1 The equation

$$8x^3 + 12x^2 + 4x - 1 = 0$$

has roots α, β, γ. Show that the equation with roots $2\alpha + 1, 2\beta + 1, 2\gamma + 1$ is

$$y^3 - y - 1 = 0.$$ [3]

The sum $(2\alpha + 1)^n + (2\beta + 1)^n + (2\gamma + 1)^n$ is denoted by S_n. Find the values of S_3 and S_{-2}. [5]

Let, $y = 2x+1 \Rightarrow x = \frac{y-1}{2}$. Substituting in the original equation gives:

$$8\left(\frac{y-1}{2}\right)^3 + 12\left(\frac{y-1}{2}\right)^2 + 4\left(\frac{y-1}{2}\right) - 1 = 0$$

$$8\left[\frac{y^3-3y^2+3y-1}{8}\right] + 12\left[\frac{y^2-2y+1}{4}\right] + 2(y-1) - 1 = 0$$

$$y^3 - 3y^2 + 3y - 1 + 3y^2 - 6y + 3 + 2y - 2 - 1 = 0$$

$$y^3 - y - 1 = 0.$$

(Shown).

$$y^3 - y - 1 = 0$$

$$\Rightarrow S_3 - S_1 - 3 = 0$$

$$S_1 = \Sigma(2\alpha+1) = \frac{-0}{1} = 0$$

$$\therefore S_3 - 0 - 3 = 0$$

$$S_3 = 3.$$

Dividing $y^3 - y - 1 = 0$ by y^2 gives:

$$y - y^{-1} - y^{-2} = 0$$

$$\therefore S_1 - S_{-1} - S_{-2} = 0.$$

$$S_{-1} = \Sigma \frac{1}{(2\alpha+1)} = \frac{\Sigma(2\alpha+1)(2\beta+1)}{(2\alpha+1)(2\beta+1)(2\gamma+1)} = \frac{\frac{-1}{1}}{-\frac{(-1)}{1}} = \frac{-1}{1} = -1.$$

$$\therefore S_1 - S_{-1} - S_{-2} = 0$$

$$0 - (-1) - S_{-2} = 0 \Rightarrow 1 = S_{-2}$$

i.e. $S_{-2} = 1.$

2 The roots of the equation

$$x^3 - x - 1 = 0$$

are α, β, γ, and

$$S_n = \alpha^n + \beta^n + \gamma^n.$$

(i) Use the relation $y = x^2$ to show that α^2, β^2, γ^2 are the roots of the equation

$$y^3 - 2y^2 + y - 1 = 0.$$ [3]

(ii) Hence, or otherwise, find the value of S_4. [2]

(iii) Find the values of S_8, S_{12} and S_{16}. [9]

i) $y = x^2 \Rightarrow x = \sqrt{y}$. Substituting in the original equation gives:

$$(\sqrt{y})^3 - \sqrt{y} - 1 = 0$$
$$y\sqrt{y} - \sqrt{y} = 1$$
$$\sqrt{y}\,(y-1) = 1$$
$$y\,(y^2 - 2y + 1) = 1$$
$$y^3 - 2y^2 + y - 1 = 0 \quad \text{(Shown)}.$$

ii) $\Sigma\alpha^4 = (\Sigma\alpha^2)^2 - 2\,\Sigma\alpha^2\beta^2 = \left[\frac{-(-2)}{1}\right]^2 - 2\left[\frac{1}{1}\right] = 4 - 2 = 2.$

i.e. $S_4 = 2$.

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iii) Let, $z = y^2 \Rightarrow y = \sqrt{z}$. Substituting in the equation $y^3 - 2y^2 + y - 1 = 0$:

$$(\sqrt{z})^3 - 2(\sqrt{z})^2 + \sqrt{z} - 1 = 0$$
$$z\sqrt{z} - 2z + \sqrt{z} - 1 = 0$$
$$[\sqrt{z}(z+1)]^2 = [1 + 2z]^2$$
$$z(z^2 + 2z + 1) = 1 + 4z + 4z^2$$
$$z^3 + 2z^2 + z = 1 + 4z + 4z^2$$
$$z^3 - 2z^2 - 3z - 1 = 0. \text{ has roots } \alpha^4, \beta^4, \gamma^4.$$

$S_8 = \Sigma\alpha^8 = (\Sigma\alpha^4)^2 - 2\,\Sigma\alpha^4\beta^4 = \left[\frac{-(-2)}{1}\right]^2 - 2\left[\frac{-3}{1}\right] = 4 + 6 = 10.$

$\therefore S_8 = 10.$

$$z^3 - 2z^2 - 3z - 1 = 0$$

$$\Rightarrow S_{12} - 2S_8 - 3S_4 - 3 = 0$$

$$S_{12} - 2(10) - 3(2) - 3 = 0$$

$$S_{12} = 29.$$

Multiplying $z^3 - 2z^2 - 3z - 1 = 0$ by z gives:

$$z^4 - 2z^3 - 3z^2 - z = 0$$

$$\Rightarrow S_{16} - 2S_{12} - 3S_8 - S_4 = 0$$

$$S_{16} - 2(29) - 3(10) - 2 = 0$$

$$S_{16} - 58 - 30 - 2 = 0$$

$$\underline{S_{16} = 90}.$$

3 Show that the sum of the cubes of the roots of the equation

$$x^3 + \lambda x + 1 = 0$$

is -3. [3]

Show also that there is no real value of λ for which the sum of the fourth powers of the roots is negative. [3]

$$x^3 + \lambda x + 1 = 0$$

$$\Rightarrow S_3 + \lambda S_1 + 3 = 0$$

$$S_1 = \Sigma\alpha = \frac{-b}{a} = \frac{-0}{1} = 0$$

$$\therefore S_3 + \lambda(0) + 3 = 0$$

$$S_3 = -3.$$

Multiplying the equation by x gives:

$$x^4 + \lambda x^2 + x = 0$$

$$\Rightarrow S_4 + \lambda S_2 + S_1 = 0$$

$$S_2 = \Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$$

$$= (0)^2 - 2(\lambda) = -2\lambda.$$

$$\therefore S_4 + \lambda(-2\lambda) + 0 = 0$$

$$S_4 = 2\lambda^2.$$

Then if λ is real, $\lambda^2 \geq 0 \Rightarrow 2\lambda^2 \geq 0 \Rightarrow S_4 \geq 0$

i.e. S_4 is greater than or equal to 0 if λ is real.

4 Obtain the sum of the squares of the roots of the equation

$$x^4 + 3x^3 + 5x^2 + 12x + 4 = 0.$$ [2]

Deduce that this equation does not have more than 2 real roots. [3]

Show that, in fact, the equation has exactly 2 real roots in the interval $-3 < x < 0$. [5]

Denoting these roots by α and β, and the other 2 roots by γ and δ, show that $|\gamma| = |\delta| = \frac{2}{\sqrt{(\alpha\beta)}}$. [4]

$\sum\alpha^2 = (\sum\alpha)^2 - 2\sum\alpha\beta$
$= (-3)^2 - 2(5) = 9-10 = -1.$

Let, all roots are real. But then $\sum\alpha^2 > 0$. We have a contradiction since $\sum\alpha^2 < 0$. Therefore, all roots are not real.
The quartic equation has real coefficients, the complex roots occur in conjugate pairs. Then as $\sum\alpha^2 < 0$, there are atleast two complex roots. Therefore, this equation can not have more than two real roots.

Let, $f(x) = x^4+3x^3+5x^2+12x+4$.
$f(-3) = (-3)^4+3(-3)^3+5(-3)^2+12(-3)+4 = 13$.
$f(-2) = (-2)^4+3(-2)^3+5(-2)^2+12(-2)+4 = -8$.
$f(-1) = (-1)^4+3(-1)^3+5(-1)^2+12(-1)+4 = -5$
$f(0) = 0^4+3(0)^3+5(0)^2+12(0)+4 = 4$

$f(-3) > 0$ and $f(-2) < 0$ $\therefore$ there exists a real root α such that

$-3 < \alpha < -2$.

$f(-1) < 0$ and $f(0) > 0$ $\therefore$ there exists a real root β such that

$-1 < \beta < 0$.

$\therefore$ There are exactly 2 roots in the interval $-3 < x < 0$.

α, β are real roots $\Rightarrow$ γ, δ are complex roots.
Since γ, δ form a conjugate pair: $|\gamma| = |\delta|$.

$$\alpha\beta\gamma\delta = \frac{4}{1} = 4$$

$$\therefore |\alpha\beta\gamma\delta| = 4$$

$$|\gamma\delta| = \frac{4}{\alpha\beta}. \quad \text{Since } |\gamma| = |\delta|,$$

$$|\gamma||\gamma| = \frac{4}{\alpha\beta}$$

$$|\gamma|^2 = \frac{4}{\alpha\beta}$$

$$|\gamma| = \frac{2}{\sqrt{\alpha\beta}}.$$

Then $|\gamma| = |\delta| = \dfrac{2}{\sqrt{\alpha\beta}}$.

5 The equation

$$x^3 + 3x - 1 = 0$$

has roots α, β, γ. Use the substitution $y = x^3$ to show that the equation whose roots are α^3, β^3, γ^3 is

$$y^3 - 3y^2 + 30y - 1 = 0.$$ [2]

Find the value of $\alpha^9 + \beta^9 + \gamma^9$. [5]

$y = x^3 \Rightarrow x = \sqrt[3]{y}$.

Substituting in the original equation gives:

$$(\sqrt[3]{y})^3 + 3(\sqrt[3]{y}) - 1 = 0$$

$$y + 3\sqrt[3]{y} - 1 = 0$$

$$[3\sqrt[3]{y}]^3 = [1-y]^3$$

$$27y = 1 - 3y + 3y^2 - y^3$$

$y^3 - 3y^2 + 30y - 1 = 0$ has roots $\alpha^3, \beta^3, \gamma^3$.

(Shown).

$$\Sigma\alpha^3 = S_3 = \frac{-B}{A} = \frac{-(-3)}{1} = 3.$$

$$S_6 = \Sigma\alpha^6 = (\Sigma\alpha^3)^2 - 2\Sigma\alpha^3\beta^3 = (3)^2 - 2\left(\frac{30}{1}\right) = 9 - 60 = -51.$$

$$y^3 - 3y^2 + 30y - 1 = 0$$

$$\Rightarrow S_9 - 3S_6 + 30S_3 - 3 = 0$$

$$S_9 - 3(-51) + 30(3) - 3 = 0$$

$$S_9 = -240$$

$$\therefore \alpha^9 + \beta^9 + \gamma^9 = -240.$$

6 The equation

$$x^3 + x - 1 = 0$$

has roots α, β, γ. Show that the equation with roots $\alpha^3, \beta^3, \gamma^3$ is [4]

$$y^3 - 3y^2 + 4y - 1 = 0.$$

[3]

Hence find the value of $\alpha^6 + \beta^6 + \gamma^6$.

Let, $y = x^3 \Rightarrow x = \sqrt[3]{y}$.

Substituting into the original equation gives:

$$\left(\sqrt[3]{y}\right)^3 + \sqrt[3]{y} - 1 = 0$$

$$y + \sqrt[3]{y} - 1 = 0$$

$$\left[\sqrt[3]{y}\right]^3 = \left[1 - y\right]^3$$

$$y = 1 - 3y + 3y^2 - y^3$$

$y^3 - 3y^2 + 4y - 1 = 0$. has roots $\alpha^3, \beta^3, \gamma^3$.

$\alpha^6 + \beta^6 + \gamma^6$

$= \Sigma\alpha^6 = (\Sigma\alpha^3)^2 - 2\Sigma\alpha^3\beta^3$

$= \left[\frac{-(-3)}{1}\right]^2 - 2\left[\frac{4}{1}\right] = 9 - 8 = \underline{1}.$

7 The equation

$$x^4 - x^3 - 1 = 0$$

has roots $\alpha, \beta, \gamma, \delta$. By using the substitution $y = x^3$, or by any other method, find the exact value of $\alpha^6 + \beta^6 + \gamma^6 + \delta^6$. [5]

Let, $y = x^3 \Rightarrow x = \sqrt[3]{y}$.

Substituting into the original equation gives:

$$\left(\sqrt[3]{y}\right)^4 - \left(\sqrt[3]{y}\right)^3 - 1 = 0$$

$$y\sqrt[3]{y} - y - 1 = 0$$

$$\left[y\sqrt[3]{y}\right]^3 = \left[y+1\right]^3$$

$$y^4 = y^3 + 3y^2 + 3y + 1$$

$y^4 - y^3 - 3y^2 - 3y - 1 = 0$ has roots $\alpha^3, \beta^3, \gamma^3, \delta^3$.

$\alpha^6 + \beta^6 + \gamma^6 + \delta^6$

$= \Sigma\alpha^6 = \left(\Sigma\alpha^3\right)^2 - 2\Sigma\alpha^3\beta^3$

$= \left[\frac{-(-1)}{1}\right]^2 - 2\left[\frac{-3}{1}\right] = 1 + 6 = 7.$

i.e. $S_6 = 7$.

8 The equation

$$x^3 + x - 1 = 0$$

has roots α, β, γ. Use the relation $x = \sqrt{y}$ to show that the equation

$$y^3 + 2y^2 + y - 1 = 0$$

has roots $\alpha^2, \beta^2, \gamma^2$. [2]

Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(i) Write down the value of S_2 and show that $S_4 = 2$. [3]

(ii) Find the values of S_6 and S_8. [4]

$x = \sqrt{y} \Rightarrow y = x^2.$

Substituting into the original equation gives:

$$(\sqrt{y})^3 + \sqrt{y} - 1 = 0$$

$$y\sqrt{y} + \sqrt{y} = 1$$

$$[\sqrt{y}(y+1)]^2 = 1^2$$

$$y(y+1)^2 = 1$$

$$y(y^2 + 2y + 1) = 1$$

$$y^3 + 2y^2 + y - 1 = 0 \quad \text{(Shown)}.$$

i) $S_2 = \Sigma\alpha^2 = -\frac{2}{1} = -2.$

$S_4 = \Sigma\alpha^4 = (\Sigma\alpha^2) - 2\,\Sigma\alpha^2\beta^2 = (-2)^2 - 2\left(\frac{1}{1}\right) = 4 - 2 = 2.$ (Shown).

ii)

$$y^3 + 2y^2 + y - 1 = 0$$

$$S_6 + 2S_4 + S_2 - 3 = 0$$

$$S_6 + 2(2) + (-2) - 3 = 0$$

$$S_6 = 1.$$

Multiplying $y^3 + 2y^2 + y - 1 = 0$ by y gives:

$$y^4 + 2y^3 + y^2 - y = 0$$

$$\therefore S_8 + 2S_6 + S_4 - S_2 = 0$$

$$S_8 + 2(1) + 2 - (-2) = 0$$

$$S_8 = -6.$$

9 The equation

$$x^4 + x^3 + cx^2 + 4x - 2 = 0,$$

where c is a constant, has roots $\alpha, \beta, \gamma, \delta$.

