Past Paper Questions: Polar Curves

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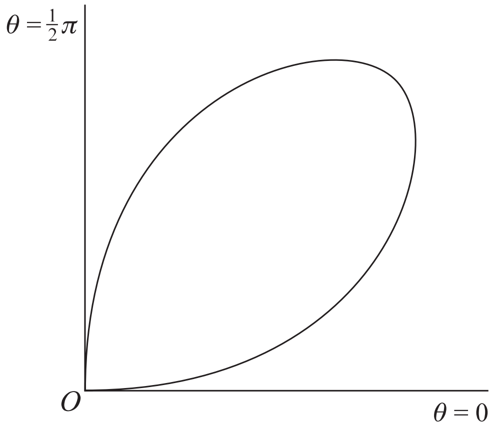

The diagram shows one loop of the curve whose polar equation is $r = a \sin 2\theta$, where a is a positive constant. Find the area of the loop, giving your answer in terms of a and π. [4]

2 The curve C has equation

$$(x^2 + y^2)^2 = 4xy.$$

(i) Show that the polar equation of C is $r^2 = 2\sin 2\theta$. [2]

(ii) Draw a sketch of C, indicating any lines of symmetry as well as the form of C at the pole. [5]

(iii) Write down the maximum possible distance of a point of C from the pole. [1]

3 The curve C has polar equation

$$r = \frac{\pi - \theta}{\theta},$$

where $\frac{1}{2}\pi \leqslant \theta \leqslant \pi$.

(i) Draw a sketch of C. [3]

(ii) Show that the area of the region bounded by the line $\theta = \frac{1}{2}\pi$ and C is

$$\pi\left(\tfrac{3}{4} - \ln 2\right).$$ [5]

4 Draw a diagram to illustrate the region R which is bounded by the curve whose polar equation is $r = \cos 2\theta$ and the lines $\theta = 0$ and $\theta = \frac{1}{6}\pi$. [2]

Determine the exact area of R. [4]

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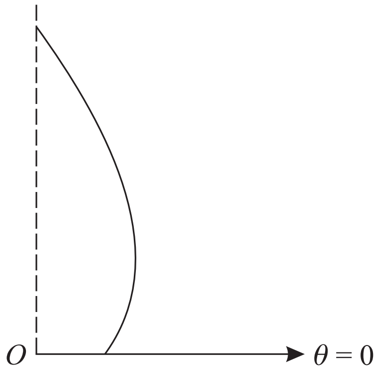


The diagram shows the curve C with polar equation $r = e^{\theta}$, where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. Find the maximum distance of a point of C from the line $\theta = \frac{1}{2}\pi$, giving the answer in exact form. [4]

Find the area of the finite region bounded by C and the lines $\theta = 0$ and $\theta = \frac{1}{2}\pi$, giving the answer in exact form. [3]

6 The curves C_1 and C_2 have polar equations

$$r = \theta + 2 \quad \text{and} \quad r = \theta^2$$

respectively, where $0 \leqslant \theta \leqslant \pi$.

(i) Find the polar coordinates of the point of intersection of C_1 and C_2. [2]

(ii) Sketch C_1 and C_2 on the same diagram. [2]

(iii) Find the area bounded by C_1, C_2 and the line $\theta = 0$. [3]

7 Draw a sketch of the curve C whose polar equation is $r = \theta$, for $0 \leqslant \theta \leqslant \dfrac{\pi}{2}$ [2]

On the same diagram draw the line $\theta = \alpha$, where $0 < \alpha < \frac{1}{2}\pi$. [1]

The region bounded by C and the line $\theta = \frac{1}{2}\pi$ is denoted by R. Find the exact value of α for which the line $\theta = \alpha$ divides R into two regions of equal area. [4]

8 The curve C has polar equation

$$r = a\left(1 - e^{-\theta}\right),$$

where a is a positive constant and $0 \leqslant \theta < 2\pi$.

