Past Paper Questions: Summation of Series

1 Let $v_1, v_2, v_3, \ldots$ be a sequence and let

$$u_n = nv_n - (n+1)v_{n+1},$$

for $n = 1, 2, 3, \ldots$. Find $\sum_{n=1}^{N} u_n$. [2]

In each of the following cases determine whether the series $u_1 + u_2 + u_3 + \ldots$ is convergent, and justify your conclusion. Give the sum to infinity where this exists.

(i) $v_n = n^{-\frac{1}{2}}$. [2]

(ii) $v_n = n^{-\frac{3}{2}}$. [2]

$$\sum_{n=1}^{N} u_n = \sum_{n=1}^{N} nv_n - (n+1)v_{n+1}$$

$$= \Big[(1v_1 - \cancel{2v_2}) + (\cancel{2v_2} - \cancel{3v_3}) + (\cancel{3v_3} - \cancel{4v_4}) + (\cancel{4v_4} - \cancel{5v_5}) + \cdots$$
$$\cdots + [\cancel{(N-2)v_{N-2}} - \cancel{(N-1)v_{N-1}}] + [\cancel{(N-1)v_{N-1}} - \cancel{Nv_N}] + [\cancel{Nv_N} - (N+1)v_{N+1}]\Big]$$

$$= v_1 - (N+1)v_{N+1}.$$

i) $v_n = n^{-1/2}$.

$$u_1 + u_2 + u_3 + \cdots = \sum_{n=1}^{\infty} u_n = \lim_{N \to \infty} \left[v_1 - (N+1)v_{N+1}\right]$$

$$= \lim_{N \to \infty} \left[(1)^{-1/2} - (N+1)(N+1)^{-1/2}\right] = \lim_{N \to \infty} \left[\frac{1}{1^{1/2}} - \frac{N+1}{(N+1)^{\frac{1}{2}}}\right]$$

$$= \lim_{N \to \infty} \left[1 - (N+1)^{1/2}\right] = -\infty$$

$\therefore$ Series does not converge $\Rightarrow$ Sum to infinity $= -\infty$.

ii) $v_n = n^{-3/2}$

$$u_1 + u_2 + u_3 + \cdots = \sum_{n=1}^{\infty} u_n = \lim_{N \to \infty} \left[v_1 - (N+1)v_{N+1}\right]$$

$$= \lim_{N \to \infty} \left[(1)^{-3/2} - (N+1) \times (N+1)^{-3/2}\right] = \lim_{N \to \infty} \left[1 - \frac{1}{N+1}\right] = 1.$$

$\therefore \sum_{n=1}^{\infty} u_n$ is convergent $\Rightarrow$ Sum to infinity $= 1$.

2 Use the relevant standard results in the List of Formulae to prove that

$$S_N = \sum_{n=1}^{N}(8n^3 - 6n^2) = N(N+1)(2N^2 - 1).$$ [2]

Hence show that

$$\sum_{n=N+1}^{2N}(8n^3 - 6n^2)$$

can be expressed in the form

$$N(aN^3 + bN^2 + cN + d),$$

where the constants a, b, c, d are to be determined. [2]

$$S_N = \sum_{n=1}^{N} 8n^3 - 6n^2$$

$$= 8\left[\frac{N^2(N+1)^2}{4}\right] - 6\left[\frac{N(N+1)(2N+1)}{6}\right] = 2N^2(N+1)^2 - N(N+1)(2N+1)$$

$$= N(N+1)\left[2N(N+1) - (2N+1)\right] = N(N+1)\left[2N^2 + 2N - 2N - 1\right]$$

$$= N(N+1)(2N^2 - 1).$$

$$\sum_{n=N+1}^{2N}(8n^3 - 6n^2) = \sum_{n=1}^{2N}(8n^3 - 6n^2) - \sum_{n=1}^{N}(8n^3 - 6n^2)$$

$$= 2N(2N+1)(2\times(2N)^2 - 1) - (N)(N+1)(2N^2 - 1)$$

$$= 2N(2N+1)(8N^2 - 1) - N(N+1)(2N^2 - 1)$$

$$= N\left[2(2N+1)(8N^2 - 1) - (N+1)(2N^2 - 1)\right]$$

$$= N\left[2(16N^3 + 8N^2 - 2N - 1) - (2N^3 + 2N^2 - N - 1)\right]$$

$$= N\left[32N^3 + 16N^2 - 4N - 2 - 2N^3 - 2N^2 + N + 1\right]$$

$$= N\left[30N^3 + 14N^2 - 3N - 1\right].$$

3 Use the method of differences to find S_N, where

$$S_N = \sum_{n=N}^{N^2} \frac{1}{n(n+1)}.$$ [3]

Deduce the value of $\lim_{N\to\infty} S_N$. [1]

$$\frac{1}{n(n+1)} \equiv \frac{A}{n} + \frac{B}{n+1}$$

$$1 \equiv A(n+1) + Bn$$

$$n=0 \Rightarrow 1 = A(0+1) \Rightarrow A = 1$$

$$n=-1 \Rightarrow 1 = B(-1) \Rightarrow B = -1.$$

$$\therefore \frac{1}{n(n+1)} \equiv \frac{1}{n} - \frac{1}{n+1}.$$

$$\sum_{n=1}^{N} \frac{1}{n} - \frac{1}{n+1} = \left[\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{N-2}} - \cancel{\frac{1}{N-1}}\right) + \left(\cancel{\frac{1}{N-1}} - \cancel{\frac{1}{N}}\right) + \left(\cancel{\frac{1}{N}} - \frac{1}{N+1}\right)\right]$$

$$= \frac{1}{1} - \frac{1}{N+1} = 1 - \frac{1}{N+1}.$$

$$\sum_{n=N}^{N^2} \frac{1}{n(n+1)} = \sum_{n=1}^{N^2} \frac{1}{n(n+1)} - \sum_{n=1}^{N-1} \frac{1}{n(n+1)}$$

$$= \left[1 - \frac{1}{N^2+1}\right] - \left[1 - \frac{1}{N-1+1}\right] = \frac{1}{N-1+1} - \frac{1}{N^2+1} = \frac{1}{N} - \frac{1}{N^2+1}$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \left[\frac{1}{N} - \frac{1}{N^2+1}\right] = 0 - 0 = 0.$$

4 Express

$$u_n = \frac{1}{4n^2 - 1}$$

in partial fractions, and hence find $\sum_{n=1}^{N} u_n$ in terms of N. [4]

Deduce that the infinite series $u_1 + u_2 + u_3 + \ldots$ is convergent and state the sum to infinity. [2]

$$\frac{1}{4n^2-1} \equiv \frac{1}{(2n-1)(2n+1)} \equiv \frac{A}{2n-1} + \frac{B}{2n+1}$$

$$1 \equiv A(2n+1) + B(2n-1)$$

$$n = \frac{1}{2} \Rightarrow 1 = A(2) \Rightarrow A = \frac{1}{2}.$$

$$n = \frac{-1}{2} \Rightarrow 1 = B(-2) \Rightarrow B = \frac{-1}{2}.$$

$$\therefore u_n = \frac{\frac{1}{2}}{2n-1} - \frac{\frac{1}{2}}{2n+1} = \frac{1}{2}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right].$$

$$\Rightarrow \sum_{n=1}^{N} u_n = \sum_{n=1}^{N} \frac{1}{2}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$$

$$= \frac{1}{2}\sum_{n=1}^{N} \frac{1}{2n-1} - \frac{1}{2n+1}$$

$$= \frac{1}{2}\left[\left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right) + \left(\cancel{\frac{1}{7}} - \cancel{\frac{1}{9}}\right) + \cdots \right.$$
$$\left. \cdots + \left(\cancel{\frac{1}{2N-5}} - \cancel{\frac{1}{2N-3}}\right) + \left(\cancel{\frac{1}{2N-3}} - \cancel{\frac{1}{2N-1}}\right) + \left(\cancel{\frac{1}{2N-1}} - \frac{1}{2N+1}\right)\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{2N+1}\right] = \frac{1}{2} - \frac{1}{2(2N+1)}.$$

$$S_\infty = \sum_{n=1}^{\infty} u_n = \lim_{N \to \infty}\left[\frac{1}{2} - \frac{1}{2(2N+1)}\right] = \frac{1}{2} - 0 = \frac{1}{2}.$$

5 Verify that

$$\frac{1}{n^2+1} - \frac{1}{(n+1)^2+1} = \frac{2n+1}{(n^2+1)(n^2+2n+2)}.$$ [1]

Use the method of differences to show that, for all $N \geqslant 1$,

$$\sum_{n=1}^{N} \frac{2n+1}{(n^2+1)(n^2+2n+2)} < \frac{1}{2}.$$ [3]

Write down the value of

$$\sum_{n=1}^{\infty} \frac{2n+1}{(n^2+1)(n^2+2n+2)}.$$ [1]

$$\frac{1}{n^2+1} - \frac{1}{(n+1)^2+1} = \frac{(n+1)^2+1-(n^2+1)}{(n^2+1)[(n+1)^2+1]} = \frac{n^2+2n+2-n^2-1}{(n^2+1)[n^2+2n+1+1]}$$

$$= \frac{2n+1}{(n^2+1)(n^2+2n+2)}. \text{ (Shown).}$$

$$\sum_{n=1}^{N} \frac{2n+1}{(n^2+1)(n^2+2n+2)} = \sum_{n=1}^{N} \frac{1}{n^2+1} - \frac{1}{(n+1)^2+1}$$

$$= \left[\left(\frac{1}{1^2+1} - \cancel{\frac{1}{2^2+1}}\right) + \left(\cancel{\frac{1}{2^2+1}} - \cancel{\frac{1}{3^2+1}}\right) + \left(\cancel{\frac{1}{3^2+1}} - \cancel{\frac{1}{4^2+1}}\right) + \left(\cancel{\frac{1}{4^2+1}} + \cdots\right) \right.$$
$$\left. \cdots + \left[\cancel{\frac{1}{(N-2)^2+1}} - \cancel{\frac{1}{(N-1)^2+1}}\right] + \left[\cancel{\frac{1}{(N-1)^2+1}} - \cancel{\frac{1}{N^2+1}}\right] + \left[\cancel{\frac{1}{N^2+1}} - \frac{1}{(N+1)^2+1}\right]\right]$$

$$= \frac{1}{2} - \frac{1}{(N+1)^2+1} < \frac{1}{2}. \text{ since } \frac{1}{(N+1)^2+1} > 0.$$

$$\Rightarrow \sum_{n=1}^{N} \frac{2n+1}{(n^2+1)(n^2+2n+2)} < \frac{1}{2}.$$

$$\sum_{n=1}^{\infty} \frac{2n+1}{(n^2+1)(n^2+2n+2)} = \lim_{N\to\infty}\left[\frac{1}{2} - \frac{1}{(N+1)^2+1}\right] = \frac{1}{2} - 0 = \frac{1}{2}.$$

6 Given that

$$u_n = \ln\left(\frac{1+x^{n+1}}{1+x^n}\right),$$

where $x > -1$, find $\sum_{n=1}^{N} u_n$ in terms of N and x. [3]

Find the sum to infinity of the series

$$u_1 + u_2 + u_3 + \dots$$

when

(i) $-1 < x < 1$, [1]

(ii) $x = 1$. [1]

$$\sum_{n=1}^{N} u_n = \sum_{n=1}^{N} \ln\left[\frac{1+x^{n+1}}{1+x^n}\right] = \sum_{n=1}^{N} \ln\left[1+x^{n+1}\right] - \ln\left[1+x^n\right]$$

$$= \Big[\left(\cancel{\ln[1+x^2]} - \ln[1+x]\right) + \left(\cancel{\ln[1+x^3]} - \cancel{\ln(1+x^2)}\right) + \left(\cancel{\ln(1+x^4)} - \cancel{\ln(1+x^3)}\right) + \dots$$

$$\dots + \left\{\cancel{\ln(1+x^{N-1})} - \cancel{\ln(1+x^{N-2})}\right\} + \left[\cancel{\ln(1+x^N)} - \cancel{\ln(1+x^{N-1})}\right] + \left[\ln(1+x^{N+1}) - \cancel{\ln(1+x^N)}\right]\Big]$$

$$= -\ln[1+x] + \ln[1+x^{N+1}] = \ln\left[\frac{1+x^{N+1}}{1+x}\right].$$

i) $-1 < x < 1$

$$S_\infty = \lim_{N\to\infty} \sum_{n=1}^{N} u_n = \lim_{N\to\infty}\left[\ln\left[\frac{1+x^{N+1}}{1+x}\right]\right] = \ln\left[\frac{1+0}{1+x}\right] = \ln\left[\frac{1}{1+x}\right].$$

ii) $x = 1$

$$S_\infty = \lim_{N\to\infty} \sum_{n=1}^{N} u_n = \lim_{N\to\infty}\left[\ln\left[\frac{1+x^{N+1}}{1+x}\right]\right] = \lim_{N\to\infty} \ln\left[\frac{1+(1)^{N+1}}{1+1}\right] = \ln\left[\frac{1+1}{1+1}\right] = \ln\left[\frac{2}{2}\right] = \ln(1)$$

$$= 0.$$

7 Verify that, for all positive values of n,

$$\frac{1}{(n+2)(2n+3)} - \frac{1}{(n+3)(2n+5)} = \frac{4n+9}{(n+2)(n+3)(2n+3)(2n+5)}.$$ [2]

For the series

$$\sum_{n=1}^{N} \frac{4n+9}{(n+2)(n+3)(2n+3)(2n+5)},$$

find

(i) the sum to N terms, [3]

(ii) the sum to infinity. [1]

$$\frac{1}{(n+2)(2n+3)} - \frac{1}{(n+3)(2n+5)} = \frac{(n+3)(2n+5)-(n+2)(2n+3)}{(n+2)(2n+3)(n+3)(2n+5)}$$

$$= \frac{(2n^2+6n+5n+15)-(2n^2+4n+3n+6)}{(n+2)(n+3)(2n+3)(2n+5)} = \frac{4n+9}{(n+2)(n+3)(2n+3)(2n+5)}.$$

i) $$\sum_{n=1}^{N} \frac{4n+9}{(n+2)(n+3)(2n+3)(2n+5)} = \sum_{n=1}^{N} \frac{1}{(n+2)(2n+3)} - \frac{1}{(n+3)(2n+5)}$$

$$= \left[\left(\frac{1}{3(5)} - \cancel{\frac{1}{4(7)}}\right) + \left(\cancel{\frac{1}{4(7)}} - \cancel{\frac{1}{5(9)}}\right) + \left(\cancel{\frac{1}{5(9)}} - \cancel{\frac{1}{6(11)}}\right) + \cdots \right.$$
$$\left. \cdots + \left[\cancel{\frac{1}{(N+1)(2N+3)}} - \cancel{\frac{1}{(N+2)(2N+3)}}\right] + \left[\cancel{\frac{1}{(N+2)(2N+3)}} - \frac{1}{(N+3)(2N+5)}\right]\right]$$

$$= \frac{1}{15} - \frac{1}{(N+3)(2N+5)}.$$

ii) $$S_\infty = \sum_{n=1}^{\infty} \frac{4n+9}{(n+2)(n+3)(2n+3)(2n+5)} = \lim_{N\to\infty}\left[\frac{1}{15} - \frac{1}{(N+3)(2N+5)}\right] = \frac{1}{15}.$$

8 The sum S_N is defined by $S_N = \sum_{n=1}^{N} n^5$. Using the identity

$$\left(n+\tfrac{1}{2}\right)^6 - \left(n-\tfrac{1}{2}\right)^6 \equiv 6n^5 + 5n^3 + \tfrac{3}{8}n,$$

find S_N in terms of N. [You need not simplify your result.] [4]