(i) Use the substitution $y = \frac{1}{x}$ to find an equation which has roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$. [2]

(ii) Find, in terms of c, the values of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ and $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$. [3]

(iii) Hence find

$$\left(\alpha - \frac{1}{\alpha}\right)^2 + \left(\beta - \frac{1}{\beta}\right)^2 + \left(\gamma - \frac{1}{\gamma}\right)^2 + \left(\delta - \frac{1}{\delta}\right)^2$$

in terms of c. [2]

(iv) Deduce that when $c = -3$ the roots of the given equation are not all real. [3]

i) Let, $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$.

Substituting in the original equation gives:

$$\left(\frac{1}{y}\right)^4 + \left(\frac{1}{y}\right)^3 + c\left(\frac{1}{y}\right)^2 + 4\left(\frac{1}{y}\right) - 2 = 0$$

$$\frac{1}{y^4} + \frac{1}{y^3} + \frac{c}{y^2} + \frac{4}{y} - 2 = 0$$

$$1 + y + cy^2 + 4y^3 - 2y^4 = 0$$

$$\Rightarrow 2y^4 - 4y^3 - cy^2 - y - 1 = 0$$ has roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$.

ii) From original equation,

$$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\,\Sigma\alpha\beta = \left(\frac{-1}{1}\right)^2 - 2\left(\frac{c}{1}\right) = 1 - 2c.$$

From derived equation,

$$\Sigma\frac{1}{\alpha^2} = \left(\Sigma\frac{1}{\alpha}\right)^2 - 2\,\Sigma\left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right) = \left[\frac{-(-4)}{2}\right]^2 - 2\left[\frac{-c}{2}\right] = 4 + c.$$

iii)
$$\left(\alpha - \frac{1}{\alpha}\right)^2 + \left(\beta - \frac{1}{\beta}\right)^2 + \left(\gamma - \frac{1}{\gamma}\right)^2 + \left(\delta - \frac{1}{\delta}\right)^2$$

$$= \alpha^2 - 2 + \frac{1}{\alpha^2} + \beta^2 - 2 + \frac{1}{\beta^2} + \gamma^2 - 2 + \frac{1}{\gamma^2} + \delta^2 - 2 + \frac{1}{\delta^2}$$

$$= (\alpha^2 + \beta^2 + \gamma^2 + \delta^2) + \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}\right) - 8$$

$$= 1 - 2c + 4 + c - 8$$

$$= -c - 3.$$

iv) If $c = -3$,

$$\left(\alpha - \frac{1}{\alpha}\right)^2 + \left(\beta - \frac{1}{\beta}\right)^2 + \left(\gamma - \frac{1}{\gamma}\right)^2 + \left(\delta - \frac{1}{\delta}\right)^2 = -(-3) - 3 = 0$$

Then ~~since~~ if $\alpha, \beta, \gamma, \delta$ are real, then

$$\left(\alpha - \frac{1}{\alpha}\right)^2 = \left(\beta - \frac{1}{\beta}\right)^2 = \left(\gamma - \frac{1}{\gamma}\right)^2 = \left(\delta - \frac{1}{\delta}\right)^2 = 0$$

$$\Rightarrow \quad \alpha = \beta = \gamma = \delta = \pm 1.$$

But $\alpha\beta\gamma\delta = \frac{-2}{1} = -2.$

This is not possible if $\alpha = \beta = \gamma = \delta = \pm 1.$

Therefore, we have a contradiction $\therefore$ Not all roots are real.

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10 The roots of the equation

$$x^3 + px^2 + qx + r = 0$$

are $\frac{\beta}{k}$, β, $k\beta$, where p, q, r, k and β are non-zero real constants. Show that $\beta = -\frac{q}{p}$. [4]

Deduce that $rp^3 = q^3$. [2]

$$\sum \alpha = \frac{-b}{a} = \frac{-p}{1} = -p$$

$$\therefore \frac{\beta}{k} + \beta + k\beta = -p$$

$$\beta\left(\frac{1}{k} + 1 + k\right) = -p \rightarrow (1)$$

$$\sum \alpha\beta = \frac{c}{a} = \frac{q}{1} = q$$

$$\left(\frac{\beta}{k}\right)\beta + \beta(k\beta) + \left(\frac{\beta}{k}\right)(k\beta) = q$$

$$\frac{\beta^2}{k} + k\beta^2 + \beta^2 = q$$

$$\beta^2\left(\frac{1}{k} + 1 + k\right) = q \rightarrow (2).$$

Dividing (2) by (1) gives:

$$\frac{\beta^2\left(\frac{1}{k} + 1 + k\right)}{\beta\left(\frac{1}{k} + 1 + k\right)} = \frac{q}{-p}$$

$$\beta = -\frac{q}{p}. \text{ (Shown)}.$$

$$\alpha\beta\gamma = \frac{-d}{a} = \frac{-r}{1}$$

$$\frac{\beta}{k} \times \beta \times k\beta = -r$$

$$\beta^3 = -r.$$

Substituting $\beta = \frac{-q}{p}$ gives:

$$\left(\frac{-q}{p}\right)^3 = -r$$

$$\frac{-q^3}{p^3} = -r \Rightarrow q^3 = rp^3 \text{ (Shown)}.$$

11 Find a cubic equation with roots α, β and γ, given that

$$\alpha + \beta + \gamma = -6, \qquad \alpha^2 + \beta^2 + \gamma^2 = 38, \qquad \alpha\beta\gamma = 30.$$ [3]

Hence find the numerical values of the roots. [3]

Let, $x^3 + bx^2 + cx + d = 0$ be the equation with roots α, β, γ.

$$\Sigma\alpha = \frac{-b}{a} = \frac{-b}{1} = -b$$

$$-6 = -b$$

$$\therefore b = 6.$$

$$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$$

$$38 = (-6)^2 - 2\left(\frac{c}{1}\right)$$

$$38 = 36 - 2c$$

$$2c = -2$$

$$c = -1.$$

$$\alpha\beta\gamma = \frac{-d}{1}$$

$$30 = -d$$

$$\therefore d = -30$$

$\therefore$ Cubic equation with roots α, β, γ: $x^3 + 6x^2 - x - 30 = 0$.

Let, $f(x) = x^3 + 6x^2 - x - 30 = 0$

$$f(2) = 2^3 + 6(2)^2 - 2 - 30 = 8 + 24 - 2 - 30 = 0$$

$\therefore x - 2$ is a factor of $f(x)$.

2	1	6	-1	-30
	$\downarrow$	2	16	30
	1	8	15	X

$$\therefore f(x) = (x-2)(x^2 + 8x + 15)$$

$$= (x-2)(x^2 + 3x + 5x + 15)$$

$$0 = (x-2)(x+3)(x+5)$$

$$\Rightarrow \quad x = -5, -3, 2.$$

$$\therefore \alpha = -5, \beta = -3, \gamma = 2.$$

12 The roots of the cubic equation $x^3 - 7x^2 + 2x - 3 = 0$ are α, β, γ. Find the values of

(i) $\alpha^2 + \beta^2 + \gamma^2$, [2]

(ii) $\alpha^3 + \beta^3 + \gamma^3$. [3]

i) $\sum\alpha^2 = \left(\sum\alpha\right)^2 - 2\sum\alpha\beta$

$= \left[\frac{-(-7)}{1}\right]^2 - 2\left[\frac{2}{1}\right] = 49 - 4 = 45.$

$x^3 - 7x^2 + 2x - 3 = 0$

ii) $\sum\alpha^3 - 7\sum\alpha^2 + 2\sum\alpha - 9 = 0$

$\sum\alpha^3 - 7(45) + 2\left[\frac{-(-7)}{1}\right] - 9 = 0$

$\sum\alpha^3 - 315 + 14 - 9 = 0$

$\therefore \sum\alpha^3 = 310.$

13 The cubic equation $x^3 - x^2 - 3x - 10 = 0$ has roots α, β, γ.

(i) Let $u = -\alpha + \beta + \gamma$. Show that $u + 2\alpha = 1$, and hence find a cubic equation having roots $-\alpha + \beta + \gamma$, $\alpha - \beta + \gamma$, $\alpha + \beta - \gamma$. [5]

(ii) State the value of $\alpha\beta\gamma$ and hence find a cubic equation having roots $\frac{1}{\beta\gamma}, \frac{1}{\gamma\alpha}, \frac{1}{\alpha\beta}$. [5]

i) $u + 2\alpha = (-\alpha + \beta + \gamma) + 2\alpha = \alpha + \beta + \gamma = \sum\alpha = \frac{-b}{a} = \frac{-(-1)}{1} = 1.$

i.e. $u + 2\alpha = 1$ (shown).

$u_1 = 1 - 2\alpha$. Similar working shows ~~$u+2\beta=1$~~,

$u_2 = \alpha - \beta + \gamma \Rightarrow (\alpha - \beta + \gamma) + 2\beta = 1 \Rightarrow u_2 = 1 - 2\beta$

$u_3 = \alpha + \beta - \gamma \Rightarrow$ ~~and~~ $(\alpha + \beta - \gamma) + 2\gamma = 1. \Rightarrow u_3 = 1 - 2\gamma.$

Then we want an equation in u with roots $1-2\alpha, 1-2\beta, 1-2\gamma$.

Let, $u = 1 - 2x \Rightarrow x = \frac{1-u}{2}$.

Substituting in the original equation gives:

$$\left(\frac{1-u}{2}\right)^3 - \left(\frac{1-u}{2}\right)^2 - 3\left(\frac{1-u}{2}\right) - 10 = 0$$

$$\frac{1-3u+3u^2-u^3}{8} - \frac{1-2u+u^2}{4} - \frac{3-3u}{2} - 10 = 0$$

$$1-3u+3u^2-u^3 - 2(1-2u+u^2) - 4(3-3u) - 10(8) = 0$$

$$1-3u+3u^2-u^3-2+4u-2u^2-12+12u-80 = 0$$

$$-u^3+u^2+13u-93 = 0$$

$$\Rightarrow u^3 - u^2 - 13u + 93 = 0.$$

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ii) $\alpha\beta\gamma = \frac{-d}{a} = \frac{-(-10)}{1} = 10.$

Note $\frac{1}{\beta\gamma} = \frac{\alpha}{\alpha\beta\gamma} = \frac{\alpha}{10}$.

Similarly $\frac{1}{\gamma\alpha} = \frac{\beta}{\alpha\beta\gamma} = \frac{\beta}{10}$.

and $\frac{1}{\alpha\beta} = \frac{\gamma}{\alpha\beta\gamma} = \frac{\gamma}{10}$.

~~ie~~ Then, we want a cubic equation with roots $\frac{\alpha}{10}, \frac{\beta}{10}, \frac{\gamma}{10}$.

Let, $y = \frac{x}{10} \Rightarrow x = 10y$.

Substituting into the original equation gives:

$$(10y)^3 - (10y)^2 - 3(10y) - 10 = 0$$

$$1000y^3 - 100y^2 - 30y - 10 = 0$$

$$\Rightarrow 100y^3 - 10y^2 - 3y - 1 = 0.$$

14 The cubic equation $x^3 - 2x^2 - 3x + 4 = 0$ has roots α, β, γ. Given that $c = \alpha + \beta + \gamma$, state the value of c. [1]

Use the substitution $y = c - x$ to find a cubic equation whose roots are $\alpha + \beta, \beta + \gamma, \gamma + \alpha$. [3]

Find a cubic equation whose roots are $\frac{1}{\alpha+\beta}, \frac{1}{\beta+\gamma}, \frac{1}{\gamma+\alpha}$. [2]

Hence evaluate $\frac{1}{(\alpha+\beta)^2} + \frac{1}{(\beta+\gamma)^2} + \frac{1}{(\gamma+\alpha)^2}$. [2]

$c = \alpha+\beta+\gamma = \sum\alpha = \frac{-B}{A} = \frac{-(-2)}{1} = 2.$

$y = c - x = (\alpha+\beta+\gamma) - x.$

When $x = \alpha$, $y = \alpha+\beta+\gamma-\alpha = \beta+\gamma$

When $x = \beta$, $y = \alpha+\beta+\gamma-\beta = \alpha+\gamma.$

When $x = \gamma$, $y = \alpha+\beta+\gamma-\gamma = \alpha+\beta.$

Then using $y = c - x = 2 - x$ will give an equation with roots $\alpha+\beta, \beta+\gamma, \gamma+\alpha$.

$y = 2 - x \Rightarrow x = 2 - y.$

Substituting into the original equation gives:

$$(2-y)^3 - 2(2-y)^2 - 3(2-y) + 4 = 0$$

$$8 - 12y + 6y^2 - y^3 - 2[4 - 4y + y^2] - 6 + 3y + 4 = 0$$

$$8 - 12y + 6y^2 - y^3 - 8 + 8y - 2y^2 - 6 + 3y + 4 = 0$$

$$-y^3 + 4y^2 - y - 2 = 0$$

$$\therefore y^3 - 4y^2 + y + 2 = 0.$$

For equation with roots $\frac{1}{\alpha+\beta}, \frac{1}{\beta+\gamma}, \frac{1}{\gamma+\alpha}$, let, $z = \frac{1}{y} \Rightarrow y = \frac{1}{z}$.