(i) Draw a sketch of C. [3]

(ii) Show that the area of the region bounded by C and the lines $\theta = \ln 2$ and $\theta = \ln 4$ is

$$\tfrac{1}{2}a^2\left(\ln 2 - \tfrac{13}{32}\right).$$ [4]

9 The curve C has polar equation

$$r = \frac{a}{1 + \theta},$$

where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(i) Show that r decreases as θ increases. [2]

(ii) The point P of C is further from the initial line than any other point of C. Show that, at P,

$$\tan \theta = 1 + \theta,$$

and verify that this equation has a root between 1.1 and 1.2. [4]

(iii) Draw a sketch of C. [3]

(iv) Find the area of the region bounded by the initial line, the line $\theta = \frac{1}{2}\pi$ and C, leaving your answer in terms of π and a. [3]

10 The curve C has polar equation $r = 2\cos 2\theta$. Sketch the curve for $0 \leqslant \theta < 2\pi$. [4]

Find the exact area of one loop of the curve. [4]

11 The curves C_1 and C_2 have polar equations

$$C_1: \quad r = a,$$
$$C_2: \quad r = 2a\cos 2\theta, \text{ for } 0 \leqslant \theta \leqslant \tfrac{1}{4}\pi,$$

where a is a positive constant. Sketch C_1 and C_2 on the same diagram. [3]

The curves C_1 and C_2 intersect at the point with polar coordinates (a, β). State the value of β. [1]

Show that the area of the region bounded by the initial line, the arc of C_1 from $\theta = 0$ to $\theta = \beta$, and the arc of C_2 from $\theta = \beta$ to $\theta = \frac{1}{4}\pi$ is

$$a^2\left(\tfrac{1}{6}\pi - \tfrac{1}{8}\sqrt{3}\right).$$ [4]

12 The curve C has polar equation $r = 2 + 2\cos\theta$, for $0 \leqslant \theta \leqslant \pi$. Sketch the graph of C. [2]

Find the area of the region R enclosed by C and the initial line. [4]

The half-line $\theta = \frac{1}{5}\pi$ divides R into two parts. Find the area of each part, correct to 3 decimal places. [3]

13 The curve C has cartesian equation

$$(x^2 + y^2)^2 = a^2(x^2 - y^2),$$

where a is a positive constant. Show that C has polar equation

$$r^2 = a^2 \cos 2\theta.$$ [2]

Sketch C for $-\pi < \theta \leqslant \pi$. [2]

Find the area of the sector between $\theta = -\frac{1}{4}\pi$ and $\theta = \frac{1}{4}\pi$. [3]

Find the polar coordinates of all points of C where the tangent is parallel to the initial line. [7]

14 Find the area of the region enclosed by the curve with polar equation $r = 2(1 + \cos\theta)$, for $0 \leqslant \theta < 2\pi$. [4]

15 Use the identity $2\sin P\cos Q \equiv \sin(P+Q) + \sin(P-Q)$ to show that

$$2\sin\theta\cos(\theta - \tfrac{1}{4}\pi) \equiv \cos(2\theta - \tfrac{3}{4}\pi) + \frac{1}{\sqrt{2}}.$$ [3]

A curve has polar equation $r = 2\sin\theta\cos(\theta - \frac{1}{4}\pi)$, for $0 \leqslant \theta \leqslant \frac{3}{4}\pi$. Sketch the curve and state the polar equation of its line of symmetry, justifying your answer. [3]

Show that the area of the region enclosed by the curve is $\frac{3}{8}(\pi + 1)$. [6]

16 The curve C has polar equation $r = a(1 + \sin\theta)$, where a is a positive constant and $0 \leqslant \theta < 2\pi$. Draw a sketch of C. [2]

Find the exact value of the area of the region enclosed by C and the half-lines $\theta = \frac{1}{3}\pi$ and $\theta = \frac{2}{3}\pi$. [4]

