Past Paper Questions: Summation of Series

1 Let $v_1, v_2, v_3, \ldots$ be a sequence and let

$$u_n = nv_n - (n+1)v_{n+1},$$

for $n = 1, 2, 3, \ldots$. Find $\sum_{n=1}^{N} u_n$. [2]

In each of the following cases determine whether the series $u_1 + u_2 + u_3 + \ldots$ is convergent, and justify your conclusion. Give the sum to infinity where this exists.

(i) $v_n = n^{-\frac{1}{2}}$. [2]

(ii) $v_n = n^{-\frac{3}{2}}$. [2]

2 Use the relevant standard results in the List of Formulae to prove that

$$S_N = \sum_{n=1}^{N}(8n^3 - 6n^2) = N(N+1)(2N^2 - 1).$$ [2]

Hence show that

$$\sum_{n=N+1}^{2N}(8n^3 - 6n^2)$$

can be expressed in the form

$$N(aN^3 + bN^2 + cN + d),$$

where the constants a, b, c, d are to be determined. [2]

3 Use the method of differences to find S_N, where

$$S_N = \sum_{n=N}^{N^2} \frac{1}{n(n+1)}.$$ [3]

Deduce the value of $\lim_{N \to \infty} S_N$. [1]

4 Express

$$u_n = \frac{1}{4n^2 - 1}$$

in partial fractions, and hence find $\sum_{n=1}^{N} u_n$ in terms of N. [4]

Deduce that the infinite series $u_1 + u_2 + u_3 + \ldots$ is convergent and state the sum to infinity. [2]

5 Verify that

$$\frac{1}{n^2+1} - \frac{1}{(n+1)^2+1} = \frac{2n+1}{(n^2+1)(n^2+2n+2)}.$$ [1]

Use the method of differences to show that, for all $N \geqslant 1$,

$$\sum_{n=1}^{N} \frac{2n+1}{(n^2+1)(n^2+2n+2)} < \frac{1}{2}.$$ [3]

Write down the value of

$$\sum_{n=1}^{\infty} \frac{2n+1}{(n^2+1)(n^2+2n+2)}.$$ [1]

6 Given that

$$u_n = \ln\left(\frac{1 + x^{n+1}}{1 + x^n}\right),$$

where $x > -1$, find $\sum_{n=1}^{N} u_n$ in terms of N and x. [3]

Find the sum to infinity of the series

$$u_1 + u_2 + u_3 + \dots$$

when

(i) $-1 < x < 1$, [1]

(ii) $x = 1$. [1]

7 Verify that, for all positive values of n,

$$\frac{1}{(n+2)(2n+3)} - \frac{1}{(n+3)(2n+5)} = \frac{4n+9}{(n+2)(n+3)(2n+3)(2n+5)}.$$ [2]

For the series

$$\sum_{n=1}^{N} \frac{4n+9}{(n+2)(n+3)(2n+3)(2n+5)},$$

find

(i) the sum to N terms, [3]

(ii) the sum to infinity. [1]

8 The sum S_N is defined by $S_N = \sum_{n=1}^{N} n^5$. Using the identity

$$\left(n + \tfrac{1}{2}\right)^6 - \left(n - \tfrac{1}{2}\right)^6 \equiv 6n^5 + 5n^3 + \tfrac{3}{8}n,$$

find S_N in terms of N. [You need not simplify your result.] [4]

Hence find $\lim_{N \to \infty} N^{-\lambda} S_N$, for each of the two cases

(i) $\lambda = 6$,

(ii) $\lambda > 6$.

