AS Level Further Mathematics

Topic:

Roots of Polynomials

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Q.1 Oct/Nov/P11+P13/22

1 The cubic equation $x^3 + bx^2 + d = 0$ has roots α, β, γ, where $\alpha = \beta$ and $d \neq 0$.

(a) Show that $4b^3 + 27d = 0$. [5]

(b) Given that $2\alpha^2 + \gamma^2 = 3b$, find the values of b and d. [3]

(i) $x^3 + bx^2 + d = 0$ has roots α, β, γ and $\alpha = \beta$, $d \neq 0$

$\alpha + \beta + \gamma = -b$

$2\alpha + \gamma = -b$ — (1)

$\alpha\beta\gamma = -d$

$\alpha^2\gamma = -d$ — (2)

$\alpha\beta + \beta\gamma + \alpha\gamma = 0$

$\alpha^2 + 2\alpha\gamma = 0$ — (3)

$\alpha(\alpha + 2\gamma) = 0$

$\alpha = 0$, $\alpha = -2\gamma$

(Ignore as $d \neq 0$)

putting the value of α in (1)

$2(-2\gamma) + \gamma = -b$

$3\gamma = b$

$\gamma = \frac{1}{3}b$

$\alpha = -\frac{2}{3}b$ as $\alpha = -2\gamma$

putting the value of α, γ in (2)

$\left(-\frac{2}{3}b\right)^2\left(\frac{1}{3}b\right) = -d$

$4b^3 = -27d$

$4b^3 + 27d = 0$ Proved

→

iii) Given $2\alpha^2 + \gamma^2 = 3b$

$$2\left(-\frac{2}{3}b\right)^2 + \left(\frac{1}{3}b\right)^2 = 3b$$

$$b^2 - 3b = 0$$

$$b(b-3) = 0$$

$$b = 0 \quad , \quad b = 3$$

Since $4b^3 + 27d = 0$

$$4(3)^3 + 27d = 0$$

$$27d = -108$$

$$d = -4$$

Q.2 Oct/Nov/P12/22

2 The equation $x^4+3x^2+2x+6=0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}, \frac{1}{\delta^2}$ and state the value of $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}$. [4]

(b) Find the value of $\beta^2\gamma^2\delta^2+\alpha^2\gamma^2\delta^2+\alpha^2\beta^2\delta^2+\alpha^2\beta^2\gamma^2$. [3]

(c) Find the value of $\frac{1}{\alpha^4}+\frac{1}{\beta^4}+\frac{1}{\gamma^4}+\frac{1}{\delta^4}$. [2]

(a) Let $y=\frac{1}{x^2}$

$$y=x^{-2}$$

$$x=y^{-\frac{1}{2}}$$

$$(y^{-\frac{1}{2}})^4+3(y^{-\frac{1}{2}})^2+2(y^{-\frac{1}{2}})+6=0$$

$$y^{-2}+3y^{-1}+2y^{-\frac{1}{2}}+6=0$$

$$\frac{1}{y^2}+\frac{3}{y}+\frac{2}{y^{1/2}}+6=0$$

$$1+3y+2y^{\frac{3}{2}}+6y^2=0$$

$$2y^{\frac{3}{2}}=-[(1+3y)+6y^2]$$

$$4y^3=[(1+3y)+6y^2]^2$$

$$=(1+3y)^2+(6y^2)^2+2(6y^2)(1+3y)$$

$$4y^3=1+9y^2+6y+36y^4+12y^2+36y^3$$

$$36y^4+32y^3+21y^2+6y+1=0$$

This equation has $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}, \frac{1}{\delta^2}$

(b) $$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}=\frac{\beta^2\gamma^2\delta^2+\alpha^2\gamma^2\delta^2+\alpha^2\beta^2\delta^2+\alpha^2\beta^2\gamma^2}{(\alpha\beta\gamma\delta)^2}$$

$$\frac{-32}{36}=\beta^2\gamma^2\delta^2+\alpha^2\gamma^2\delta^2+\alpha^2\beta^2\delta^2+\alpha^2\beta^2\gamma^2/36$$

