Past Paper Questions: Vectors

1 Find the acute angle between the planes with equations

$$x - 2y + z - 9 = 0 \qquad \text{and} \qquad x + y - z + 2 = 0. \qquad [3]$$

The planes meet in the line l, and A is the point on l whose position vector is $p\mathbf{i} + q\mathbf{j} + \mathbf{k}$.

(i) Find p and q. [2]

(ii) Find a vector equation for l. [3]

The non-coincident planes Π_1 and Π_2 are both perpendicular to l. The perpendicular distance from A to Π_1 is $\sqrt{14}$ and the perpendicular distance from A to Π_2 is also $\sqrt{14}$. Find equations for Π_1 and Π_2 in the form $ax + by + cz = d$. [5]

2 **(i)** Find the acute angle beween the line l whose equation is

$$\mathbf{r} = s(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

and the plane Π_1 whose equation is

$$x - z = 0.$$ [3]

(ii) Find, in the form $ax + by + cz = 0$, the equation of the plane Π_2 which contains l and is perpendicular to Π_1. [3]

(iii) Find a vector equation of the line of intersection of the planes Π_1 and Π_2 and hence, or otherwise, show that the vectors $\mathbf{i} - \mathbf{k}$, $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $3\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ are linearly dependent. [3]

(iv) The variable line m passes through the point with position vector $4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$ and is perpendicular to l. The line m meets Π_1 at Q. Find the minimum distance of Q from the origin, as m varies, giving your answer correct to 3 significant figures. [5]

3 The points A, B, C have position vectors $a\mathbf{i}$, $b\mathbf{j}$, $c\mathbf{k}$ respectively, where a, b, c are all positive. The plane containing A, B, C is denoted by Π.

(i) Find a vector perpendicular to Π. [3]

(ii) Find the perpendicular distance from the origin to Π, in terms of a, b, c. [3]

4 The planes Π_1 and Π_2 have equations

$$x + 2y - 3z + 4 = 0 \qquad \text{and} \qquad 2x + y - 4z - 3 = 0$$

respectively. Show that, for all values of λ, every point which is in both Π_1 and Π_2 is also in the plane

$$x + 2y - 3z + 4 + \lambda(2x + y - 4z - 3) = 0.$$ [2]

The planes Π_1 and Π_2 meet in the line l.

(i) Find the equation of the plane Π_3 which passes through l and the point whose position vector is $a\mathbf{k}$. [3]

(ii) Find the value of a if Π_2 is perpendicular to Π_3. [3]

5 The equation of the plane Π is

$$2x + 3y + 4z = 48.$$

Obtain a vector equation of Π in the form

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c},$$

where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are of the form $p\mathbf{i}$, $q\mathbf{i} + r\mathbf{j}$ and $s\mathbf{i} + t\mathbf{k}$ respectively, and where p, q, r, s, t are integers. [6]

The line l has vector equation $\mathbf{r} = 29\mathbf{i} - 2\mathbf{j} - \mathbf{k} + \theta(5\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})$. Show that l lies in Π. [3]

Find, in the form $ax + by + cz = d$, the equation of the plane which contains l and is perpendicular to Π. [4]

6 The line l_1 passes through the points P and Q whose position vectors are $\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $-2\mathbf{i} + 5\mathbf{j} + 13\mathbf{k}$ respectively. The line l_2 passes through the point S whose position vector is $\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}$ and is parallel to the vector $\mathbf{i} - \mathbf{j} - 3\mathbf{k}$. The point whose position vector is $-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ is on the line l_3, the common perpendicular to l_1 and l_2.

(i) Find, in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, an equation for l_3. [3]

(ii) Find the perpendicular distance from S to l_3. [4]

(iii) Find the perpendicular distance from S to the plane which contains l_3 and passes through P. [4]

7 The position vectors of the points A, B, C, D are

$$7\mathbf{i} + 4\mathbf{j} - \mathbf{k}, \qquad 3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}, \qquad 2\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}, \qquad 2\mathbf{i} + 7\mathbf{j} + \lambda\mathbf{k}$$

respectively. It is given that the shortest distance between the line AB and the line CD is 3.

(i) Show that $\lambda^2 - 5\lambda + 4 = 0$. [7]

(ii) Find the acute angle between the planes through A, B, D corresponding to the values of λ satisfying the equation in part **(i)**. [7]

8 The line l_1 is parallel to the vector $4\mathbf{j} - \mathbf{k}$ and passes through the point A whose position vector is $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$. The variable line l_2 is parallel to the vector $\mathbf{i} - (2\sin t)\mathbf{j}$, where $0 \leqslant t < 2\pi$, and passes through the point B whose position vector is $\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. The points P and Q are on l_1 and l_2, respectively, and PQ is perpendicular to both l_1 and l_2.