[3]

9 By considering the identity

$$\cos[(2n-1)\alpha] - \cos[(2n+1)\alpha] \equiv 2\sin\alpha\sin 2n\alpha,$$

show that if α is not an integer multiple of π then

$$\sum_{n=1}^{N} \sin(2n\alpha) = \tfrac{1}{2}\cot\alpha - \tfrac{1}{2}\operatorname{cosec}\alpha\cos[(2N+1)\alpha].$$ [4]

Deduce that the infinite series

$$\sum_{n=1}^{\infty} \sin\left(\tfrac{2}{3}n\pi\right)$$

does not converge. [1]

10 Express $\dfrac{1}{(2r+1)(2r+3)}$ in partial fractions and hence use the method of differences to find

$$\sum_{r=1}^{n} \frac{1}{(2r+1)(2r+3)}.$$ [4]

Deduce the value of

$$\sum_{r=1}^{\infty} \frac{1}{(2r+1)(2r+3)}.$$ [1]

11 Find $2^2 + 4^2 + \ldots + (2n)^2$. [2]

Hence find $1^2 - 2^2 + 3^2 - 4^2 + \ldots - (2n)^2$, simplifying your answer. [3]

12 Given that $f(r) = \frac{1}{(r+1)(r+2)}$, show that

$$f(r-1) - f(r) = \frac{2}{r(r+1)(r+2)}.$$ [2]

Hence find $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$. [3]

Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)}$. [1]

13 Find the sum of the first n terms of the series

$$\frac{1}{1 \times 3} + \frac{1}{2 \times 4} + \frac{1}{3 \times 5} + \dots$$

and deduce the sum to infinity. [5]

14 Use the method of differences to show that $\sum_{r=1}^{N} \frac{1}{(2r+1)(2r+3)} = \frac{1}{6} - \frac{1}{2(2N+3)}$. [5]

Deduce that $\sum_{r=N+1}^{2N} \frac{1}{(2r+1)(2r+3)} < \frac{1}{8N}$. [4]

15 Let $\mathrm{f}(r) = r!(r-1)$. Simplify $\mathrm{f}(r+1) - \mathrm{f}(r)$ and hence find $\displaystyle\sum_{r=n+1}^{2n} r!(r^2+1)$. [5]

16 Expand and simplify $(r+1)^4 - r^4$. [1]

Use the method of differences together with the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2.$$ [4]

17 Show that the difference between the squares of consecutive integers is an odd integer. [1]

Find the sum to n terms of the series

$$\frac{3}{1^2 \times 2^2} + \frac{5}{2^2 \times 3^2} + \frac{7}{3^2 \times 4^2} + \dots + \frac{2r+1}{r^2(r+1)^2} + \dots$$

and deduce the sum to infinity of the series. [5]

18 Use the List of Formulae (MF10) to show that $\sum_{r=1}^{13}(3r^2 - 5r + 1)$ and $\sum_{r=0}^{9}(r^3 - 1)$ have the same numerical value. [4]

19 Use the formula for $\tan(A - B)$ in the List of Formulae (MF10) to show that

$$\tan^{-1}(x+1) - \tan^{-1}(x-1) = \tan^{-1}\left(\frac{2}{x^2}\right).$$ [3]

Deduce the sum to n terms of the series

$$\tan^{-1}\left(\frac{2}{1^2}\right) + \tan^{-1}\left(\frac{2}{2^2}\right) + \tan^{-1}\left(\frac{2}{3^2}\right) + \dots .$$ [4]

20 Express $\frac{4}{r(r+1)(r+2)}$ in partial fractions and hence find $\sum_{r=1}^{n} \frac{4}{r(r+1)(r+2)}$. [5]

Deduce the value of $\sum_{r=1}^{\infty} \frac{4}{r(r+1)(r+2)}$. [1]

21 Verify that $\dfrac{1}{(3r+1)(3r+4)} = \dfrac{1}{3}\left(\dfrac{1}{3r+1} - \dfrac{1}{3r+4}\right)$. [1]

Let S_N denote $\displaystyle\sum_{r=1}^{N} \frac{1}{(3r+1)(3r+4)}$ and let S denote $\displaystyle\sum_{r=1}^{\infty} \frac{1}{(3r+1)(3r+4)}$. Find the least value of N such that $S - S_N < \frac{1}{10\,000}$. [5]

22 It is given that $\sum_{r=1}^{n} u_r = n^2(2n + 3)$, where n is a positive integer.

(i) Find $\sum_{r=n+1}^{2n} u_r$. [2]