$\longrightarrow$

$$\beta^2\gamma^2\delta^2 + \alpha^2\gamma^2\delta^2 + \alpha^2\beta^2\delta^2 + \alpha^2\beta^2\gamma^2 = -32$$

(c) $$\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} + \frac{1}{\delta^4} = \Sigma\left(\frac{1}{\alpha^2}\right)^2 - 2\,\Sigma\left(\frac{1}{\alpha^2}\right)\left(\frac{1}{\beta^2}\right)$$

$$= \left(\Sigma\frac{1}{\alpha^2}\right)^2 - 2\,\Sigma\left(\frac{1}{\alpha^2\beta^2}\right)$$

$$= \left(-\frac{32}{36}\right)^2 - 2\left(\frac{21}{36}\right)$$

$$= -\frac{61}{162}$$

Q.3 May/June/P11/22

The cubic equation $2x^3+5x^2-6=0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\frac{1}{\alpha^3}, \frac{1}{\beta^3}, \frac{1}{\gamma^3}$. [3]

(b) Find the value of $\frac{1}{\alpha^6}+\frac{1}{\beta^6}+\frac{1}{\gamma^6}$. [3]

(c) Find also the value of $\frac{1}{\alpha^9}+\frac{1}{\beta^9}+\frac{1}{\gamma^9}$. [2]

(a) $2x^3+5x^2-6=0$ has α, β, γ roots

Let $y = \frac{1}{x^3}$

$\Rightarrow x = y^{-\frac{1}{3}}$

$$2\left(y^{-\frac{1}{3}}\right)^3+5\left(y^{-\frac{1}{3}}\right)^2-6=0$$

$$5y^{-\frac{2}{3}} = 6-2y^{-1}$$

$$125y^{-2} = (6-2y^{-1})^3$$

$$= {}^3C_0(6)^3 - {}^3C_1(6)^2(2y^{-1}) + {}^3C_2(6)^1(2y^{-1})^2 - {}^3C_3(2y^{-1})^3$$

$$\frac{125}{y^2} = 216 - \frac{216}{y} + \frac{72}{y^2} - \frac{8}{y^3}$$

$$125y = 216y^3 - 216y^2 + 72y - 8$$

$$216y^3 - 216y^2 - 53y - 8 = 0$$

This equation has roots $\frac{1}{\alpha^3}, \frac{1}{\beta^3}, \frac{1}{\gamma^3}$

(b) $$\frac{1}{\alpha^6}+\frac{1}{\beta^6}+\frac{1}{\gamma^6} = \Sigma\left(\frac{1}{\alpha^3}\right)^2$$

$$= \left(\Sigma\frac{1}{\alpha^3}\right)^2 - 2\left(\Sigma\left(\frac{1}{\alpha^3\beta^3}\right)\right)$$

$$= 1 - 2\left(\frac{-53}{216}\right)$$

$$= \frac{161}{108}$$

→

(c) $216y^3 - 216y^2 - 53y - 8 = 0$ has roots $\frac{1}{\alpha^3}$, $\frac{1}{\beta^3}$ and $\frac{1}{\gamma^3}$

$$216y^3 = 216y^2 + 53y + 8$$

$$216 S_3 = 216 S_2 + 53 S_1 + 24$$

$$216 S_3 = 216 \times \frac{161}{108} + 53(1) + 24$$

$$S_3 = \frac{133}{72}$$

$$\frac{1}{\alpha^9} + \frac{1}{\beta^9} + \frac{1}{\gamma^9} = \frac{133}{72}$$

Q.4 May/June/P13/22

The cubic equation $x^3+5x^2+10x-2=0$ has roots α, β, γ.