(i) Find the length of PQ in terms of t. [5]

(ii) Hence find the values of t for which l_1 and l_2 intersect. [2]

(iii) For the case $t = \frac{1}{4}\pi$, find the perpendicular distance from A to the plane BPQ, giving your answer correct to 3 decimal places. [5]

9 The lines l_1 and l_2 have vector equations

$$\mathbf{r} = 4\mathbf{i} - 2\mathbf{j} + \lambda(2\mathbf{i} + \mathbf{j} - 4\mathbf{k}) \qquad \text{and} \qquad \mathbf{r} = 4\mathbf{i} - 5\mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} - \mathbf{j} - \mathbf{k})$$

respectively.

(i) Show that l_1 and l_2 intersect. [3]

(ii) Find the perpendicular distance from the point P whose position vector is $3\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$ to the plane containing l_1 and l_2. [3]

(iii) Find the perpendicular distance from P to l_1. [4]

10 The line l_1 passes through the point A whose position vector is $3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and is parallel to the vector $\mathbf{i} + \mathbf{j}$. The line l_2 passes through the point B whose position vector is $-\mathbf{i} - \mathbf{k}$ and is parallel to the vector $\mathbf{j} + 2\mathbf{k}$. The point P is on l_1 and the point Q is on l_2 and PQ is perpendicular to both l_1 and l_2.

(i) Find the length of PQ. [4]

(ii) Find the position vector of Q. [5]

(iii) Show that the perpendicular distance from Q to the plane containing AB and the line l_1 is $\sqrt{3}$. [4]

11 The line l_1 passes through the point with position vector $8\mathbf{i} + 8\mathbf{j} - 7\mathbf{k}$ and is parallel to the vector $4\mathbf{i} + 3\mathbf{j}$. The line l_2 passes through the point with position vector $7\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and is parallel to the vector $4\mathbf{i} - \mathbf{k}$. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. In either order,

(i) show that $PQ = 13$,

(ii) find the position vectors of P and Q.

[9]

12 The lines l_1 and l_2 have equations

$$l_1:\ \mathbf{r} = 6\mathbf{i} + 5\mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) \quad \text{and} \quad l_2:\ \mathbf{r} = 6\mathbf{i} + 5\mathbf{j} + 4\mathbf{k} + \mu(4\mathbf{i} + 6\mathbf{j} + \mathbf{k}).$$

Find a cartesian equation of the plane Π containing l_1 and l_2. [4]

Find the position vector of the foot of the perpendicular from the point with position vector $\mathbf{i} + 10\mathbf{j} + 3\mathbf{k}$ to Π. [4]

The line l_3 has equation $\mathbf{r} = \mathbf{i} + 10\mathbf{j} + 3\mathbf{k} + \nu(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$. Find the shortest distance between l_1 and l_3. [5]

13 The position vectors of the points A, B, C, D are

$$2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}, \quad -2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}, \quad \mathbf{i} + 4\mathbf{j} + \mathbf{k}, \quad \mathbf{i} + 5\mathbf{j} + m\mathbf{k},$$

respectively, where m is an integer. It is given that the shortest distance between the line through A and B and the line through C and D is 3. Show that the only possible value of m is 2. [7]

Find the shortest distance of D from the line through A and C. [3]

Show that the acute angle between the planes ACD and BCD is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$. [4]

14 The plane Π_1 has parametric equation

$$\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} - \mathbf{k}) + \mu(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}).$$

Find a cartesian equation of Π_1. [4]

The plane Π_2 has cartesian equation $3x - 2y - 3z = 4$. Find the acute angle between Π_1 and Π_2. [3]

Find a vector equation of the line of intersection of Π_1 and Π_2. [4]

15 The points A, B, C and D have coordinates as follows:

$$A(2, 1, -2), \quad B(4, 1, -1), \quad C(3, -2, -1) \quad \text{and} \quad D(3, 6, 2).$$

The plane Π_1 passes through the points A, B and C. Find a cartesian equation of Π_1. [4]

Find the area of triangle ABC and hence, or otherwise, find the volume of the tetrahedron $ABCD$.
[The volume of a tetrahedron is $\frac{1}{3} \times$ area of base $\times$ perpendicular height.] [6]

The plane Π_2 passes through the points A, B and D. Find the acute angle between Π_1 and Π_2. [4]

