AS Level Further Mathematics

Topic: Momentum

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Q.1 Oct/Nov/P31+P33/2022

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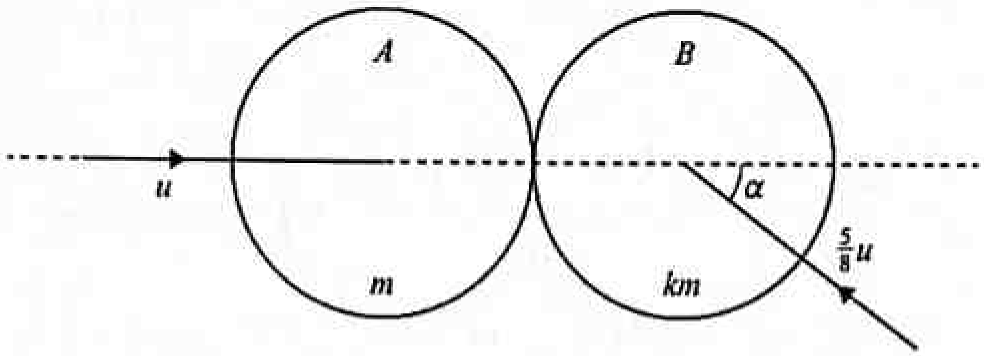
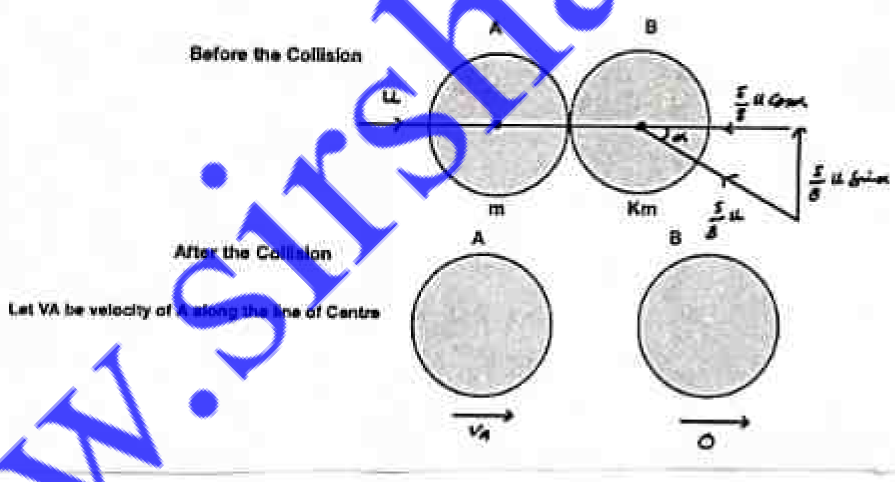


Two uniform smooth spheres A and B of equal radii have masses m and km respectively. The two spheres are moving on a horizontal surface with speeds u and $\frac{5}{8}u$ respectively. Immediately before the spheres collide, A is travelling along the line of centres, and B's direction of motion makes an angle α with the line of centres (see diagram). The coefficient of restitution between the spheres is $\frac{2}{3}$ and $\tan\alpha = \frac{3}{4}$.

After the collision, the direction of motion of B is perpendicular to the line of centres.

(a) Find the value of k. [4]

(b) Find the loss in the total kinetic energy as a result of the collision. [4]

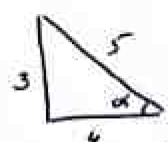


By conservation law of Momentum

$\xrightarrow{+ve}$

$$mu - km\left(\frac{5}{8}u\cos\alpha\right) = mV_A$$

Given $\tan\alpha = \frac{3}{4}$



$$V_A = u - k\left[\frac{5}{8}u\cdot\frac{4}{5}\right] = u - \frac{ku}{2}$$

$\sin\alpha = \frac{3}{5}$

$\cos\alpha = \frac{4}{5}$

By Newton law of Restitution

$$e = \frac{\text{speed of separation}}{\text{speed of approach}}$$

$$\frac{2}{3} = \frac{0 - V_A}{u + \frac{5}{8}u\cos\alpha} = \frac{-V_A}{u + \frac{u}{2}} = \frac{-2V_A}{3u}$$

$$\frac{2}{3} = \frac{-2\left(u - \frac{ku}{2}\right)}{3u}$$ By substituting V_A

$$2 = -2\left(1 - \frac{k}{2}\right)$$

$$2 = -2 + k$$

$$\boxed{k = 4}$$

(b)

Loss of K.E = K.E Before − K.E After

velocities after the collision

$$V_A = u - \frac{ku}{2} = u - \frac{4u}{2} = -u$$

$$V_B = \frac{5}{8}u\sin\alpha = \frac{5}{8}u\left(\frac{3}{5}\right) = \frac{3}{8}u$$

K.E Before the collision $= \frac{1}{2}mu^2 + \frac{1}{2}km\left(-\frac{5}{8}u\right)^2 = \frac{41}{32}mu^2$

K.E After the collision $= \frac{1}{2}mV_A^2 + \frac{1}{2}kmV_B^2$

$$= \frac{1}{2}m(-u)^2 + \frac{1}{2}(4)m\left(\frac{3}{8}u\right)^2 = \frac{25}{32}mu^2$$

Loss in K.E $= \frac{41}{32}mu^2 - \frac{25}{32}mu^2$

$$= \frac{1}{2}mu^2$$

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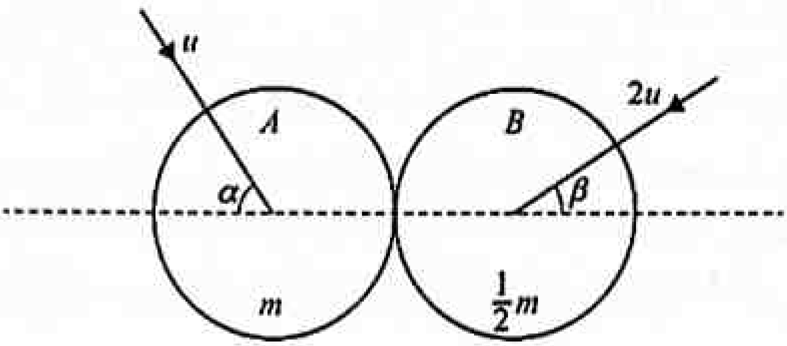
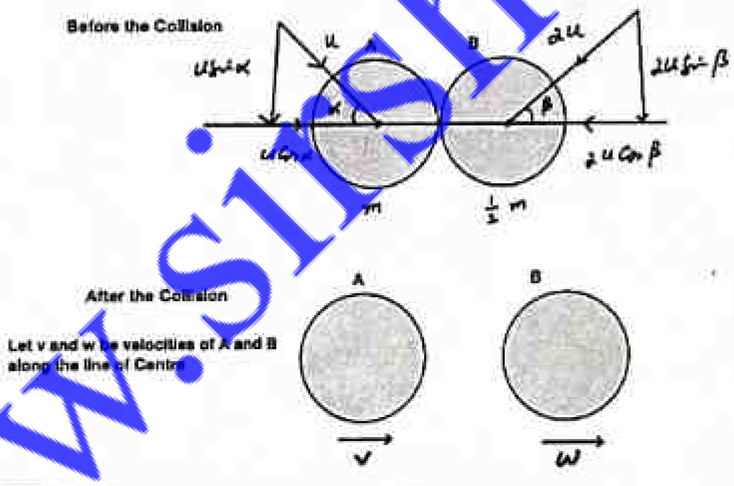


Two uniform smooth spheres A and B of equal radii have masses m and $\frac{1}{2}m$ respectively. The two spheres are moving on a horizontal surface when they collide. Immediately before the collision, sphere A is travelling with speed u and its direction of motion makes an angle α with the line of centres. Sphere B is travelling with speed $2u$ and its direction of motion makes an angle β with the line of centres (see diagram). The coefficient of restitution between the spheres is $\frac{5}{8}$ and $\alpha + \beta = 90°$.

(a) Find the component of the velocity of B parallel to the line of centres after the collision, giving your answer in terms of u and α. [4]

The direction of motion of B after the collision is parallel to the direction of motion of A before the collision.

(b) Find the value of $\tan\alpha$. [5]



By conservation law of momentum

$$mv + \frac{1}{2}mw = mu\cos\alpha + \frac{1}{2}m(-2u\cos\alpha)$$

$$2v + w = 2u\cos\alpha - 2u\cos\beta \quad \text{— ①}$$

By Newton law of Restitution $e = \dfrac{\text{speed of separation}}{\text{speed of approach}}$

$$\frac{5}{8} = \frac{w - v}{u\cos\alpha + 2u\cos\beta}$$

$$-8v + 8w = 5u\cos\alpha + 10u\sin\alpha \quad —① $$

given $\alpha + \beta = 90$

$\beta = 90 - \alpha$

$\cos\beta = \cos(90-\alpha)$

$= \sin\alpha$

By solving ① and ②

$$8v + 4w = 8u\cos\alpha - 8u\sin\alpha$$

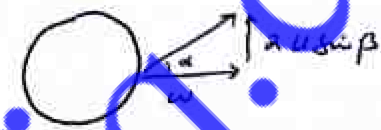
$$-8v + 8w = 5u\cos\alpha + 10u\sin\alpha$$

$$12w = 13u\cos\alpha + 2u\sin\alpha$$

$$w = \frac{13}{12}u\cos\alpha + \frac{1}{6}u\sin\alpha$$

ⓑ

$$\tan\alpha = \frac{2u\sin\beta}{w}$$



$$= \frac{2u\sin(90-\alpha)}{w}$$

$$\frac{\sin\alpha}{\cos\alpha} = \frac{2u\cos\alpha}{w}$$

$$(\sin\alpha)w = 2u\cos^2\alpha$$

$$\sin\alpha\left[\frac{13}{12}u\cos\alpha + \frac{1}{6}u\sin\alpha\right] = 2u\cos^2\alpha$$