(a) Find the value of $\alpha^2+\beta^2+\gamma^2$. [3]

$$x^3+5x^2+10x-2=0$$

$$\Sigma\alpha = \alpha+\beta+\gamma = -5$$

$$\Sigma\alpha\beta = \alpha\beta+\alpha\gamma+\beta\gamma = +10$$

$$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$$

$$= (5)^2 - 2(10)$$

$$= 25-20$$

$$= 5$$

Q.5 May/June/P11/21

The equation $x^4 - 2x^3 - 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\alpha^3, \beta^3, \gamma^3, \delta^3$ and state the value of $\alpha^3 + \beta^3 + \gamma^3 + \delta^3$. [4]

(b) Find the value of $\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} + \frac{1}{\delta^3}$. [3]

(c) Find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [2]

(a) $x^4 - 2x^3 - 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$

Let $y = x^3$

$x = y^{1/3}$

$y^{\frac{4}{3}} - 2y - 1 = 0$

$y^4 = (2y+1)^3$

$y^4 = 8y^3 + 12y^2 + 6y + 1$

$y^4 - 8y^3 - 12y^2 - 6y - 1 = 0$

This equation has roots $\alpha^3, \beta^3, \gamma^3, \delta^3$

$\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = \sum \alpha^3 = 8$

(b) $\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} + \frac{1}{\delta^3} = \frac{\beta^3\gamma^3\delta^3 + \alpha^3\gamma^3\delta^3 + \alpha^3\beta^3\delta^3 + \alpha^3\beta^3\gamma^3}{(\alpha\beta\gamma\delta)^3}$

$= \frac{\sum \alpha^3\beta^3\gamma^3}{(\alpha\beta\gamma\delta)^3} = \frac{6}{-1} = -6$

(c) $x^4 - 2x^3 - 1 = 0$

$x^4 = 2x^3 + 1$

$S_4 = 2S_3 + 4$

$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = 2(8) + 4$

$= 20$

Q.6 May/June/P13/21

The cubic equation $2x^3 - 4x^2 + 3 = 0$ has roots α, β, γ. Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(a) State the value of S_1 and find the value of S_2. [3]

(b) (i) Express S_{n+3} in terms of S_{n+2} and S_n. [1]

(ii) Hence, or otherwise, find the value of S_4. [2]

(c) Use the substitution $y = S_1 - x$, where S_1 is the numerical value found in part (a), to find and simplify an equation whose roots are $\alpha+\beta, \beta+\gamma, \gamma+\alpha$. [3]

(a) $2x^3 - 4x^2 + 3 = 0$ has roots α, β, γ

$S_1 = \Sigma\alpha = \frac{4}{2} = 2$

$S_2 = \Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$

$= (2)^2 - 2(0)$

$= 4$

(b) (i) $2x^3 - 4x^2 + 3 = 0$

$x^3 = 2x^2 - \frac{3}{2}$

$x^{n+3} = 2x^{n+2} - \frac{3}{2}x^n$

$S_{n+3} = 2S_{n+2} - \frac{3}{2}S_n$

(ii) $x^3 = 2x^2 - \frac{3}{2}$

$x^4 = 2x^3 - \frac{3}{2}x$

$S_4 = 2S_3 - \frac{3}{2}S_1$

$S_4 = 2(2S_2 - \frac{3}{2}S_0) - \frac{3}{2}S_1$

$= 2[8 - \frac{3}{2}\times 3] - \frac{3}{2}(2)$

$= 7 - 3$

$= 4$

(c) $y = S_1 - x$

$y = 2 - x$

$x = 2 - y$

$2x^3 - 4x^2 + 3 = 0$

$2(2-y)^3 - 4(2-y)^2 + 3 = 0$

$2\left[{}^3C_0 8 - {}^3C_1 4y + {}^3C_2 2y^2 - {}^3C_3 y^3 \right] - 4(4 - 4y + y^2) + 3 = 0$

$2(8 - 12y + 6y^2 - y^3) - 16 + 16y - 4y^2 + 3 = 0$

$-2y^3 + 8y^2 - 8y + 3 = 0$

$2y^3 - 8y^2 + 8y - 3 = 0$

Q.7 Oct/Nov/P11/21

It is given that

$$\alpha+\beta+\gamma=3,\quad \alpha^2+\beta^2+\gamma^2=5,\quad \alpha^3+\beta^3+\gamma^3=6.$$

The cubic equation $x^3+bx^2+cx+d=0$ has roots α,β,γ.