Hence find $\lim_{N\to\infty} N^{-\lambda} S_N$, for each of the two cases

(i) $\lambda = 6$,

(ii) $\lambda > 6$. [3]

$$\sum_{n=1}^{N} 6n^5 + 5n^3 + \frac{3n}{8} = \sum_{n=1}^{N} \left(n+\frac{1}{2}\right)^6 - \left(n-\frac{1}{2}\right)^6$$

$$= \Bigg[\left[\cancel{\left(\frac{3}{2}\right)^6} - \left(\frac{1}{2}\right)^6\right] + \left[\cancel{\left(\frac{5}{2}\right)^6} - \cancel{\left(\frac{3}{2}\right)^6}\right] + \left[\cancel{\left(\frac{7}{2}\right)^6} - \cancel{\left(\frac{5}{2}\right)^6}\right] + \left[\cancel{\left(\frac{9}{2}\right)^6} - \cancel{\left(\frac{7}{2}\right)^6}\right] + \cdots$$

$$\cdots + \cancel{\left(N-\frac{3}{2}\right)^6} + \cancel{\left(N-\frac{5}{2}\right)^6}\Big] + \left[\cancel{\left(N-\frac{1}{2}\right)^6} - \cancel{\left(N-\frac{3}{2}\right)^6}\right] + \left[\left(N+\frac{1}{2}\right)^6 - \cancel{\left(N-\frac{1}{2}\right)^6}\right]\Bigg]$$

$$= -\left(\frac{+1}{2}\right)^6 + \left(N+\frac{1}{2}\right)^6 = \left(N+\frac{1}{2}\right)^6 - \frac{1}{64}.$$

$$\therefore\ 6\sum_{n=1}^{N} n^5 + 5\sum_{n=1}^{N} n^3 + \frac{3}{8}\sum_{n=1}^{N} n = \left(N+\frac{1}{2}\right)^6 - \frac{1}{64}.$$

$$6S_N + 5\frac{(N)^2(N+1)^2}{4} + \frac{3}{8}\frac{N(N+1)}{2} = \left(N+\frac{1}{2}\right)^6 - \frac{1}{64}$$

$$\Rightarrow\ S_N = \frac{1}{6}\left[\left(N+\frac{1}{2}\right)^6 - \frac{1}{64} - \frac{5}{4}\left(\frac{N^2(N+1)^2}{1}\right) - \frac{3}{8}\left[\frac{N(N+1)}{2}\right]\right]$$

$$S_N = \frac{1}{6}\left\{\left(N+\frac{1}{2}\right)^6 - \frac{1}{64} - \frac{5N^2(N+1)^2}{4} - \frac{3N(N+1)}{16}\right]$$

i) $\lambda = 6$.

$$\lim_{N\to\infty} N^{-\lambda} S_N = \lim_{N\to\infty} N^{-6} S_N$$

$$= \lim_{N\to\infty} \frac{1}{6}\left\{\left(1+\frac{1}{2N}\right)^6 - \frac{1}{64N^6} - \frac{5\left[1+\frac{1}{N^2}\right]}{4N^2} - \frac{3\left(1+\frac{1}{N}\right)}{16N^4}\right]$$

$$= \frac{1}{6}\left\{(1+0)^6 - 0 - 5(0) - 3(0)\right\} = \frac{1}{6}.$$

II) For $\lambda > 6$,

$$\lim_{N\to\infty} N^{-\lambda} S_N = \lim_{N\to\infty} \frac{1}{6}\left\{\left[\frac{1}{N^{\lambda-6}}\left(1+\frac{1}{2N}\right)^6 - \frac{1}{64N^{\lambda}} - \frac{5\left(1+\frac{1}{N^2}\right)}{4N^{\lambda-4}} - \frac{3\left(1+\frac{1}{N}\right)}{16N^{\lambda-2}}\right]\right\}$$

$$= \frac{1}{6}\left\{0-0-0-0\right\} = 0.$$

9 By considering the identity

$$\cos[(2n-1)\alpha] - \cos[(2n+1)\alpha] \equiv 2\sin\alpha\sin 2n\alpha,$$

show that if α is not an integer multiple of π then

$$\sum_{n=1}^{N}\sin(2n\alpha) = \tfrac{1}{2}\cot\alpha - \tfrac{1}{2}\operatorname{cosec}\alpha\cos[(2N+1)\alpha]. \qquad [4]$$

Deduce that the infinite series

$$\sum_{n=1}^{\infty}\sin(\tfrac{2}{3}n\pi)$$

does not converge. [1]

$$2\sin\alpha\,\sin 2n\alpha \equiv \cos[(2n-1)\alpha] - \cos[(2n+1)\alpha]$$

$$\Rightarrow \sin 2n\alpha \equiv \frac{\cos[(2n-1)\alpha] - \cos[(2n+1)\alpha]}{2\sin\alpha} \quad \text{since } \alpha \neq k\pi \therefore \sin\alpha \neq 0.$$

$$\sum_{n=1}^{N}\sin(2n\alpha) = \sum_{n=1}^{N}\frac{\cos[(2n-1)\alpha]-\cos[(2n+1)\alpha]}{2\sin\alpha} = \frac{1}{2\sin\alpha}\sum_{n=1}^{N}[\cos(2n-1)\alpha]-[\cos(2n+1)\alpha]$$

$$= \frac{1}{2\sin\alpha}\Big[(\cos\alpha - \cancel{\cos 3\alpha}) + (\cancel{\cos 3\alpha} - \cancel{\cos 5\alpha}) + (\cancel{\cos 5\alpha} - \cancel{\cos 7\alpha}) + (\cancel{\cos 7\alpha} - \cancel{\cos 9\alpha}) + \cdots$$
$$\cdots + [\cancel{\cos(2N-5)\alpha} - \cancel{\cos(2N-3)\alpha}] + [\cancel{\cos(2N-3)\alpha} - \cancel{\cos(2N-1)\alpha}] + [\cancel{\cos(2N-1)\alpha} - \cos(2N+1)\alpha]\Big]$$

$$= \frac{1}{2\sin\alpha}[\cos\alpha - \cos(2N+1)\alpha] = \frac{\cos\alpha}{2\sin\alpha} - \frac{\cos(2N+1)\alpha}{2\sin\alpha}$$

$$= \frac{1}{2}\cot\alpha - \frac{1}{2}\operatorname{cosec}\alpha\cos(2N+1)\alpha.$$

$$\sum_{n=1}^{\infty}\sin\left(\frac{2n\pi}{3}\right) = \lim_{N\to\infty}\left[\frac{1}{2}\cot\left(\frac{\pi}{3}\right) - \frac{1}{2}\operatorname{cosec}\left(\frac{\pi}{3}\right)\cos\left[(2N+1)\frac{\pi}{3}\right]\right]$$

$$= \lim_{N\to\infty}\left[\frac{\sqrt{3}}{6} - \frac{1}{2}\times 2\times\cos\left(\frac{2N+1}{3}\right)\pi\right] = \lim_{N\to\infty}\left[\frac{\sqrt{3}}{6} - \cos\left(\frac{2N+1}{3}\right)\pi\right]$$

Note $\cos\left(\frac{2N+1}{3}\right)\pi$ oscillates between -1 and $\frac{1}{2}$

$\therefore \sum_{n=1}^{\infty}\sin\left(\frac{2n\pi}{3}\right)$ oscillates as well. and so does not converge.

10 Express $\frac{1}{(2r+1)(2r+3)}$ in partial fractions and hence use the method of differences to find

$$\sum_{r=1}^{n} \frac{1}{(2r+1)(2r+3)}.$$ [4]

Deduce the value of

$$\sum_{r=1}^{\infty} \frac{1}{(2r+1)(2r+3)}.$$ [1]

$$\frac{1}{(2r+1)(2r+3)} \equiv \frac{A}{2r+1} + \frac{B}{2r+3}$$

$$1 \equiv A(2r+3) + B(2r+1)$$

$$r = \frac{-1}{2} \Rightarrow 1 = A(2) \Rightarrow A = \frac{1}{2}.$$

$$r = \frac{-3}{2} \Rightarrow 1 = B(-2) \Rightarrow B = \frac{-1}{2}.$$

$$\sum_{r=1}^{n} \frac{1}{(2r+1)(2r+3)} = \sum_{r=1}^{n} \frac{\frac{1}{2}}{2r+1} - \frac{\frac{1}{2}}{2r+3}$$

$$= \frac{1}{2} \sum_{r=1}^{n} \frac{1}{2r+1} - \frac{1}{2r+3}$$

$$= \frac{1}{2}\left[\left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right) + \left(\cancel{\frac{1}{7}} - \cancel{\frac{1}{9}}\right) + \left(\cancel{\frac{1}{9}} - \cancel{\frac{1}{11}}\right) + \cdots \right.$$
$$\left. \cdots + \left(\cancel{\frac{1}{2n-3}} - \cancel{\frac{1}{2n-1}}\right) + \left(\cancel{\frac{1}{2n-1}} - \cancel{\frac{1}{2n+1}}\right) + \left(\cancel{\frac{1}{2n+1}} - \frac{1}{2n+3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{3} - \frac{1}{2n+3}\right] = \frac{1}{6} - \frac{1}{2(2n+3)}.$$

$$\sum_{r=1}^{\infty} \frac{1}{(2r+1)(2r+3)} = \lim_{n\to\infty} \left(\frac{1}{6} - \frac{1}{2(2n+3)}\right) = \frac{1}{6} - 0 = \frac{1}{6}.$$

11 Find $2^2 + 4^2 + \ldots + (2n)^2$. [2]

Hence find $1^2 - 2^2 + 3^2 - 4^2 + \ldots - (2n)^2$, simplifying your answer. [3]

$$2^2 + 4^2 + \cdots + (2n)^2$$
$$= \sum_{r=1}^{n} (2r)^2 = 4 \sum_{r=1}^{n} r^2 = 4\left[\frac{n(n+1)(2n+1)}{6}\right] = \frac{2n(n+1)(2n+1)}{3}.$$

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (2n-1)^2 - (2n)^2$$
$$= \left[1^2 + 2^2 + 3^2 + \cdots + (2n)^2\right] - 2\left[2^2 + 4^2 + \cdots + (2n)^2\right]$$
$$= \sum_{r=1}^{2n} r^2 - 2\left(\frac{2n(n+1)(2n+1)}{3}\right)$$
$$= \frac{2n(2n+1)(4n+1)}{6} - \frac{4n(n+1)(2n+1)}{3}$$
$$= \frac{n(2n+1)}{3}\left[4n+1 - 4(n+1)\right] = \frac{n(2n+1)}{3}\left[-3\right] = -n(2n+1)$$
$$= -2n^2 - n.$$

12 Given that $f(r) = \dfrac{1}{(r+1)(r+2)}$, show that

$$f(r-1) - f(r) = \frac{2}{r(r+1)(r+2)}.$$ [2]

Hence find $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$. [3]

Deduce the value of $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)}$. [1]

$$f(r-1) - f(r) = \frac{1}{(r-1+1)(r-1+2)} - \frac{1}{(r+1)(r+2)} = \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}$$

$$= \frac{r+2-r}{r(r+1)(r+2)} = \frac{2}{r(r+1)(r+2)} \quad \text{(shown)}.$$

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{2}\sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)} = \frac{1}{2}\sum_{r=1}^{n} f(r-1) - f(r)$$

$$= \frac{1}{2}\Big[[f(0) - \cancel{f(1)}] + [\cancel{f(1)} - \cancel{f(2)}] + [\cancel{f(2)} - \cancel{f(3)}] + [\cancel{f(3)} - \cancel{f(4)}] + \cdots$$
$$\cdots + [\cancel{f(n-3)} - \cancel{f(n-2)}] + [\cancel{f(n-2)} - \cancel{f(n-1)}] + [\cancel{f(n-1)} - f(n)] \Big]$$

$$= \frac{1}{2}[f(0) - f(n)] = \frac{1}{2}\left[\frac{1}{(1)(2)} - \frac{1}{(n+1)(n+2)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right] = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

$$\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)} = \lim_{n\to\infty}\left(\frac{1}{4} - \frac{1}{2(n+1)(n+2)}\right) = \frac{1}{4} - 0 = \frac{1}{4}.$$

13 Find the sum of the first n terms of the series

$$\frac{1}{1\times3}+\frac{1}{2\times4}+\frac{1}{3\times5}+\dots$$

and deduce the sum to infinity. [5]

$$\frac{1}{1\times3}+\frac{1}{2\times4}+\frac{1}{3\times5}+\dots+\frac{1}{n\times(n+2)}=\sum_{r=1}^{n}\frac{1}{r(r+2)}.$$

$$\frac{1}{r(r+2)}\equiv\frac{A}{r}+\frac{B}{r+2}$$

$$1\equiv A(r+2)+Br$$

$$r=0 \Rightarrow 1=A(2)\Rightarrow A=\frac{1}{2}.$$

$$r=-2 \Rightarrow 1=B(-2)\Rightarrow B=\frac{-1}{2}.$$

$$\Rightarrow \sum_{r=1}^{n}\frac{\frac{1}{2}}{r}-\frac{\frac{1}{2}}{r+2}=\frac{1}{2}\sum_{r=1}^{n}\frac{1}{r}-\frac{1}{r+2}$$

$$=\frac{1}{2}\left[\left(\frac{1}{1}-\cancel{\frac{1}{3}}\right)+\left(\frac{1}{2}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{6}}\right)+\left(\cancel{\frac{1}{5}}-\cancel{\frac{1}{7}}\right)+\dots \\ \dots+\left(\cancel{\frac{1}{n-3}}-\cancel{\frac{1}{n-1}}\right)+\left(\cancel{\frac{1}{n-2}}-\cancel{\frac{1}{n}}\right)+\left(\cancel{\frac{1}{n-1}}-\frac{1}{n+1}\right)+\left(\cancel{\frac{1}{n}}-\frac{1}{n+2}\right)\right]$$

$$=\frac{1}{2}\left[1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right]=\frac{1}{2}\left[\frac{3}{2}-\left(\frac{1}{n+1}+\frac{1}{n+2}\right)\right]$$

$$=\frac{3}{4}-\frac{1}{2}\left(\frac{(2n+3)}{(n+1)(n+2)}\right)=\frac{3}{4}-\frac{2n+3}{2(n+1)(n+2)}$$

$$\sum_{r=1}^{\infty}\frac{1}{r(r+2)}=\lim_{n\to\infty}\left(\frac{3}{4}-\frac{2n+3}{2(n+1)(n+2)}\right)=\frac{3}{4}-0=\frac{3}{4}.$$

14 Use the method of differences to show that $\displaystyle\sum_{r=1}^{N} \frac{1}{(2r+1)(2r+3)} = \frac{1}{6} - \frac{1}{2(2N+3)}$. [5]