17 The curve C has cartesian equation $(x^2+y^2)^2 = 2a^2xy$, where a is a positive constant. Show that the polar equation of C is $r^2 = a^2 \sin 2\theta$. [3]

Sketch C for $-\pi < \theta \leqslant \pi$. [2]

Find the area enclosed by one loop of C. [2]

18 The curves C_1 and C_2 have polar equations

$$C_1: \quad r = \frac{1}{\sqrt{2}}, \quad \text{for } 0 \leqslant \theta < 2\pi,$$
$$C_2: \quad r = \sqrt{\left(\sin \tfrac{1}{2}\theta\right)}, \quad \text{for } 0 \leqslant \theta \leqslant \pi.$$

Find the polar coordinates of the point of intersection of C_1 and C_2. [2]

Sketch C_1 and C_2 on the same diagram. [3]

Find the exact value of the area of the region enclosed by C_1, C_2 and the half-line $\theta = 0$. [4]

19 The curve C has polar equation $r = \mathrm{e}^{4\theta}$ for $0 \leqslant \theta \leqslant \alpha$, where α is measured in radians. The length of C is 2015. Find the value of α. [6]

20 A curve C has polar equation $r^2 = 8 \operatorname{cosec} 2\theta$ for $0 < \theta < \frac{\pi}{2}$. Find a cartesian equation of C. [3]

Sketch C. [2]

Determine the exact area of the sector bounded by the arc of C between $\theta = \frac{1}{6}\pi$ and $\theta = \frac{1}{3}\pi$, the half-line $\theta = \frac{1}{6}\pi$ and the half-line $\theta = \frac{1}{3}\pi$. [3]

[It is given that $\int \operatorname{cosec} x \, dx = \ln \left|\tan \frac{1}{2}x\right| + c$.]

21 A curve has polar equation $r = \dfrac{1}{1 - \cos\theta}$, for $0 < \theta < 2\pi$. Find, in the form $y^2 = \mathrm{f}(x)$, the cartesian equation of the curve. [3]

Hence sketch the curve, and shade the region whose area is given by $\dfrac{1}{2}\displaystyle\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \dfrac{1}{(1 - \cos\theta)^2}\,\mathrm{d}\theta$. [3]

Using the cartesian equation of the curve, find the area of this region. [3]

22 The curve C has polar equation $r = a(1 + \sin\theta)$ for $-\pi < \theta \leqslant \pi$, where $a$is a positive constant.

(i) Sketch C. [2]

(ii) Find the area of the region enclosed by C. [4]

(iii) Show that the length of the arc of C from the pole to the point furthest from the pole is given by

$$s = (\sqrt{2})a \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sqrt{(1 + \sin\theta)}\, d\theta.$$ [3]

(iv) Show that the substitution $u = 1 + \sin\theta$ reduces this integral for s to $(\sqrt{2})a \int_0^2 \frac{1}{\sqrt{(2-u)}}\,\mathrm{d}u$. Hence evaluate s. [4]

23 A curve C has polar equation $r = 2a\cos\left(2\theta + \frac{1}{2}\pi\right)$ for $0 \leqslant \theta < 2\pi$, where a is a positive constant.

(i) Show that $r = -2a\sin 2\theta$ and sketch C.

[4]

(ii) Deduce that the cartesian equation of C is

$$(x^2 + y^2)^{\frac{3}{2}} = -4axy.$$

[2]

(iii) Find the area of one loop of C. [5]

(iv) Show that, at the points (other than the pole) at which a tangent to C is parallel to the initial line,

$$2\tan\theta = -\tan 2\theta.$$

[3]

24 The curve C has polar equation $r = \cos 2\theta$, for $-\frac{1}{4}\pi \leqslant \theta \leqslant \frac{1}{4}\pi$

(i) Sketch C. [2]

(ii) Find the area of the region enclosed by C, showing full working. [3]

(iii) Find a cartesian equation of C. [3]

25 The curves C_1 and C_2 have polar equations, for $0 \leqslant \theta \leqslant \pi$, as follows:

$$C_1\colon r = a,$$
$$C_2\colon r = 2a|\cos\theta|,$$

where a is a positive constant. The curves intersect at the points P_1 and P_2.