$$13\sin\alpha\cos\alpha + 2\sin^2\alpha = 24\cos^2\alpha$$

$$2\tan^2\alpha + 13\tan\alpha - 24 = 0$$

$\tan\alpha = \frac{3}{2}$, $\tan\alpha = -8$ (Ignore)

as α is acute

So $\boxed{\tan\alpha = \frac{3}{2}}$

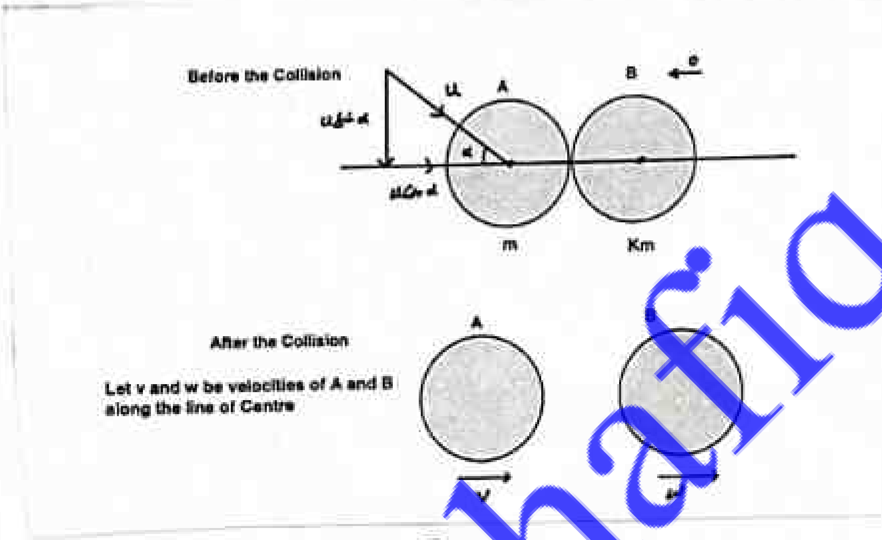
Q.3 May/June/P31+P32/2022

6 Two uniform smooth spheres A and B of equal radii have masses m and km respectively. The two spheres are on a horizontal surface. Sphere A is travelling with speed u towards sphere B which is at rest. The spheres collide. Immediately before the collision, the direction of motion of A makes an angle α with the line of centres. The coefficient of restitution between the spheres is $\frac{1}{2}$.

(a) Show that the speed of B after the collision is $\frac{3u\cos\alpha}{2(1+k)}$ and find also an expression for the speed of A along the line of centres after the collision, in terms of k, u and α. [4]

After the collision, the kinetic energy of A is equal to the kinetic energy of B.

(b) Given that $\tan\alpha = \frac{2}{3}$, find the possible values of k. [5]



By conservation law of Momentum

$$mv + Kmw = mu\cos\alpha$$

$$v + Kw = u\cos\alpha \quad —①$$

By Newton law of Restitution

$$e = \frac{\text{speed of separation}}{\text{speed of approach}}$$

$$\frac{1}{2} = \frac{w - v}{u\cos\alpha}$$

$$-2v + 2w = u\cos\alpha \quad —②$$

By solving ① and ②

$$2v + 2Kw = 2u\cos\alpha \qquad 2① + ②$$

$$-2v + 2w = u\cos\alpha$$

$$2(1+K)w = 3u\cos\alpha$$

$$w = \frac{3u\cos\alpha}{2(1+K)}, \quad v = \frac{(2-K)u\cos\alpha}{2(1+K)}$$

(b) velocity of A After collision $= \sqrt{(u\sin\alpha)^2 + \left(\dfrac{(2-k)u\cos\alpha}{2(1+k)}\right)^2}$

velocity of B After collision $= \dfrac{3u\cos\alpha}{2(1+k)}$

Given

K.E of A after the collision = K.E of B after the collision

$$\frac{1}{2}m\left[(u\sin\alpha)^2 + \left(\frac{(2-k)u\cos\alpha}{2(1+k)}\right)^2\right] = \frac{1}{2}km\left(\frac{3u\cos\alpha}{2(1+k)}\right)^2$$

$$4(1+k)^2\sin^2\alpha + (2-k)^2\cos^2\alpha = 9k\cos^2\alpha$$

$$4(1+k)^2\tan^2\alpha + (2-k)^2 = 9k$$

$$4(1+k)^2\left(\frac{4}{9}\right) + (2-k)^2 = 9k$$

$$16(1+2k+k^2) + 9(4-4k+k^2) = 81k$$

$$25k^2 - 85k + 52 = 0$$

$$\boxed{k = \frac{4}{5}} \text{ or } \boxed{k = \frac{13}{5}}$$

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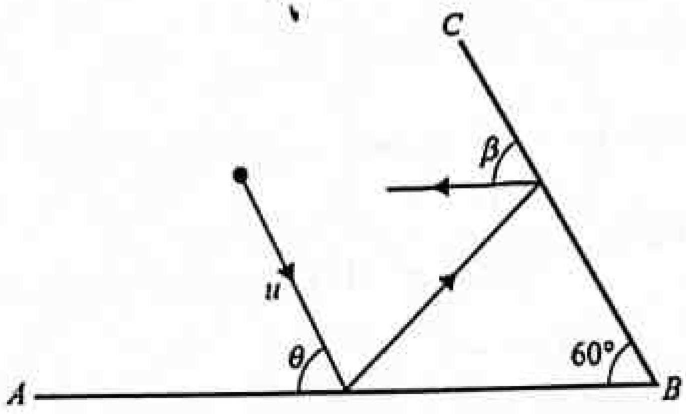
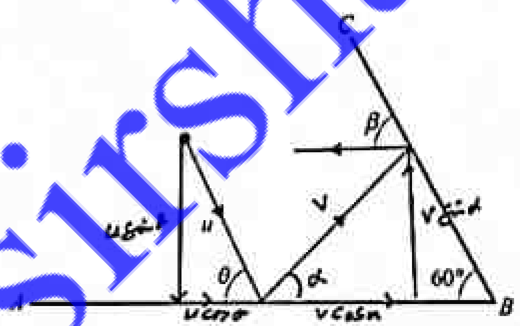


AB and BC are two fixed smooth vertical barriers on a smooth horizontal surface, with angle $ABC = 60°$. A particle of mass m is moving with speed u on the surface. The particle strikes AB at an angle θ with AB. It then strikes BC and rebounds at an angle β with BC (see diagram). The coefficient of restitution between the particle and each barrier is e and $\tan\theta = 2$.

The kinetic energy of the particle after the first collision is 40% of its kinetic energy before the first collision.

(a) Find the value of e. [4]

(b) Find the size of angle β. [4]



Let v be the speed after first collision

Given

$$\frac{1}{2}mv^2 = 40\% \text{ of } \frac{1}{2}mu^2$$

$$\frac{1}{2}mv^2 = \frac{2}{5} \times \frac{1}{2}mu^2$$

$$v^2 = \frac{2}{5}u^2$$

By Newton law of Restitution

$$e = \frac{\text{speed of separation}}{\text{speed of approach}}$$

$$e = \frac{v \sin\alpha}{u \sin\theta}$$

$$v \sin\alpha = e u \sin\theta \quad — ①$$

also $$v \cos\alpha = u \cos\theta \quad — ②$$

$$v^2(\sin^2\alpha + \cos^2\alpha) = u^2(e^2 \sin^2\theta + \cos^2\theta)$$

given $\tan\theta = 2$

$\sin\theta = \frac{2}{\sqrt{5}}$

$\cos\theta = \frac{1}{\sqrt{5}}$

$$\frac{2}{5}u^2 = u^2\left(e^2 \times \frac{4}{5} + \frac{1}{5}\right)$$

$$\frac{2}{5} = \frac{4}{5}e^2 + \frac{1}{5}$$

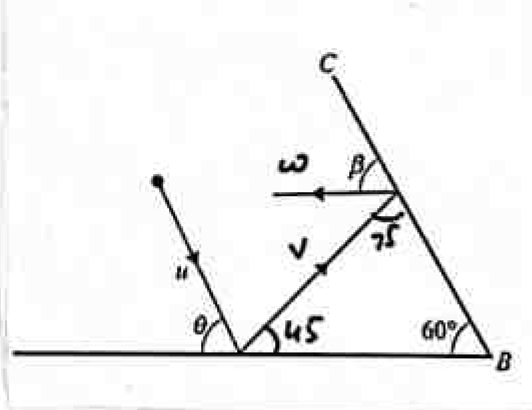
$$e^2 = \frac{1}{4} \Rightarrow \boxed{e = \frac{1}{2}}$$