(ii) Find u_r. [3]

23 **(i)** Verify that $\dfrac{2r+1}{r(r+1)(r+2)} = \dfrac{1}{2}\left\{\dfrac{(2r+1)(2r+3)}{(r+1)(r+2)} - \dfrac{(2r-1)(2r+1)}{r(r+1)}\right\}$. [2]

(ii) Hence show that $\displaystyle\sum_{r=1}^{n} \frac{2r+1}{r(r+1)(r+2)} = \frac{1}{2}\left\{\frac{(2n+1)(2n+3)}{(n+1)(n+2)} - \frac{3}{2}\right\}$. [2]

(iii) Deduce the value of $\displaystyle\sum_{r=1}^{\infty} \frac{2r+1}{r(r+1)(r+2)}$. [2]

24 Let $S_n = \sum_{r=1}^{n} (-1)^{r-1} r^2$.

(i) Use the standard result for $\sum_{r=1}^{n} r^2$ given in the List of Formulae (MF10) to show that

$$S_{2n} = -n(2n+1).$$ [4]

(ii) State the value of $\lim_{n\to\infty} \frac{S_{2n}}{n^2}$ and find $\lim_{n\to\infty} \frac{S_{2n+1}}{n^2}$. [4]

25 **(i)** Verify that

$$\frac{n(\mathrm{e}-1)+\mathrm{e}}{n(n+1)\mathrm{e}^{n+1}} = \frac{1}{n\mathrm{e}^{n}} - \frac{1}{(n+1)\mathrm{e}^{n+1}}.$$ [1]

Let $S_N = \sum_{n=1}^{N} \frac{n(\mathrm{e}-1)+\mathrm{e}}{n(n+1)\mathrm{e}^{n+1}}$.

(ii) Express S_N in terms of N and e. [2]

Let $S = \lim_{N \to \infty} S_N$.

(iii) Find the least value of N such that $(N + 1)(S - S_N) < 10^{-3}$. [3]

26 Let $u_n = \dfrac{4\sin\left(n - \frac{1}{2}\right)\sin\frac{1}{2}}{\cos(2n-1) + \cos 1}$.

(i) Using the formulae for $\cos P \pm \cos Q$ given in the List of Formulae MF10, show that

$$u_n = \frac{1}{\cos n} - \frac{1}{\cos(n-1)}.$$ [2]

(ii) Use the method of differences to find $\sum_{n=1}^{N} u_n$. [2]

(iii) Explain why the infinite series $u_1 + u_2 + u_3 + \dots$ does not converge. [1]

27 **(i)** Use the method of differences to show that $\sum_{r=1}^{N} \frac{1}{(3r+1)(3r-2)} = \frac{1}{3} - \frac{1}{3(3N+1)}$. [4]

(ii) Find the limit, as $N \to \infty$, of $\displaystyle\sum_{r=N+1}^{N^2} \frac{N}{(3r+1)(3r-2)}$. [4]

28 Given that

$$u_n = \frac{1}{n^2 - n + 1} - \frac{1}{n^2 + n + 1},$$

find $S_N = \sum_{n=N+1}^{2N} u_n$ in terms of N. [3]

Find a number M such that $S_N < 10^{-20}$ for all $N > M$. [3]

29 Let

$$S_N = \sum_{n=1}^{N} (-1)^{n-1} n^3.$$

Find S_{2N} in terms of N, simplifying your answer as far as possible. [4]

Hence write down an expression for S_{2N+1} and find the limit, as $N \to \infty$, of $\frac{S_{2N+1}}{N^3}$. [3]

30 Show that $\left(n + \frac{1}{2}\right)^3 - \left(n - \frac{1}{2}\right)^3 \equiv 3n^2 + \frac{1}{4}$. [1]

Use this result to prove that $\sum_{n=1}^{N} n^2 = \frac{1}{6}N(N+1)(2N+1)$. [2]