Substituting in equation $y^3 - 4y^2 + y + 2 = 0$ gives:

$$\left(\frac{1}{z}\right)^3 - 4\left(\frac{1}{z}\right)^2 + \frac{1}{z} + 2 = 0$$

$$\frac{1}{z^3} - \frac{4}{z^2} + \frac{1}{z} + 2 = 0$$

$$\frac{1 - 4z + z^2 + 2z^3}{z^3} = 0$$

$$2z^3 + z^2 - 4z + 1 = 0.$$

$$\Sigma \frac{1}{(\alpha+\beta)^2} = \left(\Sigma \frac{1}{\alpha+\beta}\right)^2 - 2\left[\Sigma \left(\frac{1}{\alpha+\beta}\right)\left(\frac{1}{\beta+\gamma}\right)\right]$$

$$= \left(\frac{-1}{2}\right)^2 - 2\left(\frac{-4}{2}\right)$$

$$\Sigma \frac{1}{(\alpha+\beta)^2} = \frac{1}{4} + 4 = 4\frac{1}{4}.$$

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15 The roots of the equation $x^4 - 4x^2 + 3x - 2 = 0$ are α, β, γ and δ; the sum $\alpha^n + \beta^n + \gamma^n + \delta^n$ is denoted by S_n. By using the relation $y = x^2$, or otherwise, show that α^2, β^2, γ^2 and δ^2 are the roots of the equation

$$y^4 - 8y^3 + 12y^2 + 7y + 4 = 0.$$ [3]

State the value of S_2 and hence show that

$$S_8 = 8S_6 - 12S_4 - 72.$$ [3]

Let, $y = x^2 \Rightarrow x = \sqrt{y}$.

Substituting in the original equation gives:

$$(\sqrt{y})^4 - 4(\sqrt{y})^2 + 3\sqrt{y} - 2 = 0$$

$$y^2 - \cancel{8y\sqrt{y}} + 3\sqrt{y} - 2 = 0$$

$$\cancel{\left[\sqrt{y}(3-8y)\right]^2 = \left[2-y^2\right]^2}$$

$$\cancel{y\left[9 - 48y + 64y^2\right] = 4 + y^4 - 4y^2}$$

$$\cancel{9y - 48y^2 + 64y^3 = 4 + y^4 - 4y^2}$$

$$\cancel{= y^4 - 64y^3}$$

$$y^2 - 4y + 3\sqrt{y} - 2 = 0$$

$$(3\sqrt{y})^2 = (2 + 4y - y^2)^2$$

$$9y = 4 + 2(2)(4y - y^2) + (4y - y^2)^2$$

$$9y = 4 + 16y - 4y^2 + 16y^2 + y^4 - 8y^3$$

$$9y = y^4 - 8y^3 + 12y^2 + 16y + 4$$

$$0 = y^4 - 8y^3 + 12y^2 + 7y + 4$$

(shown).

$$\Sigma\alpha^2 = \frac{-B}{A} = \frac{-(-8)}{1} = 8. \quad \text{i.e. } S_2 = 8.$$

From $y^4 - 8y^3 + 12y^2 + 7y + 4 = 0$

$$S_8 - 8S_6 + 12S_4 + 7S_2 + 16 = 0$$

$$S_8 - 8S_6 + 12S_4 + 7(8) + 16 = 0$$

$$S_8 = 8S_6 - 12S_4 - 72. \quad \text{(Shown)}.$$

16 The equation $x^3 + px + q = 0$, where p and q are constants, with $q \neq 0$, has one root which is the reciprocal of another root. Prove that $p + q^2 = 1$. [5]

Since one root is a reciprocal of the other, let roots be $\alpha, \frac{1}{\alpha}, \beta$.

$$\Sigma\alpha = \frac{-b}{a} = \frac{-0}{1} = 0$$

$$\therefore \alpha + \frac{1}{\alpha} + \beta = 0 \rightarrow (1)$$

$$\Sigma\alpha\beta = \frac{c}{a} = \frac{p}{1} = p$$

$$\therefore \alpha\left(\frac{1}{\alpha}\right) + \alpha\beta + \frac{1}{\alpha}(\beta) = p$$

$$1 + \alpha\beta + \frac{\beta}{\alpha} = p \rightarrow (2)$$

$$\alpha\beta\gamma = \frac{-d}{a} = \frac{-q}{1}$$

$$\alpha\left(\frac{1}{\alpha}\right)\beta = -q$$

$$\beta = -q \rightarrow (3)$$

Substituting (3) in (1) gives:

$$\alpha + \frac{1}{\alpha} - q = 0$$

$$\alpha + \frac{1}{\alpha} = q \rightarrow (4)$$

Substituting (3) in (2) gives:

$$1 + \alpha\beta + \frac{\beta}{\alpha} = p$$

$$1 + \alpha(-q) + \frac{-q}{\alpha} = p$$

$$1 - q\left(\alpha + \frac{1}{\alpha}\right) = p \rightarrow (5)$$

Substituting (4) in (5) gives:

$$1 - q(q) = p$$

$$1 - q^2 = p$$

$$\Rightarrow 1 = p + q^2.$$

i.e. $p + q^2 = 1$ (shown).

17 The roots of the cubic equation $x^3 - 7x^2 + 2x - 3 = 0$ are α, β and γ. Find the values of

(i) $\dfrac{1}{(\alpha\beta)(\beta\gamma)(\gamma\alpha)}$,

(ii) $\dfrac{1}{\alpha\beta} + \dfrac{1}{\beta\gamma} + \dfrac{1}{\gamma\alpha}$,

(iii) $\dfrac{1}{\alpha^2\beta\gamma} + \dfrac{1}{\alpha\beta^2\gamma} + \dfrac{1}{\alpha\beta\gamma^2}$. [6]

Deduce a cubic equation, with integer coefficients, having roots $\dfrac{1}{\alpha\beta}$, $\dfrac{1}{\beta\gamma}$ and $\dfrac{1}{\gamma\alpha}$. [2]

i) $\dfrac{1}{(\alpha\beta)(\beta\gamma)(\gamma\alpha)} = \dfrac{1}{(\alpha\beta\gamma)^2} = \dfrac{1}{\left(\frac{-d}{a}\right)^2} = \dfrac{1}{\left(\frac{-(-3)}{1}\right)^2} = \dfrac{1}{3^2} = \dfrac{1}{9}$.

ii) $\dfrac{1}{\alpha\beta} + \dfrac{1}{\beta\gamma} + \dfrac{1}{\gamma\alpha} = \dfrac{\gamma + \alpha + \beta}{\alpha\beta\gamma} = \dfrac{\Sigma\alpha}{\alpha\beta\gamma} = \dfrac{\frac{-b}{a}}{\frac{-d}{a}}$

$= \dfrac{\frac{-(-7)}{1}}{\frac{-(-3)}{1}} = \dfrac{7}{3}$

iii) $\dfrac{1}{\alpha^2\beta\gamma} + \dfrac{1}{\alpha\beta^2\gamma} + \dfrac{1}{\alpha\beta\gamma^2} = \dfrac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha^2\beta^2\gamma^2} = \dfrac{\Sigma\alpha\beta}{(\alpha\beta\gamma)^2} = \dfrac{\frac{c}{a}}{\left(\frac{-d}{a}\right)^2} = \dfrac{\left(\frac{2}{1}\right)}{\left[\frac{-(-3)}{1}\right]^2}$

$= \dfrac{2}{3^2} = \dfrac{2}{9}$.

For a cubic equation with roots $\dfrac{1}{\alpha\beta}, \dfrac{1}{\beta\gamma}, \dfrac{1}{\gamma\alpha}$: $x^3 + bx^2 + cx + d = 0$

$\Sigma \dfrac{1}{\alpha\beta} = \dfrac{7}{3} \therefore \dfrac{-b}{1} = \dfrac{7}{3} \Rightarrow b = \dfrac{-7}{3}$.

$\Sigma\left(\dfrac{1}{\alpha\beta}\right)\left(\dfrac{1}{\beta\gamma}\right) = \dfrac{2}{9} \therefore \dfrac{c}{1} = \dfrac{2}{9} \Rightarrow c = \dfrac{2}{9}$.

$\dfrac{1}{(\alpha\beta)(\beta\gamma)(\gamma\alpha)} = \dfrac{1}{9} \therefore \dfrac{-d}{1} = \dfrac{1}{9} \Rightarrow d = \dfrac{-1}{9}$.

Then $x^3 - \dfrac{7}{3}x^2 + \dfrac{2}{9}x - \dfrac{1}{9} = 0$.

18 The quartic equation $x^4 - px^2 + qx - r = 0$, where p, q and r are real constants, has two pairs of equal roots. Show that $p^2 + 4r = 0$ and state the value of q. [6]

Since, the quartic has two equal roots, let the roots be $\alpha, \alpha, \beta, \beta$.

$$\therefore \sum \alpha = \frac{-b}{a} = \frac{-0}{1} = 0$$

$$\therefore 2\alpha + 2\beta = 0$$

$$\alpha = -\beta \rightarrow (1)$$

$$\sum \alpha\beta = \frac{c}{a}$$

$$\alpha^2 + \alpha\beta + \alpha\beta + \alpha\beta + \alpha\beta + \beta^2 = \frac{-p}{1} = -p$$

$$\alpha^2 + 4\alpha\beta + \beta^2 = -p \rightarrow (2)$$

$$\sum \alpha\beta\gamma = \frac{-d}{a}$$

$$\alpha^2\beta + \alpha^2\beta + \alpha\beta^2 + \alpha\beta^2 = \frac{-q}{1}$$

$$2\alpha^2\beta + 2\alpha\beta^2 = -q \rightarrow (3)$$

$$\alpha\beta\gamma\delta = \frac{e}{a} = \frac{-r}{1}$$

$$\alpha^2\beta^2 = -r \rightarrow (4)$$

Substituting (1) in (2) gives:

$$(-\beta)^2 + 4(-\beta)\beta + \beta^2 = -p$$

$$-2\beta^2 = -p$$

$$\therefore p = 2\beta^2 \rightarrow (5)$$

Substituting (1) in (4) gives:

$$(-\beta)^2\beta^2 = -r$$

$$\beta^4 = -r$$

$$\therefore r = -\beta^4 \rightarrow (6)$$

Substituting (5) and (6) in $p^2 + 4r$ gives:

$$p^2 + 4r = (2\beta^2)^2 + 4(-\beta^4) = 4\beta^4 - 4\beta^4 = 0$$

$$\therefore p^2 + 4r = 0. \text{ (Shown)}.$$

Substituting (1) in (3) gives:

$$2(-\beta)^2\beta + 2(-\beta)(\beta^2) = -q$$

$$+2\beta^3 \cancel{+} -2\beta^3 = -q$$

$$0 = -q$$

$$\Rightarrow q = 0$$

19 The roots of the cubic equation $2x^3 + x^2 - 7 = 0$ are α, β and γ. Using the substitution $y = 1 + \dfrac{1}{x}$, or otherwise, find the cubic equation whose roots are $1 + \dfrac{1}{\alpha}$, $1 + \dfrac{1}{\beta}$ and $1 + \dfrac{1}{\gamma}$, giving your answer in the form $ay^3 + by^2 + cy + d = 0$, where a, b, c and d are constants to be found. [4]

$$y = 1 + \frac{1}{x} \Rightarrow y - 1 = \frac{1}{x} \Rightarrow x = \frac{1}{y-1}$$

Substituting in the original equation gives:

$$2\left(\frac{1}{y-1}\right)^3 + \left(\frac{1}{y-1}\right)^2 - 7 = 0$$

$$\frac{2}{(y-1)^3} + \frac{1}{(y-1)^2} - 7 = 0$$

$$2 + y - 1 - 7(y-1)^3 = 0$$

$$2 + y - 1 - 7\left[y^3 - 3y^2 + 3y - 1\right] = 0$$

$$2 + y - 1 - 7y^3 + 21y^2 - 21y + 7 = 0$$

$$-7y^3 + 21y^2 - 20y + 8 = 0$$

$$\therefore\ 7y^3 - 21y^2 + 20y - 8 = 0$$

$a = 7$, $b = -21$, $c = 20$, $d = -8$.

20 The cubic equation

$$z^3 - z^2 - z - 5 = 0$$

has roots α, β and γ. Show that the value of $\alpha^3 + \beta^3 + \gamma^3$ is 19. [4]

Find the value of $\alpha^4 + \beta^4 + \gamma^4$. [2]

Show that the cubic equation with roots $\frac{\alpha - 1}{\alpha}$, $\frac{\beta - 1}{\beta}$ and $\frac{\gamma - 1}{\gamma}$ may be found using the substitution $z = \frac{1}{1-x}$, and find this equation, giving your answer in the form $px^3 + qx^2 + rx + s = 0$, where p, q, r and s are constants to be determined. [4]

$$z^3 - z^2 - z - 5 = 0.$$

$$S_1 = \frac{-b}{a} = \frac{-(-1)}{1} = 1.$$

$$S_2 = \Sigma\alpha^2 = (\Sigma\alpha)^2 - 2(\Sigma\alpha\beta) = 1^2 - 2(-1) = 3.$$

$$\therefore S_3 - S_2 - S_1 - 15 = 0$$

$$S_3 - 3 - 1 - 15 = 0$$

$$S_3 = 19.$$

$$\text{i.e. } \alpha^3 + \beta^3 + \gamma^3 = 19.$$

Multiplying $z^3 - z^2 - z - 5 = 0$ by z gives:

$$z^4 - z^3 - z^2 - 5z = 0$$

$$\Rightarrow S_4 - S_3 - S_2 - 5S_1 = 0$$

$$S_4 - 19 - 3 - 5(1) = 0$$

$$S_4 = 27.$$

We require roots $\frac{\alpha-1}{\alpha}, \frac{\beta-1}{\beta}, \frac{\gamma-1}{\gamma}$, then let

$$x = \frac{z-1}{z} \Rightarrow xz = z - 1 \Rightarrow xz - z = -1 \Rightarrow z(x-1) = -1 \Rightarrow z = \frac{-1}{x-1}$$

Then $z = \frac{1}{1-x}$.