(b) From ① and ②, we can write

$$\tan\alpha = e \tan\theta = \frac{1}{2} \cdot 2 = 1$$

$$\alpha = 45^\circ$$

In second collision



$$w \cos\beta = v \cos 75^\circ \quad — ③$$

$$w \sin\beta = e v \sin 75^\circ \quad — ④$$

④ ÷ ③

$$\tan\beta = e \tan 75^\circ$$

$$\beta = \tan^{-1}\left(\frac{1}{2} \times \tan 75^\circ\right)$$

$$\beta = 61.8^\circ$$

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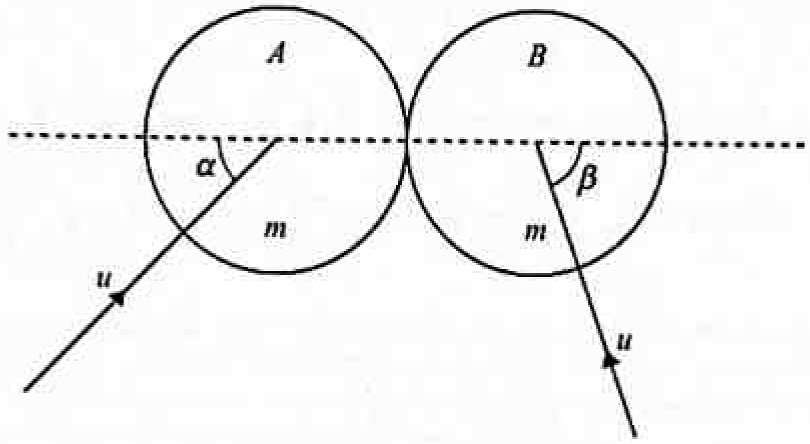
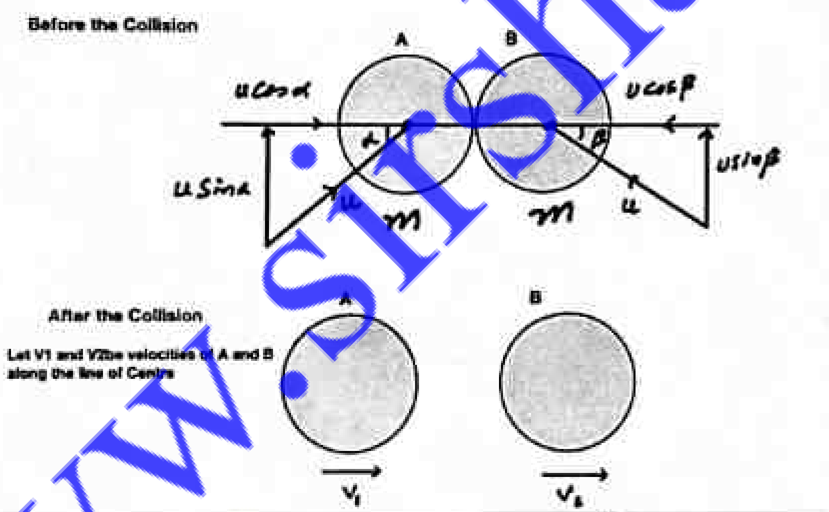


Two uniform smooth spheres A and B of equal radii each have mass m. The two spheres are each moving with speed u on a horizontal surface when they collide. Immediately before the collision, A's direction of motion makes an angle α with the line of centres, and B's direction of motion makes an angle β with the line of centres (see diagram). The coefficient of restitution between the spheres is $\frac{1}{3}$ and $2\cos\beta = \cos\alpha$.

(a) Show that the direction of motion of A after the collision is perpendicular to the line of centres. **[4]**

The total kinetic energy of the spheres after the collision is $\frac{3}{4}mu^2$.

(b) Find the value of α. **[4]**



By conservation law of momentum

$\xrightarrow{+ve}$

$$m_1V_1 + m_2V_2 = m\,u\cos\alpha + m(-u\cos\beta)$$

$$V_1 + V_2 = u\cos\alpha - u\cos\beta \quad \text{—①}$$

By Newton law of Restitution

$$e = \frac{V_2 - V_1}{u\cos\alpha + u\cos\beta}$$

$$\frac{1}{3} = \frac{V_2 - V_1}{u\cos\alpha + u\cos\beta}$$

$$-3V_1 + 3V_2 = u\cos\alpha + u\cos\beta \quad -(2)$$

Given $2\cos\beta = \cos\alpha$

$$V_1 + V_2 = u\cos\alpha - u\left(\tfrac{1}{2}\cos\alpha\right) = \tfrac{u}{2}\cos\alpha \quad -(1)$$

$$-3V_1 + 3V_2 = u\cos\alpha + u\left(\tfrac{u}{2}\cos\alpha\right) = \tfrac{3u}{2}\cos\alpha \quad -(2)$$

(1) − (2)

$$3V_1 + 3V_2 = \frac{3u}{2}\cos\alpha$$

$$\underline{\begin{array}{l} -3V_1 + 3V_2 = \frac{3u}{2}\cos\alpha \\ + \quad\quad - \quad\quad\quad - \end{array}}$$

$$6V_1 = 0$$

$$V_1 = 0$$

So A has no speed along line of centres.

A will move perpendicular to line of centres

(b) putting the value of $V_1 = 0$ in any equation (1) or (2)

$$V_2 = \tfrac{1}{2}u\cos\alpha$$

K.E of A after collision $= \frac{1}{2}m(u\sin\alpha)^2$

K.E of B after collision $= \frac{1}{2}m\left(\sqrt{\left(\frac{1}{2}u\cos\alpha\right)^2 + (u\sin\beta)^2}\right)^2$

given

$$\tfrac{1}{2}mu^2\sin^2\alpha + \tfrac{1}{2}m\left(\tfrac{u^2}{4}\cos^2\alpha\right) + \tfrac{1}{2}mu^2\sin^2\beta = \tfrac{3}{4}mu^2$$

$$2\sin^2\alpha - \tfrac{1}{2}\cos^2\alpha + 2\sin^2\beta = 3$$

$$2\sin^2\alpha - \tfrac{1}{2}\cos^2\alpha + 2(1 - \cos^2\beta) = 3$$

$$2\sin^2\alpha - \tfrac{1}{2}\cos^2\alpha + 2 - \tfrac{1}{2}\cos^2\alpha = 3 \quad \text{as } 2\cos\beta = \cos\alpha$$

$$\sin^2\alpha = \tfrac{1}{2}$$

$$\sin\alpha = \tfrac{1}{\sqrt{2}}$$

$$\alpha = 45^\circ$$

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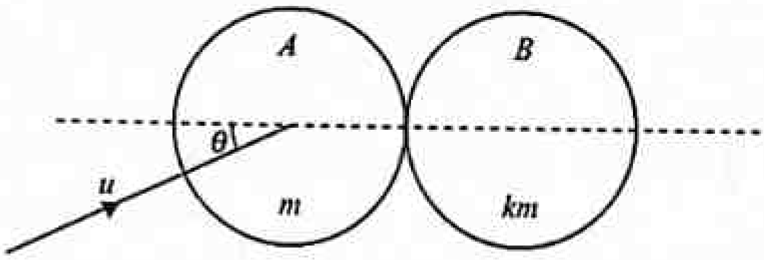
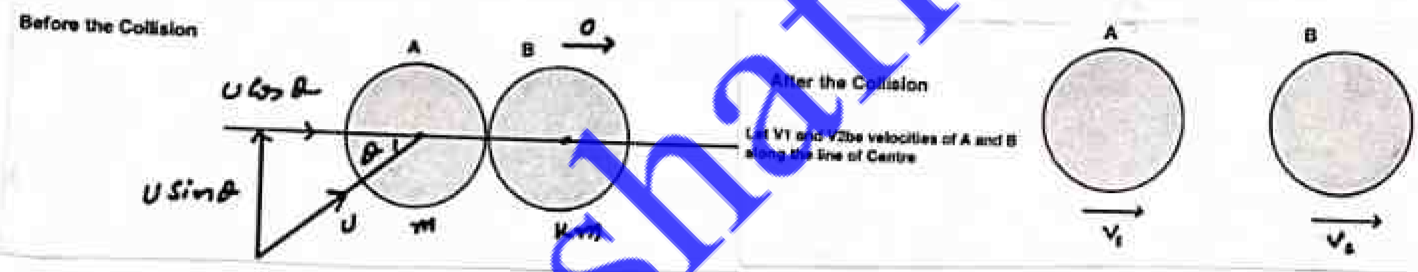


Two uniform smooth spheres A and B of equal radii have masses m and km respectively. Sphere A is moving with speed u on a smooth horizontal surface when it collides with sphere B which is at rest. Immediately before the collision, A's direction of motion makes an angle θ with the line of centres (see diagram). The coefficient of restitution between the spheres is $\frac{1}{3}$.

(a) Show that the speed of B after the collision is $\dfrac{4u\cos\theta}{3(1+k)}$. [3]

70% of the total kinetic energy of the spheres is lost as a result of the collision.

(b) Given that $\tan\theta = \frac{1}{3}$, find the value of k. [6]