The sums S, T and U are defined as follows:

$$S = 1^2 + 2^2 + 3^2 + 4^2 + \ldots + (2N)^2 + (2N+1)^2,$$
$$T = 1^2 + 3^2 + 5^2 + 7^2 + \ldots + (2N-1)^2 + (2N+1)^2,$$
$$U = 1^2 - 2^2 + 3^2 - 4^2 + \ldots - (2N)^2 + (2N+1)^2.$$

Find and simplify expressions in terms of N for each of S, T and U. [5]

Hence

(i) describe the behaviour of $\frac{S}{T}$ as $N \to \infty$, [1]

(ii) prove that if $\frac{S}{U}$ is an integer then $\frac{T}{U}$ is an integer. [3]

31 Verify that if

$$v_n = n(n+1)(n+2)\ \dots\ (n+m),$$

then

$$v_{n+1} - v_n = (m+1)(n+1)(n+2)\ \dots\ (n+m).$$ [2]

Given now that

$$u_n = (n+1)(n+2)\ \dots\ (n+m),$$

find $\sum_{n=1}^{N} u_n$ in terms of m and N. [3]

32 Express

$$\frac{2n+3}{n(n+1)}$$

in partial fractions and hence use the method of differences to find

$$\sum_{n=1}^{N} \frac{2n+3}{n(n+1)} \left(\frac{1}{3}\right)^{n+1}$$

in terms of N. [4]

Deduce the value of

$$\sum_{n=1}^{\infty} \frac{2n+3}{n(n+1)} \left(\frac{1}{3}\right)^{n+1}.$$ [1]

33 Use the method of differences to find S_N, where

$$S_N = \sum_{n=1}^{N} \frac{1}{n(n+2)}.$$ [4]

Deduce the value of $\lim_{N \to \infty} S_N$. [1]

34 Verify that $\dfrac{1}{n^2} - \dfrac{1}{(n+1)^2} = \dfrac{2n+1}{n^2(n+1)^2}$. [1]

Let $S_N = \sum_{r=1}^{N} \dfrac{2r+1}{r^2(r+1)^2}$. Express S_N in terms of N. [2]

Let $S = \lim_{N \to \infty} S_N$. Find the least value of N such that $S - S_N < 10^{-16}$. [3]

35 Let $f(r) = r(r+1)(r+2)$. Show that

$$f(r) - f(r-1) = 3r(r+1).$$ [1]

Hence show that $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$. [2]

Using the standard result for $\sum_{r=1}^{n} r$, deduce that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$. [2]

Find the sum of the series

$$1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + 5^2 + 2 \times 6^2 + \ldots + 2(n-1)^2 + n^2,$$

where n is odd. [3]

36 Show that $\sum_{r=n+1}^{2n} r^2 = \frac{1}{6}n(2n+1)(7n+1)$. [4]

37 It is given that

$$S_n = \sum_{r=1}^{n} u_r = 2n^2 + n.$$

Write down the values of S_1, S_2, S_3, S_4. Express u_r in terms of r, justifying your answer. [4]

Find

$$\sum_{r=n+1}^{2n} u_r.$$ [3]

38 Express $\frac{1}{r(r+1)(r-1)}$ in partial fractions. [1]

Find

$$\sum_{r=2}^{n} \frac{1}{r(r+1)(r-1)}.$$ [4]

State the value of

$$\sum_{r=2}^{\infty} \frac{1}{r(r+1)(r-1)}.$$ [1]

39 Given that

$$u_k = \frac{1}{\sqrt{(2k-1)}} - \frac{1}{\sqrt{(2k+1)}},$$

express $\sum_{k=13}^{n} u_k$ in terms of n. [4]

Deduce the value of $\sum_{k=13}^{\infty} u_k$. [1]

40 The sequence $a_1, a_2, a_3, \ldots$ is such that, for all positive integers n,

$$a_n = \frac{n+5}{\sqrt{(n^2-n+1)}} - \frac{n+6}{\sqrt{(n^2+n+1)}}.$$

The sum $\sum_{n=1}^{N} a_n$ is denoted by S_N. Find

(i) the value of S_{30} correct to 3 decimal places, [3]