(i) Find the polar coordinates of P_1 and P_2. [2]

(ii) In a single diagram, sketch C_1, C_2 and their line of symmetry. [3]

(iii) The region R enclosed by C_1 and C_2 is bounded by the arcs OP_1, P_1P_2 and P_2O, where O is the pole. Find the area of R, giving your answer in exact form. [5]

26 The curve C_1 has polar equation $r^2 = 2\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(i) The point on C_1 furthest from the line $\theta = \frac{1}{2}\pi$ is denoted by P. Show that, at P,

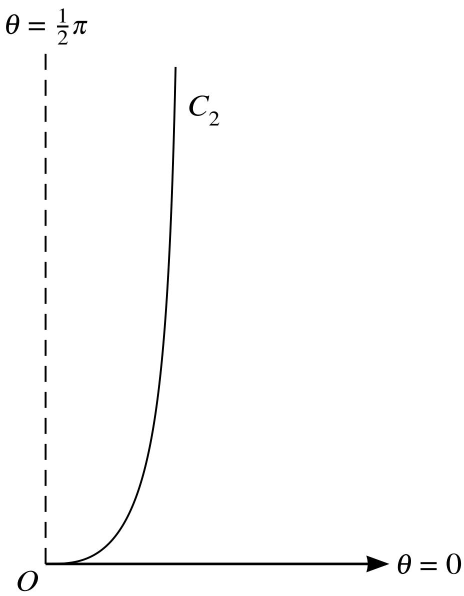
$$2\theta \tan \theta = 1$$

and verify that this equation has a root between 0.6 and 0.7. [5]

The curve C_2 has polar equation $r^2 = \theta \sec^2 \theta$, for $0 \leqslant \theta < \frac{1}{2}\pi$. The curves C_1 and C_2 intersect at the pole, denoted by O, and at another point Q.

(ii) Find the exact value of θ at Q. [2]

(iii) The diagram below shows the curve C_2. Sketch C_1 on this diagram. [2]



(iv) Find, in exact form, the area of the region OPQ enclosed by C_1 and C_2. [5]

27 The curve C has polar equation $r^2 = \ln(1 + \theta)$, for $0 \leqslant \theta \leqslant 2\pi$

. **(i)** Sketch C. [2]

(ii) Using the substitution $u = 1 + \theta$, or otherwise, find the area of the region bounded by C and the initial line, leaving your answer in an exact form. [5]

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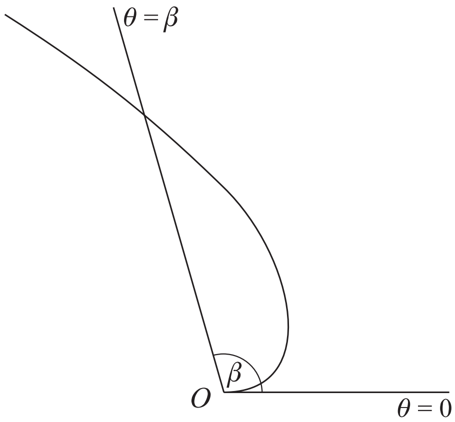


The curve C has polar equation

$$r = \theta^{\frac{1}{2}} e^{\theta^2/\pi},$$

where $0 \leqslant \theta \leqslant \pi$. The area of the finite region bounded by C and the line $\theta = \beta$ is π (see diagram). Show that

$$\beta = (\pi \ln 3)^{\frac{1}{2}}. \qquad [6]$$

29 The curve C has polar equation

$$r = e^{\frac{1}{5}\theta}, \qquad 0 \leqslant \theta \leqslant \tfrac{3}{2}\pi.$$

(i) Draw a sketch of C. [2]

(ii) Find the length of C, correct to 3 significant figures. [4]

30 The curves C_1 and C_2 have polar equations

$$r = 4\cos\theta \qquad \text{and} \qquad r = 1 + \cos\theta$$

respectively, where $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$.