(Shown)

Substituting in the original equation gives:

$$\left(\frac{1}{1-x}\right)^3 - \left(\frac{1}{1-x}\right)^2 - \left(\frac{1}{1-x}\right) - 5 = 0$$

$$\frac{1}{(1-x)^3} - \frac{1}{(1-x)^2} - \frac{1}{(1-x)} - 5 = 0$$

$$\frac{1-(1-x)-(1-x)^2-5(1-x)^3}{(1-x)^3}=0$$

$$1-1+x-(1-2x+x^2)-5(1-3x+3x^2-x^3)=0$$

$$1-1+x-1+2x-x^2-5+15x-15x^2+5x^3=0$$

$$\Rightarrow\ 5x^3-16x^2+18x-6=0$$

$\therefore p=5,\ q=-16,\ r=18,\ s=-6.$

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21 By finding a cubic equation whose roots are α, β and γ, solve the set of simultaneous equations

$$\alpha+\beta+\gamma=-1,$$
$$\alpha^2+\beta^2+\gamma^2=29,$$
$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=-1.$$

[8]

Let, cubic equation be of the form $x^3+bx^2+cx+d=0$.

$$\Sigma\alpha=\frac{-b}{1}$$
$$-1=-b$$
$$\Rightarrow b=1.$$

$$\Sigma\alpha^2=(\Sigma\alpha)^2-2(\Sigma\alpha\beta)$$
$$29=(-1)^2-2\left(\frac{c}{1}\right)$$
$$29=1-2c$$
$$2c=-28$$
$$c=-14.$$

$$\Sigma\frac{1}{\alpha}=\frac{\Sigma\alpha\beta}{\alpha\beta\gamma}$$
$$-1=\frac{\frac{c}{a}}{\frac{-d}{a}}$$
$$-1=\frac{\frac{-14}{1}}{\frac{-d}{1}} \Rightarrow -1=\frac{14}{d} \therefore d=-14$$

i.e. Cubic equation with roots α,β,γ: $x^3+x^2-14x-14=0$.

Let, $f(x)=x^3+x^2-14x-14=0$

$f(-1)=(-1)^3+(-1)^2-14(-1)-14=-1+1+14-14=0$

$\therefore x+1$ is a factor of $f(x)$.

Using synthetic division,

$$\begin{array}{r|rrrr} -1 & 1 & 1 & -14 & -14 \\ & & -1 & 0 & 14 \\ \hline & 1 & 0 & -14 & \times \end{array}$$

$\therefore f(x)=(x+1)(x^2-14)$.

$$(x+1)(x^2-14)=0$$

$$\Rightarrow \quad x=-1 \quad \text{or} \quad x=\pm\sqrt{14}.$$

$$\therefore \quad \alpha=-1, \ \beta=\sqrt{14}, \ \gamma=-\sqrt{14}.$$

22 The roots of the cubic equation $x^3 + 2x^2 - 3 = 0$ are α, β and γ.

(i) By using the substitution $y = \frac{1}{x^2}$, find the cubic equation with roots $\frac{1}{\alpha^2}$, $\frac{1}{\beta^2}$ and $\frac{1}{\gamma^2}$. [3]

$$y = \frac{1}{x^2} \Rightarrow x^2 = \frac{1}{y} \Rightarrow x = \sqrt{\frac{1}{y}} = \frac{1}{\sqrt{y}}.$$

Substituting in original equation gives:

$$\left(\frac{1}{\sqrt{y}}\right)^3 + 2\left(\frac{1}{\sqrt{y}}\right)^2 - 3 = 0$$

$$\frac{1}{y\sqrt{y}} + \frac{2}{y} - 3 = 0$$

$$\left(\frac{1}{y\sqrt{y}}\right)^2 = \left(3 - \frac{2}{y}\right)^2$$

$$\frac{1}{y^3} = 9 - \frac{12}{y} + \frac{4}{y^2}$$

$$\frac{1}{y^3} = \frac{9y^2 - 12y + 4}{y^2}$$

$$1 = 9y^3 - 12y^2 + 4y$$

$$0 = 9y^3 - 12y^2 + 4y - 1$$

(ii) Hence find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$. [1]

$$\sum \frac{1}{\alpha^2} = \frac{-b}{a} = \frac{-(-12)}{9} = \frac{12}{9} = \frac{4}{3}.$$

(iii) Find also the value of $\frac{1}{\alpha^2\beta^2} + \frac{1}{\beta^2\gamma^2} + \frac{1}{\gamma^2\alpha^2}$. [1]

$$\sum \frac{1}{\alpha^2\beta^2} = \frac{c}{a} = \frac{4}{9}.$$

23 It is given that the equation

$$x^3 - 21x^2 + kx - 216 = 0,$$

where k is a constant, has real roots a, ar and ar^{-1}.

(i) Find the numerical values of the roots. [6]

From the equation, $\alpha\beta\gamma = \frac{-d}{a} = \frac{-(-216)}{1}$

$$(a)(ar)(ar^{-1}) = 216$$

$$a^3 = 216$$

$$\Rightarrow a = \sqrt[3]{216} = 6.$$

$$\Sigma\alpha = \frac{-b}{a}$$

$$a + ar + ar^{-1} = \frac{-(-21)}{1}$$

$$a\left[1 + r + \frac{1}{r}\right] = 21$$

$$6\left[\frac{r + r^2 + 1}{r}\right] = 21$$

$$\frac{r^2 + r + 1}{r} = \frac{21}{6} = \frac{7}{2}$$

$$\therefore 2r^2 + 2r + 2 = 7r$$

$$\Rightarrow 2r^2 - 5r + 2 = 0$$

$$2r^2 - 4r - r + 2 = 0$$

$$(2r - 1)(r - 2) = 0$$

$\therefore 2r - 1 = 0$ or $r - 2 = 0$

$r = \frac{1}{2}$ $\quad$ $r = 2.$

When $r = \frac{1}{2}$,

Roots: $6, 6\left(\frac{1}{2}\right), 6\left(\frac{1}{2}\right)^{-1} = 6, 3, 12$

When $r = 2$,

Roots: $6, 6(2), 6(2)^{-1} = 6, 12, 3.$

$\therefore$ Roots are $\underline{3, 6, 12}.$

(ii) Deduce the value of k. [2]

$$\Sigma \alpha\beta = \frac{c}{a} = \frac{k}{1} = k$$

$$3(6)+3(12)+6(12) = k$$

$$18 + 36 + 72 = k$$

$$\therefore \quad k = 126.$$

24 The equation

$$9x^3 - 9x^2 + x - 2 = 0$$

has roots α, β, γ.

(i) Use the substitution $y = 3x - 1$ to show that $3\alpha - 1$, $3\beta - 1$, $3\gamma - 1$ are the roots of the equation

$$y^3 - 2y - 7 = 0.$$ [2]

$\Rightarrow$ $y = 3x - 1 \Rightarrow x = \frac{y+1}{3}$.

Substituting in original equation gives:

$$9\left(\frac{y+1}{3}\right)^3 - 9\left(\frac{y+1}{3}\right)^2 + \frac{y+1}{3} - 2 = 0$$

$$\cancel{9}\left[\frac{y^3+3y^2+3y+1}{\cancel{27}\,3}\right] - \cancel{9}\left[\frac{y^2+2y+1}{1\cancel{9}}\right] + \frac{y+1}{3} - 2 = 0$$

$$\frac{y^3+3y^2+3y+1 - 3(y^2+2y+1) + y+1 - 6}{3} = 0$$

$$y^3 + 3y^2 + 3y + 1 - 3y^2 - 6y - 3 + y + 1 - 6 = 0$$

$$y^3 - 2y - 7 = 0.$$

The sum $(3\alpha - 1)^n + (3\beta - 1)^n + (3\gamma - 1)^n$ is denoted by S_n.

(ii) Find the value of S_3. [2]

From $y^3 - 2y - 7 = 0$

$$S_3 - 2S_1 - 21 = 0$$

$$S_3 - 2\left[\frac{-0}{1}\right] - 21 = 0$$

$$S_3 - 21 = 0$$

$$S_3 = 21.$$

(iii) Find the value of S_{-2}. **[4]**

Dividing $y^3 - 2y - 7 = 0$ by y^2 gives:

$$y^1 - 2y^{-1} - 7y^{-2} = 0$$

$$\Rightarrow S_1 - 2S_{-1} - 7S_{-2} = 0.$$

$$S_1 = \frac{-b}{a} = \frac{-0}{1} = 0.$$

$$S_{-1} = \frac{\Sigma(3\alpha-1)(3\beta-1)}{(3\alpha-1)(3\beta-1)(3\gamma-1)} = \frac{\frac{c}{a}}{\frac{-d}{a}} = \frac{\frac{-2}{1}}{\frac{-(-7)}{1}} = \frac{-2}{7}.$$

$$\therefore 0 - 2\left(-\frac{2}{7}\right) - 7S_{-2} = 0$$

$$\frac{4}{7} = 7S_{-2}$$

$$\therefore S_{-2} = \frac{4}{49}.$$

25 The equation

$$x^3 - x + 1 = 0$$

has roots α, β, γ.

(i) Use the relation $x = y^{\frac{1}{3}}$ to show that the equation

$$y^3 + 3y^2 + 2y + 1 = 0$$

has roots α^3, β^3, γ^3. Hence write down the value of $\alpha^3 + \beta^3 + \gamma^3$. [3]

i) $x = y^{\frac{1}{3}}$

Substituting in the original equation gives:

$$\left(y^{\frac{1}{3}}\right)^3 - y^{\frac{1}{3}} + 1 = 0$$

$$y - y^{\frac{1}{3}} + 1 = 0$$

$$\left[y + 1\right]^3 = \left[y^{\frac{1}{3}}\right]^3$$

$$y^3 + 3y^2 + 3y + 1 = y$$

$$y^3 + 3y^2 + 2y + 1 = 0.$$ has roots $\alpha^3, \beta^3, \gamma^3$.

$$\therefore \Sigma\alpha^3 = \frac{-b}{a} = \frac{-3}{1} = -3.$$

Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(ii) Find the value of S_{-3}. [2]

$$S_{-3} = \Sigma \frac{1}{\alpha^3} = \frac{\Sigma \alpha^3\beta^3}{\alpha^3\beta^3\gamma^3} = \frac{\frac{c}{a}}{\frac{-d}{a}} = \frac{\frac{2}{1}}{\frac{-1}{1}} = \frac{2}{-1} = -2.$$

(iii) Show that $S_6 = 5$ and find the value of S_9. [4]

$$S_6 = \Sigma(\alpha^3)^2 = (\Sigma\alpha^3)^2 - 2\Sigma\alpha^3\beta^3$$
$$= (-3)^2 - 2(2)$$
$$= 9 - 4$$
$$S_6 = 5 \text{ (shown)}.$$

$$y^3 + 3y^2 + 2y + 1 = 0$$
$$(\Sigma\alpha^9) + 3(\Sigma\alpha^6) + 2(\Sigma\alpha^3) + 3 = 0$$
$$S_9 + 3(5) + 2(-3) + 3 = 0$$
$$S_9 + 15 - 6 + 3 = 0$$
$$S_9 = -12.$$

26 A cubic equation $x^3 + bx^2 + cx + d = 0$ has real roots α, β and γ such that

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = -\frac{5}{12},$$
$$\alpha\beta\gamma = -12,$$
$$\alpha^3 + \beta^3 + \gamma^3 = 90.$$

(i) Find the values of c and d. [3]

$$\Sigma \frac{1}{\alpha} = \frac{\Sigma \alpha\beta}{\alpha\beta\gamma}$$
$$\frac{-5}{12} = \frac{\frac{c}{1}}{-12}$$
$$\frac{-5}{12} = \frac{c}{-12}$$
$$c = 5$$

$$x^3 + bx^2 + cx + d = 0$$
$$\Rightarrow \alpha\beta\gamma = \frac{-d}{1}$$
$$-12 = -d$$
$$d = 12.$$

(ii) Express $\alpha^2 + \beta^2 + \gamma^2$ in terms of b. [2]

$$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\,\Sigma\alpha\beta$$
$$= \left(\frac{-b}{1}\right)^2 - 2\left(\frac{c}{1}\right) = b^2 - 2c$$
$$= b^2 - 2(5)$$
$$\Sigma\alpha^2 = b^2 - 10$$

(iii) Show that $b^3 - 15b + 126 = 0$. [4]

$$x^3 + bx^2 + cx + d = 0$$
$$\Rightarrow x^3 + bx^2 + 5x + 12 = 0$$
$$\therefore S_3 + bS_2 + 5S_1 + 36 = 0$$
$$90 + b(b^2 - 10) + 5\left(\frac{-b}{1}\right) + 36 = 0$$
$$90 + b^3 - 10b - 5b + 36 = 0$$
$$b^3 - 15b + 126 = 0.$$

(iv) Given that $3 + i\sqrt{(12)}$ is a root of $y^3 - 15y + 126 = 0$, deduce the value of b. [2]

For the equation $b^3 - 15b + 126 = 0$, complex roots exist as conjugate pairs.

$$\sum \alpha = \frac{-(0)}{1} = 0.$$

Let, α be the real root. β, γ are complex. If $\beta = 3 + \sqrt{12}\,i$, then $\gamma = 3 - \sqrt{12}\,i$.

$$\therefore \alpha + 3 + \sqrt{12}\,i + 3 - \sqrt{12}\,i = 0$$

$$\alpha + 6 = 0$$

$$\alpha = -6$$

i.e Real root of equation $= -6$.

Since b is real, ignore complex roots so $b = -6$.

27 Find the sum of the squares of the roots of the equation

$$x^3 + x + 12 = 0,$$

and deduce that only one of the roots is real. [4]

The real root of the equation is denoted by α. Prove that $-3 < \alpha < -2$, and hence prove that the modulus of each of the other roots lies between 2 and $\sqrt{6}$. [5]

$$x^3 + x + 12 = 0.$$

$$\Sigma\alpha = \frac{-b}{a} = \frac{-0}{1} = 0, \quad \Sigma\alpha\beta = \frac{c}{a} = \frac{1}{1} = 1.$$

$$\therefore \Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta = 0^2 - 2(1) = -2.$$

In a cubic equation, the number of complex roots is at most 3. However, when a cubic equation has real coefficients, complex roots exist as conjugate pairs. For the equation $x^3+x+12=0$, $\Sigma\alpha^2 = -2 < 0$ $\therefore$ there exists complex roots. Then 2 roots must be complex and form a conjugate pair while the third ~~root~~ root can be real.