Deduce that $\displaystyle\sum_{r=N+1}^{2N} \frac{1}{(2r+1)(2r+3)} < \frac{1}{8N}$. [4]

$$\frac{1}{(2r+1)(2r+3)} \equiv \frac{A}{2r+1} + \frac{B}{2r+3}$$

$$1 = A(2r+3) + B(2r+1)$$

$$r = \frac{-1}{2} \Rightarrow 1 = A(2) \Rightarrow A = \frac{1}{2}.$$

$$r = \frac{-3}{2} \Rightarrow 1 = B(-2) \Rightarrow B = \frac{-1}{2}.$$

$$\frac{1}{(2r+1)(2r+3)} = \frac{\frac{1}{2}}{2r+1} - \frac{\frac{1}{2}}{2r+3}.$$

$$\sum_{r=1}^{N} \frac{1}{(2r+1)(2r+3)} = \sum_{r=1}^{N} \frac{1}{2}\left(\frac{1}{2r+1} - \frac{1}{2r+3}\right)$$

$$= \frac{1}{2} \sum_{r=1}^{N} \frac{1}{2r+1} - \frac{1}{2r+3}$$

$$= \frac{1}{2}\left[\left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right) + \left(\cancel{\frac{1}{7}} - \cancel{\frac{1}{9}}\right) + \left(\cancel{\frac{1}{9}} - \cancel{\frac{1}{11}}\right) + \cdots \right.$$
$$\left. \cdots + \left(\cancel{\frac{1}{2N-3}} - \cancel{\frac{1}{2N-1}}\right) + \left(\cancel{\frac{1}{2N-1}} - \cancel{\frac{1}{2N+1}}\right) + \left(\cancel{\frac{1}{2N+1}} - \frac{1}{2N+3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{3} - \frac{1}{2N+3}\right] = \frac{1}{6} - \frac{1}{2(2N+3)}. \text{ (Shown).}$$

$$\sum_{r=N+1}^{2N} \frac{1}{(2r+1)(2r+3)}$$

$$= \sum_{r=1}^{2N} \frac{1}{(2r+1)(2r+3)} - \sum_{r=1}^{N} \frac{1}{(2r+1)(2r+3)}$$

$$= \left[\frac{1}{6} - \frac{1}{2(4N+3)}\right] - \left[\frac{1}{6} - \frac{1}{2(2N+3)}\right] = \frac{1}{2(2N+3)} - \frac{1}{2(4N+3)}.$$

$$= \frac{4N+3-(2N+3)}{2(2N+3)(4N+3)} = \frac{2N}{2(2N+3)(4N+3)} = \frac{N}{(2N+3)(4N+3)} < \frac{N}{2N(4N)} = \frac{1}{8N}$$

$$\Rightarrow \sum_{r=N+1}^{2N} \frac{1}{(2r+1)(2r+3)} < \frac{1}{8N}.$$

15 Let $f(r) = r!(r-1)$. Simplify $f(r+1) - f(r)$ and hence find $\sum_{r=n+1}^{2n} r!(r^2+1)$. [5]

$$f(r+1) - f(r)$$

$$= (r+1)!(r+1-1) - r!(r-1)$$

$$= (r+1)!\,r - r!(r-1) = r(r+1)r! - (r-1)r!$$

$$= r!\left[r^2+r-r+1\right] = r!(r^2+1).$$

$$\sum_{r=1}^{n} r!(r^2+1) = \sum_{r=1}^{n} f(r+1) - f(r)$$

$$= \Big[[\cancel{f(2)} - f(1)] + [\cancel{f(3)} - \cancel{f(2)}] + [\cancel{f(4)} - \cancel{f(3)}] + [\cancel{f(5)} - \cancel{f(4)}] + \cdots$$
$$\cdots + [\cancel{f(n-1)} - \cancel{f(n-2)}] + [\cancel{f(n)} - \cancel{f(n-1)}] + [f(n+1) - \cancel{f(n)}] \Big]$$

$$= -f(1) + f(n+1) = -1!(1-1) + (n+1)!(n+1-1) = n(n+1)! - 0$$

$$= n(n+1)!$$

$$\sum_{r=n+1}^{2n} r!(r^2+1) = \sum_{r=1}^{2n} r!(r^2+1) - \sum_{r=1}^{n} r!(r^2+1)$$

$$= 2n(2n+1)! - n(n+1)!$$

16 Expand and simplify $(r+1)^4 - r^4$. [1]

Use the method of differences together with the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2.$$ [4]

$$(r+1)^4 - r^4$$
$$= r^4 + 4r^3 + 6r^2 + 4r + 1 - r^4$$
$$= 4r^3 + 6r^2 + 4r + 1.$$

$$\sum_{r=1}^{n} 4r^3 + 6r^2 + 4r + 1 = \sum_{r=1}^{n} (r+1)^4 - r^4$$

$$= \left[\left(\cancel{2^4} - 1^4\right) + \left(\cancel{3^4} - \cancel{2^4}\right) + \left(\cancel{4^4} - \cancel{3^4}\right) + \left(\cancel{5^4} - \cancel{4^4}\right) + \cdots \right.$$
$$\left. \cdots + \left[\cancel{(n-1)^4} - \cancel{(n-2)^4}\right] + \left[\cancel{n^4} - \cancel{(n-1)^4}\right] + \left((n+1)^4 - \cancel{n^4}\right) \right]$$

$$= -1^4 + (n+1)^4 = (n+1)^4 - 1.$$

$$\therefore \sum_{r=1}^{n} 4r^3 + 6r^2 + 4r + 1 = (n+1)^4 - 1$$

$$4\sum_{r=1}^{n} r^3 + \frac{6n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} + n = (n+1)^4 - 1$$

$$4\sum_{r=1}^{n} r^3 = (n+1)^4 - 1 - n(n+1)(2n+1) - 2(n)(n+1) - n$$

$$4\sum_{r=1}^{n} r^3 = n^4 + 4n^3 + 6n^2 + 4n + 1 - 1 - n(2n^2 + 3n + 1) - 2n^2 - 2n - n$$

$$4\sum_{r=1}^{n} r^3 = n^4 + 4n^3 + 6n^2 + 4n - 2n^3 - 3n^2 - n - 2n^2 - 3n$$

$$4\sum_{r=1}^{n} r^3 = n^4 + 2n^3 + n^2$$

$$\sum_{r=1}^{n} r^3 = \frac{n^2(n^2 + 2n + 1)}{4} = \frac{n^2(n+1)^2}{4}$$

(Shown)

17 Show that the difference between the squares of consecutive integers is an odd integer. [1]

Find the sum to n terms of the series

$$\frac{3}{1^2 \times 2^2} + \frac{5}{2^2 \times 3^2} + \frac{7}{3^2 \times 4^2} + \ldots + \frac{2r+1}{r^2(r+1)^2} + \ldots$$

and deduce the sum to infinity of the series. [5]

Difference between squares of consecutive integers

$= (n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n+1.$

which is an odd integer.

$$\frac{3}{1^2\times2^2} + \frac{5}{2^2\times3^2} + \frac{7}{3^2\times4^2} + \ldots + \frac{2r+1}{r^2(r+1)^2}$$

$$= \sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = \sum_{r=1}^{n} \frac{(r+1)^2 - r^2}{r^2(r+1)^2} = \sum_{r=1}^{n} \frac{(r+1)^2}{r^2(r+1)^2} - \frac{r^2}{r^2(r+1)^2}$$

$$= \sum_{r=1}^{n} \frac{1}{r^2} - \frac{1}{(r+1)^2}$$

$$= \left[\left(\frac{1}{1^2} - \cancel{\frac{1}{2^2}}\right) + \left(\cancel{\frac{1}{2^2}} - \cancel{\frac{1}{3^2}}\right) + \left(\cancel{\frac{1}{3^2}} - \cancel{\frac{1}{4^2}}\right) + \left(\cancel{\frac{1}{4^2}} - \cancel{\frac{1}{5^2}}\right) + \cdots \right.$$
$$\left. \cdots + \left(\cancel{\frac{1}{(n-2)^2}} - \cancel{\frac{1}{(n-1)^2}}\right) + \left(\cancel{\frac{1}{(n-1)^2}} - \cancel{\frac{1}{n^2}}\right) + \left(\cancel{\frac{1}{n^2}} - \frac{1}{(n+1)^2}\right) \right]$$

$$= 1 - \frac{1}{(n+1)^2}.$$

$$\sum_{r=1}^{\infty} \frac{2r+1}{r^2(r+1)^2} = \lim_{n\to\infty} \left(1 - \frac{1}{(n+1)^2}\right) = 1 - 0 = \underline{1}.$$

18 Use the List of Formulae (MF10) to show that $\sum_{r=1}^{13}(3r^2 - 5r + 1)$ and $\sum_{r=0}^{9}(r^3 - 1)$ have the same numerical value. [4]

$$\sum_{r=1}^{13}(3r^2-5r+1) = 3\sum_{r=1}^{13} r^2 - 5\sum_{r=1}^{13} r + \sum_{r=1}^{13} 1$$

$$= 3\left[\frac{13(13+1)(2\times13+1)}{6}\right] - 5\left[\frac{13(13+1)}{2}\right] + 13$$

$$= 2015.$$

$$\sum_{r=0}^{9} r^3 - 1 = \sum_{r=1}^{9} r^3 - 1 + (0^3 - 1)$$

$$= \sum_{r=1}^{9} r^3 - \sum_{r=1}^{9} 1 + (-1)$$

$$= \frac{9^2(9+1)^2}{4} - 9 - 1 = 2015.$$

(Shown).

19 Use the formula for $\tan(A-B)$ in the List of Formulae (MF10) to show that

$$\tan^{-1}(x+1)-\tan^{-1}(x-1)=\tan^{-1}\left(\frac{2}{x^2}\right).$$ [3]

Deduce the sum to n terms of the series

$$\tan^{-1}\left(\frac{2}{1^2}\right)+\tan^{-1}\left(\frac{2}{2^2}\right)+\tan^{-1}\left(\frac{2}{3^2}\right)+\dots.$$ [4]

Let, $A=\tan^{-1}(x+1)$ and $B=\tan^{-1}(x-1)$.

$\Rightarrow \tan A = x+1$ $\qquad \tan B = x-1$.

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}=\frac{(x+1)-(x-1)}{1+(x+1)(x-1)}=\frac{x+1-x+1}{1+x^2-1}$$

$$\tan(A-B)=\frac{2}{x^2}$$

$$\Rightarrow A-B=\tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\therefore \tan^{-1}(x+1)-\tan^{-1}(x-1)=\tan^{-1}\left(\frac{2}{x^2}\right).$$

$$\tan^{-1}\left(\frac{2}{1^2}\right)+\tan^{-1}\left(\frac{2}{2^2}\right)+\tan^{-1}\left(\frac{2}{3^2}\right)+\dots$$

$$=\sum_{r=1}^{\infty}\tan^{-1}\left(\frac{2}{r^2}\right)$$

$$\sum_{r=1}^{N}\tan^{-1}\left(\frac{2}{r^2}\right)=\sum_{r=1}^{N}\tan^{-1}(r+1)-\tan^{-1}(r-1).$$

$$=\Big[\big[\cancel{\tan^{-1}(2)}-\tan^{-1}(0)\big]+\big[\cancel{\tan^{-1}(3)}-\tan^{-1}(1)\big]+\big[\cancel{\tan^{-1}(4)}-\cancel{\tan^{-1}(2)}\big]+\big[\cancel{\tan^{-1}(5)}-\cancel{\tan^{-1}(3)}\big]+\dots$$
$$+\big[\cancel{\tan^{-1}(N-2)}-\cancel{\tan^{-1}(N-4)}\big]+\big[\cancel{\tan^{-1}(N-1)}-\cancel{\tan^{-1}(N-3)}\big]+\big[\tan^{-1}(N)-\cancel{\tan^{-1}(N-2)}\big]+\big[\tan^{-1}(N+1)-\cancel{\tan^{-1}(N-1)}\big]\Big]$$

$$=-\tan^{-1}(0)-\tan^{-1}(1)+\tan^{-1}(N)+\tan^{-1}(N+1)$$

$$=-0-\frac{\pi}{4}+\tan^{-1}(N)+\tan^{-1}(N+1)$$

$$=\tan^{-1}(N+1)+\tan^{-1}(N)-\frac{\pi}{4}.$$

20 Express $\dfrac{4}{r(r+1)(r+2)}$ in partial fractions and hence find $\displaystyle\sum_{r=1}^{n} \frac{4}{r(r+1)(r+2)}$. [5]

Deduce the value of $\displaystyle\sum_{r=1}^{\infty} \frac{4}{r(r+1)(r+2)}$. [1]

$$\frac{4}{r(r+1)(r+2)} \equiv \frac{A}{r} + \frac{B}{r+1} + \frac{C}{r+2}$$

$$4 \equiv A(r+1)(r+2) + B(r)(r+2) + Cr(r+1)$$

$$r = 0 \Rightarrow \quad 4 = A(1)(2) \Rightarrow A = 2$$

$$r = -1 \Rightarrow \quad 4 = B(-1)(1) \Rightarrow B = -4$$

$$r = -2 \Rightarrow \quad 4 = C(-2)(-1) \Rightarrow C = 2$$

$$\therefore \sum_{r=1}^{n} \frac{4}{r(r+1)(r+2)} = \sum_{r=1}^{n} \frac{2}{r} - \frac{4}{r+1} + \frac{2}{r+2}$$

$$= \left[\begin{array}{l} \left(\frac{2}{1} - \frac{4}{2} + \cancel{\frac{2}{3}}\right) + \left(\frac{2}{2} - \cancel{\frac{4}{3}} + \cancel{\frac{2}{4}}\right) + \left(\cancel{\frac{2}{3}} - \cancel{\frac{4}{4}} + \cancel{\frac{2}{5}}\right) + \left(\cancel{\frac{2}{4}} - \cancel{\frac{4}{5}} + \cancel{\frac{2}{6}}\right) + \left(\cancel{\frac{2}{5}} - \cancel{\frac{4}{6}} + \cancel{\frac{2}{7}}\right) + \cdots \\ \cdots + \left(\cancel{\frac{2}{n-3}} - \cancel{\frac{4}{n-2}} + \cancel{\frac{2}{n-1}}\right) + \left(\cancel{\frac{2}{n-2}} - \cancel{\frac{4}{n-1}} + \cancel{\frac{2}{n}}\right) + \left(\cancel{\frac{2}{n-1}} - \cancel{\frac{4}{n}} + \frac{2}{n+1}\right) + \left(\cancel{\frac{2}{n}} - \frac{4}{n+1} + \frac{2}{n+2}\right) \end{array} \right]$$

$$= 2 - \frac{4}{2} + \frac{2}{2} + \frac{2}{n+1} - \frac{4}{n+1} + \frac{2}{n+2}$$

$$= 1 - \frac{2}{n+1} + \frac{2}{n+2}.$$

$$\sum_{r=1}^{\infty} \frac{4}{r(r+1)(r+2)} = \lim_{n\to\infty} \left(1 - \frac{2}{n+1} + \frac{2}{n+2}\right)$$

$$= 1 - 0 + 0 = \underline{1}.$$

21 Verify that $\dfrac{1}{(3r+1)(3r+4)} = \dfrac{1}{3}\left(\dfrac{1}{3r+1} - \dfrac{1}{3r+4}\right)$. [1]

Let S_N denote $\displaystyle\sum_{r=1}^{N} \frac{1}{(3r+1)(3r+4)}$ and let S denote $\displaystyle\sum_{r=1}^{\infty} \frac{1}{(3r+1)(3r+4)}$. Find the least value of N such that $S - S_N < \frac{1}{10\,000}$. [5]

$$\frac{1}{(3r+1)(3r+4)} \equiv \frac{A}{3r+1} + \frac{B}{3r+4}$$

$$1 \equiv A(3r+4) + B(3r+1)$$

$$r = \frac{-1}{3} \Rightarrow 1 = A(3) \Rightarrow A = \frac{1}{3}.$$

$$r = \frac{-4}{3} \Rightarrow 1 = B(-3) \Rightarrow B = \frac{-1}{3}.$$

$$\therefore \frac{1}{(3r+1)(3r+4)} \equiv \frac{\frac{1}{3}}{3r+1} - \frac{\frac{1}{3}}{3r+4} = \frac{1}{3}\left[\frac{1}{3r+1} - \frac{1}{3r+4}\right] \quad \text{(Verified)}$$

$$S_N = \sum_{r=1}^{N} \frac{1}{(3r+1)(3r+4)} = \sum_{r=1}^{N} \frac{1}{3}\left[\frac{1}{3r+1} - \frac{1}{3r+4}\right]$$

$$= \frac{1}{3}\left[\left(\frac{1}{4} - \cancel{\frac{1}{7}}\right) + \left(\cancel{\frac{1}{7}} - \cancel{\frac{1}{10}}\right) + \left(\cancel{\frac{1}{10}} - \cancel{\frac{1}{13}}\right) + \left(\cancel{\frac{1}{13}} + \dots\right) \dots + \left[\cancel{\frac{1}{3N-5}} - \cancel{\frac{1}{3N-2}}\right] + \left[\cancel{\frac{1}{3N-2}} - \cancel{\frac{1}{3N+1}}\right] + \left[\cancel{\frac{1}{3N+1}} - \frac{1}{3N+4}\right]\right]$$

$$= \frac{1}{3}\left[\frac{1}{4} - \frac{1}{3N+4}\right] = \frac{1}{12} - \frac{1}{3(3N+4)}.$$

$$S = \sum_{r=1}^{\infty} \frac{1}{(3r+1)(3r+4)} = \lim_{N\to\infty}\left(\frac{1}{12} - \frac{1}{3(3N+4)}\right) = \frac{1}{12}.$$

$$S - S_N < \frac{1}{10\,000}$$

$$\frac{1}{12} - \left(\frac{1}{12} - \frac{1}{3(3N+4)}\right) < \frac{1}{10\,000}$$

$$\frac{1}{3(3N+4)} < \frac{1}{10\,000}$$

$$10\,000 < 3(3N+4)$$

$$10000 < 9N + 12$$

$$9N > 9988 \Rightarrow N > 1109.77$$

$\therefore$ Least value of $N = 1110$.

