C. [3]

32 The curve C has equation

$$y = \frac{x^2 + 2\lambda x}{x^2 - 2x + \lambda},$$

where λ is a constant and $\lambda \neq -1$.

(i) Show that C has at most two stationary points. [3]

(ii) Show that if C has **exactly** two stationary points then $\lambda > -\frac{5}{4}$. [2]

(iii) Find the set of values of λ such that C has two vertical asymptotes. [2]

(iv) Find the x-coordinates of the points of intersection of C with

(a) the x-axis,

(b) the horizontal asymptote.

[3]

(v) Sketch C in each of the cases

(a) $\lambda < -2$,

(b) $\lambda > 2$.

[4]

33 The curve C has equation $y = \dfrac{x^2 + px + 1}{x - 2}$, where p is a constant. Given that C has two asymptotes, find the equation of each asymptote. [3]

Find the set of values of p for which C has two distinct turning points. [5]

Sketch C in the case $p = -1$. Your sketch should indicate the coordinates of any intersections with the axes, but need not show the coordinates of any turning points. [3]

34 A curve C has equation

$$y = \frac{5(x^2 - x - 2)}{x^2 + 5x + 10}.$$

Find the coordinates of the points of intersection of C with the axes. [2]

Show that, for all real values of x, $-1 \leqslant y \leqslant 15$. [4]

Sketch C, stating the coordinates of any turning points and the equation of the horizontal asymptote. [7]

35 The curve C has equation

$$y = \lambda x + \frac{x}{x-2},$$

where λ is a non-zero constant. Find the equations of the asymptotes of C. [3]

Show that C has no turning points if $\lambda < 0$. [3]

Sketch C in the case $\lambda = -1$, stating the coordinates of the intersections with the axes. [3]

36 The curve C has equation $y = \dfrac{x^2 - 3x + 3}{x - 2}$. Find the equations of the asymptotes of C. [3]

Show that there are no points on C for which $-1 < y < 3$. [4]

Find the coordinates of the turning points of C. [3]

Sketch C. [2]

37 The curve C has equation

$$y = \frac{px^2 + 4x + 1}{x + 1},$$

where p is a positive constant and $p \neq 3$.

(i) Obtain the equations of the asymptotes of C. [3]

(ii) Find the value of p for which the x-axis is a tangent to C, and sketch C in this case. [4]

(iii) For the case $p = 1$, show that C has no turning points, and sketch C, giving the exact coordinates of the points of intersection of C with the x-axis. [5]

38 The curve C has equation

$$y = \frac{2x^2 + 5x - 1}{x + 2}.$$

Find the equations of the asymptotes of C. [3]

Show that $\frac{dy}{dx} > 2$ at all points on C. [3]

Sketch C. [3]

39 A curve C has equation $y = \dfrac{2x^2 + x - 1}{x - 1}$. Find the equations of the asymptotes of C. [3]

Show that there is no point on C for which $1 < y < 9$. [4]

40 The curve C has equation $y = \dfrac{2x^2 + kx}{x + 1}$, where k is a constant. Find the set of values of k for which C has no stationary points. [5]

For the case $k = 4$, find the equations of the asymptotes of C and sketch C, indicating the coordinates of the points where C intersects the coordinate axes. [6]

41 The curve C has equation

$$y = \frac{3x - 9}{(x - 2)(x + 1)}.$$

(i) Find the equations of the asymptotes of C. [2]

(ii) Show that there is no point on C for which $\frac{1}{3} < y < 3$. [4]

(iii) Find the coordinates of the turning points of C. [3]

(iv) Sketch C. [3]

42 The curve C has equation

$$y = \frac{x^2 + ax - 1}{x + 1},$$

where a is constant and $a > 1$.

(i) Find the equations of the asymptotes of C. [3]

(ii) Show that C intersects the x-axis twice. [1]

(iii) Justifying your answer, find the number of stationary points on C. [2]

(iv) Sketch C, stating the coordinates of its point of intersection with the y-axis. [3]

43 The curve C has equation

$$y = \frac{5x^2 + 5x + 1}{x^2 + x + 1}.$$

(i) Find the equation of the asymptote of C. [2]

(ii) Show that, for all real values of x, $-\frac{1}{3} \leqslant y < 5$. [4]

(iii) Find the coordinates of any stationary points of C. [2]

(iv) Sketch C, stating the coordinates of any intersections with the y-axis. [2]

44 The line $y = 2x + 1$ is an asymptote of the curve C with equation

$$y = \frac{x^2 + 1}{ax + b}.$$

(i) Find the values of the constants a and b. [3]

(ii) State the equation of the other asymptote of C. [1]

(iii) Sketch C. [Your sketch should indicate the coordinates of any points of intersection with the y-axis. You do not need to find the coordinates of any stationary points.] [3]

45 The curve C has equation $y = \dfrac{x^2 + x - 1}{x - 1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that there is no point on C for which $1 < y < 5$. [4]

(c) Find the coordinates of the intersections of C with the axes, and sketch C. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{x^2 + x - 1}{x - 1}\right|$. [2]

46 Let a be a positive constant.

(a) The curve C_1 has equation $y = \frac{x-a}{x-2a}$. [2]

Sketch C_1.

The curve C_2 has equation $y = \left(\frac{x-a}{x-2a}\right)^2$. The curve C_3 has equation $y = \left|\frac{x-a}{x-2a}\right|$.

(b) (i) Find the coordinates of any stationary points of C_2. [3]

(ii) Find also the coordinates of any points of intersection of C_2 and C_3. [3]

(c) Sketch C_2 and C_3 on a single diagram, clearly identifying each curve. Hence find the set of values of x for which $\left(\frac{x-a}{x-2a}\right)^2 \leqslant \left|\frac{x-a}{x-2a}\right|$. [5]

47 The curve C has equation $y = \frac{x^2}{2x+1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the coordinates of the stationary points on C. [3]

(c) Sketch C. [3]

48 The curve C has equation $y = \dfrac{10 + x - 2x^2}{2x - 3}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that C has no turning points. [3]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{10+x-2x^2}{2x-3}\right|$ and find in exact form the set of values of x for which $\left|\dfrac{10+x-2x^2}{2x-3}\right| < 4$. [6]