Let, $f(x) = x^3 + x + 12.$

$$f(-3) = (-3)^3 + (-3) + 12 = -18.$$

$$f(-2) = (-2)^3 + (-2) + 12 = 2.$$

Since $f(-3) < 0$ and $f(-2) > 0$, then there exists a root α such that $-3 < \alpha < 2$. (Shown).

Since β, γ are complex and exist as conjugate pairs, $\gamma = \beta^*$.

Then $\alpha\beta\gamma = \frac{-d}{a} = \frac{-12}{1} = -12$

i.e. $\alpha\beta\beta^* = -12.$

Note that $\beta\beta^* = |\beta|^2.$

$$\therefore \alpha\beta\beta^* = -12$$

$$\Rightarrow \alpha|\beta|^2 = -12$$

i.e. $\alpha = \dfrac{-12}{|\beta|^2}.$

Then $$-3 < \frac{-12}{|\beta|^2} < -2$$

$$-3 < \frac{-12}{|\beta|^2}$$

$$-3|\beta|^2 < -12$$

$$|\beta|^2 > 4$$

$$|\beta| > 2$$

$$\frac{-12}{|\beta|^2} < -2$$

$$6 > |\beta|^2$$

$$\sqrt{6} > |\beta|$$

(Ignore negative range since $|\beta| > 0$)

$$\therefore \quad \underline{2 < |\beta| < \sqrt{6}}$$

Note $|\beta| = |\beta^*| = |\gamma|$ $\therefore$ $\underline{2 < |\gamma| < \sqrt{6}}$

28 Given that

$$\alpha+\beta+\gamma=0, \quad \alpha^2+\beta^2+\gamma^2=14, \quad \alpha^3+\beta^3+\gamma^3=-18,$$

find a cubic equation whose roots are α, β, γ. [4]

Hence find possible values for α, β, γ. [2]

Let, cubic equation be of the form $x^3+bx^2+cx+d=0$.

$$\Sigma\alpha = \frac{-b}{1} = 0$$

$$\Rightarrow b=0$$

$$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$$

$$14 = 0^2 - 2\left(\frac{c}{1}\right)$$

$$-7 = c$$

$$\therefore x^3+0x^2-7x+d=0$$

$$\Rightarrow x^3-7x+d=0$$

Then $S_3 - 7S_1 + 3d = 0$

$$-18 - 7(0) + 3d = 0$$

$$3d = 18$$

$$d = 6$$

$\therefore$ Cubic equation: $x^3-7x+6=0$.

Let, $f(x) = x^3-7x+6=0$

$f(1) = 1^3-7(1)+6=0$

$\therefore x-1$ is a factor of $f(x)$.

1	1	0	-7	6
		1	1	-6
	1	1	-6	x

$$f(x) = (x-1)(x^2+x-6)$$

$$0 = (x-1)(x+3)(x-2)$$

$$\Rightarrow x = 1, 2, -3.$$

i.e. $\alpha=1, \beta=2, \gamma=-3$.

29 In the equation

$$x^3 + ax^2 + bx + c = 0,$$

the coefficients a, b and c are real. It is given that all the roots are real and greater than 1.

(i) Prove that $a < -3$. [1]

(ii) By considering the sum of the squares of the roots, prove that $a^2 > 2b + 3$. [2]

(iii) By considering the sum of the cubes of the roots, prove that $a^3 < -9b - 3c - 3$. [4]

i) $\alpha > 1, \beta > 1, \gamma > 1 \quad \therefore \alpha + \beta + \gamma > 3$

i.e. $\Sigma\alpha > 3$

$\frac{-B}{A} > 3$

$\frac{-a}{1} > 3$

$a < -3$.

(Shown).

ii) $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\,\Sigma\alpha\beta = \left(\frac{-a}{1}\right)^2 - 2\left(\frac{b}{1}\right) = a^2 - 2b$.

$$\left.\begin{array}{l} \alpha > 1 \Rightarrow \alpha^2 > 1 \\ \beta > 1 \Rightarrow \beta^2 > 1 \\ \gamma > 1 \Rightarrow \gamma^2 > 1 \end{array}\right\} \alpha^2 + \beta^2 + \gamma^2 > 3 \quad \text{i.e. } \Sigma\alpha^2 > 3$$

$\therefore a^2 - 2b > 3$

$a^2 > 2b + 3$.

(Shown).

iii) $x^3 + ax^2 + bx + c = 0$

$\Rightarrow S_3 + aS_2 + bS_1 + 3c = 0$

$S_3 + a(a^2 - 2b) + b(-a) + 3c = 0$

$S_3 + a^3 - 2ab - ab + 3c = 0$

$S_3 + a^3 - 3ab + 3c = 0$

$S_3 = -a^3 + 3ab - 3c$.

$$\left.\begin{array}{l} \alpha > 1 \Rightarrow \alpha^3 > 1 \\ \beta > 1 \Rightarrow \beta^3 > 1 \\ \gamma > 1 \Rightarrow \gamma^3 > 1 \end{array}\right\} \alpha^3 + \beta^3 + \gamma^3 > 3 \quad \text{i.e. } \Sigma\alpha^3 = S_3 > 3.$$

$$-a^3+3ab-3c > 3$$

$$3ab-3c-3 > a^3$$

$$\text{i.e. } a^3 < 3ab-3c-3.$$

Note $\Sigma\alpha\beta = \alpha\beta+\alpha\gamma+\beta\gamma > 0$ because $\alpha\beta > 1, \alpha\gamma > 1$ and $\beta\gamma > 1$.

$$\therefore \frac{C}{A} = \frac{b}{1} > 0 \Rightarrow b > 0$$

Then $a < -3 \Rightarrow ab < -3b$

$$\therefore 3ab < -9b.$$

Then $a^3 < 3ab-3c-3 < -9b-3c-3.$

Then $a^3 < -9b-3c-3.$

(Shown).

30 The roots of the equation

$$x^3 + x + 1 = 0$$

are α, β, γ. Show that the equation whose roots are

$$\frac{4\alpha+1}{\alpha+1}, \quad \frac{4\beta+1}{\beta+1}, \quad \frac{4\gamma+1}{\gamma+1}$$

is of the form

$$y^3 + py + q = 0,$$

where the numbers p and q are to be determined. [5]

Hence find the value of

$$\left(\frac{4\alpha+1}{\alpha+1}\right)^n + \left(\frac{4\beta+1}{\beta+1}\right)^n + \left(\frac{4\gamma+1}{\gamma+1}\right)^n,$$

for $n = 2$ and for $n = 3$. [4]

Let, $y = \frac{4x+1}{x+1} \Rightarrow xy + y = 4x + 1 \Rightarrow xy - 4x = 1 - y \Rightarrow x(y-4) = 1 - y$

$\Rightarrow x = \frac{1-y}{y-4}$.

Substituting into the original equation gives:

$$\left(\frac{1-y}{y-4}\right)^3 + \left(\frac{1-y}{y-4}\right) + 1 = 0$$

$$\frac{(1-y)^3}{(y-4)^3} + \frac{1-y}{y-4} + 1 = 0$$

$$1 - 3y + 3y^2 - y^3 + (1-y)(y^2 - 8y + 16) + (y-4)^3 = 0$$

$$1 - 3y + 3y^2 - y^3 + y^2 - y^3 - 8y + 8y^2 + 16 - 16y + y^3 - 12y^2 + 48y - 64 = 0$$

$$-y^3 + 21y - 47 = 0$$

$$y^3 - 21y + 47 = 0.$$

(Shown).

Let, $S_n = \left(\frac{4\alpha+1}{\alpha+1}\right)^n + \left(\frac{4\beta+1}{\beta+1}\right)^n + \left(\frac{4\gamma+1}{\gamma+1}\right)^n$.

$$S_2 = \Sigma\left(\frac{4\alpha+1}{\alpha+1}\right)^2 = \left[\Sigma\left(\frac{4\alpha+1}{\alpha+1}\right)\right]^2 - 2\Sigma\left(\frac{4\alpha+1}{\alpha+1}\right)\left(\frac{4\beta+1}{\beta+1}\right)$$

$$= \left[\frac{-0}{1}\right]^2 - 2\left[\frac{-21}{1}\right] = 0 + 42 = 42.$$

From derived equation $y^3 - 21y + 47 = 0$

$$S_3 - 21 S_1 + 141 = 0$$

$$S_3 - 21\left[\frac{-0}{1}\right] + 141 = 0$$

$$S_3 = -141.$$

31 The roots of the equation

$$x^3 - 8x^2 + 5 = 0$$

are α, β, γ. Show that

$$\alpha^2 = \frac{5}{\beta + \gamma}.$$ [4]

It is given that the roots are all real. Without reference to a graph, show that one of the roots is negative and the other two roots are positive. [3]

From $x^3 - 8x^2 + 5 = 0$,

$$\alpha + \beta + \gamma = \frac{-b}{a} = \frac{-(-8)}{1} = 8.$$

$$\therefore \beta + \gamma = 8 - \alpha \rightarrow (1)$$

Since α is a root of the equation $x^3 - 8x^2 + 5 = 0$

$$\therefore \alpha^3 - 8\alpha^2 + 5 = 0$$

$$\alpha^2(\alpha - 8) + 5 = 0$$

$$\alpha^2 = \frac{-5}{\alpha - 8}$$

$$\alpha^2 = \frac{5}{8 - \alpha} \rightarrow (2)$$

Substituting (1) in (2) gives:

$$\alpha^2 = \frac{5}{\beta + \gamma} \text{ (Shown).}$$

Product of roots $= \alpha\beta\gamma = \frac{-d}{a} = \frac{-5}{1} = -5.$

$\alpha\beta\gamma$ is negative $\therefore$ either all three roots are negative or one root is negative ~~positive~~ and two roots are ~~negativ~~ positive.

Let, α, β, γ are all negative. But $\alpha + \beta + \gamma = 8$. This is not possible if all three roots are negative. Then we have a contradiction. So α, β, γ are not all negative. Therefore, one root is negative and the other two are positive.

32 The roots of the equation

$$x^4 - 5x^2 + 2x - 1 = 0$$

are $\alpha, \beta, \gamma, \delta$. Let $S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$.

(i) Show that

$$S_{n+4} - 5S_{n+2} + 2S_{n+1} - S_n = 0.$$ [2]

(ii) Find the values of S_2 and S_4. [3]

(iii) Find the value of S_3 and hence find the value of S_6. [6]

(iv) Hence find the value of

$$\alpha^2(\beta^4 + \gamma^4 + \delta^4) + \beta^2(\gamma^4 + \delta^4 + \alpha^4) + \gamma^2(\delta^4 + \alpha^4 + \beta^4) + \delta^2(\alpha^4 + \beta^4 + \gamma^4).$$ [3]

i)

$$x^4 - 5x^2 + 2x - 1 = 0$$

Multiplying by x^n gives:

$$x^{n+4} - 5x^{n+2} + 2x^{n+1} - x^n = 0$$

Since α is a root,

$$\alpha^{n+4} - 5\alpha^{n+2} + 2\alpha^{n+1} - \alpha^n = 0$$

Summing ~~the~~ over the equations with β, γ and δ gives:

$$S_{n+4} - 5S_{n+2} + 2S_{n+1} - S_n = 0.$$

ii) $S_2 = \left(\Sigma\alpha\right)^2 - 2\left(\Sigma\alpha\beta\right) = \left(\frac{-0}{1}\right)^2 - 2\left(\frac{-5}{1}\right) = 10.$

Substituting $n=0$ gives:

$$S_4 - 5S_2 + 2S_1 - S_0 = 0$$

$$S_4 - 5(10) + 2\left(\frac{-0}{1}\right) - 4 = 0$$

$$S_4 - 50 + 0 - 4 = 0$$

$$S_4 = 54.$$

iii) Substituting $n = -1$ gives:

$$S_3 - 5S_1 + 2S_0 - S_{-1} = 0.$$

$$S_{-1} = \Sigma \frac{1}{\alpha} = \frac{\Sigma\alpha\beta\gamma}{\alpha\beta\gamma\delta} = \frac{\frac{-2}{1}}{\frac{-1}{1}} = 2.$$

$$\therefore S_3 - 5(0) + 2(4) - 2 = 0$$

$$S_3 = -6.$$

Substituting $n = 2$ gives:

$$S_6 - 5S_4 + 2S_3 - S_2 = 0$$

$$S_6 - 5(54) + 2(-6) - 10 = 0$$

$$S_6 = 10 + 12 + 270$$

$$S_6 = 292.$$

iv) $$\alpha^2(\beta^4+\gamma^4+\delta^4)+\beta^2(\alpha^4+\gamma^4+\delta^4)+\gamma^2(\alpha^4+\beta^4+\delta^4)+\delta^2(\alpha^4+\beta^4+\gamma^4)=\Sigma\alpha^2\beta^4$$

$$S_2S_4=(\alpha^2+\beta^2+\gamma^2+\delta^2)(\alpha^4+\beta^4+\gamma^4+\delta^4)$$

$$S_2S_4=(\alpha^6+\beta^6+\gamma^6+\delta^6)+\Sigma\alpha^2\beta^4$$

$$S_2S_4=S_6+\Sigma\alpha^2\beta^4$$

$$\therefore\ \Sigma\alpha^2\beta^4=S_2S_4-S_6$$

$$=10(54)-292=248.$$

33 The equation

$$x^3 + 5x + 3 = 0$$

has roots α, β, γ. Use the substitution $x = -\frac{3}{y}$ to find a cubic equation in y and show that the roots of this equation are $\beta\gamma, \gamma\alpha, \alpha\beta$. [4]

Find the exact values of $\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2$ and $\beta^3\gamma^3 + \gamma^3\alpha^3 + \alpha^3\beta^3$. [5]

$x = \frac{-3}{y} \Rightarrow y = \frac{-3}{x}$

Substituting gives:

$$x^3 + 5x + 3 = 0$$

$$\left(\frac{-3}{y}\right)^3 + 5\left(\frac{-3}{y}\right) + 3 = 0$$

$$\frac{-27}{y^3} - \frac{15}{y} + 3 = 0$$

$$\frac{-27 - 15y^2 + 3y^3}{y^3} = 0$$

$$3y^3 - 15y^2 - 27 = 0$$

$$y^3 - 5y^2 - 9 = 0.$$

Note $\alpha\beta\gamma = \frac{-d}{a} = -\frac{3}{1} = -3$

$\therefore y = \frac{-3}{x} = \frac{\alpha\beta\gamma}{x}.$

When $x = \alpha$, $y = \frac{\alpha\beta\gamma}{\alpha} = \beta\gamma$,

when $x = \beta$, $y = \frac{\alpha\beta\gamma}{\beta} = \alpha\gamma$,

when $x = \gamma$, $y = \frac{\alpha\beta\gamma}{\gamma} = \alpha\beta$.