16 The line l_1 passes through the point A whose position vector is $4\mathbf{i} + 7\mathbf{j} - \mathbf{k}$ and is parallel to the vector $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. The line l_2 passes through the point B whose position vector is $\mathbf{i} + 7\mathbf{j} + 11\mathbf{k}$ and is parallel to the vector $\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$. The points P on l_1 and Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Find the position vectors of P and Q. [8]

Find the shortest distance between the line through A and B and the line through P and Q, giving your answer correct to 3 significant figures. [6]

17 The line l_1 passes through the points $A(2, 3, -5)$ and $B(8, 7, -13)$. The line l_2 passes through the points $C(-2, 1, 8)$ and $D(3, -1, 4)$. Find the shortest distance between the lines l_1 and l_2. [5]

The plane Π_1 passes through the points A, B and D. The plane Π_2 passes though the points A, C and D. Find the acute angle between Π_1 and Π_2, giving your answer in degrees. [6]

18 With respect to an origin O, the point A has position vector $4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and the plane Π_1 has equation

$$\mathbf{r} = (4 + \lambda + 3\mu)\mathbf{i} + (-2 + 7\lambda + \mu)\mathbf{j} + (2 + \lambda - \mu)\mathbf{k},$$

where λ and μ are real. The point L is such that $\overrightarrow{OL} = 3\overrightarrow{OA}$ and Π_2 is the plane through L which is parallel to Π_1. The point M is such that $\overrightarrow{AM} = 3\overrightarrow{ML}$.

(i) Show that A is in Π_1. [1]

(ii) Find a vector perpendicular to Π_2. [2]

(iii) Find the position vector of the point N in Π_2 such that ON is perpendicular to Π_2. [5]

(iv) Show that the position vector of M is $10\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$ and find the perpendicular distance of M from the line through O and N, giving your answer correct to 3 significant figures. [6]

19 The lines l_1 and l_2 have equations $\mathbf{r} = 8\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j})$ and $\mathbf{r} = 5\mathbf{i} + 3\mathbf{j} - 14\mathbf{k} + \mu(2\mathbf{j} - 3\mathbf{k})$ respectively. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Find the position vector of the point P and the position vector of the point Q. [8]

The points with position vectors $8\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $5\mathbf{i} + 3\mathbf{j} - 14\mathbf{k}$ are denoted by A and B respectively. Find

(i) $\overrightarrow{AP} \times \overrightarrow{AQ}$ and hence the area of the triangle APQ,

(ii) the volume of the tetrahedron $APQB$. (You are given that the volume of a tetrahedron is $\frac{1}{3} \times$ area of base $\times$ perpendicular height.)

[6]

20 A line, passing through the point A (3, 0, 2), has vector equation $\mathbf{r} = 3\mathbf{i} + 2\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$. It meets the plane Π, which has equation $\mathbf{r} . (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 3$, at the point P. Find the coordinates of P. [3]

Write down a vector $\mathbf{n}$ which is perpendicular to Π, and calculate the vector $\mathbf{w}$, where

$$\mathbf{w} = \mathbf{n} \times (2\mathbf{i} + \mathbf{j} - 2\mathbf{k}).$$ [3]

The point Q lies in Π and is the foot of the perpendicular from A to Π. Use the vector $\mathbf{w}$ to determine an equation of the line PQ in the form $\mathbf{r} = \mathbf{u} + \mu\mathbf{v}$. [4]

21 Find a cartesian equation of the plane Π_1 passing through the points with coordinates (2, −1, 3), (4, 2, −5) and (−1, 3, −2). [4]

The plane Π_2 has cartesian equation $3x - y + 2z = 5$. Find the acute angle between Π_1 and Π_2. [3]

Find a vector equation of the line of intersection of the planes Π_1 and Π_2. [4]

22 The position vectors of the points A, B, C, D are

$$\mathbf{a} = 2\mathbf{i} + \lambda\mathbf{j} - 3\mathbf{k},\ \mathbf{b} = 6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k},\ \mathbf{c} = \mathbf{i} + 2\mathbf{j} - \mathbf{k},\ \mathbf{d} = \mathbf{i} + 7\mathbf{j} + 4\mathbf{k} \text{ respectively.}$$

It is given that the shortest distance between the lines AB and CD is 3.