Find the values of b, c and d. [6]

$x^3+bx^2+cx+d=0$ has roots α,β,γ

$\alpha+\beta+\gamma=-b$

$-b=3$ Given $\alpha+\beta+\gamma=3$

$\boxed{b=-3}$

$\alpha^2+\beta^2+\gamma^2=(\Sigma\alpha)^2-2\Sigma\alpha\beta$

$5=(3)^2-2(c)$

$2c=9-5$

$\boxed{c=2}$

$S_3=-bS_2-cS_1-3d$

$S_3=(3)(5)-2(3)-3d$

$6=15-6-3d$

$-3=-3d$

$\boxed{d=1}$

Q.8 Oct/Nov/P12/21

The cubic equation $x^3+2x^2+3x+3=0$ has roots α, β, γ.

(a) Find the value of $\alpha^2+\beta^2+\gamma^2$. [2]

(b) Show that $\alpha^3+\beta^3+\gamma^3=1$. [2]

(c) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}\left((\alpha+r)^3+(\beta+r)^3+(\gamma+r)^3\right)=n+\tfrac{1}{4}n(n+1)\left(an^2+bn+c\right),$$

where a, b and c are constants to be determined. [6]

(a) $x^3+2x^2+3x+3=0$ has roots α, β, γ

$\Sigma\alpha = -2$

$\alpha^2+\beta^2+\gamma^2 = \Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$

$= (-2)^2 - 2(3)$

$= -2$

(b)

$x^3 = -2x^2 - 3x - 3$

$S_3 = -2S_2 - 3S_1 - 9$

$S_3 = -2(-2) - 3(-2) - 9$

$\alpha^3+\beta^3+\gamma^3 = 4+6-9$

$= 1$

(c) $(\alpha+r)^3 = \alpha^3 + 3\alpha^2 r + 3\alpha r^2 + r^3$

$(\beta+r)^3 = \beta^3 + 3\beta^2 r + 3\beta r^2 + r^3$

$(\gamma+r)^3 = \gamma^3 + 3\gamma^2 r + 3\gamma r^2 + r^3$

$(\alpha+r)^3+(\beta+r)^3+(\gamma+r)^3 = (\alpha^3+\beta^3+\gamma^3) + 3(\alpha^2+\beta^2+\gamma^2)r + 3(\alpha+\beta+\gamma)r^2 + 3r^3$

$= 1 + 3(-2)r + 3(-2)r^2 + 3r^3$

$\sum_{r=1}^{n} (\alpha+r)^3+(\beta+r)^3+(\gamma+r)^3 = \sum_{r=1}^{n} 1 - 6r - 6r^2 + 3r^3$

$= n - 6\times\frac{1}{2}n(n+1) - 6\times\frac{1}{6}n(n+1)(2n+1)$

$+ 3\times\frac{1}{4}n^2(n+1)^2$

$$= n - 3n(n+1) - n(n+1)(2n+1) + \frac{3}{4}n^2(n+1)^2$$

$$= n + \frac{1}{4}n(n+1)\left[-12 - 8n - 4 + 3n^2 + 3n\right]$$

$$= n + \frac{1}{4}n(n+1)(3n^2 - 5n - 16)$$

Q.9 May/June/P11/20

The cubic equation $6x^3 + px^2 - 3x - 5 = 0$, where p is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^2, \beta^2, \gamma^2$. [3]

(b) It is given that $\alpha^2 + \beta^2 + \gamma^2 = 2(\alpha + \beta + \gamma)$.