$\therefore$ Roots are $\alpha\beta, \alpha\gamma$ and $\beta\gamma$.

Define $S_n = (\alpha\beta)^n + (\alpha\gamma)^n + (\beta\gamma)^n$.

$\Sigma\alpha\beta = S_1 = \frac{-b}{a} = \frac{-(-5)}{1} = 5.$, $\Sigma(\alpha\beta)(\alpha\gamma) = \frac{c}{a} = \frac{0}{1} = 0.$

$\therefore S_2 = (\Sigma\alpha\beta)^2 - 2(\Sigma(\alpha\beta)(\alpha\gamma)) = 5^2 - 2(0) = 25.$

From the equation $y^3 - 5y^2 - 9 = 0$

$$S_3 - 5S_2 - 27 = 0$$

$$S_3 - 5(25) - 27 = 0$$

$$\underline{S_3 = 152.}$$

34 The roots of the equation $x^3 + 4x - 1 = 0$ are α, β and γ. Use the substitution $y = \frac{1}{1+x}$ to show that the equation $6y^3 - 7y^2 + 3y - 1 = 0$ has roots $\frac{1}{\alpha+1}$, $\frac{1}{\beta+1}$ and $\frac{1}{\gamma+1}$. [2]

For the cases $n = 1$ and $n = 2$, find the value of

$$\frac{1}{(\alpha+1)^n} + \frac{1}{(\beta+1)^n} + \frac{1}{(\gamma+1)^n}.$$ [2]

Deduce the value of $\frac{1}{(\alpha+1)^3} + \frac{1}{(\beta+1)^3} + \frac{1}{(\gamma+1)^3}$. [2]

Hence show that $\frac{(\beta+1)(\gamma+1)}{(\alpha+1)^2} + \frac{(\gamma+1)(\alpha+1)}{(\beta+1)^2} + \frac{(\alpha+1)(\beta+1)}{(\gamma+1)^2} = \frac{73}{36}$. [3]

$$x^3+4x-1=0.$$

Let, $y=\frac{1}{1+x} \Rightarrow 1+x=\frac{1}{y} \Rightarrow x=\frac{1}{y}-1 \Rightarrow x=\frac{1-y}{y}.$

$$x^3+4x-1=0$$

$$\left(\frac{1-y}{y}\right)^3+4\left(\frac{1-y}{y}\right)-1=0$$

$$\frac{1-3y+3y^2-y^3}{y^3}+\frac{4-4y}{y}-1=0$$

$$\frac{1-3y+3y^2-y^3+4y^2-4y^3-y^3}{y^3}=0$$

$$-6y^3+7y^2-3y+1=0$$

$$\therefore 6y^3-7y^2+3y-1=0. \quad] \text{ Roots}$$

$x=\alpha \Rightarrow y=\frac{1}{\alpha+1}$, $x=\beta \Rightarrow y=\frac{1}{\beta+1}$, $x=\gamma \Rightarrow y=\frac{1}{\gamma+1}$.

(Shown).

Let, $S_n=\frac{1}{(\alpha+1)^n}+\frac{1}{(\beta+1)^n}+\frac{1}{(\gamma+1)^n}.$

$$S_1=\frac{-b}{a}=\frac{-(-7)}{6}=\frac{7}{6}.$$

$$S_2=\left(\Sigma\frac{1}{\alpha+1}\right)^2-2\left(\Sigma\frac{1}{\alpha+1}\cdot\frac{1}{\beta+1}\right)=\left(\frac{7}{6}\right)^2-2\left(\frac{3}{6}\right)=\frac{49}{36}-1=\frac{13}{36}.$$

$$6y^3-7y^2+3y-1=0$$

$$\Rightarrow 6S_3-7S_2+3S_1-3=0$$

$$6S_3-7\left(\frac{13}{36}\right)+3\left(\frac{7}{6}\right)-3=0 \Rightarrow S_3=\frac{7\left(\frac{13}{36}\right)-3\left(\frac{7}{6}\right)+3}{6}$$

$$S_3=\frac{73}{216}.$$

$$\frac{(\beta+1)(\gamma+1)}{(\alpha+1)^2} + \frac{(\gamma+1)(\alpha+1)}{(\beta+1)^2} + \frac{(\alpha+1)(\beta+1)}{(\gamma+1)^2}$$

$$= \frac{(\alpha+1)(\beta+1)(\gamma+1)}{(\alpha+1)^3} + \frac{(\alpha+1)(\beta+1)(\gamma+1)}{(\beta+1)^3} + \frac{(\alpha+1)(\beta+1)(\gamma+1)}{(\gamma+1)^3}$$

$$= (\alpha+1)(\beta+1)(\gamma+1)\left[\frac{1}{(\alpha+1)^3} + \frac{1}{(\beta+1)^3} + \frac{1}{(\gamma+1)^3}\right]$$

$$= \frac{1}{\frac{1}{(\alpha+1)(\beta+1)(\gamma+1)}}\left[S_3\right]$$

Note $\frac{1}{(\alpha+1)(\beta+1)(\gamma+1)}$ = Product of roots of equation $6y^3-7y^2+3y-1=0$

$$= \frac{-d}{a} = \frac{-(-1)}{6} = \frac{1}{6}.$$

$$\therefore \frac{1}{\frac{1}{(\alpha+1)(\beta+1)(\gamma+1)}} \times S_3 = \frac{1}{\frac{1}{6}} \times \frac{73}{216} = 6 \times \frac{73}{216} = \frac{73}{36}.$$

(Shown).

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35 The equation $x^3 + px + q = 0$ has a repeated root. Prove that $4p^3 + 27q^2 = 0$. [5]

Let, the roots be α, α, β.

$\sum \alpha = \frac{-b}{a} = \frac{-0}{1} = 0 \therefore \alpha + \alpha + \beta = 0$

$\beta = -2\alpha \rightarrow (1).$

$\sum \alpha\beta = \frac{c}{a} = \frac{p}{1} = p \therefore \alpha^2 + \alpha\beta + \alpha\beta = p$

$\alpha^2 + 2\alpha\beta = p \rightarrow (2)$

Substituting (1) in (2) gives:

$\alpha^2 + 2\alpha(-2\alpha) = p$

$\alpha^2 - 4\alpha^2 = p$

$-3\alpha^2 = p$

$\sum \alpha\beta\gamma = \frac{-d}{a} = \frac{-q}{1} \therefore \alpha^2\beta = -q \rightarrow (3)$

Substituting (1) in (3) gives:

$\alpha^2(-2\alpha) = -q$

$2\alpha^3 = q$

Then $4p^3 + 27q^2 = 4\left[-3\alpha^2\right]^3 + 27\left[2\alpha^3\right]^2 = 4\left[-27\alpha^6\right] + 27(4\alpha^6)$

$= -108\alpha^6 + 108\alpha^6$

$= 0.$

$\therefore 4p^3 + 27q^2 = 0.$

(Proven).

36 The equation

$$x^3 + 5x^2 - 3x - 15 = 0$$

has roots α, β, γ. Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [3]

Hence show that the matrix $\begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{pmatrix}$ is singular. [4]

$$x^3 + 5x^2 - 3x - 15 = 0$$

$$\Sigma\alpha = \frac{-b}{a} = \frac{-5}{1} = -5, \quad \Sigma\alpha\beta = \frac{c}{a} = \frac{-3}{1} = -3.$$

$$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta = (-5)^2 - 2(-3) = 25 + 6 = 31$$

$$\text{i.e. } \alpha^2 + \beta^2 + \gamma^2 = 31.$$

For matrix to be singular, determinant is zero:

$$\begin{vmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{vmatrix} = 1[1-\gamma^2] - \alpha[\alpha - \beta\gamma] + \beta[\alpha\gamma - \beta]$$

$$= 1 - \gamma^2 - \alpha^2 + \alpha\beta\gamma + \alpha\beta\gamma - \beta^2 = 1 - \alpha^2 - \beta^2 - \gamma^2 + 2\alpha\beta\gamma$$

$$= 1 - (\alpha^2 + \beta^2 + \gamma^2) + 2\alpha\beta\gamma = 1 - [31] + 2\left[-\frac{(-15)}{1}\right] = 1 - 31 + 30 = 0$$

$\therefore$ determinant $= 0$ $\therefore$ matrix is singular.

37 The roots of the equation $x^4 - 3x^2 + 5x - 2 = 0$ are $\alpha, \beta, \gamma, \delta$, and $\alpha^n + \beta^n + \gamma^n + \delta^n$ is denoted by S_n. Show that

$$S_{n+4} - 3S_{n+2} + 5S_{n+1} - 2S_n = 0. \qquad [2]$$

Find the values of

(i) S_2 and S_4, [3]

(ii) S_3 and S_5. [6]

Hence find the value of

$$\alpha^2(\beta^3 + \gamma^3 + \delta^3) + \beta^2(\gamma^3 + \delta^3 + \alpha^3) + \gamma^2(\delta^3 + \alpha^3 + \beta^3) + \delta^2(\alpha^3 + \beta^3 + \gamma^3). \qquad [3]$$

$$x^4 - 3x^2 + 5x - 2 = 0$$

Multiplying by x^n gives:

$$x^n(x^4 - 3x^2 + 5x - 2) = x^n(0)$$

$$x^{n+4} - 3x^{n+2} + 5x^{n+1} - 2x^n = 0$$

$$\therefore \; S_{n+4} - 3S_{n+2} + 5S_{n+1} - 2S_n = 0. \text{ (Shown).}$$

i) $S_1 = \Sigma\alpha = \frac{-b}{a} = \frac{-0}{1} = 0$, $\Sigma\alpha\beta = \frac{c}{a} = \frac{-3}{1} = -3$.

$$\therefore S_2 = \Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta = 0^2 - 2(-3) = 6.$$

$n=0 \Rightarrow S_4 - 3S_2 + 5S_1 - 2S_0 = 0 \Rightarrow S_4 - 3S_2 + 5S_1 - 8 = 0$

$$S_4 - 3(6) + 5(0) - 8 = 0$$

$$S_4 = 26.$$

ii) Substituting $n = -1$ gives:

$$S_3 - 3S_1 + 5S_0 - 2S_{-1} = 0.$$

$$S_{-1} = \Sigma\frac{1}{\alpha} = \frac{\Sigma\alpha\beta\gamma}{\alpha\beta\gamma\delta} = \frac{\frac{-d}{a}}{\frac{e}{a}} = \frac{\frac{-5}{1}}{\frac{-2}{1}} = \frac{5}{2}.$$

$$\therefore \; S_3 - 3(0) + 5(4) - 2\left(\frac{5}{2}\right) = 0$$

$$S_3 = -15.$$

Substituting $n = 1$ gives:

$$S_5 - 3S_3 + 5S_2 - 2S_1 = 0$$

$$S_5 - 3(-15) + 5(6) - 2(0) = 0$$

$$S_5 = -75.$$

Let, $\alpha^2(\beta^3+\gamma^3+\delta^3)+\beta^2(\alpha^3+\gamma^3+\delta^3)+\gamma^2(\alpha^3+\beta^3+\delta^3)+\delta^2(\alpha^3+\beta^3+\gamma^3)=\sum\alpha^2\beta^3$.

$$S_2 S_3 = (\alpha^2+\beta^2+\gamma^2+\delta^2)(\alpha^3+\beta^3+\gamma^3+\delta^3)$$

$$S_2 S_3 = (\alpha^5+\beta^5+\gamma^5+\delta^5)+\sum\alpha^2\beta^3$$

$$S_2 S_3 = S_5+\sum\alpha^2\beta^3$$

$$\therefore \sum\alpha^2\beta^3 = S_2 S_3 - S_5$$

$$= 6(-15)-(-75)$$

$$= -90+75$$

$$\sum\alpha^2\beta^3 = -15$$

38 A cubic equation has roots α, β and γ such that

$$\alpha + \beta + \gamma = 4,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 14,$$
$$\alpha^3 + \beta^3 + \gamma^3 = 34.$$

Find the value of $\alpha\beta + \beta\gamma + \gamma\alpha$. [2]

Show that the cubic equation is

$$x^3 - 4x^2 + x + 6 = 0,$$

and solve this equation. [6]

$\sum\alpha^2 = (\sum\alpha)^2 - 2\sum\alpha\beta$

$14 = 4^2 - 2\sum\alpha\beta$

$14 = 16 - 2\sum\alpha\beta$

$2\sum\alpha\beta = 2$

$\therefore \sum\alpha\beta = 1.$

i.e. $\alpha\beta + \alpha\gamma + \beta\gamma = \underline{1}.$

Let, cubic equation is $x^3 + bx^2 + cx + d = 0$

$\sum\alpha = 4 \therefore -b = 4 \Rightarrow b = -4.$

$\sum\alpha\beta = 1 \quad \therefore c = 1$

Then equation is of the form $x^3 - 4x^2 + x + d = 0$

$\Rightarrow S_3 - 4S_2 + S_1 + 3d = 0$

$34 - 4(14) + 4 + 3d = 0$

$d = 6$

$\therefore$ Equation is $x^3 - 4x^2 + x + 6 = 0.$

Let, $f(x) = x^3 - 4x^2 + x + 6 = 0$

$f(-1) = (-1)^3 - 4(-1)^2 + (-1) + 6 = -1 - 4 - 1 + 6 = 0$

$\therefore x+1$ is a factor of $f(x)$.

$$\begin{array}{r|rrrr} -1 & 1 & -4 & 1 & 6 \\ & & -1 & 5 & -6 \\ \hline & 1 & -5 & 6 & \times \end{array}$$

$$\therefore \; f(x) = (x+1)(x^2-5x+6)$$

$$0 = (x+1)(x-2)(x-3)$$

$\therefore \; x+1=0$ or $x-2=0$ or $x-3=0$

$x=-1$ $\qquad x=2$ or $x=3$.