(i) Show that C_1 and C_2 meet at the points $A\left(\frac{4}{3}, \alpha\right)$ and $B\left(\frac{4}{3}, -\alpha\right)$, where α is the acute angle such that $\cos\alpha = \frac{1}{3}$. [2]

(ii) In a single diagram, draw sketch graphs of C_1 and C_2. [3]

(iii) Show that the area of the region bounded by the arcs OA and OB of C_1, and the arc AB of C_2, is

$$4\pi - \frac{1}{3}\sqrt{2} - \frac{13}{2}\alpha.$$ [7]

31 The curve C has equation

$$r = 10\ln(1 + \theta),$$

where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. Draw a sketch of C. [2]

Use the substitution $w = \ln(1 + \theta)$ to show that the area of the sector bounded by the line $\theta = \frac{1}{2}\pi$ and the arc of C joining the origin to the point where $\theta = \frac{1}{2}\pi$ is

$$50(b^2 - 2b + 2)e^b - 100,$$

where $b = \ln(1 + \frac{1}{2}\pi)$. [6]

32 The curve C has polar equation

$$r = \theta \sin \theta,$$

where $0 \leqslant \theta \leqslant \pi$. Draw a sketch of C. [2]

Find the area of the region enclosed by C, leaving your answer in terms of π. [7]

33 The curve C has polar equation

$$r = \left(\tfrac{1}{2}\pi - \theta\right)^2,$$

where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. Draw a sketch of C. [3]

Find the area of the region bounded by C and the initial line, leaving your answer in terms of π. [3]

34 The curve C has polar equation

$$r = a \sin 3\theta,$$

where $0 \leqslant \theta \leqslant \frac{1}{3}\pi$.

(i) Show that the area of the region enclosed by C is $\frac{1}{12}\pi a^2$. [3]

(ii) Show that, at the point of C at maximum distance from the initial line,

$$\tan 3\theta + 3 \tan \theta = 0.$$ [3]

(iii) Use the formula

$$\tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$$

to find this maximum distance. [4]

(iv) Draw a sketch of C. [2]

35 The curves C_1 and C_2 have polar equations given by

$$C_1: \quad r = 3\sin\theta, \qquad 0 \leqslant \theta < \pi,$$
$$C_2: \quad r = 1 + \sin\theta, \quad -\pi < \theta \leqslant \pi.$$

(i) Find the polar coordinates of the points, other than the pole, where C_1 and C_2 meet. [2]

(ii) In a single diagram, draw sketch graphs of C_1 and C_2. [3]

(iii) Show that the area of the region which is inside C_1 but outside C_2 is π. [5]

36 The curve C has polar equation $r = 3 + 2\cos\theta$, for $-\pi < \theta \leqslant \pi$. The straight line l has polar equation $r\cos\theta = 2$. Sketch both C and l on a single diagram. [3]

Find the polar coordinates of the points of intersection of C and l. [4]

The region R is enclosed by C and l, and contains the pole. Find the area of R. [6]

37 The curve C has polar equation $r = 1 + \sin\theta$ for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$. Draw a sketch of C. [2]

The area of the region enclosed by the initial line, the half-line $\theta = \frac{1}{2}\pi$, and the part of C for which θ is positive, is denoted by A_1. The area of the region enclosed by the initial line, and the part of C for which θ is negative, is denoted by A_2. Find the ratio $A_1 : A_2$, giving your answer correct to 1 decimal place. [8]

38 Find the cartesian equation corresponding to the polar equation $r = (\sqrt{2})\sec(\theta - \frac{\pi}{4})$. [3]

Sketch the the graph of $r = (\sqrt{2})\sec(\theta - \frac{1}{4}\pi)$, for $-\frac{1}{4}\pi < \theta < \frac{3}{4}\pi$, indicating clearly the polar coordinates of the intersection with the initial line. [2]