22 It is given that $\sum_{r=1}^{n} u_r = n^2(2n+3)$, where n is a positive integer.

(i) Find $\sum_{r=n+1}^{2n} u_r$. [2]

$$= \sum_{r=1}^{2n} u_r - \sum_{r=1}^{n} u_r$$

$$= (2n)^2(2\times 2n+3) - n^2(2n+3)$$

$$= 4n^2(4n+3) - n^2(2n+3)$$

$$= n^2\left[4(4n+3) - 2n - 3\right] = n^2\left[16n+12-2n-3\right]$$

$$= n^2\left[14n+9\right] = 14n^3 + 9n^2.$$

(ii) Find u_r. [3]

$$u_r = S_r - S_{r-1}$$

$$= r^2(2r+3) - (r-1)^2(2\times(r-1)+3)$$

$$= r^2(2r+3) - (r-1)^2(2r+1)$$

$$= 2r^3 + 3r^2 - (r^2-2r+1)(2r+1)$$

$$= 2r^3+3r^2 - (2r^3 - 4r^2 + 2r + r^2 - 2r + 1)$$

$$= 2r^3 + 3r^2 - (2r^3 - 3r^2 + 1)$$

$$= 2r^3 + 3r^2 - 2r^3 + 3r^2 - 1$$

$$= 6r^2 - 1.$$

23 (i) Verify that $\frac{2r+1}{r(r+1)(r+2)} = \frac{1}{2}\left\{\frac{(2r+1)(2r+3)}{(r+1)(r+2)} - \frac{(2r-1)(2r+1)}{r(r+1)}\right\}$. [2]

R.H.S. $\frac{1}{2}\left\{\frac{(2r+1)(2r+3)}{(r+1)(r+2)} - \frac{(2r-1)(2r+1)}{r(r+1)}\right\}$

$= \frac{1}{2}\left\{\frac{r(2r+1)(2r+3) - (2r-1)(2r+1)(r+2)}{r(r+1)(r+2)}\right\}$

$= \frac{1}{2}\left\{\frac{r(4r^2+8r+3) - (4r^2-1)(r+2)}{r(r+1)(r+2)}\right\} = \frac{1}{2}\left\{\frac{4r^3+8r^2+3r-4r^3+r-8r^2+2}{r(r+1)(r+2)}\right\}$

$= \frac{1}{2}\left\{\frac{4r+2}{r(r+1)(r+2)}\right\} = \frac{2r+1}{r(r+1)(r+2)}$. (Shown).

(ii) Hence show that $\sum_{r=1}^{n} \frac{2r+1}{r(r+1)(r+2)} = \frac{1}{2}\left\{\frac{(2n+1)(2n+3)}{(n+1)(n+2)} - \frac{3}{2}\right\}$. [2]

$\sum_{r=1}^{n} \frac{2r+1}{r(r+1)(r+2)} = \sum_{r=1}^{n} \frac{1}{2}\left\{\frac{(2r+1)(2r+3)}{(r+1)(r+2)} - \frac{(2r-1)(2r+1)}{r(r+1)}\right\}$

$= \frac{1}{2}\left\{\left(\cancel{\frac{3(5)}{2(3)}} - \frac{(1)(3)}{1(2)}\right) + \left(\cancel{\frac{5(7)}{3(4)}} - \cancel{\frac{(3)(5)}{2(3)}}\right) + \left(\cancel{\frac{7(9)}{4(5)}} - \cancel{\frac{5(7)}{3(4)}}\right) + \cdots \right.$

$\left. \cdots - \cancel{\frac{(2n-5)(2n-3)}{(n-2)(n-1)}} + \left(\cancel{\frac{(2n-1)(2n+1)}{n(n-1)}} - \cancel{\frac{(2n-3)(2n-1)}{(n-1)(n)}}\right) + \left(\frac{(2n+1)(2n+3)}{(n+1)(n+2)} - \cancel{\frac{(2n-1)(2n+1)}{n(n+1)}}\right)\right\}$

$= \frac{1}{2}\left[-\frac{3}{2} + \frac{(2n+1)(2n+3)}{(n+1)(n+2)}\right] = \frac{1}{2}\left\{\frac{(2n+1)(2n+3)}{(n+1)(n+2)} - \frac{3}{2}\right\}$

(iii) Deduce the value of $\sum_{r=1}^{\infty} \frac{2r+1}{r(r+1)(r+2)}$. [2]

$\sum_{r=1}^{\infty} \frac{2r+1}{r(r+1)(r+2)} = \lim_{n\to\infty}\left[\frac{1}{2}\left(\frac{(2n+1)(2n+3)}{(n+1)(n+2)} - \frac{3}{2}\right)\right]$

$= \frac{1}{2}\left(\frac{4}{1} - \frac{3}{2}\right) = \frac{5}{4}$.

24 Let $S_n = \sum_{r=1}^{n} (-1)^{r-1} r^2$.

(i) Use the standard result for $\sum_{r=1}^{n} r^2$ given in the List of Formulae (MF10) to show that

$$S_{2n} = -n(2n+1). \qquad [4]$$

$$S_{2n} = \sum_{r=1}^{2n} (-1)^{r-1} r^2$$

$$= (-1)^0(1)^2 + (-1)^1(2)^2 + (-1)^2(3)^2 + (-1)^3(4)^2 + \cdots + (-1)^{2n-1}(2n-1)^2 + (-1)^{2n}(2n)^2$$

$$= 1^2 - 2^2 + 3^2 - 4^2 + \cdots - (2n-1)^2 + (2n)^2$$

$$= \left[1^2 + 2^2 + 3^2 + 4^2 + \cdots + (2n)^2\right] - 2\left[2^2 + 4^2 + \cdots + (2n)^2\right]$$

$$= \sum_{r=1}^{2n} r^2 - 2\sum_{r=1}^{n} (2r)^2$$

$$= \frac{2n(2n+1)(4n+1)}{6} - 8\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{n}{3}(2n+1)\left[4n+1-4(n+1)\right] = \frac{n(2n+1)}{3}\left[4n+1-4n-4\right]$$

$$S_{2n} = \frac{n(2n+1)}{3}\left[-3\right] = -n(2n+1)$$

(ii) State the value of $\lim_{n\to\infty} \frac{S_{2n}}{n^2}$ and find $\lim_{n\to\infty} \frac{S_{2n+1}}{n^2}$. [4]

$$\frac{S_{2n}}{n^2} = \frac{-n(2n+1)}{n^2} = \frac{-(2n+1)}{n}$$

$$\lim_{n\to\infty} \frac{S_{2n}}{n^2} = \lim_{n\to\infty} \frac{-(2n+1)}{n} = -2.$$

$$S_{2n+1} = S_{2n} + (2n+1)^{th} \text{ term}$$
$$= -n(2n+1) + (-1)^{2n+1-1}(2n+1)^2$$
$$= -n(2n+1) + (2n+1)^2$$
$$S_{2n+1} = (2n+1)[-n+2n+1] = (2n+1)(n+1).$$

$$\frac{S_{2n+1}}{n^2} = \frac{(2n+1)(n+1)}{n^2} = \left(2+\frac{1}{n}\right)\left(1+\frac{1}{n}\right).$$

$$\lim_{n\to\infty} \frac{S_{2n+1}}{n^2} = \lim_{n\to\infty} \left(2+\frac{1}{n}\right)\left(1+\frac{1}{n}\right) = (2+0)(1+0) = 2.$$

25 **(i)** Verify that

$$\frac{n(\mathrm{e}-1)+\mathrm{e}}{n(n+1)\mathrm{e}^{n+1}} = \frac{1}{n\mathrm{e}^n} - \frac{1}{(n+1)\mathrm{e}^{n+1}}.$$ [1]

$$\frac{1}{ne^n} - \frac{1}{(n+1)e^{n+1}} = \frac{(n+1)e - n}{n(n+1)e^{n+1}} = \frac{ne+e-n}{n(n+1)e^{n+1}}$$

$$= \frac{n(e-1)+e}{n(n+1)e^{n+1}}.$$

Let $S_N = \sum_{n=1}^{N} \frac{n(\mathrm{e}-1)+\mathrm{e}}{n(n+1)\mathrm{e}^{n+1}}$.

(ii) Express S_N in terms of N and e. [2]

$$S_N = \sum_{n=1}^{N} \frac{n(e-1)+e}{n(n+1)e^{n+1}} = \sum_{n=1}^{N} \frac{1}{ne^n} - \frac{1}{(n+1)e^{n+1}}$$

$$= \left[\left(\frac{1}{1e^1} - \cancel{\frac{1}{2e^2}}\right) + \left(\cancel{\frac{1}{2e^2}} - \cancel{\frac{1}{3e^3}}\right) + \left(\cancel{\frac{1}{3e^3}} - \cancel{\frac{1}{4e^4}}\right) + \left(\cancel{\frac{1}{4e^4}} - \cancel{\frac{1}{5e^5}}\right) + \cdots \right.$$
$$\left. \cdots - \cancel{\frac{1}{(N-1)e^{N-1}}}\right) + \left(\cancel{\frac{1}{(N-1)e^{N-1}}} - \cancel{\frac{1}{Ne^N}}\right) + \left(\cancel{\frac{1}{Ne^N}} - \frac{1}{(N+1)e^{N+1}}\right) \right]$$

$$= \frac{1}{e} - \frac{1}{(N+1)e^{N+1}}.$$

Let $S = \lim_{N \to \infty} S_N$.

(iii) Find the least value of N such that $(N+1)(S - S_N) < 10^{-3}$. [3]

$$S = \lim_{N \to \infty} S_N = \lim_{N \to \infty} \left[\frac{1}{e} - \frac{1}{(N+1)e^{N+1}} \right] = \frac{1}{e} - 0 = \frac{1}{e}.$$

$$(N+1)(S - S_N) < 10^{-3}$$

$$(N+1)\left[\frac{1}{e} - \left(\frac{1}{e} - \frac{1}{(N+1)e^{N+1}} \right) \right] < 10^{-3}$$

$$(N+1)\left[\frac{1}{(N+1)e^{N+1}} \right] < 10^{-3}$$

$$\frac{1}{e^{N+1}} < 10^{-3}$$

$$\frac{1}{e^{N+1}} < \frac{1}{10^3}$$

$$10^3 < e^{N+1}$$

$$1000 < e^{N+1}$$

$$\ln(1000) < N+1$$

$$-1 + \ln(1000) < N$$

$$5.907 < N$$

$$\Rightarrow N > 5.91$$

$\therefore$ Least value of $N \approx 6$.

26 Let $u_n = \dfrac{4\sin(n-\frac{1}{2})\sin\frac{1}{2}}{\cos(2n-1)+\cos 1}$.

(i) Using the formulae for $\cos P \pm \cos Q$ given in the List of Formulae MF10, show that

$$u_n = \frac{1}{\cos n} - \frac{1}{\cos(n-1)}.$$ [2]

$$u_n = \frac{4\sin\left(n-\frac{1}{2}\right)\sin\left(\frac{1}{2}\right)}{\cos(2n-1)+\cos 1} = \frac{2\left[2\sin\left(n-\frac{1}{2}\right)\sin\left(\frac{1}{2}\right)\right]}{\cos(2n-1)+\cos 1} = \frac{2\left[2\sin\left(\frac{2n-1}{2}\right)\sin\left(\frac{1}{2}\right)\right]}{\cos(2n-1)+\cos 1}$$

$$= \frac{2\left[-2\sin\left(\frac{n-1+n}{2}\right)\sin\left(\frac{n-1-n}{2}\right)\right]}{2\cos\left(\frac{2n-1+1}{2}\right)\cos\left(\frac{2n-1-1}{2}\right)} = \frac{2\left[\cos(n-1)-\cos(n)\right]}{2\cos(n)\cos(n-1)}$$

$$= \frac{\cos(n-1)-\cos(n)}{\cos(n)\cdot\cos(n-1)} = \frac{\cos(n-1)}{\cos(n)\cos(n-1)} - \frac{\cos(n)}{\cos(n)\cos(n-1)}$$

$$u_n = \frac{1}{\cos n} - \frac{1}{\cos(n-1)} \quad \text{(Shown)}$$

(ii) Use the method of differences to find $\sum_{n=1}^{N} u_n$. [2]

$$\sum_{n=1}^{N} u_n = \sum_{n=1}^{N} \frac{1}{\cos n} - \frac{1}{\cos(n-1)}$$

$$= \Bigg[\left[\cancel{\frac{1}{\cos(1)}} - \frac{1}{\cos(0)}\right] + \left[\cancel{\frac{1}{\cos(2)}} - \cancel{\frac{1}{\cos(1)}}\right] + \left[\cancel{\frac{1}{\cos(3)}} - \cancel{\frac{1}{\cos(2)}}\right] + \left[\cancel{\frac{1}{\cos(4)}} - \cancel{\frac{1}{\cos(3)}}\right] + \dots$$

$$\dots + \left[\cancel{\frac{1}{\cos(N-3)}} - \cancel{\frac{1}{\cos(N-4)}}\right] + \left[\cancel{\frac{1}{\cos(N-2)}} - \cancel{\frac{1}{\cos(N-3)}}\right] + \left[\cancel{\frac{1}{\cos(N-1)}} - \cancel{\frac{1}{\cos(N-2)}}\right] + \left[\frac{1}{\cos N} - \cancel{\frac{1}{\cos(N-1)}}\right]\Bigg]$$

$$= -\frac{1}{\cos(0)} + \frac{1}{\cos(N)} = \frac{1}{\cos(N)} - \frac{1}{1} = \frac{1}{\cos(N)} - 1.$$

(iii) Explain why the infinite series $u_1 + u_2 + u_3 + \dots$ does not converge. [1]

Note that as $N \mapsto \infty$, $\cos(N)$ does not approach any value

That is, $\cos N$ oscillates.

$u_1 + u_2 + u_3 + \dots$ does not converge as $\lim_{N\mapsto\infty}\left[\frac{1}{\cos(N)} - 1\right]$ does not exist.

$\Rightarrow$

27 (i) Use the method of differences to show that $\sum_{r=1}^{N} \frac{1}{(3r+1)(3r-2)} = \frac{1}{3} - \frac{1}{3(3N+1)}$. [4]

$$\frac{1}{(3r+1)(3r-2)} \equiv \frac{A}{3r+1} + \frac{B}{3r-2}$$

$$1 \equiv A(3r-2) + B(3r+1)$$

$$r = \frac{-1}{3} \Rightarrow 1 = A(-3) \Rightarrow A = \frac{-1}{3}.$$

$$r = \frac{2}{3} \Rightarrow 1 = B(3) \Rightarrow B = \frac{1}{3}.$$

$$\therefore \sum_{r=1}^{N} \frac{1}{(3r+1)(3r-2)} \equiv \sum_{r=1}^{N} \frac{\frac{-1}{3}}{3r+1} + \frac{\frac{1}{3}}{3r-2} \equiv \frac{1}{3}\sum_{r=1}^{N} \frac{1}{3r-2} - \frac{1}{3r+1}$$

$$= \frac{1}{3}\Big[\Big(\frac{1}{1} - \cancel{\frac{1}{4}}\Big) + \Big(\cancel{\frac{1}{4}} - \cancel{\frac{1}{7}}\Big) + \Big(\cancel{\frac{1}{7}} - \cancel{\frac{1}{10}}\Big) + \Big(\cancel{\frac{1}{10}} - \cancel{\frac{1}{13}}\Big) + \cdots$$

$$\cdots - \cancel{\frac{1}{3N-5}}\Big) + \Big(\cancel{\frac{1}{3N-5}} - \cancel{\frac{1}{3N-2}}\Big) + \Big(\cancel{\frac{1}{3N-2}} - \frac{1}{3N+1}\Big)\Big]$$

$$= \frac{1}{3}\Big[1 - \frac{1}{3N+1}\Big]$$

$$= \frac{1}{3} - \frac{1}{3(3N+1)} \quad \text{(Shown)}.$$

(ii) Find the limit, as $N \to \infty$, of $\sum_{r=N+1}^{N^2} \frac{N}{(3r+1)(3r-2)}$. [4]

$$\sum_{r=N+1}^{N^2} \frac{N}{(3r+1)(3r-2)} = \sum_{r=1}^{N^2} \frac{N}{(3r+1)(3r-2)} - \sum_{r=1}^{N} \frac{N}{(3r+1)(3r-2)}$$

$$= N\left[\frac{1}{3} - \frac{1}{3(3N^2+1)}\right] - N\left[\frac{1}{3} - \frac{1}{3(3N+1)}\right]$$

$$= \frac{N}{3} - \frac{N}{3(3N^2+1)} - \frac{N}{3} + \frac{N}{3(3N+1)}$$

$$= \frac{N}{3(3N+1)} - \frac{N}{3(3N^2+1)}.$$

$\Rightarrow$ As $N \mapsto \infty$, $\frac{N}{3(3N+1)} - \frac{N}{3(3N^2+1)} \mapsto \frac{1}{9} - 0 = \frac{1}{9}$.