By conservation law of momentum

$$m v_1 + K m v_2 = m u \cos\theta$$

$$v_1 + K v_2 = u\cos\theta \quad \text{--- (1)}$$

By Newton law of Restitution

$$\frac{1}{3} = \frac{v_2 - v_1}{u\cos\theta}$$

$$-3v_1 + 3v_2 = u\cos\theta \quad \text{--- (2)}$$

3 (1) + (2)

$$3v_1 + 3Kv_2 = 3u\cos\theta$$

$$-3v_1 + 3v_2 = u\cos\theta$$

$$3(1+K)v_2 = 4u\cos\theta$$

$$v_2 = \frac{4u\cos\theta}{3(1+K)}$$

putting the value of v_2 in (1)

$$V_1 = u\cos\theta - \frac{4Ku\cos\theta}{3(1+K)}$$

$$V_1' = \frac{(3-K)u\cos\theta}{3(1+K)}$$

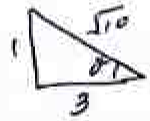
given

K.E of A after collision + K.E of B after collision = 30% of $\frac{1}{2}mu^2$

velocity of A after collision $= \sqrt{V_1^2 + (u\sin\theta)^2}$

velocity of B after collision $= \dfrac{4u\cos\theta}{3(1+K)}$

given $\tan\theta = \frac{1}{3}$



$$\sin\theta = \frac{1}{\sqrt{10}}$$

$$\cos\theta = \frac{3}{\sqrt{10}}$$

$$\frac{1}{2}m\left(V_1^2 + (u\sin\theta)^2\right) + \frac{1}{2}KmV_2^2 = \frac{3}{10} \times \frac{1}{2}mu^2$$

$$\frac{(3-K)^2u^2}{9(1+K)^2}\cdot\frac{1}{10} + u^2\cdot\frac{1}{10} + K\frac{16u^2}{9(1+K)^2}\cdot\frac{1}{10} = \frac{3}{10}u^2$$

$$(3-K)^2 + 16K = 2(1+K)^2$$

$$9 - 6K + K^2 + 16K = 2 + 4K + 2K^2$$

$$K^2 - 6K - 7 = 0$$

$$\boxed{K = 7}$$

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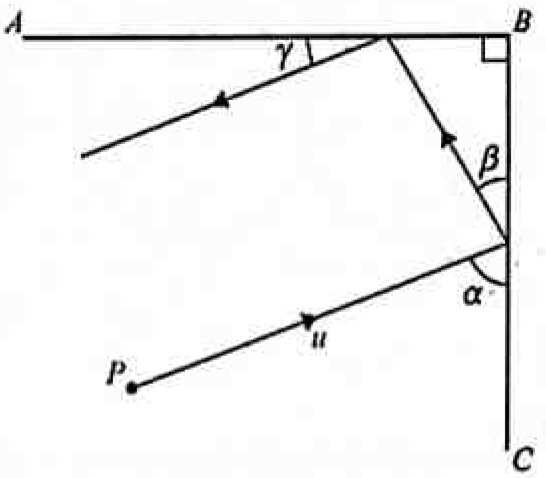


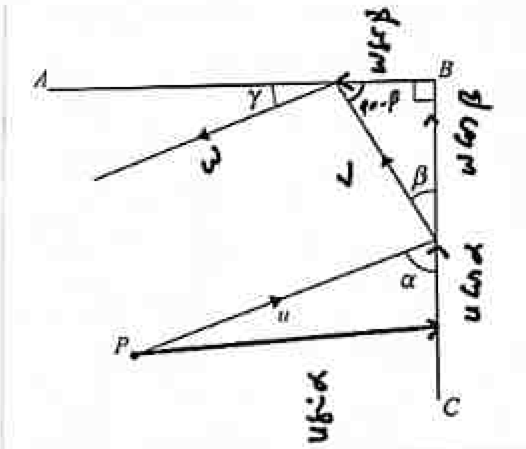
The smooth vertical walls AB and CB are at right angles to each other. A particle P is moving with speed u on a smooth horizontal floor and strikes the wall CB at an angle α. It rebounds at an angle β to the wall CB. The particle then strikes the wall AB and rebounds at an angle γ to that wall (see diagram). The coefficient of restitution between each wall and P is e.

(a) Show that $\tan\beta = e\tan\alpha$. [3]

(b) Express γ in terms of α and explain what this result means about the final direction of motion of P. [4]

As a result of the two impacts the particle loses $\frac{8}{9}$ of its initial kinetic energy.

(c) Given that $\alpha + \beta = 90°$, find the value of e and the value of $\tan\alpha$. [4]



(a) $V\cos\beta = u\cos\alpha$ — ①

$$e = \frac{V\sin\beta}{u\sin\alpha}$$

$V\sin\beta = e\,u\sin\alpha$ — ②

② ÷ ①
$$\frac{V\sin\beta}{V\cos\beta} = \frac{e\,u\sin\alpha}{u\cos\alpha}$$

$$\tan\beta = e\tan\alpha$$

(b)

$$w\cos\gamma = v\cos(90-\beta)$$

$$w\cos\gamma = v\sin\beta$$

$$w\cos\gamma = eu\sin\alpha \quad \text{--- (3)}$$

$$e = \frac{w\sin\gamma}{v\sin(90-\beta)}$$

$$w\sin\gamma = ev\cos\beta$$

$$w\sin\gamma = eu\cos\alpha \quad \text{--- (4)}$$

(4) ÷ (3)

$$\frac{w\sin\gamma}{w\cos\gamma} = \frac{eu\cos\alpha}{eu\sin\alpha}$$

$$\tan\gamma = \frac{1}{\tan\alpha}$$

$$\tan\gamma = \tan(90-\alpha)$$

$$\boxed{\gamma = 90-\alpha}$$

(c)

$$w = \sqrt{(w\sin\gamma)^2 + (w\cos\gamma)^2}$$

$$= \sqrt{(eu\cos\alpha)^2 + (eu\sin\alpha)^2}$$

$$= eu$$

Given

Final K.E $= \frac{1}{9}$ of initial K.E

$$\frac{1}{2}mw^2 = \frac{1}{9}\cdot\frac{1}{2}mu^2$$

$$\frac{1}{2}me^2u^2 = \frac{1}{18}mu^2$$

$$e^2 = \frac{1}{9} \Rightarrow e = \frac{1}{3}$$

From part (a) $\quad \tan\beta = e\tan\alpha \qquad$ given $\alpha+\beta = 90$

$$\tan(90-\alpha) = e\tan\alpha$$

$$\tan^2\alpha = \frac{1}{e} \Rightarrow \tan\alpha = \sqrt{3}$$

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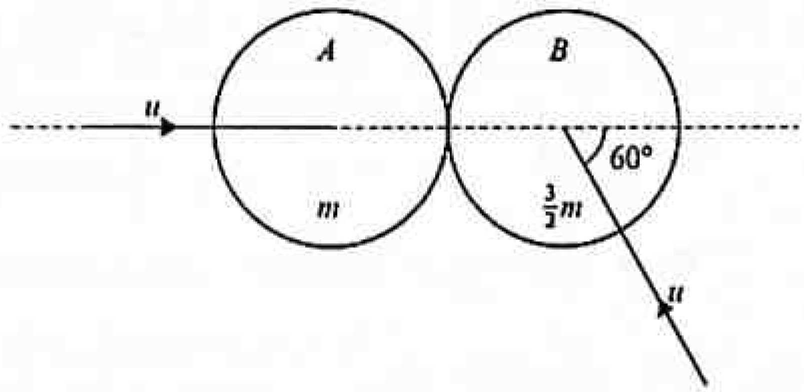
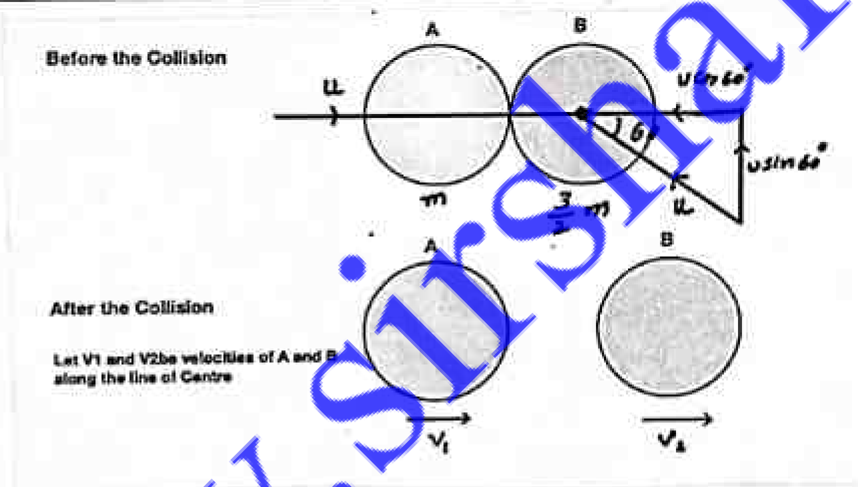


Two uniform smooth spheres A and B of equal radii have masses m and $\frac{3}{2}m$ respectively. The two spheres are each moving with speed u on a horizontal surface when they collide. Immediately before the collision A's direction of motion is along the line of centres, and B's direction of motion makes an angle of 60° with the line of centres (see diagram). The coefficient of restitution between the spheres is $\frac{2}{3}$.

(a) Find the angle through which the direction of motion of B is deflected by the collision. [6]

(b) Find the loss in the total kinetic energy of the system as a result of the collision. [3]



By conservation law of Momentum

$$mV_1 + \frac{3}{2}mV_2 = mu - \frac{3}{2}mu\cos 60^\circ$$

$$V_1 + \frac{3}{2}V_2 = u - \frac{3}{4}u$$

$$4V_1 + 6V_2 = u \quad \text{—— (1)}$$

By Newton law of Restitution

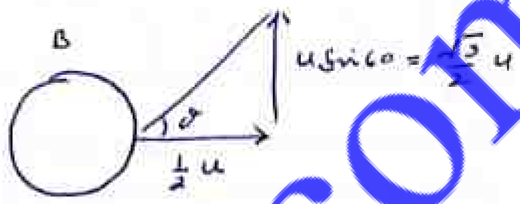
$$\frac{2}{3} = \frac{V_2 - V_1}{u + u\cos 60^\circ}$$

$$-V_1 + V_2 = u \quad \text{—— (2)}$$

$$4V_1 + 6V_2 = u$$
$$-4V_1 + 4V_2 = 4u$$
$$10V_2 = 5u$$
$$V_2 = \frac{1}{2}u$$

and $$V_1 = -\frac{1}{2}u$$

Let θ be the angle of deflection of B



$$\tan\theta = \frac{\frac{\sqrt{3}}{2}u}{\frac{1}{2}u}$$
$$\tan\theta = \sqrt{3}$$
$$\theta = \tan^{-1}(\sqrt{3}) = 60^\circ \text{ (above line of centres)}$$

(b)

$$\text{K.E Before the collision} = \frac{1}{2}mu^2 + \frac{1}{2}\cdot\frac{3}{2}mu^2 = \frac{5}{4}mu^2$$

$$\text{K.E After the collision} = \frac{1}{2}m\left(\frac{u}{2}\right)^2 + \frac{1}{2}\cdot\frac{3}{2}m\left[\left(\frac{u}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}u\right)^2\right]$$
$$= \frac{mu^2}{8} + \frac{3}{4}m(u^2)$$
$$= \frac{7}{8}mu^2$$

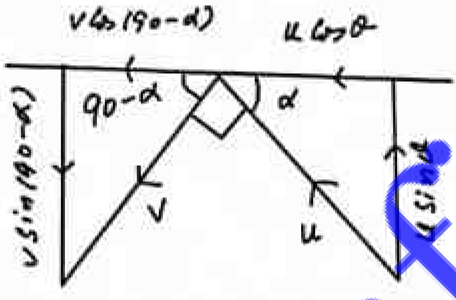
$$\text{Loss in K.E} = \text{K.E Before} - \text{K.E After}$$
$$= \frac{5}{4}mu^2 - \frac{7}{8}mu^2$$
$$= \frac{3}{8}mu^2$$

6 A particle P of mass m is moving with speed u on a fixed smooth horizontal surface. The particle strikes a fixed vertical barrier. At the instant of impact the direction of motion of P makes an angle α with the barrier. The coefficient of restitution between P and the barrier is e. As a result of the impact, the direction of motion of P is turned through 90°.