39 The curve C has polar equation $r = 1 + 2\cos\theta$. Sketch the curve for $-\frac{2}{3}\pi \leqslant \theta < \frac{2}{3}\pi$. [2]

Find the area bounded by C and the half-lines $\theta = -\frac{1}{3}\pi$, $\theta = \frac{1}{3}\pi$. [4]

40 The curve C has polar equation $r = 2e^{\theta}$, for $\frac{1}{6}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$. Find

(i) the area of the region bounded by the half-lines $\theta = \frac{1}{6}\pi$, $\theta = \frac{1}{2}\pi$ and C, [2]

(ii) the length of C. [3]

41 The curve C has polar equation $r = 2\sin\theta(1 - \cos\theta)$, for $0 \leqslant \theta \leqslant \pi$. Find $\frac{dr}{d\theta}$ and hence find the polar coordinates of the point of C that is furthest from the pole. [5]

Sketch C. [2]

Find the exact area of the sector from $\theta = 0$ to $\theta = \frac{1}{4}\pi$. [6]

42 A circle has polar equation $r = a$, for $0 \leqslant \theta < 2\pi$, and a cardioid has polar equation $r = a(1 - \cos\theta)$, for $0 \leqslant \theta < 2\pi$, where a is a positive constant. Draw sketches of the circle and the cardioid on the same diagram. [3]

Write down the polar coordinates of the points of intersection of the circle and the cardioid. [2]

Show that the area of the region that is both inside the circle and inside the cardioid is

$$\left(\tfrac{5}{4}\pi - 2\right)a^2.$$ [6]

43 The curve C has polar equation $r = a(1 - \cos\theta)$ for $0 \leqslant \theta < 2\pi$. Sketch C. [2]

Find the area of the region enclosed by the arc of C for which $\frac{1}{2}\pi \leqslant \theta \leqslant \frac{3}{2}\pi$, the half-line $\theta = \frac{1}{2}\pi$ and the half-line $\theta = \frac{3}{2}\pi$. [5]

Show that

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^2 = 4a^2 \sin^2\left(\tfrac{1}{2}\theta\right),$$

where s denotes arc length, and find the length of the arc of C for which $\frac{1}{2}\pi \leqslant \theta \leqslant \frac{3}{2}\pi$. [7]

44 The polar equation of a curve C is $r = a(1 + \cos\theta)$ for $0 \leqslant \theta < 2\pi$, where a is a positive constant.

(i) Sketch C. [2]

(ii) Show that the cartesian equation of C is

$$x^2 + y^2 = a\big(x + \sqrt{(x^2 + y^2)}\big).$$ [2]

(iii) Find the area of the sector of C between $\theta = 0$ and $\theta = \frac{1}{3}\pi$. [4]

(iv) Find the arc length of C between the point where $\theta = 0$ and the point where $\theta = \frac{1}{3}\pi$. [5]

45 The curve C has polar equation

$$r = 5\sqrt{(\cot \theta)},$$

where $0.01 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(i) Find the area of the finite region bounded by C and the line $\theta = 0.01$, showing full working. Give your answer correct to 1 decimal place. [3]

Let P be the point on C where $\theta = 0.01$.

(ii) Find the distance of P from the initial line, giving your answer correct to 1 decimal place. [2]

(iii) Find the maximum distance of C from the initial line. [3]

(iv) Sketch C. [2]

46 The curve C has polar equation $r = a\cos 3\theta$, for $-\frac{1}{6}\pi \leqslant \theta \leqslant \frac{1}{6}\pi$, where a is a positive constant.

(i) Sketch C. [2]

(ii) Find the area of the region enclosed by C, showing full working. [3]

(iii) Using the identity $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$, find a cartesian equation of C. [3]

47 The curves C_1 and C_2 have polar equations, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$, as follows:

$$C_1 : r = 2\left(e^{\theta} + e^{-\theta}\right),$$
$$C_2 : r = e^{2\theta} - e^{-2\theta}.$$

The curves intersect at the point P where $\theta = \alpha$.