28 Given that

$$u_n = \frac{1}{n^2 - n + 1} - \frac{1}{n^2 + n + 1},$$

find $S_N = \sum_{n=N+1}^{2N} u_n$ in terms of N. [3]

Find a number M such that $S_N < 10^{-20}$ for all $N > M$. [3]

$$\sum_{n=1}^{N} u_n = \sum_{n=1}^{N} \frac{1}{n^2-n+1} - \frac{1}{n^2+n+1}$$

$$= \left[\left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{7}}\right) + \left(\cancel{\frac{1}{7}} - \cancel{\frac{1}{13}}\right) + \left(\cancel{\frac{1}{13}} + \cdots\right) \right.$$
$$\left. \left(\cdots - \cancel{\frac{1}{N^2-3N+2}}\right) + \left(\cancel{\frac{1}{N^2-3N+2}} - \cancel{\frac{1}{N^2-N+1}}\right) + \left(\cancel{\frac{1}{N^2-N+1}} - \frac{1}{N^2+N+1}\right)\right]$$

$$= 1 - \frac{1}{N^2+N+1}.$$

$$\sum_{n=N+1}^{2N} u_n = \sum_{n=1}^{2N} u_n - \sum_{n=1}^{N} u_n$$
$$= \left[1 - \frac{1}{(2N)^2 + 2N + 1}\right] - \left[1 - \frac{1}{N^2+N+1}\right] = \frac{1}{N^2+N+1} - \frac{1}{4N^2+2N+1}.$$

$$S_N = \frac{1}{N^2+N+1} - \frac{1}{4N^2+2N+1} < \frac{1}{N^2+N+1} < \frac{1}{N^2}.$$

We require $S_N < 10^{-20}$

Then $\frac{1}{N^2} < 10^{-20}$

$$\frac{1}{N^2} < \frac{1}{10^{20}}$$

$$10^{20} < N^2$$

$$10^{10} < N$$

Then $S_N < 10^{-20}$ for all $N > 10^{10}$

$\therefore M = 10^{10}.$

29 Let

$$S_N = \sum_{n=1}^{N} (-1)^{n-1} n^3.$$

Find S_{2N} in terms of N, simplifying your answer as far as possible. [4]

Hence write down an expression for S_{2N+1} and find the limit, as $N \to \infty$, of $\frac{S_{2N+1}}{N^3}$. [3]

$$S_{2N} = \sum_{n=1}^{2N} (-1)^{n-1} n^3$$

$$= (-1)^0 (1)^3 + (-1)^1 (2)^3 + (-1)^2 (3)^3 + \dots + (-1)^{2N-2} (2N-1)^3 + (-1)^{2N-1} (2N)^3$$

$$= 1^3 - 2^3 + 3^3 - 4^3 \dots + (2N-1)^3 - (2N)^3$$

$$= \left[1^3 + 2^3 + 3^3 + 4^3 + \dots + (2N-1)^3 + (2N)^3\right] - 2\left[2^3 + 4^3 + 6^3 + \dots + (2N)^3\right]$$

$$= \sum_{n=1}^{2N} n^3 - 2\sum_{n=1}^{N} (2n)^3 = \frac{(2N)^2 (2N+1)^2}{4} - \left[\frac{8N^2 (N+1)^2}{4}\right] \times 2$$

$$= \frac{4N^2 (2N+1)^2 - 2(8N^2)(N+1)^2}{4} = \frac{N^2}{4}\left[4(4N^2+4N+1) - 16(N^2+2N+1)\right]$$

$$= \frac{N^2}{4}\left[16N^2 + 16N + 4 - 16N^2 - 32N - 16\right] = \cancel{\frac{N^2(15N^3+14N+3)}{4}} \quad \cancel{\frac{N^2}{4}\left[8N^2 - 4\right.}$$

$$= \frac{N^2}{4}\left[-16N - 12\right] = \frac{-4N^2 (4N+3)}{4} = -N^2 (4N+3).$$

$$S_{2N+1} = S_N + (2N+1)^{th} \text{ term}$$

$$= -N^2 (4N+3) + (-1)^{2N+1-1} \cdot (2N+1)^3$$

$$= -N^2 (4N+3) + (2N+1)^3$$

$$\frac{S_{2N+1}}{N^3} = \frac{-N^2 (4N+3) + (2N+1)^3}{N^3}$$

$$\frac{S_{2N+1}}{N^3} = -\left(4 + \frac{3}{N}\right) + \left(2 + \frac{1}{N}\right)^3$$

As $N \mapsto \infty$, $\frac{S_{2N+1}}{N^3} \mapsto -(4+0) + (2+0)^3 = -4 + 8 = 4$

10 Show that $(n+\frac{1}{2})^3-(n-\frac{1}{2})^3 \equiv 3n^2+\frac{1}{4}$. [1]

Use this result to prove that $\sum_{n=1}^{N} n^2 = \frac{1}{6}N(N+1)(2N+1)$. [2]

The sums S, T and U are defined as follows:

$$S = 1^2+2^2+3^2+4^2+\ldots+(2N)^2+(2N+1)^2,$$
$$T = 1^2+3^2+5^2+7^2+\ldots+(2N-1)^2+(2N+1)^2,$$
$$U = 1^2-2^2+3^2-4^2+\ldots-(2N)^2+(2N+1)^2.$$

Find and simplify expressions in terms of N for each of S, T and U. [5]

Hence

(i) describe the behaviour of $\dfrac{S}{T}$ as $N\to\infty$, [1]

(ii) prove that if $\dfrac{S}{U}$ is an integer then $\dfrac{T}{U}$ is an integer. [3]

$$\left(n+\frac{1}{2}\right)^3-\left(n-\frac{1}{2}\right)^3$$
$$=\left(n^3+\frac{3n^2}{2}+\frac{3n}{4}+\frac{1}{8}\right)-\left(n^3-\frac{3n^2}{2}+\frac{3n}{4}-\frac{1}{8}\right)=\frac{3n^2}{2}+\frac{3n^2}{2}+\frac{1}{8}+\frac{1}{8}=3n^2+\frac{1}{4}. \text{ (Shown).}$$

$$\sum_{n=1}^{N} 3n^2+\frac{1}{4} = \sum_{n=1}^{N}\left(n+\frac{1}{2}\right)^3-\left(n-\frac{1}{2}\right)^3$$
$$=\left[\left[\cancel{\left(\frac{3}{2}\right)^3}-\left(\frac{1}{2}\right)^3\right]+\left[\cancel{\left(\frac{5}{2}\right)^3}-\cancel{\left(\frac{3}{2}\right)^3}\right]+\left[\cancel{\left(\frac{7}{2}\right)^3}-\cancel{\left(\frac{5}{2}\right)^3}\right]+\cdots\right.$$
$$\left.\cdots+\left[\cancel{\left(N-\frac{3}{2}\right)^3}-\cancel{\left(N-\frac{5}{2}\right)^3}\right]+\left[\cancel{\left(N-\frac{1}{2}\right)^3}-\cancel{\left(N-\frac{3}{2}\right)^3}\right]+\left[\left(N+\frac{1}{2}\right)^3-\cancel{\left(N-\frac{1}{2}\right)^3}\right]\right]$$
$$=-\left(\frac{1}{2}\right)^3+\left(N+\frac{1}{2}\right)^3=-\frac{1}{8}+N^3+\frac{3N^2}{2}+\frac{3N}{4}+\frac{1}{8}=N^3+\frac{3N^2}{2}+\frac{3N}{4}.$$

$$\Rightarrow \sum_{n=1}^{N} 3n^2+\frac{1}{4} = N^3+\frac{3N^2}{2}+\frac{3N}{4}$$
$$3\sum_{n=1}^{N} n^2+\frac{N}{4} = N^3+\frac{3N^2}{2}+\frac{3N}{4}$$
$$3\sum_{n=1}^{N} n^2 = N^3+\frac{3N^2}{2}+\frac{N}{2}$$
$$\sum_{n=1}^{N} n^2 = \frac{2N^3+3N^2+N}{2\times 3}$$
$$\sum_{n=1}^{N} n^2 = \frac{N(2N^2+3N+1)}{6} = \frac{N(2N^2+2N+N+1)}{6}$$
$$\sum_{n=1}^{N} n^2 = \frac{N(2N+1)(N+1)}{6} \text{ (Proven).}$$

$$S = 1^2+2^2+3^2+\cdots+(2N)^2+(2N+1)^2 = \sum_{n=1} n^2$$

$$= \frac{(2N+1)(2N+1+1)(2\times(2N+1)+1)}{6} = \frac{(2N+1)(N+1)(4N+3)}{3}$$

$$T = 1^2+3^2+5^2+\cdots(2N-1)^2+(2N+1)^2$$

$$= \sum_{r=1}^{N+1}(2n-1)^2 = \sum_{r=1}^{N+1} 4n^2-4n+1 = 4\sum_{r=1}^{N+1} n^2 - 4\sum_{r=1}^{N+1} n + \sum_{r=1}^{N+1} 1$$

$$= \overset{2}{\cancel{4}}\left[\frac{(N+1)(N+1+1)(2\times(N+1)+1)}{\overset{3}{\cancel{6}}}\right] - 4\left[\frac{(N+1)(N+1+1)}{2}\right] + N+1$$

$$= N+1\left[\frac{2(N+2)(2N+3)}{3} - 2(N+2)+1\right] = (N+1)\left[\frac{2(2N^2+7N+6)-6(N+2)+3}{3}\right]$$

$$= (N+1)\left[\frac{4N^2+14N+12-6N-12+3}{3}\right] = \frac{(N+1)(4N^2+8N+3)}{3} = \frac{(N+1)(2N+1)(2N+3)}{3}$$

$$U = 1^2-2^2+3^2-4^2+\cdots+(2N-1)^2-(2N)^2+(2N+1)^2$$

Note $S+U = 2T$

$$\Rightarrow U = 2T-S = 2\left[\frac{(N+1)(2N+1)(2N+3)}{3}\right] - \left[\frac{(2N+1)(N+1)(4N+3)}{3}\right]$$

$$= \frac{(N+1)(2N+1)}{3}\left[4N+6-4N-3\right] = \frac{(N+1)(2N+1)\times\cancel{3}}{\cancel{3}} = (N+1)(2N+1).$$

i) $$\frac{S}{T} = \frac{\frac{(2N+1)(N+1)(4N+3)}{3}}{\frac{(N+1)(2N+1)(2N+3)}{3}} = \frac{4N+3}{2N+3}.$$

As $N \mapsto \infty$, $\frac{S}{T} \mapsto \frac{4}{2} = 2$.

ii) $$\frac{S}{U} = \frac{(2N+1)(N+1)(4N+3)}{3} \div (N+1)(2N+1) = \frac{4N+3}{3}$$ is an integer.

$\frac{4N}{3}+\frac{3}{3} = \frac{4N}{3}+1$ is an integer if N is a multiple of 3.

$$\frac{T}{U} = \frac{(N+1)(2N+1)(2N+3)}{3} \div (N+1)(2N+1) = \frac{2N+3}{3} = \frac{2N}{3}+\frac{3}{3} = \frac{2N}{3}+1.$$

If N is a multiple of 3, $\frac{2N}{3}$ is an integer

$\therefore \frac{2N}{3}+1$ is an integer

$\Rightarrow \frac{T}{U}$ is an integer.

31 Verify that if

$$v_n = n(n+1)(n+2)\ldots(n+m),$$

then

$$v_{n+1} - v_n = (m+1)(n+1)(n+2)\ldots(n+m).$$ [2]

Given now that

$$u_n = (n+1)(n+2)\ldots(n+m),$$

find $\sum_{n=1}^{N} u_n$ in terms of m and N. [3]

$$v_{n+1} - v_n$$

$$= (n+1)(n+2)(n+3)\cdots(n+m)(n+1+m) - n(n+1)(n+2)\cdots(n+m)$$

$$= (n+1)(n+2)\cdots(n+m)[n+1+m-n]$$

$$= (n+1)(n+2)\cdots(n+m)(m+1) = (m+1)(n+1)(n+2)\cdots(n+m).$$

(shown).

$$\sum_{n=1}^{N} u_n = \sum_{n=1}^{N}(n+1)(n+2)\cdots(n+m)$$

$$= \frac{1}{m+1}\sum_{n=1}^{N}(m+1)(n+1)(n+2)\cdots(n+m)$$

$$= \frac{1}{m+1}\sum_{n=1}^{N} v_{n+1} - v_n$$

$$= \frac{1}{m+1}\left[(\cancel{v_2}-v_1)+(\cancel{v_3}-\cancel{v_2})+(\cancel{v_4}-\cancel{v_3})+\cdots \cdots(\cancel{v_{N-1}}-\cancel{v_{N-2}})+(\cancel{v_N}-\cancel{v_{N-1}})+(v_{N+1}-\cancel{v_N})\right]$$

$$= \frac{1}{m+1}\left[v_{N+1} - v_1\right]$$

$$= \frac{1}{m+1}\left[(N+1)(N+2)(N+3)\cdots(N+1+m) - 1(2)(3)\cdots(m)(1+m)\right]$$

$$= \frac{(N+1)(N+2)(N+3)\ldots(N+1+m)}{m+1} - \frac{(m+1)!}{(m+1)}$$

$$= \frac{(N+1)(N+2)(N+3)\cdots(N+1+m)}{m+1} - m!$$

32 Express

$$\frac{2n+3}{n(n+1)}$$

in partial fractions and hence use the method of differences to find

$$\sum_{n=1}^{N} \frac{2n+3}{n(n+1)}\left(\frac{1}{3}\right)^{n+1}$$

in terms of N. [4]

Deduce the value of

$$\sum_{n=1}^{\infty} \frac{2n+3}{n(n+1)}\left(\frac{1}{3}\right)^{n+1}.$$ [1]

$$\frac{2n+3}{n(n+1)} \equiv \frac{A}{n} + \frac{B}{n+1}$$

$$2n+3 \equiv A(n+1) + Bn$$

$n = 0 \Rightarrow \quad 3 = A(1) \Rightarrow A = 3.$

$n = -1 \Rightarrow \quad 1 = B(-1) \Rightarrow B = -1.$

$$\therefore \frac{2n+3}{n(n+1)} = \frac{3}{n} - \frac{1}{n+1}.$$

$$\sum_{n=1}^{N} \frac{2n+3}{n(n+1)}\left(\frac{1}{3}\right)^{n+1} = \sum_{n=1}^{N}\left(\frac{3}{n} - \frac{1}{n+1}\right)\left(\frac{1}{3}\right)^{n+1}$$

$$= \sum_{n=1}^{N} \frac{3}{n}\cdot\left(\frac{1}{3}\right)^{n+1} - \frac{1}{n+1}\cdot\left(\frac{1}{3}\right)^{n+1} = \sum_{n=1}^{N} \frac{1}{3^n n} - \frac{1}{(n+1)3^{n+1}}$$

$$= \Bigg[\left[\frac{1}{3^1(1)} - \cancel{\frac{1}{2(3^2)}}\right] + \left[\cancel{\frac{1}{3^2(2)}} - \cancel{\frac{1}{3.3^3}}\right] + \left[\cancel{\frac{1}{3^3.3}} - \cancel{\frac{1}{4.3^4}}\right] + \cdots$$

$$- \cancel{\frac{1}{3^{N-2}\cdot(N-2)}}\Bigg] + \left[\cancel{\frac{1}{3^{N-2}(N-2)}} - \cancel{\frac{1}{(N-1)3^{N-1}}}\right] + \left[\cancel{\frac{1}{3^{N-1}\cdot(N-1)}} - \cancel{\frac{1}{N\cdot 3^N}}\right] + \left[\cancel{\frac{1}{3^N\cdot N}} - \frac{1}{(N+1)3^{N+1}}\right]\Bigg]$$

$$= \frac{1}{3} - \frac{1}{(N+1)3^{N+1}}.$$

$$\sum_{n=1}^{\infty} \frac{2n+3}{n(n+1)}\left(\frac{1}{3}\right)^{n+1} = \lim_{N\to\infty}\left[\frac{1}{3} - \frac{1}{(N+1)3^{N+1}}\right] = \frac{1}{3} - 0 = \frac{1}{3}.$$

33 Use the method of differences to find S_N, where

$$S_N = \sum_{n=1}^{N} \frac{1}{n(n+2)}.$$ [4]