(i) Show that $\lambda^2 + \lambda - 20 = 0$. [7]

(ii) The planes p_1 and p_2 are the planes through A, B and D corresponding to the two values of λ satisfying the equation in part **(i)**. Find the acute angle between p_1 and p_2. [7]

23 The position vectors of the points A, B, C, D are

$$\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \quad 3\mathbf{i} - \mathbf{j} + 5\mathbf{k}, \quad 3\mathbf{i} - \mathbf{j} + \mathbf{k}, \quad 5\mathbf{i} - 5\mathbf{j} + \alpha\mathbf{k},$$

respectively, where α is a positive integer. It is given that the shortest distance between the line AB and the line CD is equal to $2\sqrt{2}$.

(i) Show that the possible values of α are 3 and 5. [7]

(ii) Using $\alpha = 3$, find the shortest distance of the point D from the line AC, giving your answer correct to 3 significant figures. [3]

(iii) Using $\alpha = 3$, find the acute angle between the planes ABC and ABD, giving your answer in degrees. [4]

24 The plane Π_1 passes through the points $(1, 2, 1)$ and $(5, -2, 9)$ and is parallel to the vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

(i) Find the cartesian equation of Π_1. [4]

The plane Π_2 contains the lines

$$\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \quad \text{and} \quad \mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} + \mu(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}).$$

(ii) Find the cartesian equation of Π_2. [4]

(iii) Find the acute angle between Π_1 and Π_2. [3]

25 The line l_1 is parallel to the vector $a\mathbf{i} - \mathbf{j} + \mathbf{k}$, where a is a constant, and passes through the point whose position vector is $9\mathbf{j} + 2\mathbf{k}$. The line l_2 is parallel to the vector $-a\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and passes through the point whose position vector is $-6\mathbf{i} - 5\mathbf{j} + 10\mathbf{k}$.

(i) It is given that l_1 and l_2 intersect.

(a) Show that $a = -\frac{6}{13}$. [3]

(b) Find a cartesian equation of the plane containing l_1 and l_2. [4]

(ii) Given instead that the perpendicular distance between l_1 and l_2 is $3\sqrt{(30)}$, find the value of a. [5]

26 The lines l_1 and l_2 have vector equations

$$\mathbf{r} = a\mathbf{i} + 9\mathbf{j} + 13\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \quad \text{and} \quad \mathbf{r} = -3\mathbf{i} + 7\mathbf{j} - 2\mathbf{k} + \mu(-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$$

respectively. It is given that l_1 and l_2 intersect.

(i) Find the value of the constant a. [3]

The point P has position vector $3\mathbf{i} + \mathbf{j} + 6\mathbf{k}$.

(ii) Find the perpendicular distance from P to the plane containing l_1 and l_2. [4]

(iii) Find the perpendicular distance from P to l_2. [4]

27 The lines l_1 and l_2 have equations $\mathbf{r} = 6\mathbf{i} + 2\mathbf{j} + 7\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j})$ and $\mathbf{r} = 4\mathbf{i} + 4\mathbf{j} + \mu(-6\mathbf{j} + \mathbf{k})$ respectively. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Find the position vectors of P and Q. [8]

28 The line l_1 passes through the points $A(-3, 1, 4)$ and $B(-1, 5, 9)$. The line l_2 passes through the points $C(-2, 6, 5)$ and $D(-1, 7, 5)$.

(i) Find the shortest distance between the lines l_1 and l_2. [5]

(ii) Find the acute angle between the line l_2 and the plane containing A, B and D. [5]

29 The line l_1 passes through the point A with position vector $\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and is parallel to the vector $3\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$. The variable line l_2 passes through the point $(1 + 5\cos t)\mathbf{i} - (1 + 5\sin t)\mathbf{j} - 14\mathbf{k}$, where $0 \leqslant t < 2\pi$, and is parallel to the vector $15\mathbf{i} + 8\mathbf{j} - 3\mathbf{k}$. The points P and Q are on l_1 and l_2 respectively, and PQ is perpendicular to both l_1 and l_2.

(i) Find the length of PQ in terms of t. [4]

(ii) Hence show that the lines l_1 and l_2 do not intersect, and find the maximum length of PQ as t varies. [3]

(iii) The plane Π_1 contains l_1 and PQ; the plane Π_2 contains l_2 and PQ. Find the angle between the planes Π_1 and Π_2, correct to the nearest tenth of a degree. [4]

30 The line l_1 passes through the point A, whose position vector is $3\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$, and is parallel to the vector $3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$. The line l_2 passes through the point B, whose position vector is $2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, and is parallel to the vector $\mathbf{i} - \mathbf{j} - 4\mathbf{k}$. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. The plane Π_1 contains PQ and l_1, and the plane Π_2 contains PQ and l_2.