39 The cubic equation $x^3 - px - q = 0$, where p and q are constants, has roots α, β, γ. Show that

(i) $\alpha^2 + \beta^2 + \gamma^2 = 2p$, [1]

(ii) $\alpha^3 + \beta^3 + \gamma^3 = 3q$, [2]

(iii) $6(\alpha^5 + \beta^5 + \gamma^5) = 5(\alpha^3 + \beta^3 + \gamma^3)(\alpha^2 + \beta^2 + \gamma^2)$. [3]

i) $\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum \alpha\beta$

$= \left(\frac{-0}{1}\right)^2 - 2\left(\frac{-p}{1}\right) = 0 + 2p = 2p.$

i.e. $\alpha^2+\beta^2+\gamma^2 = 2p.$

ii) $x^3 - px - q = 0$

$S_3 - pS_1 - 3q = 0$

$\Rightarrow S_3 - p(0) - 3q = 0$

$\therefore S_3 = 3q.$

i.e. $\alpha^3+\beta^3+\gamma^3 = 3q.$

iii) $x^3 - px - q = 0$

Multiplying both sides of equation by x^2 gives:

$x^5 - px^3 - qx^2 = 0$

$\therefore S_5 - pS_3 - qS_2 = 0$

$S_5 = pS_3 + qS_2$

$= p(3q) + q(2p)$

$= 3pq + 2pq$

$S_5 = 5pq$

$6(\alpha^5+\beta^5+\gamma^5) = 5(\alpha^3+\beta^3+\gamma^3)(\alpha^2+\beta^2+\gamma^2)$

$6(5pq) = 5(3q)(2p)$

$30pq = 30pq$

(Shown).

40 The equation

$$8x^3 + 36x^2 + kx - 21 = 0,$$

where k is a constant, has roots $a-d$, a, $a+d$. Find the numerical values of the roots and determine the value of k. [8]

$$\Sigma\alpha = \frac{-B}{A} = \frac{-\cancel{36}^{9}}{\cancel{8}_{2}}$$

$$a-d+a+a+d = \frac{-9}{2}$$

$$3a = \frac{-9}{2} \Rightarrow a = \frac{-9}{2} \div 3 = \frac{-3}{2}.$$

$$\alpha\beta\gamma = \frac{-D}{A}$$

$$(a-d)(a)(a-d) = \frac{-(-21)}{8}$$

$$a(a^2-d^2) = \frac{21}{8}$$

Substituting $a = \frac{-3}{2}$ gives:

$$\frac{-3}{2}\left[\left(\frac{-3}{2}\right)^2 - d^2\right] = \frac{21}{8}$$

$$\frac{9}{4} - d^2 = \frac{-7}{4}$$

$$\frac{9}{4} + \frac{7}{4} = d^2$$

$$d^2 = 4$$

$$\therefore d = 2.$$

Then roots are: $a-d = \frac{-3}{2} - 2 = \frac{-7}{2}$

$a = \frac{-3}{2} = \frac{-3}{2}$

$a+d = \frac{-3}{2} + 2 = \frac{1}{2}.$

$$\Sigma\alpha\beta = \frac{C}{A} = \frac{k}{8}$$

$$\left(\frac{-7}{2}\right)\left(\frac{-3}{2}\right) + \left(\frac{-7}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{-3}{2}\right)\left(\frac{1}{2}\right) = \frac{k}{8}$$

$$\frac{11}{4} = \frac{k}{8} \Rightarrow k = 22.$$

41 The roots of the quartic equation $x^4 + 4x^3 + 2x^2 - 4x + 1 = 0$ are α, β, γ and δ. Find the values of

(i) $\alpha + \beta + \gamma + \delta$, [1]

(ii) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$, [2]

(iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$, [2]

(iv) $\frac{\alpha}{\beta\gamma\delta} + \frac{\beta}{\alpha\gamma\delta} + \frac{\gamma}{\alpha\beta\delta} + \frac{\delta}{\alpha\beta\gamma}$. [2]

Using the substitution $y = x + 1$, find a quartic equation in y. Solve this quartic equation and hence find the roots of the equation $x^4 + 4x^3 + 2x^2 - 4x + 1 = 0$. [7]

i) $\Sigma\alpha = \frac{-b}{1} = \frac{-4}{1} = -4.$

ii) $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta = (-4)^2 - 2\left(\frac{2}{1}\right) = 16 - 4 = 12.$

iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \Sigma\frac{1}{\alpha} = \frac{\Sigma\alpha\beta\gamma}{\alpha\beta\gamma\delta} = \frac{-(-4)}{1} = 4.$

iv) $\frac{\alpha}{\beta\gamma\delta} + \frac{\beta}{\alpha\gamma\delta} + \frac{\gamma}{\alpha\beta\delta} + \frac{\delta}{\alpha\beta\gamma} = \frac{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}{\alpha\beta\gamma\delta} = \frac{12}{1} = 12.$

$y = x + 1 \Rightarrow x = y - 1$

$\therefore \quad (y-1)^4 + 4(y-1)^3 + 2(y-1)^2 - 4(y-1) + 1 = 0$

$y^4 - 4y^3 + 6y^2 - 4y + 1 + 4[y^3 - 3y^2 + 3y - 1] + 2[y^2 - 2y + 1] - 4[y - 1] + 1 = 0$

$y^4 - 4y^3 + 6y^2 - 4y + 1 + 4y^3 - 12y^2 + 12y - 4 + 2y^2 - 4y + 2 - 4y + 4 + 1 = 0$

$y^4 - 4y^2 + 4 = 0$

$(y^2)^2 - 2(y^2)(2) + 2^2 = 0$

$(y^2 - 2)^2 = 0$

$\therefore \quad y^2 - 2 = 0$

$y^2 = 2$

$\Rightarrow y = \pm\sqrt{2}$ twice.

Then $x + 1 = \pm\sqrt{2}$

$\Rightarrow x = -1 \pm \sqrt{2}$ twice

42 The cubic equation $x^3 + px^2 + qx + r = 0$, where p, q and r are integers, has roots α, β and γ, such that

$$\alpha + \beta + \gamma = 15,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 83.$$

Write down the value of p and find the value of q. [3]

Given that α, β and γ are all real and that $\alpha\beta + \alpha\gamma = 36$, find α and hence find the value of r. [5]

$$\Sigma\alpha = \frac{-b}{a} = \frac{-p}{1} = 15$$
$$\therefore p = -15.$$

$$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$$
$$83 = 15^2 - 2\left(\frac{q}{1}\right)$$
$$2q = 225 - 83 = 142$$
$$q = 71$$

$$\alpha\beta + \alpha\gamma = 36$$
$$\alpha(\beta + \gamma) = 36$$
$$\therefore \beta + \gamma = \frac{36}{\alpha}.$$

$$\alpha + \beta + \gamma = 15$$
$$\therefore \alpha + \frac{36}{\alpha} = 15$$
$$\alpha^2 + 36 = 15\alpha$$
$$\alpha^2 - 15\alpha + 36 = 0$$
$$(\alpha - 3)(\alpha - 12) = 0$$
$$\therefore \alpha = 3 \text{ or } \alpha = 12.$$
(Ignore $\alpha = 12$ since $\alpha^2 = 144 > 83$).

$$\therefore \alpha = 3.$$

$$\Sigma\alpha\beta = \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = \frac{71}{1}$$
$$36 + \beta\gamma = 71$$
$$\therefore \beta\gamma = 35.$$

Then $\alpha\beta\gamma = \frac{-d}{a} = \frac{-r}{1} = -r$
$$3(35) = -r$$
$$\therefore r = -105.$$

43 Find the cubic equation with roots α, β and γ such that

$$\alpha + \beta + \gamma = 3,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 1,$$
$$\alpha^3 + \beta^3 + \gamma^3 = -30,$$

giving your answer in the form $x^3 + px^2 + qx + r = 0$, where p, q and r are integers to be found. [6]

$$\Sigma\alpha = \frac{-p}{1} = -p = 3$$
$$\therefore p = -3.$$

$$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$$
$$1 = 3^2 - 2\Sigma\alpha\beta$$
$$2\Sigma\alpha\beta = 8$$
$$\Sigma\alpha\beta = 4$$

$$\Sigma\alpha\beta = \frac{q}{1} = 4$$
$$\therefore q = 4.$$

$$x^3 + px^2 + qx + r = 0$$
$$S_3 + pS_2 + qS_1 + 3r = 0$$
$$-30 + p(1) + q(3) + 3r = 0$$
$$-30 + (-3)(1) + 4(3) + 3r = 0$$
$$r = 7$$
$$\therefore x^3 - 3x^2 + 4x + 7 = 0.$$

44 The cubic equation $2x^3 - 3x^2 + 4x - 10 = 0$ has roots α, β and γ.

(i) Find the value of $(\alpha + 1)(\beta + 1)(\gamma + 1)$. **[4]**

Let, $y = x + 1 \Rightarrow x = y - 1$

$\therefore \quad 2(y-1)^3 - 3(y-1)^2 + 4(y-1) - 10 = 0$

$2[y^3 - 3y^2 + 3y - 1] - 3[y^2 - 2y + 1] + 4y - 4 - 10 = 0$

$2y^3 - 6y^2 + 6y - 2 - 3y^2 + 6y - 3 + 4y - 4 - 10 = 0$

$2y^3 - 9y^2 + 16y - 19 = 0$ has roots $\alpha+1, \beta+1$ and $\gamma+1$.

$\therefore \ (\alpha+1)(\beta+1)(\gamma+1) = \dfrac{-d}{a} = \dfrac{-(-19)}{2} = \dfrac{19}{2} = 9\tfrac{1}{2}$.

(ii) Find the value of $(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta)$. **[4]**

Note $\alpha + \beta + \gamma = \dfrac{-(-3)}{2} = \dfrac{3}{2}$.

$\therefore \ \beta + \gamma = \frac{3}{2} - \alpha, \ \gamma + \alpha = \frac{3}{2} - \beta, \ \alpha + \beta = \frac{3}{2} - \gamma$.

Then $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta) = \left(\frac{3}{2} - \alpha\right)\left(\frac{3}{2} - \beta\right)\left(\frac{3}{2} - \gamma\right)$.

Let, $y = \frac{3}{2} - x \Rightarrow x = \frac{3}{2} - y = \frac{3-2y}{2}$.

Substituting gives:

$$2\left(\frac{3-2y}{2}\right)^3 - 3\left(\frac{3-2y}{2}\right)^2 + 4\left(\frac{3-2y}{2}\right) - 10 = 0$$

$$\cancel{2}\left[\frac{27 - 54y + 36y^2 - 8y^3}{\cancel{8}\ 4}\right] - 3\left[\frac{9 - 12y + 4y^2}{4}\right] + \frac{12 - 4y}{2} - 10 = 0$$

$$\frac{27 - 54y + 36y^2 - 8y^3 - 27 + 36y - 12y^2 + 24 - 8y - 40}{4} = 0$$

$$-8y^3 + 24y^2 - 26y + \overset{-16}{\cancel{8}} = 0$$

$$4y^3 - 12y^2 + 13y + 8 = 0.$$

Then $\sum\left(\frac{3}{2} - \alpha\right) = \dfrac{-D}{A} = \dfrac{-8}{4} = -2$.

i.e. $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta) = -2$.

45 The roots of the equation

$$x^3 + px^2 + qx + r = 0$$

are α, 2α, 4α, where p, q, r and α are non-zero real constants.

(i) Show that

$$2p\alpha + q = 0.$$

[4]

$\Sigma\alpha = \frac{-b}{a} = \frac{-p}{1}$

$\alpha + 2\alpha + 4\alpha = -p$

$7\alpha = -p \Rightarrow p = -7\alpha. \rightarrow (1)$

$\Sigma\alpha\beta = \frac{c}{a} = \frac{q}{1}$

$\alpha(2\alpha) + \alpha(4\alpha) + 2\alpha(4\alpha) = q$

$14\alpha^2 = q \rightarrow (2)$

Dividing (2) by (1) gives:

$\frac{14\alpha^2}{-7\alpha} = \frac{q}{p}$

$-2\alpha = \frac{q}{p}$

$-2p\alpha = q$

$\therefore q + 2p\alpha = 0.$

(Shown).

(ii) Show that

$$p^3 r - q^3 = 0.$$

[2]

$\alpha\beta\gamma = \frac{-r}{1}$

$\alpha(2\alpha)(4\alpha) = -r$

$8\alpha^3 = -r \rightarrow (1)$

From $q + 2p\alpha = 0 \Rightarrow \alpha = \frac{-q}{2p} \rightarrow (2)$

Substituting (2) in (1) gives:

$8\left(\frac{-q}{2p}\right)^3 = -r$

$\frac{-8q^3}{8p^3} = -r \Rightarrow \frac{q^3}{p^3} = r \Rightarrow q^3 = p^3 r$

$\therefore p^3 r - q^3 = 0$

(Shown).

46 The roots of the cubic equation

$$x^3 = 5x^2 + 13x - 4 = 0$$

are α, β, γ.