Deduce the value of $\lim_{N\to\infty} S_N$. [1]

$$\frac{1}{n(n+2)} \equiv \frac{A}{n} + \frac{B}{n+2}$$

$$1 \equiv A(n+2) + Bn$$

$$n=0 \Rightarrow \quad 1 = A(2) \Rightarrow A = \frac{1}{2}.$$

$$n=-2 \Rightarrow \quad 1 = B(-2) \Rightarrow B = \frac{-1}{2}.$$

$$S_N = \sum_{n=1}^{N}\left[\frac{\frac{1}{2}}{n} - \frac{\frac{1}{2}}{n+2}\right] = \frac{1}{2}\sum_{n=1}^{N}\left[\frac{1}{n} - \frac{1}{n+2}\right]$$

$$= \frac{1}{2}\left[\begin{array}{l}\left(\frac{1}{1}-\cancel{\frac{1}{3}}\right)+\left(\frac{1}{2}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{6}}\right)+\left(\cancel{\frac{1}{5}}+\cdots\right) \\ \cdots+\left(\cancel{\frac{1}{N-3}}-\cancel{\frac{1}{N-1}}\right)+\left(\cancel{\frac{1}{N-2}}-\cancel{\frac{1}{N}}\right)+\left(\cancel{\frac{1}{N-1}}-\frac{1}{N+1}\right)+\left(\cancel{\frac{1}{N}}-\frac{1}{N+2}\right)\end{array}\right]$$

$$= \frac{1}{2}\left[1+\frac{1}{2}-\frac{1}{N+1}-\frac{1}{N+2}\right] = \frac{1}{2}\left[\frac{3}{2}-\left(\frac{1}{N+1}+\frac{1}{N+2}\right)\right]$$

$$S_N = \frac{1}{2}\left[\frac{3}{2}-\left(\frac{2N+3}{(N+1)(N+2)}\right)\right] = \frac{3}{4} - \frac{2N+3}{2(N+1)(N+2)}$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty}\left(\frac{3}{4} - \frac{2N+3}{2(N+1)(N+2)}\right) = \frac{3}{4} - 0 = \frac{3}{4}.$$

34 Verify that $\frac{1}{n^2} - \frac{1}{(n+1)^2} = \frac{2n+1}{n^2(n+1)^2}$. [1]

Let $S_N = \sum_{r=1}^{N} \frac{2r+1}{r^2(r+1)^2}$. Express S_N in terms of N. [2]

Let $S = \lim_{N\to\infty} S_N$. Find the least value of N such that $S - S_N < 10^{-16}$. [3]

$$\frac{1}{n^2} - \frac{1}{(n+1)^2} = \frac{(n+1)^2 - n^2}{n^2(n+1)^2} = \frac{n^2+2n+1-n^2}{n^2(n+1)^2} = \frac{2n+1}{n^2(n+1)^2}.$$

(Shown)

$$S_N = \sum_{r=1}^{N} \frac{2r+1}{r^2(r+1)^2} = \sum_{r=1}^{N} \frac{1}{r^2} - \frac{1}{(r+1)^2}$$

$$= \left[\left(\frac{1}{1^2} - \cancel{\frac{1}{2^2}}\right) + \left(\cancel{\frac{1}{2^2}} - \cancel{\frac{1}{3^2}}\right) + \left(\cancel{\frac{1}{3^2}} - \cancel{\frac{1}{4^2}}\right) + \left(\cancel{\frac{1}{4^2}} - \cancel{\frac{1}{5^2}}\right) + \cdots \right.$$
$$\left. \cdots - \cancel{\frac{1}{(N-2)^2}}\right) + \left(\cancel{\frac{1}{(N-2)^2}} - \cancel{\frac{1}{(N-1)^2}}\right) + \left(\cancel{\frac{1}{(N-1)^2}} - \cancel{\frac{1}{N^2}}\right) + \left(\cancel{\frac{1}{N^2}} - \frac{1}{(N+1)^2}\right) \right]$$

$$= 1 - \frac{1}{(N+1)^2}.$$

$$S = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(1 - \frac{1}{(N+1)^2}\right) = 1 - 0 = 1.$$

$$S - S_N < 10^{-16}$$
$$1 - \left(1 - \frac{1}{(N+1)^2}\right) < 10^{-16}$$
$$\frac{1}{(N+1)^2} < \frac{1}{10^{16}}$$
$$10^{16} < (N+1)^2$$
$$10^8 < N+1$$
$$10^8 - 1 < N$$

i.e. $N > 10^8 - 1$

$\therefore$ Least value of $N = 10^8$.

35 Let $f(r) = r(r+1)(r+2)$. Show that

$$f(r) - f(r-1) = 3r(r+1).$$ [1]

Hence show that $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$. [2]

Using the standard result for $\sum_{r=1}^{n} r$, deduce that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$. [2]

Find the sum of the series

$$1^2 + 2\times 2^2 + 3^2 + 2\times 4^2 + 5^2 + 2\times 6^2 + \ldots + 2(n-1)^2 + n^2,$$

where n is odd. [3]

$$f(r) - f(r-1) = r(r+1)(r+2) - (r-1)(r-1+1)(r-1+2)$$
$$= r(r+1)(r+2) - (r-1)(r)(r+1) = r(r+1)[r+2-r+1]$$
$$f(r) - f(r-1) = 3r(r+1) \quad \text{(Shown)}.$$

$$\sum_{r=1}^{n} f(r) - f(r-1)$$
$$= \left[[\cancel{f(1)} - f(0)] + [\cancel{f(2)} - \cancel{f(1)}] + [\cancel{f(3)} - \cancel{f(2)}] + [\cancel{f(4)} - \cancel{f(3)}] + \cdots \right.$$
$$\left. \cdots + [\cancel{f(n-3)} - \cancel{f(n-4)}] + [\cancel{f(n-2)} - \cancel{f(n-3)}] + [\cancel{f(n-1)} - \cancel{f(n-2)}] + [f(n) - \cancel{f(n-1)}] \right]$$
$$= -f(0) + f(n) = f(n) - f(0).$$

$$\therefore \quad \sum_{r=1}^{n} 3r(r+1) = f(n) - f(0)$$
$$3\sum_{r=1}^{n} r(r+1) = n(n+1)(n+2) - 0$$
$$\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}.$$

$$\Rightarrow \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r = \frac{n(n+1)(n+2)}{3}$$
$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(n+2)}{3} - \frac{n(n+1)}{2}$$
$$= n(n+1)\left[\frac{n+2}{3} - \frac{1}{2}\right]$$
$$= n(n+1)\left[\frac{2n+4-3}{6}\right]$$
$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{(Shown)}.$$

$$1^2+2\times2^2+3^2+2\times4^2+5^2+2\times6^2+\cdots+2(n-1)^2+n^2$$

$$=\left[1^2+2^2+3^2+4^2+5^2+\cdots+(n-1)^2+n^2\right]+\left[2^2+4^2+6^2+\cdots+(n-1)^2\right]$$

$$=\cancel{\Sigma}\left[1^2+2^2+3^2+\cdots+(n-1)^2+n^2\right]+2^2\left[1^2+2^2+3^2+\cdots+\left(\frac{n-1}{2}\right)^2\right]$$

$$=\sum_{r=1}^{n}r^2+4\sum_{r=1}^{\frac{n-1}{2}}r^2$$

$$=\frac{n(n+1)(2n+1)}{6}+4\left[\frac{\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}+1\right)\left(2\times\frac{n-1}{2}+1\right)}{6}\right]$$

$$=\frac{n(n+1)(2n+1)}{6}+4\left[\frac{\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)(n)}{6}\right]$$

$$=\frac{n(n+1)(2n+1)}{6}+\frac{n(n-1)(n+1)}{6}$$

$$=\frac{n(n+1)}{6}\left[2n+1+n-1\right]$$

$$=\frac{3n^2(n+1)}{6}=\frac{n^2(n+1)}{2}.$$

36 Show that $\sum_{r=n+1}^{2n} r^2 = \frac{1}{6}n(2n+1)(7n+1)$. [4]

$$\sum_{r=n+1}^{2n} r^2$$

$$= \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n} r^2$$

$$= \frac{2n(2n+1)(4n+1)}{6} - \frac{n(n+1)(2n+1)}{6}$$

$$= n(2n+1)\left[\frac{2(4n+1)-(n+1)}{6}\right]$$

$$= n(2n+1)\left[\frac{8n+2-n-1}{6}\right]$$

$$= \frac{n(2n+1)(7n+1)}{6}$$

(Shown).

37 It is given that

$$S_n = \sum_{r=1}^{n} u_r = 2n^2 + n.$$

Write down the values of S_1, S_2, S_3, S_4. Express u_r in terms of r, justifying your answer. [4]

Find

$$\sum_{r=n+1}^{2n} u_r.$$ [3]

$S_1 = 2(1)^2 + 1 = 2 + 1 = 3.$

$S_2 = 2(2)^2 + 2 = 8 + 2 = 10.$

$S_3 = 2(3)^2 + 3 = 18 + 3 = 21.$

$S_4 = 2(4)^2 + 4 = 32 + 4 = 36.$

$u_1 = S_1 = 3$

$u_2 = S_2 - S_1 = 10 - 3 = 7.$

$u_3 = S_3 - S_2 = 21 - 10 = 11.$

$u_4 = S_4 - S_3 = 36 - 21 = 15.$

$\therefore \quad u_1 = 3, u_2 = 7, u_3 = 11, u_4 = 15.$

This is an arithmetic progression with $a = 3$, $d = 4$.

$\Rightarrow u_r = a + d(n-1) = 3 + 4(r-1) = 4r - 1.$

i.e. $u_r = 4r - 1.$

$$\sum_{r=n+1}^{2n} u_r = \sum_{r=1}^{2n} u_r - \sum_{r=1}^{n} u_r$$

$$= \left[2(2n)^2 + 2n\right] - \left[2(n)^2 + n\right]$$

$$= 8n^2 + 2n - 2n^2 - n$$

$$= 6n^2 + n.$$

38 Express $\dfrac{1}{r(r+1)(r-1)}$ in partial fractions. [1]

Find

$$\sum_{r=2}^{n} \frac{1}{r(r+1)(r-1)}.$$ [4]

State the value of

$$\sum_{r=2}^{\infty} \frac{1}{r(r+1)(r-1)}.$$ [1]

$$\frac{1}{r(r+1)(r-1)} \equiv \frac{A}{r} + \frac{B}{r+1} + \frac{C}{r-1}$$

$$1 \equiv A(r+1)(r-1) + Br(r-1) + Cr(r+1)$$

$$r=0 \Rightarrow 1 = A(1)(-1) \Rightarrow A = -1.$$

$$r=1 \Rightarrow 1 = C(1)(2) \Rightarrow C = \tfrac{1}{2}.$$

$$r=-1 \Rightarrow 1 = B(-1)(-2) \Rightarrow B = \tfrac{1}{2}.$$

$$\therefore \frac{1}{r(r+1)(r-1)} \equiv \frac{-1}{r} + \frac{\frac{1}{2}}{r+1} + \frac{\frac{1}{2}}{r-1} = \frac{-1}{r} + \frac{1}{2(r+1)} + \frac{1}{2(r-1)}.$$

$$\sum_{r=2}^{n} \frac{1}{r(r+1)(r-1)} = \sum_{r=2}^{n} \frac{-1}{r} + \frac{1}{2(r+1)} + \frac{1}{2(r-1)}.$$

$$= \Bigg[\left[\cancel{\frac{-1}{2}} + \cancel{\frac{1}{2(3)}} + \cancel{\frac{1}{2(1)}}\right] + \left[\cancel{\frac{-1}{3}} + \cancel{\frac{1}{2(4)}} + \cancel{\frac{1}{2(2)}}\right] + \left[\cancel{\frac{-1}{4}} + \cancel{\frac{1}{2(5)}} + \cancel{\frac{1}{2(3)}}\right] + \left[\cancel{\frac{-1}{5}} + \frac{1}{2(6)} + \cancel{\frac{1}{2(4)}}\right] + \dots$$

$$\dots + \left[\cancel{\frac{-1}{n-2}} + \cancel{\frac{1}{2(n-1)}} + \cancel{\frac{1}{2(n-3)}}\right] + \left[\cancel{\frac{-1}{n-1}} + \frac{1}{2(n)} + \cancel{\frac{1}{2(n-2)}}\right] + \left[\frac{-1}{n} + \frac{1}{2(n+1)} + \cancel{\frac{1}{2(n-1)}}\right] \Bigg]$$

$$= \frac{1}{4} + \frac{1}{2n} - \frac{1}{n} + \frac{1}{2(n+1)}$$

$$= \frac{1}{4} + \frac{1-2}{2n} + \frac{1}{2(n+1)} = \frac{1}{4} - \frac{1}{2n} + \frac{1}{2(n+1)}.$$

$$\sum_{r=2}^{\infty} \frac{1}{r(r+1)(r-1)} = \lim_{n\to\infty} \left(\frac{1}{4} - \frac{1}{2n} + \frac{1}{2(n+1)}\right) = \frac{1}{4} - 0 + 0 = \frac{1}{4}.$$

39 Given that

$$u_k = \frac{1}{\sqrt{(2k-1)}} - \frac{1}{\sqrt{(2k+1)}},$$

express $\sum_{k=13}^{n} u_k$ in terms of n. [4]

Deduce the value of $\sum_{k=13}^{\infty} u_k$. [1]

$$\sum_{k=13}^{n} u_k = \sum_{k=13}^{n} \left[\frac{1}{\sqrt{2k-1}} - \frac{1}{\sqrt{2k+1}}\right]$$

$$= \left[\left(\frac{1}{\sqrt{25}} - \cancel{\frac{1}{\sqrt{27}}}\right) + \left(\cancel{\frac{1}{\sqrt{27}}} - \cancel{\frac{1}{\sqrt{29}}}\right) + \left(\cancel{\frac{1}{\sqrt{29}}} - \cancel{\frac{1}{\sqrt{31}}}\right) + \cdots \right.$$
$$\left. \left(\cdots - \cancel{\frac{1}{\sqrt{2n-3}}}\right) + \left(\cancel{\frac{1}{\sqrt{2n-3}}} - \cancel{\frac{1}{\sqrt{2n-1}}}\right) + \left(\cancel{\frac{1}{\sqrt{2n-1}}} - \frac{1}{\sqrt{2n+1}}\right)\right]$$

$$= \frac{1}{\sqrt{25}} - \frac{1}{\sqrt{2n+1}} = \frac{1}{5} - \frac{1}{\sqrt{2n+1}}.$$

$$\sum_{k=13}^{\infty} u_k = \lim_{n\to\infty}\left[\frac{1}{5} - \frac{1}{\sqrt{2n+1}}\right] = \frac{1}{5} - 0 = \frac{1}{5}.$$

40 The sequence $a_1, a_2, a_3, \ldots$ is such that, for all positive integers n,

$$a_n = \frac{n+5}{\sqrt{(n^2 - n + 1)}} - \frac{n+6}{\sqrt{(n^2 + n + 1)}}.$$

The sum $\sum_{n=1}^{N} a_n$ is denoted by S_N. Find

(i) the value of S_{30} correct to 3 decimal places, [3]

(ii) the least value of N for which $S_N > 4.9$. [4]

i) $\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} \left[\frac{n+5}{\sqrt{n^2-n+1}} - \frac{n+6}{\sqrt{n^2+n+1}} \right]$

$$= \left[\left(\frac{6}{\sqrt{1}} - \cancel{\frac{7}{\sqrt{3}}} \right) + \left(\cancel{\frac{7}{\sqrt{3}}} - \cancel{\frac{8}{\sqrt{7}}} \right) + \left(\cancel{\frac{8}{\sqrt{7}}} - \cancel{\frac{9}{\sqrt{13}}} \right) + \left(\cancel{\frac{9}{\sqrt{13}}} + \cdots \right. \right.$$

$$\left. \cdots - \cancel{\frac{N+4}{\sqrt{N^2-3N+2}}} \right) + \left(\cancel{\frac{N+4}{\sqrt{N^2-3N+2}}} - \cancel{\frac{N+5}{\sqrt{N^2-N+1}}} \right) + \left(\cancel{\frac{N+5}{\sqrt{N^2-N+1}}} - \frac{N+6}{\sqrt{N^2+N+1}} \right) \Bigg]$$

$$= \frac{6}{\sqrt{1}} - \frac{N+6}{\sqrt{N^2+N+1}} = 6 - \frac{N+6}{\sqrt{N^2+N+1}}.$$

$$\Rightarrow S_{30} = 6 - \frac{30+6}{\sqrt{30^2+30+1}} = 6 - \frac{36}{\sqrt{931}} = 4.8201 \approx \underline{4.820}.$$

ii)

$$S_N > 4.9$$

$$\Rightarrow 6 - \frac{N+6}{\sqrt{N^2+N+1}} > 4.9$$

$$1.1 > \frac{N+6}{\sqrt{N^2+N+1}}$$

$$1.1\sqrt{N^2+N+1} > N+6$$

$$1.21(N^2+N+1) > (N+6)^2$$

$$1.21N^2 + 1.21N + 1.21 > N^2 + 12N + 36$$

$$0.21N^2 - 10.79N - 34.79 > 0$$

Critical values : $0.21N^2 - 10.79N - 34.79 = 0$

$$N = 54.42, -3.04$$

$\therefore$ $N > 54.42$ or $N < -3.04$ (Ignore)

$$\Rightarrow N > 54.42$$

$\Rightarrow$ Least value of $N = 55$.