(i) Find the length of PQ. [4]

(ii) Find a vector perpendicular to Π_1. [2]

(iii) Find the perpendicular distance from B to Π_1. [3]

(iv) Find the angle between Π_1 and Π_2. [3]

31 The planes Π_1 and Π_2 have vector equations

$$\mathbf{r} = \lambda_1(\mathbf{i} + \mathbf{j} - \mathbf{k}) + \mu_1(2\mathbf{i} - \mathbf{j} + \mathbf{k}) \qquad \text{and} \qquad \mathbf{r} = \lambda_2(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) + \mu_2(3\mathbf{i} + \mathbf{j} - \mathbf{k})$$

respectively. The line l passes through the point with position vector $4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$ and is parallel to both Π_1 and Π_2. Find a vector equation for l. [6]

Find also the shortest distance between l and the line of intersection of Π_1 and Π_2. [4]

32 With O as origin, the points A, B, C have position vectors

$$\mathbf{i}, \quad \mathbf{i} + \mathbf{j}, \quad \mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

respectively. Find a vector equation of the common perpendicular of the lines AB and OC. [6]

Show that the shortest distance between the lines AB and OC is $\frac{2}{5}\sqrt{5}$. [2]

Find, in the form $ax + by + cz = d$, an equation for the plane containing AB and the common perpendicular of the lines AB and OC. [3]

33 The points A, B and C have position vectors $2\mathbf{i}$, $3\mathbf{j}$ and $4\mathbf{k}$ respectively. Find a vector which is perpendicular to the plane Π_1 containing A, B and C. [3]

The plane Π_2 has equation

$$\mathbf{r} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j}) + \mu(\mathbf{j} - \mathbf{k}).$$

Find the acute angle between the planes Π_1 and Π_2. [5]

34 The plane Π_1 has equation

$$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \theta(2\mathbf{j} - \mathbf{k}) + \phi(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}).$$

Find a vector normal to Π_1 and hence show that the equation of Π_1 can be written as $2x + 3y + 6z = 14$. [4]

The line l has equation

$$\mathbf{r} = 3\mathbf{i} + 8\mathbf{j} + 2\mathbf{k} + t(4\mathbf{i} + 6\mathbf{j} + 5\mathbf{k}).$$

The point on l where $t = \lambda$ is denoted by P. Find the set of values of λ for which the perpendicular distance of P from Π_1 is not greater than 4. [4]

The plane Π_2 contains l and the point with position vector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Find the acute angle between Π_1 and Π_2. [4]

35 Relative to an origin O, the points A, B, C have position vectors

$$\mathbf{i}, \quad \mathbf{j} + \mathbf{k}, \quad \mathbf{i} + \mathbf{j} + \theta\mathbf{k},$$

respectively. The shortest distance between the lines AB and OC is $\frac{1}{\sqrt{2}}$. Find the value of θ. [6]

36 The plane Π_1 has equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) + \mu(-\mathbf{i} + \mathbf{k})$. Obtain a cartesian equation of Π_1 in the form $px + qy + rz = d$. [4]

The plane Π_2 has equation $\mathbf{r}.(\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}) = 12$. Find a vector equation of the line of intersection of Π_1 and Π_2. [3]

The line l passes through the point A with position vector $a\mathbf{i} + (2a + 1)\mathbf{j} - 3\mathbf{k}$ and is parallel to $3c\mathbf{i} - 3\mathbf{j} + c\mathbf{k}$, where a and c are positive constants. Given that the perpendicular distance from A to Π_1 is $\frac{15}{\sqrt{6}}$ and that the acute angle between l and Π_1 is $\sin^{-1}\left(\frac{2}{\sqrt{6}}\right)$, find the values of a and c. [7]

37 The position vectors of points A, B, C, relative to the origin O, are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, where

$$\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \qquad \mathbf{b} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}, \qquad \mathbf{c} = 3\mathbf{i} - \mathbf{j} - \mathbf{k}.$$

Find $\mathbf{a} \times \mathbf{b}$ and deduce the area of the triangle OAB. [3]

Hence find the volume of the tetrahedron $OABC$, given that the volume of a tetrahedron is $\frac{1}{3} \times$ area of base $\times$ perpendicular height. [2]

38 Find a cartesian equation of the plane Π containing the lines

$$\mathbf{r} = 3\mathbf{i} + \mathbf{k} + s(2\mathbf{i} + \mathbf{j} - \mathbf{k}) \quad \text{and} \quad \mathbf{r} = 3\mathbf{i} - 7\mathbf{j} + 10\mathbf{k} + t(\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}).$$ [4]