(i) Show that $e^{2\alpha} - 2e^{\alpha} - 1 = 0$. Hence find the exact value of α and show that the value of r at P is $4\sqrt{2}$. [6]

(ii) Sketch C_1 and C_2 on the same diagram. [3]

(iii) Find the area of the region enclosed by C_1, C_2 and the initial line, giving your answer correct to 3 significant figures. [5]

48 **(a)** Show that the curve with Cartesian equation

$$(x^2 + y^2)^{\frac{5}{2}} = 4xy(x^2 - y^2)$$

has polar equation $r = \sin 4\theta$. [4]

The curve C has polar equation $r = \sin 4\theta$, for $0 \leqslant \theta \leqslant \frac{1}{4}\pi$.

(b) Sketch C and state the equation of the line of symmetry. [3]

(c) Find the exact value of the area of the region enclosed by C. [4]

(d) Using the identity $\sin 4\theta \equiv 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta$, find the maximum distance of C from the line $\theta = \frac{1}{2}\pi$. Give your answer correct to 2 decimal places. [6]

49 The curve C has polar equation $r = \ln(1 + \pi - \theta)$, for $0 \leqslant \theta \leqslant \pi$.

(a) Sketch C and state the polar coordinates of the point of C furthest from the pole. [3]

(b) Using the substitution $u = 1 + \pi - \theta$, or otherwise, show that the area of the region enclosed by C and the initial line is

$$\tfrac{1}{2}(1+\pi)\ln(1+\pi)\big(\ln(1+\pi) - 2\big) + \pi.$$ [6]

(c) Show that, at the point of C furthest from the initial line,

$$(1+\pi-\theta)\ln(1+\pi-\theta)-\tan\theta=0$$

and verify that this equation has a root between 1.2 and 1.3. [5]

50 The curve C_1 has polar equation $r = \theta\cos\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) The point on C_1 furthest from the line $\theta = \frac{1}{2}\pi$ is denoted by P. Show that, at P,

$$2\theta\tan\theta - 1 = 0$$

and verify that this equation has a root between 0.6 and 0.7. [5]

The curve C_2 has polar equation $r = \theta\sin\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. The curves C_1 and C_2 intersect at the pole, denoted by O, and at another point Q.

(b) Find the polar coordinates of Q, giving your answers in exact form. [2]

(c) Sketch C_1 and C_2 on the same diagram. [3]

(d) Find, in terms of π, the area of the region bounded by the arc OQ of C_1 and the arc OQ of C_2. [7]

51 The curve C has polar equation $r = a\tan\theta$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{4}\pi$.

(a) Sketch C and state the greatest distance of a point on C from the pole. [2]

(b) Find the exact value of the area of the region bounded by C and the half-line $\theta = \frac{1}{4}\pi$. [4]

(c) Show that C has Cartesian equation $y = \dfrac{x^2}{\sqrt{a^2 - x^2}}$. [3]

(d) Using your answer to part **(b)**, deduce the exact value of $\displaystyle\int_0^{\frac{1}{2}a\sqrt{2}} \frac{x^2}{\sqrt{a^2 - x^2}}\,dx$. [2]

52 The curve C has polar equation $r = a\cot\left(\frac{1}{3}\pi - \theta\right)$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{6}\pi$.

It is given that the greatest distance of a point on C from the pole is $2\sqrt{3}$.