41 Use the method of differences to find $\sum_{r=1}^{n} \frac{1}{(2r)^2 - 1}$. [4]

Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{(2r)^2 - 1}$. [1]

$$\frac{1}{(2r)^2-1} \equiv \frac{1}{(2r-1)(2r+1)} \equiv \frac{A}{2r-1} + \frac{B}{2r+1}$$

$$1 \equiv A(2r+1) + B(2r-1)$$

$$\cancel{A}\ r = \frac{-1}{2} \Rightarrow \quad 1 = B(-2) \quad \Rightarrow B = \frac{-1}{2}.$$

$$r = \frac{1}{2} \Rightarrow \quad 1 = A(2) \quad \Rightarrow A = \frac{1}{2}.$$

$$\Rightarrow \sum_{r=1}^{n} \frac{1}{(2r)^2-1} = \sum_{r=1}^{n} \frac{\frac{1}{2}}{2r-1} - \frac{\frac{1}{2}}{2r+1} = \frac{1}{2} \sum_{r=1}^{n} \frac{1}{2r-1} - \frac{1}{2r+1}$$

$$= \frac{1}{2}\Bigg[\left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right) + \left(\cancel{\frac{1}{7}} - \cancel{\frac{1}{9}}\right) + \cdots$$
$$\cdots + \left(\cancel{\frac{1}{2n-5}} - \cancel{\frac{1}{2n-3}}\right) + \left(\cancel{\frac{1}{2n-3}} - \cancel{\frac{1}{2n-1}}\right) + \left(\cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}\right)\Bigg]$$

$$= \frac{1}{2}\left[1 - \frac{1}{2n+1}\right]$$

$$\sum_{r=1}^{\infty} \frac{1}{(2r)^2-1} = \lim_{n \to \infty} \left[\frac{1}{2}\left(1 - \frac{1}{2n+1}\right)\right] = \frac{1}{2}(1-0) = \frac{1}{2}.$$

42 Find $\sum_{r=1}^{n}(4r-3)(4r+1)$, giving your answer in its simplest form. [4]

$$= \sum_{r=1}^{n}\left(16r^2 - 12r + 4r - 3\right)$$

$$= \sum_{r=1}^{n}\left(16r^2 - 8r + 3\right) = 16\sum_{r=1}^{n} r^2 - 8\sum_{r=1}^{n} r - \sum_{r=1}^{n} 3$$

$$= 16\left[\frac{n(n+1)(2n+1)}{6}\right] - 8\left[\frac{n(n+1)}{2}\right] - 3n$$

$$= n\left[\frac{8(2n^2+3n+1)}{3} - 4(n+1) - 3\right]$$

$$= n\left[\frac{16n^2+24n+8-12n-12-9}{3}\right]$$

$$= n\left[\frac{16n^2+12n-13}{3}\right]$$

43 (i) By considering $(2r+1)^2 - (2r-1)^2$, use the method of differences to prove that

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1).$$ [3]

$$(2r+1)^2 - (2r-1)^2 = 4r^2+4r+1-(4r^2-4r+1) = 4r^2+4r+1-4r^2+4r-1 = 8r.$$

$$\therefore \sum_{r=1}^{n} 8r = \sum_{r=1}^{n} (2r+1)^2-(2r-1)^2$$

$$= \left[(\cancel{3^2}-1^2)+(\cancel{5^2}-\cancel{3^2})+(\cancel{7^2}-\cancel{5^2})+(\cancel{9^2}-\cancel{7^2})+\dots \right.$$
$$\left. +\cancel{[(2n-3)^2-(2n-5)^2]}+[\cancel{(2n-1)^2}-\cancel{(2n-3)^2}]+[(2n+1)^2-\cancel{(2n-1)^2}]\right]$$

$$= \left[-1^2+(2n+1)^2\right] = -1+4n^2+4n+1 = 4n^2+4n$$

$$= 4n(n+1)$$

$$\therefore \sum_{r=1}^{n} 8r = 4n(n+1)$$

$$8\sum_{r=1}^{n} r = 4n(n+1)$$

$$\Rightarrow \sum_{r=1}^{n} r = \frac{4n(n+1)}{8} = \frac{n(n+1)}{2}$$

(Shown).

(ii) By considering $(2r+1)^4 - (2r-1)^4$, use the method of differences and the result given in part **(i)** to prove that

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2.$$ [5]

$(2r+1)^4 - (2r-1)^4$

$= \left[(2r)^4 + {}^4C_1(2r)^3 + {}^4C_2(2r)^2 + {}^4C_3(2r)^1 + 1^4\right] - \left[(2r)^4 + {}^4C_1(2r)^3(-1)^1 + {}^4C_2(2r)^2(-1)^2 + {}^4C_3(2r)^1(-1)^3 + (-1)^4\right]$

$= \left[16r^4 + 4(8r^3) + 6(4r^2) + 4(2r) + 1\right] - \left[16r^4 + 4(8r^3)(-1) + 6(4r^2)(1) + 4(2r)(-1) + 1\right]$

$= 16r^4 + 32r^3 + 24r^2 + 8r + 1 - 16r^4 + 32r^3 - 24r^2 + 8r - 1$

$= 64r^3 + 16r = 16(4r^3 + r).$

$\sum_{r=1}^{n} (2r+1)^4 - (2r-1)^4$

$= \left[(\cancel{3^4} - 1^4) + (\cancel{5^4} - \cancel{3^4}) + (\cancel{7^4} - \cancel{5^4}) + (\cancel{9^4} - \cancel{7^4}) + \dots\right.$
$\left. + \left[\cancel{(2n-3)^4} - \cancel{(2n-5)^4}\right] + \left[\cancel{(2n-1)^4} - \cancel{(2n-3)^4}\right] + \left[(2n+1)^4 - \cancel{(2n-1)^4}\right]\right]$

$= -1^4 + (2n+1)^4 = (2n+1)^4 - 1 = 16n^4 + 32n^3 + 24n^2 + 8n + 1 - 1$

$= 16n^4 + 32n^3 + 24n^2 + 8n. = 8(2n^4 + 4n^3 + 3n^2 + n).$

$\Rightarrow \sum_{r=1}^{n} 16(4r^3 + r) = 8(2n^4 + 4n^3 + 3n^2 + n)$

$16 \sum_{r=1}^{n} 4r^3 + r = 8(2n^4 + 4n^3 + 3n^2 + n)$

$\sum_{r=1}^{n} 4r^3 + \sum_{r=1}^{n} r = \dfrac{2n^4 + 4n^3 + 3n^2 + n}{2}$

$\sum_{r=1}^{n} 4r^3 = \dfrac{2n^4 + 4n^3 + 3n^2 + n}{2} - \dfrac{n(n+1)}{2}$

$\sum_{r=1}^{n} 4r^3 = \dfrac{2n^4 + 4n^3 + 3n^2 + n - n^2 - n}{2}$

$4 \sum_{r=1}^{n} r^3 = \dfrac{2n^4 + 4n^3 + 2n^2}{2}$

$\sum_{r=1}^{n} r^3 = \dfrac{2n^4 + 4n^3 + 2n^2}{8} = \dfrac{2n^2(n^2 + 2n + 1)}{8} = \dfrac{n^2(n+1)^2}{4}$

(Shown).

The sums S and T are defined as follows:

$$S = 1^3 + 2^3 + 3^3 + 4^3 + \ldots + (2N)^3 + (2N+1)^3,$$
$$T = 1^3 + 3^3 + 5^3 + 7^3 + \ldots + (2N-1)^3 + (2N+1)^3.$$

(iii) Use the result given in part **(ii)** to show that $S = (2N+1)^2(N+1)^2$. [1]

$$S = \sum_{r=1}^{2N+1} r^3 = \frac{(2N+1)^2(2N+1+1)^2}{4} = \frac{(2N+1)^2(2N+2)^2}{4}$$
$$= \frac{(2N+1)^2[4(N+1)^2]}{4} = (2N+1)^2(N+1)^2.$$

(shown).

(iv) Hence, or otherwise, find an expression in terms of N for T, factorising your answer as far as possible. [2]

$$T = 1^3+3^3+5^3+7^3+\cdots+(2N-1)^3+(2N+1)^3$$
$$= [1^3+2^3+3^3+4^3+\cdots+(2N-1)^3+(2N)^3+(2N+1)^3] - [2^3+4^3+6^3+\cdots+(2N)^3]$$
$$= S - \sum_{r=1}^{N}(2r)^3 = S - \sum_{r=1}^{N} 8r^3 = S - 8\sum_{r=1}^{N} r^3$$
$$= S - 8\left[\frac{N^2(N+1)^2}{4}\right] = (2N+1)^2(N+1)^2 - 2N^2(N+1)^2$$
$$= (N+1)^2[(2N+1)^2 - 2N^2] = (N+1)^2[4N^2+4N+1-2N^2]$$
$$T = (N+1)^2(2N^2+4N+1)$$

(v) Deduce the value of $\frac{S}{T}$ as $N \to \infty$. [2]

$$\frac{S}{T} = \frac{(2N+1)^2(N+1)^2}{(2N^2+4N+1)(N+1)^2} = \frac{(4N^2+4N+1)(N+1)^2}{(2N^2+4N+1)(N+1)^2} = \frac{4N^2+4N+1}{2N^2+4N+1}.$$

As $N \mapsto \infty$, $\frac{S}{T} \mapsto \frac{4}{2} = 2.$

44 Let

$$S_N = \sum_{r=1}^{N}(3r+1)(3r+4) \quad \text{and} \quad T_N = \sum_{r=1}^{N}\frac{1}{(3r+1)(3r+4)}.$$

(i) Use standard results from the List of Formulae (MF10) to show that

$$S_N = N(3N^2 + 12N + 13).$$ [3]

$$S_N = \sum_{r=1}^{N}(3r+1)(3r+4) = \sum_{r=1}^{N}(9r^2+15r+4)$$

$$= 9\sum_{r=1}^{N}r^2 + 15\sum_{r=1}^{N}r + 4\sum_{r=1}^{N}1$$

$$= 9\left[\frac{N(N+1)(2N+1)}{6}\right] + 15\left[\frac{N(N+1)}{2}\right] + 4N$$

$$= N\left[\frac{3(2N^2+3N+1)}{2} + \frac{15(N+1)}{2} + 4\right]$$

$$= N\left[\frac{6N^2+9N+3+15N+15+8}{2}\right]$$

$$= N\left[\frac{6N^2+24N+26}{2}\right] = N(3N^2+12N+13).$$

(ii) Use the method of differences to show that

$$T_N = \frac{1}{12} - \frac{1}{3(3N+4)}.$$ [3]

$$\frac{1}{(3r+1)(3r+4)} \equiv \frac{A}{3r+1} + \frac{B}{3r+4}$$

$$1 \equiv A(3r+4) + B(3r+1)$$

$$r = \frac{-4}{3} \Rightarrow 1 = B(-3) \Rightarrow B = \frac{-1}{3}.$$

$$r = \frac{-1}{3} \Rightarrow 1 = A(3) \Rightarrow A = \frac{1}{3}$$

$$\therefore \frac{1}{(3r+1)(3r+4)} \equiv \frac{\frac{1}{3}}{3r+1} - \frac{\frac{1}{3}}{3r+4} = \frac{1}{3}\left[\frac{1}{3r+1} - \frac{1}{3r+4}\right]$$

$$T_N = \sum_{r=1}^{N} \frac{1}{(3r+1)(3r+4)} = \sum_{r=1}^{N} \frac{1}{3}\left[\frac{1}{3r+1} - \frac{1}{3r+4}\right]$$

$$= \frac{1}{3}\sum_{r=1}^{N}\left(\frac{1}{3r+1} - \frac{1}{3r+4}\right)$$

$$= \frac{1}{3}\left[\left(\frac{1}{4} - \cancel{\frac{1}{7}}\right) + \left(\cancel{\frac{1}{7}} - \cancel{\frac{1}{10}}\right) + \left(\cancel{\frac{1}{10}} - \cancel{\frac{1}{13}}\right) + \left(\cancel{\frac{1}{13}} - \cancel{\frac{1}{16}}\right) + \cdots + \left(\cancel{\frac{1}{3N-5}} - \cancel{\frac{1}{3N-2}}\right) + \left(\cancel{\frac{1}{3N-2}} - \cancel{\frac{1}{3N+1}}\right) + \left(\cancel{\frac{1}{3N+1}} - \frac{1}{3N+4}\right)\right]$$

$$= \frac{1}{3}\left(\frac{1}{4} - \frac{1}{3N+4}\right) = \frac{1}{12} - \frac{1}{3(3N+4)}.$$

(iii) Deduce that $\frac{S_N}{T_N}$ is an integer. [2]

$$\frac{S_N}{T_N} = \frac{N(3N^2+12N+13)}{\frac{1}{12} - \frac{1}{3(3N+4)}} = \frac{N(3N^2+12N+13)}{\frac{3N+4-4}{12(3N+4)}}$$

$$= \frac{N(3N^2+12N+13)}{\frac{3N}{12(3N+4)}} = \frac{12(3N+4)N(3N^2+12N+13)}{3N}$$

$$= 4(3N+4)(3N^2+12N+13)$$

Note N is an integer $\therefore$ $3N+4$, $3N^2+12N+13$ are also integers.

$\therefore \frac{S_N}{T_N}$ is also an integer.

(iv) Find $\lim_{N\to\infty} \frac{S_N}{N^3 T_N}$. [2]

$$\frac{S_N}{N^3 T_N} = \frac{1}{N^3}\left[4(3N+4)(3N^2+12N+13)\right] = 4\left[3+\frac{4}{N}\right]\left[3+\frac{12}{N}+\frac{13}{N^2}\right].$$

As $N \to \infty$, $\frac{S_N}{N^3 T_N} \to 4(3+0)(3+0+0) = 36$.