The line l passes through the point P with position vector $6\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and is parallel to the vector $2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$. Find

(i) the position vector of the point where l meets Π, [3]

(ii) the perpendicular distance from P to Π, [3]

(iii) the acute angle between l and Π. [3]

39 The plane Π has equation

$$\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) + \mu(3\mathbf{i} + \mathbf{j} - 2\mathbf{k}).$$

The line l, which does not lie in Π, has equation

$$\mathbf{r} = 3\mathbf{i} + 6\mathbf{j} + 12\mathbf{k} + t(8\mathbf{i} + 5\mathbf{j} - 8\mathbf{k}).$$

Show that l is parallel to Π. [4]

Find the position vector of the point at which the line with equation $\mathbf{r} = 5\mathbf{i} - 4\mathbf{j} + 7\mathbf{k} + s(2\mathbf{i} - \mathbf{j} + \mathbf{k})$ meets Π. [4]

Find the perpendicular distance from the point with position vector $9\mathbf{i} + 11\mathbf{j} + 2\mathbf{k}$ to l. [4]

40 The points A, B and C have position vectors $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $2\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ respectively. Find $\overrightarrow{AB} \times \overrightarrow{AC}$. [3]

Deduce, in either order, the exact value of

(i) the area of the triangle ABC,

(ii) the perpendicular distance from C to AB.

[3]

41 The plane Π_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$. Find a cartesian equation of Π_1. [3]

The plane Π_2 has equation $2x - y + z = 10$. Find the acute angle between Π_1 and Π_2. [2]

Find an equation of the line of intersection of Π_1 and Π_2, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. [5]

42 The points A, B, C have position vectors

$$4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}, \qquad 5\mathbf{i} + 7\mathbf{j} + 8\mathbf{k}, \qquad 2\mathbf{i} + 6\mathbf{j} + 4\mathbf{k},$$

respectively, relative to the origin O. Find a cartesian equation of the plane ABC. [4]

The point D has position vector $6\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$. Find the coordinates of E, the point of intersection of the line OD with the plane ABC. [4]

Find the acute angle between the line ED and the plane ABC. [3]

43 The line l_1 is parallel to the vector $\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$ and passes through the point A, whose position vector is $3\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$. The line l_2 is parallel to the vector $-2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and passes through the point B, whose position vector is $-3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Find

(i) the length PQ, [5]

(ii) the cartesian equation of the plane Π containing PQ and l_2, [4]

(iii) the perpendicular distance of A from Π. [3]

44 The points A, B and C have position vectors $\mathbf{i}$, $2\mathbf{j}$ and $4\mathbf{k}$ respectively, relative to an origin O. The point N is the foot of the perpendicular from O to the plane ABC. The point P on the line-segment ON is such that $OP = \frac{3}{4}ON$. The line AP meets the plane OBC at Q. Find a vector perpendicular to the plane ABC and show that the length of ON is $\frac{4}{\sqrt{(21)}}$. [4]

Find the position vector of the point Q. [5]

Show that the acute angle between the planes ABC and ABQ is $\cos^{-1}\left(\frac{2}{3}\right)$. [5]

45 The lines l_1 and l_2 have equations

$$\mathbf{r} = 6\mathbf{i} - 3\mathbf{j} + s(3\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}) \quad \text{and} \quad \mathbf{r} = 2\mathbf{i} - \mathbf{j} - 4\mathbf{k} + t(\mathbf{i} - 3\mathbf{j} - \mathbf{k})$$

respectively. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Show that the position vector of P is $3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and find the position vector of Q. [7]

Find, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, an equation of the plane Π which passes through P and is perpendicular to l_1. [3]

The plane Π meets the plane $\mathbf{r} = p\mathbf{i} + q\mathbf{j}$ in the line l_3. Find a vector equation of l_3. [4]

46 The plane Π_1 has equation

$$\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

(i) Find a cartesian equation of Π_1. [3]

The plane Π_2 has equation $3x + y - z = 3$.

(ii) Find the acute angle between Π_1 and Π_2, giving your answer in degrees. [2]

(iii) Find an equation of the line of intersection of Π_1 and Π_2, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. [5]

47 The position vectors of the points A, B, C, D are

$$\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \qquad 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}, \qquad -\mathbf{i} + 3\mathbf{k}, \qquad m\mathbf{j} + 4\mathbf{k},$$

respectively, where m is a constant.