(i) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. **[3]**

$$\alpha^2 + \beta^2 + \gamma^2 = \left(\Sigma\alpha\right)^2 - 2\left(\Sigma\alpha\beta\right)$$

$$= \left[\frac{-(-5)}{1}\right]^2 - 2\left[\frac{13}{1}\right] = 25 - 26$$

$$= -1$$

(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. **[2]**

$$\therefore\ x^3 - 5x^2 + 13x - 4 = 0$$

$$\Rightarrow S_3 - 5S_2 + 13S_1 - 12 = 0$$

$$S_3 - 5(-1) + 13(5) - 12 = 0$$

$$S_3 = -58.$$

47 The equation $x^3 + 2x^2 + x + 7 = 0$ has roots α, β, γ.

(i) Use the relation $x^2 = -7y$ to show that the equation

$$49y^3 + 14y^2 - 27y + 7 = 0$$

has roots $\frac{\alpha}{\beta\gamma}, \frac{\beta}{\gamma\alpha}, \frac{\gamma}{\alpha\beta}$. [4]

$x^2 = -7y \Rightarrow x = \sqrt{-7y}$.

Substituting gives: $(\sqrt{-7y})^3 + 2(\sqrt{-7y})^2 + (\sqrt{-7y}) + 7 = 0$

$-7y\sqrt{-7y} + 2(-7y) + \sqrt{-7y} + 7 = 0$

$\left[\sqrt{-7y}(-7y+1)\right]^2 = \left[14y - 7\right]^2$

$-7y(-7y+1)^2 = 49\left[4y^2 - 4y + 1\right]$

$-y\left[49y^2 - 14y + 1\right] = 7\left[4y^2 - 4y + 1\right]$

$-49y^3 + 14y^2 - y = 28y^2 - 28y + 7$

$-49y^3 - 14y^2 + 27y - 7 = 0$

$49y^3 + 14y^2 - 27y + 7 = 0$.

From original equation, $\alpha\beta\gamma = \frac{-d}{a} = \frac{-7}{1} = -7$.

$\therefore\ x^2 = -7y \Rightarrow y = \frac{x^2}{-7} = \frac{x^2}{\alpha\beta\gamma}$.

$\therefore$ Derived equation has roots $\frac{\alpha^2}{\alpha\beta\gamma} = \frac{\alpha}{\beta\gamma}$, $\frac{\beta^2}{\alpha\beta\gamma} = \frac{\beta}{\alpha\gamma}$ and $\frac{\gamma^2}{\alpha\beta\gamma} = \frac{\gamma}{\alpha\beta}$.

(shown).

(ii) Show that $\frac{\alpha^2}{\beta^2\gamma^2} + \frac{\beta^2}{\gamma^2\alpha^2} + \frac{\gamma^2}{\alpha^2\beta^2} = \frac{58}{49}$. [3]

From derived equation, $\frac{\alpha}{\beta\gamma} + \frac{\beta}{\alpha\gamma} + \frac{\gamma}{\alpha\beta} = \frac{-B}{A} = \frac{-14}{49} = \frac{-2}{7}$.

And $\sum\left(\frac{\alpha}{\beta\gamma}\right)\left(\frac{\beta}{\alpha\gamma}\right) = \frac{C}{A} = \frac{-27}{49}$.

$\therefore \sum\left(\frac{\alpha}{\beta\gamma}\right)^2 = \left(\sum \frac{\alpha}{\beta\gamma}\right)^2 - 2\sum\left(\frac{\alpha}{\beta\gamma}\right)\left(\frac{\beta}{\alpha\gamma}\right)$

$= \left(\frac{-2}{7}\right)^2 - 2\left(\frac{-27}{49}\right) = \frac{58}{49}$.

(iii) Find the exact value of $\frac{\alpha^3}{\beta^3\gamma^3} + \frac{\beta^3}{\gamma^3\alpha^3} + \frac{\gamma^3}{\alpha^3\beta^3}$. [2]

$49y^3 + 14y^2 - 27y + 7 = 0$

$\Rightarrow 49S_3 + 14S_2 - 27S_1 + 21 = 0$ when $S_n = \left(\frac{\alpha}{\beta\gamma}\right)^n + \left(\frac{\beta}{\alpha\gamma}\right)^n + \left(\frac{\gamma}{\alpha\beta}\right)^n$.

$\therefore 49S_3 + 14\left(\frac{58}{49}\right) - 27\left(\frac{-2}{7}\right) + 21 = 0$

$\therefore S_3 = \frac{-21 + 27\left(\frac{-2}{7}\right) - 14\left(\frac{58}{49}\right)}{49}$

$S_3 = \frac{-317}{343}$.

48 The cubic equation $x^3+bx^2+cx+d=0$, where b, c and d are constants, has roots α, β, γ. It is given that $\alpha\beta\gamma=-1$.

(a) State the value of d. [1]

$\alpha\beta\gamma = \frac{-d}{1} = -d \Rightarrow -1 = -d \Rightarrow d = 1.$

(b) Find a cubic equation, with coefficients in terms of b and c, whose roots are $\alpha+1$, $\beta+1$, $\gamma+1$. [3]

Let, $y = x+1 \Rightarrow x = y-1$.

Substituting, $(y-1)^3 + b(y-1)^2 + c(y-1) + d = 0$

$y^3 - 3y^2 + 3y - 1 + b[y^2 - 2y + 1] + cy - c + d = 0.$

$y^3 - 3y^2 + 3y - 1 + by^2 - 2by + b + cy - c + d = 0$

$y^3 + (b-3)y^2 + (c-2b+3)y + b - c - 1 + d = 0$

Since $d = 1$

$\therefore y^3 + (b-3)y^2 + (c-2b+3)y + b - c - 1 + 1 = 0$

$y^3 + (b-3)y^2 + (c-2b+3)y + b - c = 0.$

(c) Given also that $\gamma+1 = -\alpha-1$, deduce that $(c-2b+3)(b-3) = b-c$. [4]

From derived equation, $S_1 = \frac{-(b-3)}{1} = -(b-3)$ when $S_n = (\alpha+1)^n + (\beta+1)^n + (\gamma+1)^n$.

$\therefore (\alpha+1) + (\beta+1) + (\gamma+1) = -(b-3)$

$(\alpha+1) + (\beta+1) + -\alpha - 1 = -(b-3)$

$\beta + 1 = 3 - b \longrightarrow (1).$

$\sum(\alpha+1)(\beta+1) = \frac{C}{A} = \frac{c-2b+3}{1}$

$\sum(\alpha+1)(\beta+1) = c - 2b + 3$

$(\alpha+1)(\beta+1) + (\alpha+1)(\gamma+1) + (\beta+1)(\gamma+1) = c - 2b + 3.$

$(\alpha+1)(\beta+1) + (\alpha+1)(-(\alpha+1)) + (\beta+1)(-(\alpha+1)) = c - 2b + 3$

$\cancel{(\alpha+1)(\beta+1)} - (\alpha+1)(\alpha+1) - \cancel{(\alpha+1)(\beta+1)} = c - 2b + 3$

$-(\alpha+1)(\alpha+1) = c - 2b + 3.$

$\therefore -(\alpha+1)^2 = c - 2b + 3 \longrightarrow (2).$

$\therefore (\alpha+1)^2 = -(c-2b+3)$

$$(\alpha-1)(\beta-1)(\gamma-1) = \frac{-D}{A} = \frac{-(b-c)}{1}$$

$$(\alpha-1)(\beta-1)(-(\alpha-1)) = -(b-c)$$

$$-(\alpha+1)^2(\beta-1) = -(b-c)$$

$$\therefore (\alpha+1)^2(\beta-1) = b-c \rightarrow (3).$$

Combining (1), (2) and (3) gives:

$$-(c-2b+3)(3-b) = b-c$$

$$(c-2b+3)(b-3) = b-c.$$

(Shown).

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49 The cubic equation $7x^3+3x^2+5x+1=0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$. [3]

Let, $y = \frac{1}{x}$. $\Rightarrow x = \frac{1}{y}$

$$7\left(\frac{1}{y}\right)^3 + 3\left(\frac{1}{y}\right)^2 + 5\left(\frac{1}{y}\right) + 1 = 0$$

$$\frac{7}{y^3} + \frac{3}{y^2} + \frac{5}{y} + 1 = 0$$

$$\frac{7+3y+5y^2+y^3}{y^3} = 0$$

$$\Rightarrow y^3+5y^2+3y+7=0$$

(b) Find the value of $\alpha^{-2}+\beta^{-2}+\gamma^{-2}$. [2]

$$\Sigma\frac{1}{\alpha^2} = \left(\Sigma\frac{1}{\alpha}\right)^2 - 2\Sigma\frac{1}{\alpha\beta}$$

$$= \left(\frac{-5}{1}\right)^2 - 2\left(\frac{3}{1}\right) = 25-6$$

$$\Sigma\frac{1}{\alpha^2} = 19$$

$$\therefore \alpha^{-2}+\beta^{-2}+\gamma^{-2} = 19.$$

(c) Find the value of $\alpha^{-3}+\beta^{-3}+\gamma^{-3}$. [2]

$$y^3+5y^2+3y+7=0$$

$$\therefore S_3+5S_2+3S_1+21=0$$ where $S_n = \frac{1}{\alpha^n}+\frac{1}{\beta^n}+\frac{1}{\gamma^n}$

$$S_3 = -5(19) - 3\left(\frac{-5}{1}\right) - 21$$

$$S_3 = -101$$

$$\therefore \alpha^{-3}+\beta^{-3}+\gamma^{-3} = \underline{-101}.$$

50 The cubic equation $6x^3 + px^2 - 3x - 5 = 0$, where p is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^2, \beta^2, \gamma^2$. [3]

Let, $y = x^2 \Rightarrow x = \sqrt{y}$.

Substituting gives: $6(\sqrt{y})^3 + p(\sqrt{y})^2 - 3\sqrt{y} - 5 = 0$

$$6y\sqrt{y} + py - 3\sqrt{y} - 5 = 0$$

$$6y\sqrt{y} - 3\sqrt{y} = 5 - py$$

$$[\sqrt{y}(6y-3)]^2 = [5 - py]^2$$

$$y(36y^2 - 36y + 9) = 25 - 10py + p^2y^2$$

$$36y^3 - 36y^2 + 9y = 25 - 10py + p^2y^2$$

$$36y^3 - 36y^2 - p^2y^2 + 9y + 10py - 25 = 0$$

$$36y^3 + (-36 - p^2)y^2 + (9 + 10p)y - 25 = 0.$$

(b) It is given that $\alpha^2 + \beta^2 + \gamma^2 = 2(\alpha + \beta + \gamma)$.

(i) Find the value of p. [3]

From original equation, $\alpha + \beta + \gamma = \frac{-b}{a} = \frac{-p}{6}$.

From derived equation, $\alpha^2 + \beta^2 + \gamma^2 = \frac{-B}{A} = \frac{-(-36-p^2)}{36} = \frac{36+p^2}{36}$.

Since $\Sigma\alpha^2 = 2\Sigma\alpha$

$$\frac{36+p^2}{36} = 2\left(\frac{-p}{6}\right)$$

$$p^2 + 36 = -12p$$

$$p^2 + 12p + 36 = 0$$

$$p^2+6p+6p+36=0$$
$$(p+6)^2=0$$
$$\Rightarrow p=-6$$

(ii) Find the value of $\alpha^3+\beta^3+\gamma^3$. [2]

From original equation,

$$6x^3+px^2-3x-5=0$$
$$\Rightarrow 6S_3+pS_2-3S_1-15=0$$
$$6S_3+p\left(\frac{36+p^2}{36}\right)-3\left(\frac{-p}{6}\right)-15=0$$
$$6S_3+(-6)\left[\frac{36+(-6)^2}{36}\right]-3\left[\frac{-(-6)}{6}\right]-15=0$$
$$S_3=5$$
$$\therefore \alpha^3+\beta^3+\gamma^3=5.$$

51 The cubic equation $x^3 + cx + 1 = 0$, where c is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^3, \beta^3, \gamma^3$. [3]

Let, $y = x^3 \Rightarrow x = y^{\frac{1}{3}}$.

Substituting gives: $\left(y^{\frac{1}{3}}\right)^3 + c\,y^{\frac{1}{3}} + 1 = 0$

$$y + c\sqrt[3]{y} + 1 = 0$$

$$\left(c\sqrt[3]{y}\right)^3 = (-1-y)^3$$

$$c^3y = (-1)^3 + 3(-1)^2(-y) + 3(-1)^2(-y)^2 + (-y)^3$$

$$c^3y = -1 - 3y + 3y^2 - y^3$$

$$y^3 + 3y^2 + c^3y + 3y + 1 = 0.$$

$$y^3 + 3y^2 + (c^3+3)y + 1 = 0$$

(b) Show that $\alpha^6 + \beta^6 + \gamma^6 = 3 - 2c^3$. [3]

$\Sigma\alpha^3 = \frac{-B}{A} = \frac{-3}{1} = -3$, $\Sigma\alpha^3\beta^3 = \frac{C}{A} = \frac{c^3+3}{1} = c^3+3$.

$\therefore\ \Sigma\alpha^6 = (\Sigma\alpha^3)^2 - 2\Sigma\alpha^3\beta^3$

$= (-3)^2 - 2(c^3+3)$

$= 9 - 2c^3 - 6$

$= 3 - 2c^3$.

(c) Find the real value of c for which the matrix $\begin{pmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{pmatrix}$ is singular. [5]

For matrix to be singular, determinant $=0$

$$\begin{vmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{vmatrix} = 0$$

$$1\left[1-\gamma^6\right]-\alpha^3\left[\alpha^3-\beta^3\gamma^3\right]+\beta^3\left[\alpha^3\gamma^3-\beta^3\right]=0$$

$$1-\gamma^6-\alpha^6+\alpha^3\beta^3\gamma^3+\alpha^3\beta^3\gamma^3-\beta^6=0$$

$$1+2\alpha^3\beta^3\gamma^3-(\alpha^6+\beta^6+\gamma^6)=0$$

$$\alpha^3\beta^3\gamma^3=\frac{-D}{A}=\frac{-1}{1}=-1$$

$$\therefore \quad 1+2(-1)-(3-2c^3)=0$$

$$-4+2c^3=0$$

$$2c^3=4$$

$$c^3=2$$

$$c=\sqrt[3]{2}.$$