(a) Show that $\tan^2\alpha = \frac{1}{e}$. [3]

The particle P loses two-thirds of its kinetic energy in the impact.

(b) Find the value of α and the value of e. [5]



$$v\cos(90-\alpha) = u\cos\alpha$$

$$v\sin\alpha = u\cos\alpha \quad \text{—①}$$

By Newton law of Restitution

$$e = \frac{v\sin(90-\alpha)}{u\sin\alpha}$$

$$v\cos\alpha = eu\sin\alpha \quad \text{—②}$$

① ÷ ②
$$\frac{v\sin\alpha}{v\cos\alpha} = \frac{u\cos\alpha}{eu\sin\alpha}$$

$$\tan\alpha = \frac{1}{e\tan\alpha}$$

$$\tan^2\alpha = \frac{1}{e}$$

(b) K.E after impact $= \frac{1}{3}$ K.E of before the impact

$$\frac{1}{2}mv^2 = \frac{1}{3}\cdot\frac{1}{2}mu^2$$

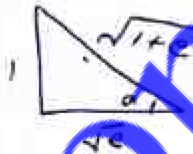
$$V^2 = (V\cos(90-\alpha))^2 + (V\sin(90-\alpha))^2$$

$$= (V\sin\alpha)^2 + (V\cos\alpha)^2$$

$$= (u\cos\alpha)^2 + (eu\sin\alpha)^2$$

$$= u^2\cos^2\alpha + e^2u^2\sin^2\alpha$$

From part (a) $\tan\alpha = \dfrac{1}{\sqrt{e}}$



$$\sin\alpha = \frac{1}{\sqrt{1+e}}$$

$$\cos\alpha = \frac{\sqrt{e}}{\sqrt{1+e}}$$

So $V^2 = u^2\left[\dfrac{e}{1+e} + e^2 \cdot \dfrac{1}{1+e}\right]$

$$= u^2\left[\frac{e(1+e)}{(1+e)}\right] = eu^2$$

$$\frac{1}{2}mV^2 = \frac{1}{3}\cdot\frac{1}{2}mu^2$$

$$\frac{1}{2}meu^2 = \frac{1}{3}\cdot\frac{1}{2}mu^2$$

$$e = \frac{1}{3}$$

we know

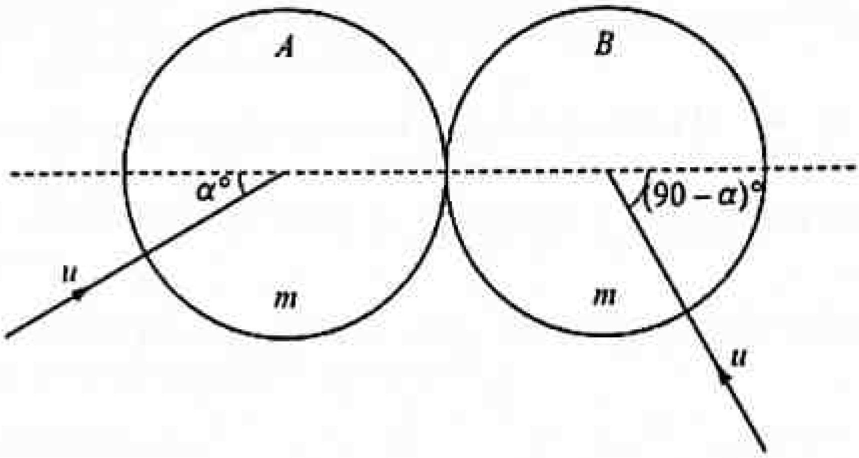
$$\tan\alpha = \frac{1}{\sqrt{e}}$$

$$\tan\alpha = \sqrt{3}$$

$$\alpha = 60^\circ$$

Q.10 May/June/P33/2020

5

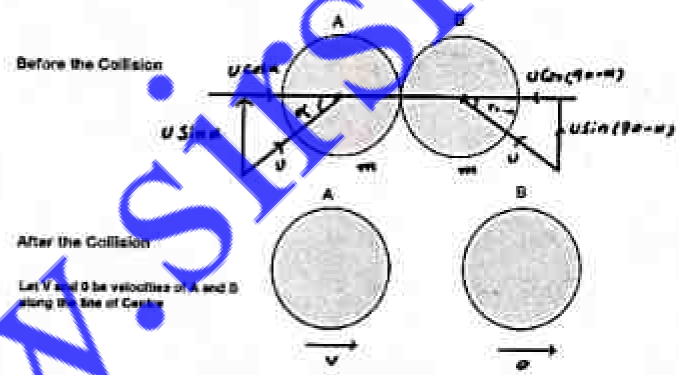


Two uniform smooth spheres A and B of equal radii each have mass m. The two spheres are each moving with speed u on a horizontal surface when they collide. Immediately before the collision A's direction of motion makes an angle of $\alpha°$ with the line of centres, and B's direction of motion is perpendicular to that of A (see diagram). The coefficient of restitution between the spheres is e.

Immediately after the collision, B moves in a direction at right angles to the line of centres.

(a) Show that $\tan\alpha = \dfrac{1+e}{1-e}$. [4]

(b) Given that $\tan\alpha = 2$, find the speed of A after the collision. [4]



By conservation law of momentum

$\xrightarrow{+ve}$ $\quad mV = mu\cos\alpha - mu\sin\alpha \qquad$ as $\cos(90-\alpha) = \sin\alpha$

$V = u\cos\alpha - u\sin\alpha$ —— (1)

By Newton law of Restitution

$$e = \frac{0 - V}{u\cos\alpha + u\sin\alpha}$$

$-V = eu\cos\alpha + eu\sin\alpha$ —— (2)

Add ① and ②

$$0 = (eu + u)\cos\alpha + (eu - u)\sin\alpha$$

$$0 = u(e+1)\cos\alpha - u(1-e)\sin\alpha$$

$$(1-e)\sin\alpha = (e+1)\cos\alpha$$

$$\tan\alpha = \frac{1+e}{1-e}$$

Speed of A after collision $= \sqrt{v^2 + (u\sin\alpha)^2}$

$$= \sqrt{(u\cos\alpha - u\sin\alpha)^2 + u^2\sin^2\alpha}$$

$$= u\sqrt{\cos^2\alpha + \sin^2\alpha - 2\sin\alpha\cos\alpha + \sin^2\alpha}$$

$$= u\sqrt{1 - 2\sin\alpha\cos\alpha + \sin^2\alpha}$$

Given $\tan\alpha = 2$

$$\sin\alpha = \frac{1}{\sqrt{5}}$$

$$\cos\alpha = \frac{2}{\sqrt{5}}$$

Speed of A after collision $= u\sqrt{1 - 2\left(\frac{1}{\sqrt{5}}\right)\left(\frac{2}{\sqrt{5}}\right) + \frac{1}{5}}$

$$= u\sqrt{1 - \frac{4}{5} + \frac{1}{5}}$$

$$= u$$

Q.11 Oct/Nov/P31+P33/2020

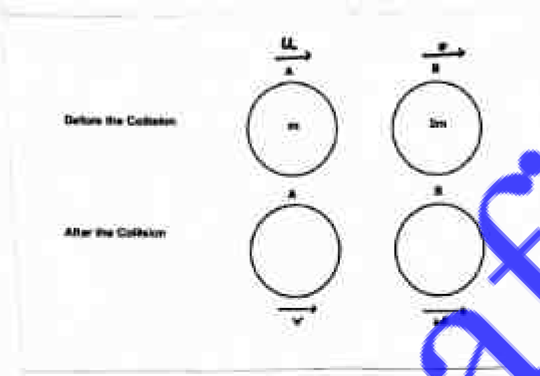
6 Two smooth spheres A and B have equal radii and masses m and $2m$ respectively. Sphere B is at rest on a smooth horizontal floor. Sphere A is moving on the floor with velocity u and collides directly with B. The coefficient of restitution between the spheres is e.

(a) Find, in terms of u and e, the velocities of A and B after the collision. [3]

Subsequently, B collides with a fixed vertical wall which makes an angle θ with the direction of motion of B, where $\tan\theta = \frac{3}{4}$.

The coefficient of restitution between B and the wall is $\frac{2}{3}$. Immediately after B collides with the wall, the kinetic energy of A is $\frac{5}{32}$ of the kinetic energy of B.