(i) Show that the lines AB and CD are parallel when $m = \frac{3}{2}$. [1]

(ii) Given that $m \neq \frac{3}{2}$, find the shortest distance between the lines AB and CD. [5]

(iii) When $m = 2$, find the acute angle between the planes ABC and ABD, giving your answer in degrees. [6]

48 With O as the origin, the points A, B, C have position vectors

$$\mathbf{i} - \mathbf{j}, \quad 2\mathbf{i} + \mathbf{j} + 7\mathbf{k}, \quad \mathbf{i} - \mathbf{j} + \mathbf{k}$$

respectively.

(i) Find the shortest distance between the lines OC and AB. [5]

(ii) Find the cartesian equation of the plane containing the line OC and the common perpendicular of the lines OC and AB. [4]

49 The points A, B, C have position vectors

$$-\mathbf{i}+\mathbf{j}+2\mathbf{k}, \quad -2\mathbf{i}-\mathbf{j}, \quad 2\mathbf{i}+2\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

(b) Find the perpendicular distance from O to the plane ABC. [2]

(c) Find the acute angle between the planes OAB and ABC. [4]

50 The points A, B, C have position vectors

$$-2\mathbf{i}+2\mathbf{j}-\mathbf{k}, \quad -2\mathbf{i}+\mathbf{j}+2\mathbf{k}, \quad -2\mathbf{j}+\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

(b) Find the acute angle between the planes OBC and ABC. [4]

The point D has position vector $t\mathbf{i} - \mathbf{j}$.

(c) Given that the shortest distance between the lines AB and CD is $\sqrt{10}$, find the value of t. [6]

51 The lines l_1 and l_2 have equations $\mathbf{r} = 3\mathbf{i} + 3\mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} + 4\mathbf{k})$ and $\mathbf{r} = 3\mathbf{i} - 5\mathbf{j} - 6\mathbf{k} + \mu(5\mathbf{j} + 6\mathbf{k})$ respectively.

(a) Find the shortest distance between l_1 and l_2. [5]

The plane Π contains l_1 and is parallel to the vector $\mathbf{i}+\mathbf{k}$.

(b) Find the equation of Π, giving your answer in the form $ax+by+cz=d$. [4]

(c) Find the acute angle between l_2 and Π. [3]

52 The lines l_1 and l_2 have equations $\mathbf{r} = -5\mathbf{j} + \lambda(5\mathbf{i} + 2\mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} + \mu(\mathbf{j} + \mathbf{k})$ respectively. The plane Π contains l_1 and is parallel to l_2.

(a) Find the equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

(b) Find the distance between l_2 and Π. [3]

The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2.

(c) Show that P has position vector $\frac{55}{27}\mathbf{i} - 5\mathbf{j} + \frac{22}{27}\mathbf{k}$ and state a vector equation for PQ. [8]

53 Let t be a positive constant.

The line l_1 passes through the point with position vector $t\mathbf{i}+\mathbf{j}$ and is parallel to the vector $-2\mathbf{i}-\mathbf{j}$. The line l_2 passes through the point with position vector $\mathbf{j}+t\mathbf{k}$ and is parallel to the vector $-2\mathbf{j}+\mathbf{k}$.

It is given that the shortest distance between the lines l_1 and l_2 is $\sqrt{21}$.

(a) Find the value of t. [5]

The plane Π_1 contains l_1 and is parallel to l_2.

(b) Write down an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a}+\lambda\mathbf{b}+\mu\mathbf{c}$. [1]

The plane Π_2 has Cartesian equation $5x - 6y + 7z = 0$.

(c) Find the acute angle between l_2 and Π_2. [3]

(d) Find the acute angle between Π_1 and Π_2. [3]

54 The lines l_1 and l_2 have equations $\mathbf{r} = -\mathbf{i} - \mathbf{j}\,2+\mathbf{k} + s(2\mathbf{i} - 3\mathbf{j})$ and $\mathbf{r} = 3\mathbf{i} - 2\ \mathbf{k} + t(3\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ respectively.

The plane Π_1 contains l_1 and the point P with position vector $-2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$.

(a) Find an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. [2]

The plane Π_2 contains l_2 and is parallel to l_1.

(b) Find an equation of Π_2, giving your answer in the form $ax + by + cz = d$. [4]

(c) Find the acute angle between Π_1 and Π_2. [5]

(d) The point Q is such that $\overrightarrow{OQ} = -5\overrightarrow{OP}$.