(ii) the least value of N for which $S_N > 4.9$. [4]

41 Use the method of differences to find $\displaystyle\sum_{r=1}^{n} \frac{1}{(2r)^2 - 1}$. [4]

Deduce the value of $\displaystyle\sum_{r=1}^{\infty} \frac{1}{(2r)^2 - 1}$. [1]

42 Find $\sum_{r=1}^{n}(4r-3)(4r+1)$, giving your answer in its simplest form. [4]

43 **(i)** By considering $(2r + 1)^2 - (2r - 1)^2$, use the method of differences to prove that

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n + 1).$$ [3]

(ii) By considering $(2r+1)^4 - (2r-1)^4$, use the method of differences and the result given in part **(i)** to prove that

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2.$$ [5]

The sums S and T are defined as follows:

$$S = 1^3 + 2^3 + 3^3 + 4^3 + \dots + (2N)^3 + (2N+1)^3,$$
$$T = 1^3 + 3^3 + 5^3 + 7^3 + \dots + (2N-1)^3 + (2N+1)^3.$$

(iii) Use the result given in part **(ii)** to show that $S = (2N+1)^2(N+1)^2$. [1]

(iv) Hence, or otherwise, find an expression in terms of N for T, factorising your answer as far as possible. [2]

(v) Deduce the value of $\dfrac{S}{T}$ as $N \to \infty$. [2]

44 Let

$$S_N = \sum_{r=1}^{N} (3r+1)(3r+4) \quad \text{and} \quad T_N = \sum_{r=1}^{N} \frac{1}{(3r+1)(3r+4)}.$$

(i) Use standard results from the List of Formulae (MF10) to show that

$$S_N = N(3N^2 + 12N + 13).$$ [3]

(ii) Use the method of differences to show that

$$T_N = \frac{1}{12} - \frac{1}{3(3N+4)}.$$ [3]

(iii) Deduce that $\dfrac{S_N}{T_N}$ is an integer. [2]

(iv) Find $\lim\limits_{N\to\infty} \dfrac{S_N}{N^3 T_N}$. [2]

45 Let $S_N = \sum_{r=1}^{N} (5r+1)(5r+6)$ and $T_N = \sum_{r=1}^{N} \frac{1}{(5r+1)(5r+6)}$.

(i) Use standard results from the List of Formulae (MF10) to show that

$$S_N = \tfrac{1}{3}N(25N^2 + 90N + 83).$$ [3]

(ii) Use the method of differences to express T_N in terms of N. [4]

(iii) Find $\lim_{N \to \infty} \left(N^{-3} S_N T_N\right)$. [2]

46 **(a)** Use standard results from the List of Formulae (MF19) to show that

$$\sum_{r=1}^{n}(7r+1)(7r+8) = an^3 + bn^2 + cn,$$

where a, b and c are constants to be determined. [3]

(b) Use the method of differences to find $\sum_{r=1}^{n} \frac{1}{(7r+1)(7r+8)}$ in terms of n. [4]

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{(7r+1)(7r+8)}$. [1]

47 **(a)** By simplifying $\left(x^n - \sqrt{x^{2n}+1}\right)\left(x^n + \sqrt{x^{2n}+1}\right)$ show $\dfrac{1}{x^n - \sqrt{x^{2n}+1}} = -x^n - \sqrt{x^{2n}+1}$. [1]

Let $u_n = x^{n+1} + \sqrt{x^{2n+2}+1} + \dfrac{1}{x^n - \sqrt{x^{2n}+1}}$.