(b) Find the possible values of e. [7]



By Conservation law of momentum

$$m w + 2 m V = m u$$

$$w + 2V = u \quad \text{—— (1)}$$

By Newton law of Restitution

$$\frac{V - w}{u} = e$$

$$-w + V = eu \quad \text{—— (2)}$$

Add (1) and (2)

$$3V = (1+e)u$$

$$V = \left(\frac{1+e}{3}\right)u$$

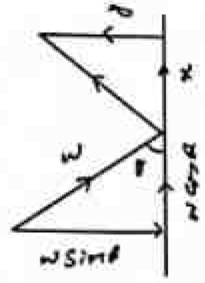
$$w = \left(\frac{1+e}{3}\right)u - eu$$

$$w = \left(\frac{1-2e}{3}\right)u$$

$x = v\cos\theta$

$\frac{y}{v\sin\theta} = e$ By law of Restitution

$y = ev\sin\theta$



Speed of B $= \sqrt{x^2 + y^2}$

$= \sqrt{v^2\cos^2\theta + e^2v^2\sin^2\theta}$

$= \sqrt{\frac{16}{25} + \frac{4}{9} \times \frac{9}{25}}\ V$

$= \sqrt{\frac{20}{25}}V = \frac{2}{\sqrt{5}}V$

given $\tan\theta = \frac{3}{4}$

$\sin\theta = \frac{3}{5}$

$\cos\theta = \frac{4}{5}$

$e = \frac{2}{3}$

K.E of B $= \frac{1}{2}(2m) \cdot \frac{4}{5}v^2 = \frac{4}{5}m\frac{(1+e)^2u^2}{9}$

K.E of A $= \frac{1}{2}m\omega^2 = \frac{1}{2}m\frac{(1-2e)^2u^2}{9}$

Given

K.E of A $= \frac{5}{32}$ K.E of B

$\frac{1}{2}m\frac{(1-2e)^2u^2}{9} = \frac{5}{32} \cdot \frac{4}{5}m\frac{(1+e)^2u^2}{9}$

$(1-2e)^2 = \frac{1}{4}(1+e)^2$

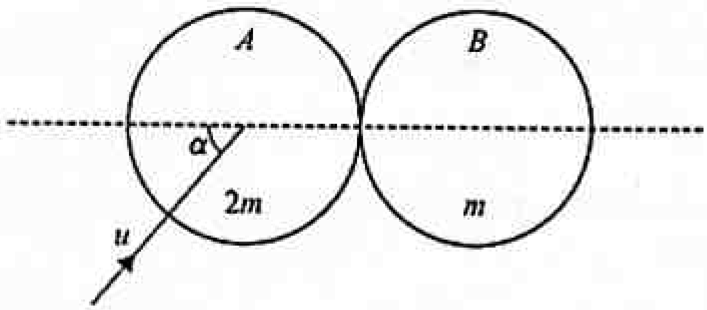
$4(1-2e)^2 = (1+e)^2$

$15e^2 - 18e + 3 = 0$

$e = \frac{1}{5}$, $e = 1$

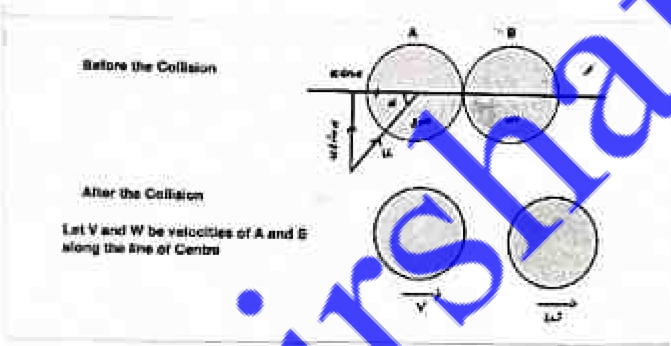
Q.12 Oct/Nov/P32/2020

2



Two uniform smooth spheres A and B of equal radii have masses $2m$ and m respectively. Sphere B is at rest on a smooth horizontal surface. Sphere A is moving on the surface with speed u and collides with B. Immediately before the collision, the direction of motion of A makes an angle α with the line of centres of the spheres, where $\tan\alpha = \frac{4}{3}$ (see diagram). The coefficient of restitution between the spheres is $\frac{1}{3}$.

Find the speed of A after the collision. [5]



By conservation law of momentum

$$2mV + mW = 2mu\cos\alpha$$

$$2V + W = 2u\cos\alpha \quad \text{—} \ (1)$$

given $\tan\alpha = \frac{4}{3}$

$\sin\alpha = \frac{4}{5}$

$\cos\alpha = \frac{3}{5}$

By Newton law of Restitution

$$\frac{W - V}{u\cos\alpha} = e$$

$$-V + W = eu\cos\alpha \quad \text{—} \ (2)$$

(1) − (2) $\quad 3V = u\cos\alpha(2 - e) = u \cdot \frac{3}{5}\left(2 - \frac{1}{3}\right) = u$

$$V = \frac{1}{3}u$$

$$\text{speed of } A = \sqrt{\left(\tfrac{1}{3}u\right)^2 + (u\sin\alpha)^2}$$

$$= \sqrt{\frac{u^2}{9} + \frac{16u^2}{25}}$$

$$= \sqrt{\frac{169}{225}u^2}$$

$$= \frac{13}{15}u$$

Q.13 May/June/P31/2019

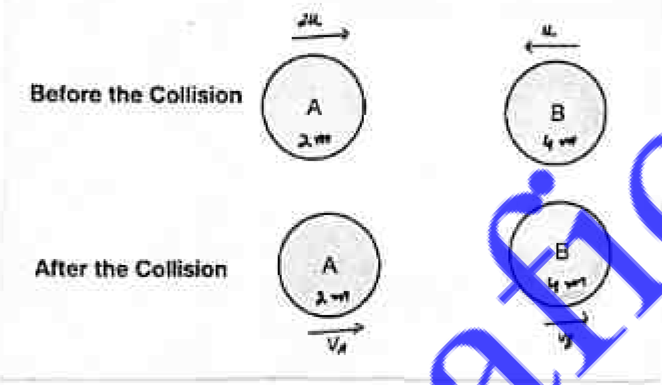
3 Three uniform small spheres A, B and C have equal radii and masses $2m$, $4m$ and m respectively. The spheres are moving in a straight line on a smooth horizontal surface, with B between A and C. The coefficient of restitution between each pair of spheres is e. Spheres A and B are moving towards each other with speeds $2u$ and u respectively. The first collision is between A and B.

(i) Find the velocities of A and B after this collision. [3]

Sphere C is moving towards B with speed $\frac{4}{3}u$ and now collides with it. As a result of this collision, B is brought to rest.

(ii) Find the value of e. [4]

(iii) Find the total kinetic energy lost by the three spheres as a result of the two collisions. [3]



(i)

By conservation law of momentum

$$2mv_A + 4mv_B = 2m(2u) + 4m(-u)$$

$$v_A + 2v_B = 0 \quad \text{— (1)}$$

By Newton law of Restitution

$$e = \frac{v_B - v_A}{2u + u}$$

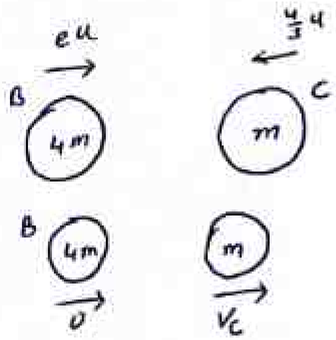
$$-v_A + v_B = 3ue \quad \text{— (2)}$$

(1) + (2)

$$3v_B = 3eu$$

$$v_B = eu$$

$$v_A = -2eu$$



(ii)

$$mv_C = 4m(eu) + m\left(-\frac{4}{3}u\right)$$

$$v_C = 4eu - \frac{4}{3}u = \left(4e - \frac{4}{3}\right)u \quad \text{— (3)}$$

$$\frac{v_C - 0}{eu + \frac{4}{3}u} = e$$

$$V_C = e(eu + \tfrac{4}{3}u) = (e^2 + \tfrac{4}{3}e)u \quad -④$$

From ③ and ④

$$e^2 + \tfrac{4}{3}e = 4e - \tfrac{4}{3}$$

$$3e^2 + 4e = 12e - 4$$

$$3e^2 - 8e + 4 = 0$$

$$e = \tfrac{2}{3}$$

(iii)

$$\text{Initial K.E} = \tfrac{1}{2}(2m)(2u)^2 + \tfrac{1}{2}(4m)(u)^2 + \tfrac{1}{2}m(\tfrac{4}{3}u)^2$$

$$= 4mu^2 + 2mu^2 + \tfrac{8}{9}mu^2$$

$$= \tfrac{62}{9}mu^2$$

$$\text{Final K.E} = \tfrac{1}{2}(2m)V_A^2 + \tfrac{1}{2}mV_C^2 \quad \text{as B becomes at rest}$$

$$= m(-2\cdot\tfrac{2}{3}u)^2 + \tfrac{1}{2}m\left[(\tfrac{4}{9} + \tfrac{8}{9})u\right]^2$$

$$= \tfrac{16}{9}u^2m + \tfrac{8}{9}u^2m$$

$$= \tfrac{24}{9}mu^2$$

$$\text{Loss in K.E} = \text{Initial K.E} - \text{Final K.E}$$

$$= \tfrac{62}{9}mu^2 - \tfrac{24}{9}mu^2$$

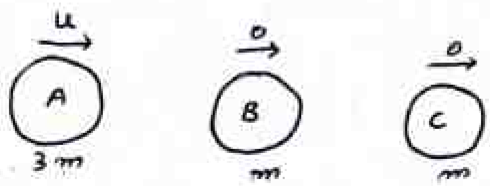
$$= \tfrac{38}{9}mu^2$$

Q.14 May/June/P32/2019

3 Three uniform small spheres A, B and C have equal radii and masses $3m$, m and m respectively. The spheres are at rest in a straight line on a smooth horizontal surface, with B between A and C. The coefficient of restitution between each pair of spheres is e. Sphere A is projected directly towards B with speed u.