Find the position vector of the foot of the perpendicular from the point Q to Π_2. [4]

55 The points A, B, C have position vectors

$$4\mathbf{i}-4\mathbf{j}+\mathbf{k}, \quad -4\mathbf{i}+3\mathbf{j}-4\mathbf{k}, \quad 4\mathbf{i}-\mathbf{j}-2\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

(b) Find the perpendicular distance from O to the plane ABC. [2]

(c) The point D has position vector $2\mathbf{i}+3\mathbf{j}-3\mathbf{k}$.

Find the coordinates of the point of intersection of the line OD with the plane ABC. [3]

56 The position vectors of the points A, B, C, D are

$$7\mathbf{i}+4\mathbf{j}-\mathbf{k}, \qquad 11\mathbf{i}+3\mathbf{j}, \qquad 2\mathbf{i}+6\mathbf{j}+3\mathbf{k}, \qquad 2\mathbf{i}+7\mathbf{j}+\lambda\mathbf{k}$$

respectively.

(a) Given that the shortest distance between the line AB and the line CD is 3, show that $\lambda^2-5\lambda+4=0$. [7]

Let Π_1 be the plane ABD when $\lambda = 1$.

Let Π_2 be the plane ABD when $\lambda = 4$.

(b) **(i)** Write down an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$. [2]

(ii) Find an equation of Π_2, giving your answer in the form $ax + by + cz = d$. [4]

(c) Find the acute angle between Π_1 and Π_2. [5]

57 The plane Π has equation $\mathbf{r} = -2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + \mathbf{k}) + \mu(2\mathbf{i} + 3\mathbf{j})$.

(a) Find a Cartesian equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

The line l passes through the point P with position vector $2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and is parallel to the vector $\mathbf{k}$.

(b) Find the position vector of the point where l meets Π. [3]

(c) Find the acute angle between l and Π. [3]

(d) Find the perpendicular distance from P to Π. [3]

58 The points A, B, C have position vectors

$$2\mathbf{i}+2\mathbf{j}, \quad -\mathbf{j}+\mathbf{k} \quad \text{and} \quad 2\mathbf{i}+\mathbf{j}-7\mathbf{k}$$

respectively, relative to the origin O.

(a) Find an equation of the plane OAB, giving your answer in the form $\mathbf{r.n} = p$. [3]

The plane Π has equation $x-3y-2z = 1$.

(b) Find the perpendicular distance of Π from the origin. [1]

(c) Find the acute angle between the planes OAB and Π. [3]

(d) Find an equation for the common perpendicular to the lines OC and AB. [10]

59 The plane Π contains the lines $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \mu(3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$.

(a) Find a Cartesian equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

The line l passes through the point P with position vector $2\mathbf{i}+3\mathbf{j}+\mathbf{k}$ and is parallel to the vector $\mathbf{j}+\mathbf{k}$.

(b) Find the acute angle between l and Π. [3]

(c) Find the position vector of the foot of the perpendicular from P to Π. [4]

60 The lines l_1 and l_2 have equations $\mathbf{r} = 2\mathbf{i} + \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = 2\mathbf{j} + 6\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ respectively.

The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2.

(a) Find the length PQ. [5]

The plane Π_1 contains PQ and l_1.

The plane Π_2 contains PQ and l_2.

(b) **(i)** Write down an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$. [1]

(ii) Find an equation of Π_2, giving your answer in the form $ax + by + cz = d$. [4]

(c) Find the acute angle between Π_1 and Π_2. [5]

61 The plane Π_1 has equation $r = -4\mathbf{j} - 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k})$.

(a) Obtain an equation of Π_1 in the form $px + qy + rz = d$. [4]

(b) The plane Π_2 has equation $\mathbf{r}.(-5\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) = 4$.

Find a vector equation of the line of intersection of Π_1 and Π_2. [4]

The line l passes through the point A with position vector $a\mathbf{i} + a\mathbf{j} + (a-7)\mathbf{k}$ and is parallel to $(1-b)\mathbf{i} + b\mathbf{j} + b\mathbf{k}$, where a and b are positive constants.

(c) Given that the perpendicular distance from A to Π_1 is $\sqrt{2}$, find the value of a. [2]

(d) Given that the obtuse angle between l and Π_1 is $\frac{3}{4}\pi$, find the exact value of b. [4]

62 The points A, B, C have position vectors

$$\mathbf{i}+\mathbf{j}, \quad -\mathbf{i}+2\mathbf{j}+4\mathbf{k}, \quad -2\mathbf{i}+\mathbf{j}+3\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

(b) Find the perpendicular distance from O to the plane ABC. [2]

(c) Find a vector equation of the common perpendicular to the lines OC and AB. [8]