(b) Use the method of differences to find $\sum_{n=1}^{N} u_n$ in terms of N and x. [3]

(c) Deduce the set of values of x for which the infinite series

$$u_1 + u_2 + u_3 + \dots$$

is convergent and give the sum to infinity when this exists. [3]

48 **(a)** By first expressing $\frac{1}{r^2-1}$ in partial fractions, show that

$$\sum_{r=2}^{n} \frac{1}{r^2-1} = \frac{3}{4} - \frac{an+b}{2n(n+1)},$$

where a and b are integers to be found. [5]

(b) Deduce the value of $\displaystyle\sum_{r=2}^{\infty} \frac{1}{r^2 - 1}$. [1]

(c) Find the limit, as $n \to \infty$, of $\displaystyle\sum_{r=n+1}^{2n} \frac{n}{r^2 - 1}$. [4]

49 Let $S_n = 2^2 + 6^2 + 10^2 + \ldots + (4n-2)^2$.

(a) Use standard results from the List of Formulae (MF19) to show that $S_n = \frac{4}{3}n(4n^2 - 1)$. [4]

(b) Express $\dfrac{n}{S_n}$ in partial fractions and find $\displaystyle\sum_{n=1}^{N} \frac{n}{S_n}$ in terms of N. [4]

(c) Deduce the value of $\displaystyle\sum_{n=1}^{\infty} \frac{n}{S_n}$. [1]

Name: ____________________ **Date:** ____________________ **Class:** ____________________

Past Paper Questions: Summation of Series

1 Let a be a positive constant.

(a) Use the method of differences to find $\sum_{r=1}^{n} \frac{1}{(ar+1)(ar+a+1)}$ in terms of n and a. [4]

(b) Find the value of a for which $\sum_{r=1}^{\infty} \frac{1}{(ar+1)(ar+a+1)} = \frac{1}{6}$. [3]

2 Let $u_r = \mathrm{e}^{rx}(\mathrm{e}^{2x} - 2\mathrm{e}^{x} + 1)$.

(a) Using the method of differences, or otherwise, find $\sum_{r=1}^{n} u_r$ in terms of n and x. [3]

(b) Deduce the set of non-zero values of x for which the infinite series

$$u_1 + u_2 + u_3 + \ldots$$

is convergent and give the sum to infinity when this exists. [3]

(c) Using a standard result from the list of formulae (MF19), find $\sum_{r=1}^{n} \ln u_r$ in terms of n and x. [3]

3 **(a)** Use standard results from the List of formulae (MF19) to find $\sum_{r=1}^{n}\left(1-r-r^2\right)$ in terms of n, simplifying your answer. [3]

(b) Show that

$$\frac{1-r-r^2}{\left(r^2+2r+2\right)\left(r^2+1\right)} = \frac{r+1}{\left(r+1\right)^2+1} - \frac{r}{r^2+1}$$

and hence use the method of differences to find $\sum_{r=1}^{n} \frac{1-r-r^2}{\left(r^2+2r+2\right)\left(r^2+1\right)}$. [5]

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1-r-r^2}{\left(r^2+2r+2\right)\left(r^2+1\right)}$. [1]

4 **(a)** Show that

$$\tan(r+1) - \tan r = \frac{\sin 1}{\cos(r+1)\cos r}.$$ [2]

Let $u_r = \dfrac{1}{\cos(r+1)\cos r}$.

(b) Use the method of differences to find $\sum_{r=1}^{n} u_r$. [3]

(c) Explain why the infinite series $u_1 + u_2 + u_3 + \ldots$ does not converge. [1]

5 **(a)** Use standard results from the list of formulae (MF19) to find $\sum_{r=1}^{n} r(r+1)(r+2)$ in terms of n, fully factorising your answer. [3]

(b) Express $\frac{1}{r(r+1)(r+2)}$ in partial fractions and hence use the method of differences to find

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}.$$ [5]

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)}$. [1]

6 Let $S_n = \sum_{r=1}^{n} \ln \frac{r(r+2)}{(r+1)^2}$.

(a) Using the method of differences, or otherwise, show that $S_n = \ln \frac{n+2}{2(n+1)}$. [4]

Let $S = \sum_{r=1}^{\infty} \ln \frac{r(r+2)}{(r+1)^2}$.

(b) Find the least value of n such that $S_n - S < 0.01$. [3]