(i) Find, in terms of u and e, expressions for the speeds of A, B and C after the first two collisions. [6]

(ii) Given that A and C are moving with equal speeds after these two collisions, find the value of e. [3]



By conservation law of momentum

$$3mV_A + mV_B = 3mu$$

$$3V_A + V_B = 3u \quad \text{—} ①$$

By Newton law of Restitution

$$e = \frac{V_B - V_A}{u}$$

$$-V_A + V_B = eu \quad \text{—} ②$$

① − ②

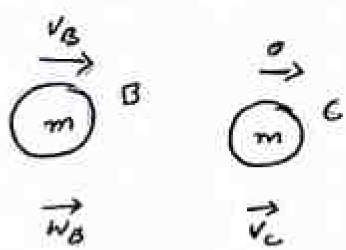
$$4V_A = (3-e)u$$

$$\boxed{V_A = \frac{(3-e)u}{4}}$$

$$V_B = eu + \frac{(3-e)u}{4}$$

$$= \frac{(4e+3-e)u}{4}$$

$$= \frac{3}{4}(1+e)u$$



By conservation law of momentum

$$mW_B + mV_C = mV_B$$

$$W_B + V_C = V_B \quad \text{—} ③$$

By Newton law of Restitution

$$e = \frac{v_C - w_B}{v_B}$$

$$-w_B + v_C = e v_B \quad -④$$

③ + ④

$$2v_C = (1+e)v_B$$

$$v_C = \frac{1}{2}(1+e)v_B$$

$$w_B = \frac{1}{2}(1+e)v_B - e v_B$$

$$w_B = \frac{1}{2}(1-e)v_B$$

$$w_B = \frac{1}{2}(1-e)\cdot\frac{3}{4}(1+e)u$$

$$\boxed{w_B = \frac{3}{8}(1-e^2)u}$$

$$v_C = \frac{1}{2}(1+e)\cdot\frac{3}{4}(1+e)u$$

$$\boxed{v_C = \frac{3}{8}(1+e)^2 u}$$

(ii) Given $v_A = v_C$

$$\frac{1}{4}(3-e)u = \frac{3}{8}(1+e)^2 u$$

$$6-2e = 3+6e+3e^2$$

$$3e^2+8e-3=0$$

$\boxed{e = 1/3}$, $e = -3$ (Ignore)

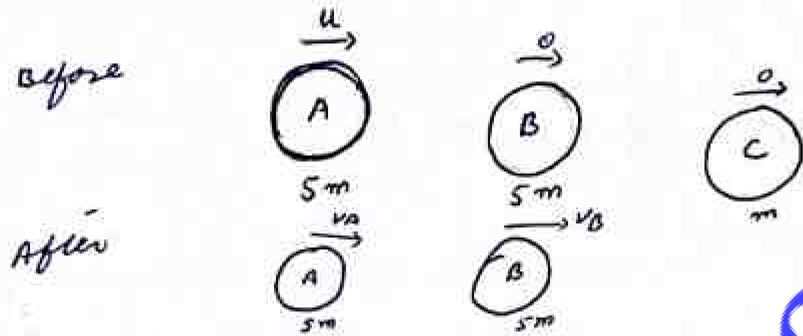
Q.15 Oct/Nov/P31+P32+P33/2019

3 Three uniform small spheres A, B and C have equal radii and masses $5m$, $5m$ and $3m$ respectively. The spheres are at rest on a smooth horizontal surface, in a straight line, with B between A and C. The coefficient of restitution between each pair of spheres is e. Sphere A is projected directly towards B with speed u.

(i) Show that the speed of A after its collision with B is $\frac{1}{2}u(1-e)$ and find the speed of B. [3]

Sphere B now collides with sphere C. Subsequently there are no further collisions between any of the spheres.

(ii) Find the set of possible values of e. [6]



By Conservation law of Momentum

$$5mv_A + 5mv_B = 5mu$$

$$v_A + v_B = u \quad — ①$$

By Newton law of Restitution

$$e = \frac{v_B - v_A}{u}$$

$$-v_A + v_B = eu \quad — ②$$

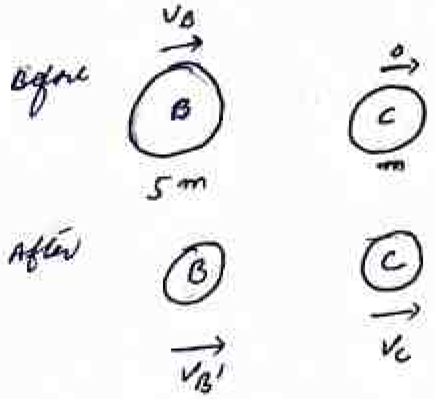
① + ②

$$2v_B = (1+e)u$$

$$v_B = \frac{1}{2}(1+e)u$$

$$v_A = -eu + \frac{1}{2}(1+e)u$$

$$v_A = \frac{(1-e)u}{2}$$



(ii) $$5mv_B' + 3mv_C = 5mv_B$$

$$5v_B' + 3v_C = 5v_B \quad — ③$$

$$e = \frac{v_C - v_B'}{v_B}$$

$$-v_B' + v_C = ev_B \quad — ④$$

③ → ④

$$5v_B' + 3v_C = 5v_B$$

$$-3v_B' + 3v_C = 3ev_B$$

$$8v_B' = (5-3e)v_B$$

$$v_B' = \frac{1}{8}(5-3e)v_B$$

$$v_C = ev_B + \frac{1}{8}(5-3e)v_B$$

$$= \frac{(5+5e)}{8}v_B$$

According to the condition, there is no further collision, so

$$v_A \le v_B'$$

$$\frac{1}{2}(1-e)u \le \frac{1}{8}(5-3e)v_B$$

$$\frac{1}{2}(1-e)u \le \frac{1}{8}(5-3e)\times\frac{1}{2}(1+e)u$$

$$8(1-e) \le (5-3e)(1+e)$$

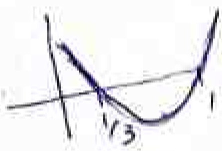
$$8-8e \le 5+5e-3e-3e^2$$

$$3e^2-10e+3 \le 0$$

$$3e^2-9e-e+3 \le 0$$

$$3e(e-3)-(e-3) \le 0$$

$$(e-3)(3e-1) \le 0$$



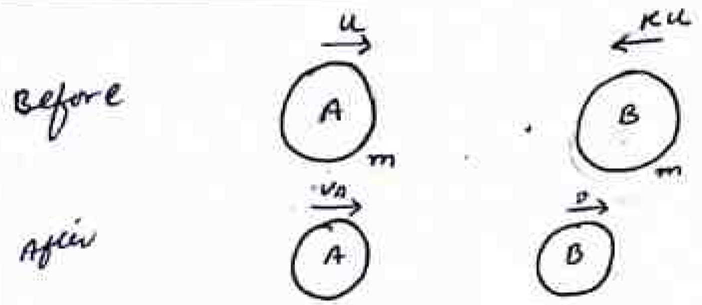
$$\frac{1}{3} \le e \le 1$$

Q.16 May/June/P31/2018

3 Two identical uniform small spheres A and B, each of mass m, are moving towards each other in a straight line on a smooth horizontal surface. Their speeds are u and ku respectively, and they collide directly. The coefficient of restitution between the spheres is e. Sphere B is brought to rest by the collision.

(i) Show that $e = \dfrac{k-1}{k+1}$. [3]

(ii) Given that 60% of the total initial kinetic energy is lost in the collision, find the values of k and e. [6]



(i) By conservation law of Momentum

$\xrightarrow{+ve}$

$$m v_A = mu - mku$$

$$v_A = (1-k)u \quad \text{--- ①}$$

By Newton law of Restitution

$$e = \frac{0 - v_A}{u + ku}$$

$$v_A = -e(u+ku) \quad \text{--- ②}$$

From ① and ②

$$-e(1+k)u = (1-k)u$$

$$\boxed{e = \frac{k-1}{1+k}}$$

(ii) K.E Final = 40% of K.E Initial

$$\frac{1}{2} m v_A^2 = \frac{40}{100} \times \left[\frac{1}{2} mu^2 + \frac{1}{2} m k^2 u^2\right]$$

$$\frac{1}{2} m (1-k)^2 u^2 = \frac{2}{5} \times \frac{1}{2} m u^2 (1+k^2)$$

$$(1-k)^2 = \frac{2}{5}(1+k^2)$$

$$5 - 10k + 5k^2 = 2 + 2k^2$$

$$3k^2 - 10k + 3 = 0$$

$$(3k-1)(k-3) = 0$$

$$k = \frac{1}{3} \quad \text{or} \quad k = 3$$

$k = \frac{1}{3}$ (Reject because e will be -ve)

when $k = 3$

$$e = \frac{3-1}{1+3}$$

$$\boxed{e = \frac{1}{2}}$$

Q.17 May/June/P33/2018

2 Two uniform small spheres A and B have equal radii and masses $4m$ and m respectively. Sphere A is moving with speed u on a smooth horizontal surface when it collides directly with sphere B which is at rest. The coefficient of restitution between the spheres is e.