45 Let $S_N = \sum_{r=1}^{N}(5r+1)(5r+6)$ and $T_N = \sum_{r=1}^{N}\frac{1}{(5r+1)(5r+6)}$.

(i) Use standard results from the List of Formulae (MF10) to show that

$$S_N = \tfrac{1}{3}N(25N^2 + 90N + 83).$$ [3]

$$S_N = \sum_{r=1}^{N}(5r+1)(5r+6) = \sum_{r=1}^{N}(25r^2+35r+6)$$

$$= 25\sum_{r=1}^{N} r^2 + 35\sum_{r=1}^{N} r + \sum_{r=1}^{N} 6$$

$$= 25\left[\frac{N(N+1)(2N+1)}{6}\right] + 35\left[\frac{N(N+1)}{2}\right] + 6N$$

$$= N\left[\frac{25(N+1)(2N+1)}{6} + \frac{35(N+1)}{2} + 6\right]$$

$$= N\left[\frac{25(2N^2+3N+1)}{6} + \frac{35N+35}{2} + 6\right]$$

$$= N\left[\frac{50N^2+75N+25+105N+105+36}{6}\right]$$

$$= N\left[\frac{50N^2+180N+166}{6}\right]$$

$$= \frac{N}{3}(25N^2+90N+83).$$

(ii) Use the method of differences to express T_N in terms of N. [4]

$$\frac{1}{(5r+1)(5r+6)} \equiv \frac{A}{5r+1} + \frac{B}{5r+6}.$$

$$1 \equiv A(5r+6) + B(5r+1)$$

$$r = \frac{-6}{5} \Rightarrow 1 = B(-5) \Rightarrow B = \frac{-1}{5}.$$

$$r = \frac{-1}{5} \Rightarrow 1 = A(5) \Rightarrow A = \frac{1}{5}.$$

$$\Rightarrow T_N = \sum_{r=1}^{N} \frac{\frac{1}{5}}{5r+1} - \frac{\frac{1}{5}}{5r+6} = \frac{1}{5}\sum_{r=1}^{N} \frac{1}{5r+1} - \frac{1}{5r+6}$$

$$= \frac{1}{5}\left[\left(\frac{1}{6} - \cancel{\frac{1}{11}}\right) + \left(\cancel{\frac{1}{11}} - \cancel{\frac{1}{16}}\right) + \left(\cancel{\frac{1}{16}} - \cancel{\frac{1}{21}}\right) + \cdots + \left(\cancel{\frac{1}{5N-9}} - \cancel{\frac{1}{5N-4}}\right) + \left(\cancel{\frac{1}{5N-4}} - \cancel{\frac{1}{5N+1}}\right) + \left(\cancel{\frac{1}{5N+1}} - \frac{1}{5N+6}\right)\right]$$

$$= \frac{1}{5}\left[\frac{1}{6} - \frac{1}{5N+6}\right] = \frac{1}{30} - \frac{1}{5(5N+6)}.$$

(iii) Find $\lim_{N\to\infty}(N^{-3}S_NT_N)$. [2]

$$N^{-3}S_N T_N = \frac{1}{N^3}\left[\frac{N}{3}(25N^2+90N+83)\right]\left[\frac{1}{30} - \frac{1}{5(5N+6)}\right]$$

$$= \frac{1}{3}\left(25 + \frac{90}{N} + \frac{83}{N^2}\right)\left(\frac{1}{30} - \frac{1}{5(5N+6)}\right)$$

$$\lim_{N\to\infty} N^{-3}S_N T_N = \frac{1}{3}(25+0+0)\left(\frac{1}{30} - 0\right) = \frac{25}{3} \times \frac{1}{30} = \frac{5}{18}.$$

46 (a) Use standard results from the List of Formulae (MF19) to show that

$$\sum_{r=1}^{n}(7r+1)(7r+8) = an^3 + bn^2 + cn,$$

where a, b and c are constants to be determined. [3]

$$\sum_{r=1}^{n}(7r+1)(7r+8) = \sum_{r=1}^{n} 49r^2 + 63r + 8$$

$$= 49\sum_{r=1}^{n} r^2 + 63\sum_{r=1}^{n} r + \sum_{r=1}^{n} 8$$

$$= 49\left[\frac{n(n+1)(2n+1)}{6}\right] + 63\left[\frac{n(n+1)}{2}\right] + 8n.$$

$$= n\left[\frac{49(n+1)(2n+1)}{6} + \frac{63(n+1)}{2} + 8\right]$$

$$= n\left[\frac{49(2n^2+3n+1) + 189(n+1) + 48}{6}\right]$$

$$= n\left[\frac{98n^2 + 147n + 49 + 189n + 189 + 48}{6}\right]$$

$$= n\left[\frac{98n^2 + 336n + 286}{6}\right] = \frac{49n^3}{3} + 56n^2 + \frac{143n}{3}$$

(b) Use the method of differences to find $\sum_{r=1}^{n} \frac{1}{(7r+1)(7r+8)}$ in terms of n. [4]

$$\frac{1}{(7r+1)(7r+8)} \equiv \frac{A}{7r+1} + \frac{B}{7r+8}$$

$$1 \equiv A(7r+8) + B(7r+1)$$

$$r = \frac{-8}{7} \Rightarrow 1 = B(-7) \Rightarrow B = \frac{-1}{7}.$$

$$r = \frac{-1}{7} \Rightarrow 1 = A(7) \Rightarrow A = \frac{1}{7}.$$

$$\therefore \sum_{r=1}^{n} \frac{1}{(7r+1)(7r+8)} = \sum_{r=1}^{n} \frac{\frac{1}{7}}{7r+1} - \frac{\frac{1}{7}}{7r+8} = \frac{1}{7}\sum_{r=1}^{n}\left(\frac{1}{7r+1} - \frac{1}{7r+8}\right)$$

$$= \frac{1}{7}\left[\left(\frac{1}{8} - \cancel{\frac{1}{15}}\right) + \left(\cancel{\frac{1}{15}} - \cancel{\frac{1}{22}}\right) + \left(\cancel{\frac{1}{22}} - \cancel{\frac{1}{29}}\right) + \left(\cancel{\frac{1}{29}} + \cdots\right.\right.$$
$$\left.\left.\cdots - \cancel{\frac{1}{7n-13}}\right) + \left(\cancel{\frac{1}{7n-13}} - \cancel{\frac{1}{7n-6}}\right) + \left(\cancel{\frac{1}{7n-6}} - \cancel{\frac{1}{7n+1}}\right) + \left(\cancel{\frac{1}{7n+1}} - \frac{1}{7n+8}\right)\right]$$

$$= \frac{1}{7}\left[\frac{1}{8} - \frac{1}{7n+8}\right]$$

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{(7r+1)(7r+8)}$. [1]

$$\sum_{r=1}^{\infty} \frac{1}{(7r+1)(7r+8)} = \lim_{n\to\infty}\left(\frac{1}{7}\left(\frac{1}{8} - \frac{1}{7n+8}\right)\right) = \frac{1}{7}\left(\frac{1}{8} - 0\right) = \frac{1}{56}.$$

47 **(a)** By simplifying $(x^n - \sqrt{x^{2n}+1})(x^n + \sqrt{x^{2n}+1})$ show $\frac{1}{x^n - \sqrt{x^{2n}+1}} = -x^n - \sqrt{x^{2n}+1}$. [1]

$$\left(x^n - \sqrt{x^{2n}+1}\right)\left(x^n + \sqrt{x^{2n}+1}\right) = (x^n)^2 - \left(\sqrt{x^{2n}+1}\right)^2 = x^{2n} - x^{2n} - 1 = -1$$

$$\Rightarrow \ -\left(x^n + \sqrt{x^{2n}+1}\right) = \frac{1}{x^n - \sqrt{x^{2n}+1}} \Rightarrow \frac{1}{x^n - \sqrt{x^{2n}+1}} = -x^n - \sqrt{x^{2n}+1}.$$

Let $u_n = x^{n+1} + \sqrt{x^{2n+2}+1} + \frac{1}{x^n - \sqrt{x^{2n}+1}}$.

(b) Use the method of differences to find $\sum_{n=1}^{N} u_n$ in terms of N and x. [3]

$$\sum_{n=1}^{N} x^{n+1} - \sqrt{x^{2n+2}+1} + \frac{1}{x^n - \sqrt{x^{2n}+1}} = \sum_{n=1}^{N} \left(x^{n+1} - \sqrt{x^{2n+2}+1}\right) - \left(x^n + \sqrt{x^{2n}+1}\right)$$

$$= \Big[\left[\left(\cancel{x^2 - \sqrt{x^4+1}}\right) - \left(x^1 + \sqrt{x^2+1}\right)\right] + \left[\left(\cancel{x^3 - \sqrt{x^6+1}}\right) - \left(\cancel{x^2 + \sqrt{x^4+1}}\right)\right] + \left[\left(\cancel{x^4 - \sqrt{x^8+1}}\right) - \left(\cancel{x^3 + \sqrt{x^6+1}}\right)\right] + \dots$$

$$\dots + \left[\left(\cancel{x^{N-1} - \sqrt{x^{2N-2}+1}}\right) - \left(\cancel{x^{N-2} + \sqrt{x^{2N-4}+1}}\right)\right] + \left[\left(\cancel{x^N - \sqrt{x^{2N}+1}}\right) - \left(\cancel{x^{N-1} + \sqrt{x^{2N-2}+1}}\right)\right] + \left[\left(x^{N+1} - \sqrt{x^{2N+2}+1}\right) - \left(\cancel{x^N + \sqrt{x^{2N}+1}}\right)\right] \Big]$$

$$= -\left(x + \sqrt{x^2+1}\right) + x^{N+1} - \sqrt{x^{2N+2}+1}.$$

$$= x^{N+1} + \sqrt{x^{2N+2}+1} - x - \sqrt{x^2+1}.$$

(c) Deduce the set of values of x for which the infinite series

$$u_1 + u_2 + u_3 + \dots$$

is convergent and give the sum to infinity when this exists. [3]

For series to be convergent, we require $-1 < x < 1$. since $x^N \mapsto 0$ as $N \mapsto \infty$ only if $-1 < x < 1$.

$$\lim_{N \mapsto \infty} \left[x^{N+1} + \sqrt{x^{2N+2}+1} - x - \sqrt{x^2+1}\right]$$

$$= 0 + \sqrt{0+1} - x - \sqrt{x^2+1}$$

$$= 1 - x - \sqrt{x^2+1}.$$

48 (a) By first expressing $\frac{1}{r^2-1}$ in partial fractions, show that

$$\sum_{r=2}^{n} \frac{1}{r^2-1} = \frac{3}{4} - \frac{an+b}{2n(n+1)},$$

where a and b are integers to be found. [5]

$$\frac{1}{r^2-1} \equiv \frac{A}{r+1} + \frac{B}{r-1}$$

$$1 \equiv A(r-1) + B(r+1)$$

$$r = 1 \Rightarrow 1 = B(2) \Rightarrow B = \frac{1}{2}.$$

$$r = -1 \Rightarrow 1 = A(-2) \Rightarrow A = \frac{-1}{2}.$$

$$\therefore \frac{1}{r^2-1} \equiv \frac{\frac{-1}{2}}{r+1} + \frac{\frac{1}{2}}{r-1} = \frac{1}{2}\left[\frac{1}{r-1} - \frac{1}{r+1}\right].$$

$$\sum_{r=2}^{n}\left(\frac{1}{r^2-1}\right) = \sum_{r=2}^{n} \frac{1}{2}\left[\frac{1}{r-1} - \frac{1}{r+1}\right]$$

$$= \frac{1}{2}\sum_{r=2}^{n}\left[\frac{1}{r-1} - \frac{1}{r+1}\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{5}} + \cdots\right) \right.$$
$$\left. \cdots + \left(\cancel{\frac{1}{n-4}} - \cancel{\frac{1}{n-2}}\right) + \left(\cancel{\frac{1}{n-3}} - \cancel{\frac{1}{n-1}}\right) + \left(\cancel{\frac{1}{n-2}} - \frac{1}{n}\right) + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right)\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right]$$

$$= \frac{1}{2}\left[\frac{3}{2} - \frac{n+1+n}{n(n+1)}\right] = \frac{1}{2}\left[\frac{3}{2} - \frac{2n+1}{n(n+1)}\right] = \frac{3}{4} - \frac{2n+1}{2n(n+1)}.$$

(b) Deduce the value of $\sum_{r=2}^{\infty} \frac{1}{r^2-1}$. [1]

$$\sum_{r=2}^{\infty} \frac{1}{r^2-1} = \lim_{n \to \infty} \left(\frac{3}{4} - \frac{2n+1}{2n(n+1)} \right) = \lim_{n \to \infty} \left(\frac{3}{4} \right) - \lim_{n \to \infty} \left(\frac{\frac{2}{n} + \frac{1}{n^2}}{2\left(1+\frac{1}{n}\right)} \right)$$

$$= \frac{3}{4} - 0 = \frac{3}{4}.$$

(c) Find the limit, as $n \to \infty$, of $\sum_{r=n+1}^{2n} \frac{n}{r^2-1}$. [4]

$$\sum_{r=n+1}^{2n} \frac{n}{r^2-1} = n \sum_{r=n+1}^{2n} \frac{1}{r^2-1} = n \left[\sum_{r=1}^{2n} \frac{1}{r^2-1} - \sum_{r=1}^{n} \frac{1}{r^2-1} \right]$$

$$= n \left[\left(\frac{3}{4} - \frac{2(2n)+1}{2(2n)(2n+1)} \right) - \left(\frac{3}{4} - \frac{2n+1}{2n(n+1)} \right) \right]$$

$$= n \left[\frac{3}{4} - \frac{4n+1}{2(2n)(2n+1)} - \frac{3}{4} + \frac{2n+1}{2n(n+1)} \right]$$

$$= n \left[\frac{2n+1}{2n(n+1)} - \frac{4n+1}{4n(2n+1)} \right] = \frac{n(2n+1)}{2n(n+1)} - \frac{n(4n+1)}{4n(2n+1)}.$$

$$\therefore \sum_{r=n+1}^{2n} \frac{2n^2+n}{2n^2+2n} - \frac{4n^2+n}{8n^2+4n}$$

$$\lim_{n \to \infty} \sum_{r=n+1}^{2n} = \lim_{n \to \infty} \frac{2n^2+n}{2n^2+2n} - \frac{4n^2+n}{8n^2+4n} = \left(\frac{2}{2} - \frac{4}{8} \right) = 1 - \frac{1}{2} = \frac{1}{2}.$$

49 Let $S_n = 2^2 + 6^2 + 10^2 + \ldots + (4n-2)^2$.

(a) Use standard results from the List of Formulae (MF19) to show that $S_n = \frac{4}{3}n(4n^2 - 1)$. [4]

$$S_n = 2^2 + 6^2 + 10^2 + \ldots + (4n-2)^2$$

$$= \sum_{r=1}^{n} (4r-2)^2 = \sum_{r=1}^{n} 16r^2 - 16r + 4$$

$$= 16\sum_{r=1}^{n} r^2 - 16\sum_{r=1}^{n} r + 4\sum_{r=1}^{n} 1$$

$$= 16\left[\frac{n(n+1)(2n+1)}{6}\right] - 16\left[\frac{n(n+1)}{2}\right] + 4n$$

$$= n\left[\frac{8(n+1)(2n+1)}{3} - 8(n+1) + 4\right].$$

$$= n\left[\frac{8(2n^2+3n+1) - 24n - 24 + 12}{3}\right]$$

$$= n\left[\frac{16n^2 + 24n + 8 - 24n - 24 + 12}{3}\right]$$

$$= n\left[\frac{16n^2 - 4}{3}\right] = \frac{4n}{3}\left[4n^2 - 1\right]$$

(Shown).

(b) Express $\frac{n}{S_n}$ in partial fractions and find $\sum_{n=1}^{N} \frac{n}{S_n}$ in terms of N. [4]

$$\frac{n}{S_n} = \frac{n}{\frac{4n}{3}(4n^2-1)} = \frac{3}{4(2n-1)(2n+1)} = \frac{3}{4}\left[\frac{1}{(2n-1)(2n+1)}\right]$$

$$\frac{1}{(2n-1)(2n+1)} \equiv \frac{A}{2n-1} + \frac{B}{2n+1}$$

$$1 \equiv A(2n+1) + B(2n-1)$$

$$n = \frac{1}{2} \Rightarrow 1 = A(2) \Rightarrow A = \frac{1}{2}.$$

$$n = \frac{-1}{2} \Rightarrow 1 = B(-2) \Rightarrow B = \frac{-1}{2}.$$

$$\therefore \frac{n}{S_n} = \frac{3}{4}\left[\frac{\frac{1}{2}}{2n-1} - \frac{\frac{1}{2}}{2n+1}\right] = \frac{3}{4}\left[\frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\right] = \frac{3}{8}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right).$$

$$\sum_{n=1}^{N} \frac{n}{S_n} = \sum_{n=1}^{N} \frac{3}{8}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = \frac{3}{8}\sum_{n=1}^{N}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

$$= \frac{3}{8}\left[\left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right) + \left(\cancel{\frac{1}{7}} + \cdots \right.\right.$$
$$\left.\cdots + \left(\cancel{\frac{1}{2N-5}} - \cancel{\frac{1}{2N-3}}\right) + \left(\cancel{\frac{1}{2N-3}} - \cancel{\frac{1}{2N-1}}\right) + \left(\cancel{\frac{1}{2N-1}} - \frac{1}{2N+1}\right)\right]$$

$$= \frac{3}{8}\left[1 - \frac{1}{2N+1}\right]$$

(c) Deduce the value of $\sum_{n=1}^{\infty} \frac{n}{S_n}$. [1]

$$\sum_{n=1}^{\infty} \frac{n}{S_n} = \lim_{n \to \infty}\left[\frac{3}{8}\left(1 - \frac{1}{2N+1}\right)\right] = \frac{3}{8}(1-0) = \frac{3}{8}.$$