(a) Sketch C and show that $a = 2$. [3]

(b) Find the exact value of the area of the region bounded by C, the initial line and the half-line $\theta = \frac{1}{6}\pi$. [4]

(c) Show that C has Cartesian equation $2\left(x+y\sqrt{3}\right)=\left(x\sqrt{3}-y\right)\sqrt{x^2+y^2}$. [3]

53 The curve C has polar equation $r = \dfrac{1}{\pi - \theta} - \dfrac{1}{\pi}$, where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Sketch C. [3]

(b) Show that the area of the region bounded by the half-line $\theta = \frac{1}{2}\pi$ and C is $\dfrac{3 - 4\ln 2}{4\pi}$. [6]

54 The curve C has polar equation $r = 2\cos\theta(1 + \sin\theta)$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

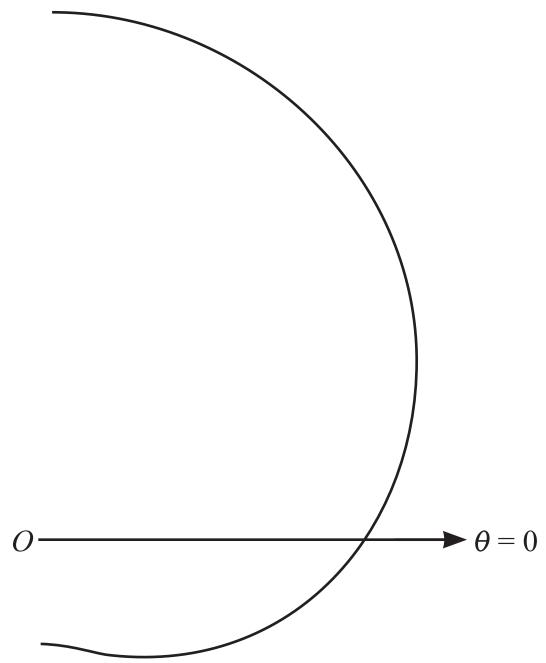
(a) Find the polar coordinates of the point on C that is furthest from the pole. [5]

(b) Sketch C. [2]

(c) Find the area of the region bounded by C and the initial line, giving your answer in exact form. [6]

55 The curve C has polar equation $r = 3 + 2\sin\theta$, for $-\pi < \theta \leqslant \pi$.

(a) The diagram shows part of C. Sketch the rest of C on the diagram. [1]



The straight line l has polar equation $r\sin\theta = 2$.

(b) Add l to the diagram in part **(a)** and find the polar coordinates of the points of intersection of C and l. [5]

(c) The region R is enclosed by C and l, and contains the pole.

Find the area of R, giving your answer in exact form. [6]

56 The curve C has polar equation $r^2 = \tan^{-1}\left(\frac{1}{2}\theta\right)$, where $0 \leqslant \theta \leqslant 2$.

(a) Sketch C and state, in exact form, the greatest distance of a point on C from the pole. [3]

(b) Find the exact value of the area of the region bounded by C and the half-line $\theta = 2$. [5]

Now consider the part of C where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(c) Show that, at the point furthest from the half-line $\theta = \frac{1}{2}\pi$,

$$(\theta^2 + 4)\tan^{-1}\left(\frac{1}{2}\theta\right)\sin\theta - \cos\theta = 0$$

and verify that this equation has a root between 0.6 and 0.7. [5]

57 The curve C has Cartesian equation $x^2+xy+y^2=a$, where a is a positive constant.

(a) Show that the polar equation of C is $r^2=\dfrac{2a}{2+\sin 2\theta}$. [3]

(b) Sketch the part of C for $0 \leqslant \theta \leqslant \frac{1}{4}\pi$. [2]

The region R is enclosed by this part of C, the initial line and the half-line $\theta = \frac{1}{4}\pi$.

(c) It is given that $\sin 2\theta$ may be expressed as $\dfrac{2\tan\theta}{1+\tan^2\theta}$. Use this result to show that the area of R is

$$\tfrac{1}{2}a\int_0^{\frac{1}{4}\pi} \frac{1+\tan^2\theta}{1+\tan\theta+\tan^2\theta}\,\mathrm{d}\theta$$

and use the substitution $t = \tan\theta$ to find the exact value of this area. [8]