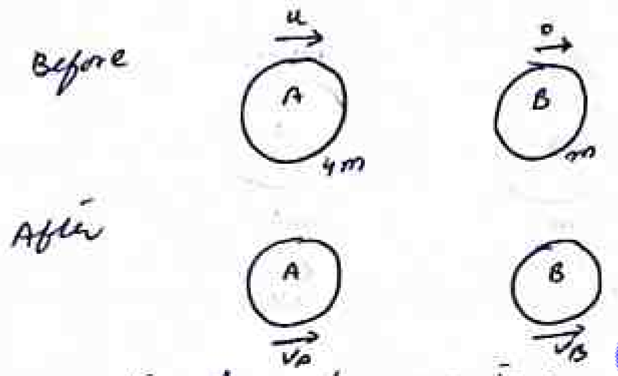
(i) Show that after the collision A moves with speed $\frac{1}{5}u(4-e)$ and find the speed of B. [4]

Sphere B continues to move until it collides with a fixed smooth vertical barrier which is perpendicular to the direction of motion of B. The coefficient of restitution between B and the barrier is $\frac{3}{4}e$. After this collision, the speeds of A and B are equal.

(ii) Find the value of e. [3]

The spheres A and B now collide directly again.

(iii) Determine whether sphere B collides with the barrier for a second time. [2]



(i)

By conservation law of momentum

$$4m v_A + m v_B = 4mu$$

$$4v_A + v_B = 4u \quad \text{--- (1)}$$

By Newton law of Restitution

$$e = \frac{v_B - v_A}{u}$$

$$-v_A + v_B = eu \quad \text{--- (2)}$$

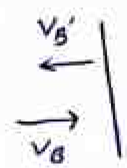
(1) - (2)

$$5v_A = (4-e)u$$

$$v_A = \frac{1}{5}(4-e)u$$

$$v_B = eu + \frac{1}{5}(4-e)u$$

$$= \frac{4}{5}(1+e)u$$



(ii)

$$\frac{3}{4}e = \frac{-v_B'}{v_B}$$

$$v_B' = -\frac{3}{4}e\, v_B$$

$$V_B' = -\frac{3}{4}e \cdot \frac{4}{5}(1+e)u$$

$$V_B' = -\frac{3}{5}e(1+e)u$$

Given $\quad V_A = V_B'$

$$\frac{1}{5}(4-e)u = \frac{3}{5}e(1+e)u$$

$$3e^2 + 4e - 4 = 0$$

$$e = \frac{2}{3}$$

(ii)

Q.18 Oct/Nov/P31+P33/2018

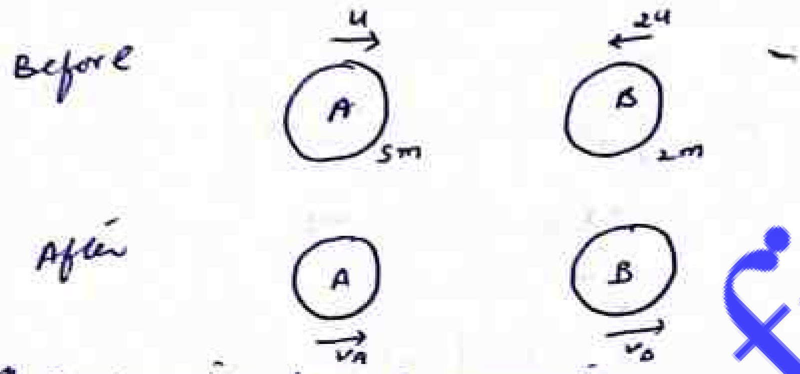
2 Two uniform small smooth spheres A and B have equal radii and masses $5m$ and $2m$ respectively. Sphere A is moving with speed u on a smooth horizontal surface when it collides directly with sphere B which is moving towards it with speed $2u$. The coefficient of restitution between the spheres is e.

(i) Show that the speed of B after the collision is $\frac{1}{7}u(1 + 15e)$ and find an expression for the speed of A. [4]

In the collision, the speed of A is halved and its direction of motion is reversed.

(ii) Find the value of e. [2]

(iii) For this collision, find the ratio of the loss of kinetic energy of A to the loss of kinetic energy of B. [3]



By Conservation law of momentum

$$5mv_A + 2mv_B = 5mu - 2m(2u)$$

$$5v_A + 2v_B = u \quad \text{—(1)}$$

By Newton law of Restitution

$$e = \frac{v_B - v_A}{u + 2u}$$

$$-v_A + v_B = 3eu \quad \text{—(2)}$$

(1)+5(2)

$$5v_A + 2v_B = u$$

$$-5v_A + 5v_B = 15eu$$

$$7v_B = (1+15e)u$$

$$v_B = \frac{1}{7}u(1+15e)$$

$$v_A = v_B - 3eu$$

$$= \frac{1}{7}u(1+15e) - 3eu$$

$$v_A = \frac{1}{7}u(1-6e)$$

(ii) $v_A = -\frac{1}{2}u$

$$\frac{1}{7}u(1-6e) = -\frac{1}{2}u$$

$$1-6e = -\frac{7}{2}$$

$$-6e = -\frac{9}{2}$$

$$e = \frac{3}{4}$$

(iii)

$$\text{Loss in KE in } A = \frac{1}{2}5m\left(u^2 - \left(-\frac{1}{2}u\right)^2\right)$$

$$= \frac{15}{8}mu^2$$

$$\text{Loss of K.E in } B = \frac{1}{2}(2m)\left[(2u)^2 - \left(\frac{7}{4}u\right)^2\right]$$

$$= \frac{15}{16}mu^2$$

$$\frac{\text{Loss of K.E in A}}{\text{Loss of K.E in B}} = \frac{15/8\, mu^2}{\frac{15}{16}mu^2} = \frac{2}{1}$$

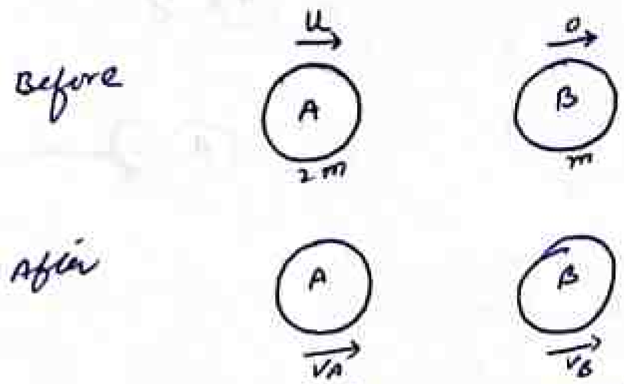
Q.19 Oct/Nov/P32/2018

2 Two uniform small smooth spheres A and B have equal radii and masses $2m$ and m respectively. Sphere A is moving with speed u on a smooth horizontal surface when it collides directly with sphere B which is at rest. The coefficient of restitution between the spheres is $\frac{2}{3}$.

(i) Find, in terms of u, the speeds of A and B after this collision. [4]

Sphere B is initially at a distance d from a fixed smooth vertical wall which is perpendicular to the direction of motion of A. The coefficient of restitution between B and the wall is $\frac{1}{2}$.

(ii) Find, in terms of d and u, the time that elapses between the first and second collisions between A and B. [5]



By conservation law of Momentum

$$2mV_A + mV_B = 2mu \Rightarrow 2V_A + V_B = 2u \quad \text{—— ①}$$

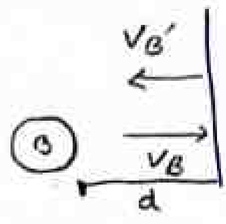
By Newton law of Restitution

$$\frac{V_B - V_A}{u} = e \Rightarrow -V_A + V_B = \frac{2}{3}u \quad \text{—— ②}$$

① − ②

$$2V_A + V_B = 2u$$
$$-V_A + V_B = \frac{2}{3}u$$
$$3V_A = \frac{4}{3}u$$
$$V_A = \frac{4}{9}u$$
$$V_B = \frac{2}{3}u + \frac{4}{9}u$$
$$V_B = \frac{10}{9}u$$

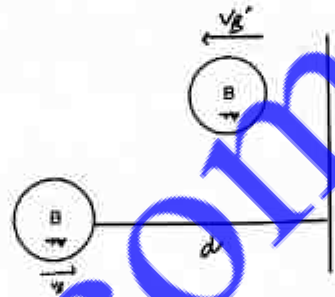
(ii) Let V_B' be the speed of B after collision and let x be the distance from wall when A and B again collides.



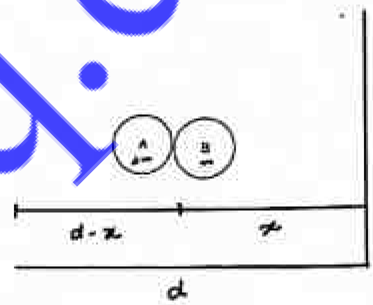
By Newton law of Restitution

$$\frac{-V_B'}{V_B} = \frac{1}{2}$$

$$V_B' = -\frac{1}{2} V_B = -\frac{1}{2}\left(\frac{10}{9}u\right) = -\frac{5}{9}u$$



A cover the distance $d-x$ and
B cover the distance d and x when
A and B collides



$$\frac{d-x}{V_A} = \frac{d}{V_B} + \frac{x}{V_B'}$$

$$\frac{d-x}{\frac{4}{9}u} = \frac{d}{\frac{10}{9}u} + \frac{x}{\frac{5}{9}u}$$

$$\frac{d-x}{4} = \frac{d}{10} + \frac{x}{5}$$

$$\frac{d-x}{4} = \frac{d+2x}{10}$$

$$10d - 10x = 4d + 8x$$

$$18x = 6d$$

$$x = \frac{d}{3}$$

$$t = \frac{d-x}{V_A}$$

$$t = \frac{d-\frac{d}{3}}{\frac{4}{9}u} = \frac{\frac{2}{3}d}{\frac{4}{9}u} = \frac{3d}{2u}$$