(i) Find the value of p. [3]

(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. [2]

(a) $6x^3 + px^2 - 3x - 5 = 0$

Let $y = x^2$

$x = \sqrt{y}$

$6y^{\frac{3}{2}} + py - 3y^{\frac{1}{2}} - 5 = 0$

$y^{\frac{1}{2}}(6y - 3) = -py + 5$

$y(6y - 3)^2 = (-py + 5)^2$

$y(36y^2 - 36y + 9) = p^2y^2 - 10py + 25$

$36y^3 - (p^2 + 36)y^2 + (10p + 9)y - 25 = 0$

(b) (i) Given $\alpha^2 + \beta^2 + \gamma^2 = 2(\alpha + \beta + \gamma)$

$\alpha + \beta + \gamma = \frac{-p}{6}$

$\alpha^2 + \beta^2 + \gamma^2 = \frac{p^2 + 36}{36}$

$\frac{p^2 + 36}{36} = 2\left(\frac{-p}{6}\right)$

$\Rightarrow p^2 + 12p + 36 = 0$

$(p + 6)^2 = 0$

$p + 6 = 0$

$\boxed{p = -6}$

(ii)

$$6S_3 = -pS_2 + 3S_1 + 15$$

$$6S_3 = -(-6)(2) + 3(1) + 15$$

$$6S_3 = 30$$

$$S_3 = 5$$

$$\alpha^3 + \beta^3 + \gamma^3 = 5$$

Q.10 May/June/P13/20

The cubic equation $7x^3+3x^2+5x+1=0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$. [3]

(b) Find the value of $\alpha^{-2}+\beta^{-2}+\gamma^{-2}$. [2]

(c) Find the value of $\alpha^{-3}+\beta^{-3}+\gamma^{-3}$. [2]

(a) $7x^3+3x^2+5x+1=0$

Let $y = x^{-1}$

$x = \frac{1}{y}$

$\frac{7}{y^3}+\frac{3}{y^2}+\frac{5}{y}+1=0$

$y^3+5y^2+3y+7=0$

This equation has roots $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$

(b) $\alpha^{-2}+\beta^{-2}+\gamma^{-2} = (\Sigma\alpha^{-1})^2 - 2\Sigma\alpha^{-1}\beta^{-1}$

$= (-5)^2 - 2(3)$

$= 19$

(c) $y^3 = -5y^2-3y-7$

$S_3 = -5S_2-3S_1-21$

$\alpha^{-3}+\beta^{-3}+\gamma^{-3} = -5(19)-3(-5)-21$

$= -101$

Q.11 Oct/Nov/P11/20

The cubic equation $x^3 + cx + 1 = 0$, where c is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^3, \beta^3, \gamma^3$. [3]

(b) Show that $\alpha^6 + \beta^6 + \gamma^6 = 3 - 2c^3$. [3]

(c) Find the real value of c for which the matrix $\begin{pmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{pmatrix}$ is singular. [5]

(a) $x^3 + cx + 1 = 0$

Let $y = x^3 \Rightarrow x = y^{1/3}$

$y + c\,y^{1/3} + 1 = 0$

$-c\,y^{1/3} = y + 1$

$-c^3 y = (y+1)^3$

$-c^3 y = y^3 + 3y^2 + 3y + 1$

$y^3 + 3y^2 + (3 + c^3)y + 1 = 0$

This equation has roots $\alpha^3, \beta^3, \gamma^3$

(b) $\alpha^6 + \beta^6 + \gamma^6 = \Sigma(\alpha^3)^2$

$= (\Sigma\alpha^3)^2 - 2\Sigma\alpha^3\beta^3$

$= (-3)^2 - 2(3 + c^3)$

$= 3 - 2c^3$

(c) Given matrix is singular, so

$$\begin{vmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{vmatrix} = 0$$

$1(1 - \gamma^6) - \alpha^3(\alpha^3 - \beta^3\gamma^3) + \beta^3(\alpha^3\gamma^3 - \beta^3) = 0$

$1 - (\alpha^6 + \beta^6 + \gamma^6) + 2\alpha^3\beta^3\gamma^3 = 0$

$1 - (3 - 2c^3) + 2(-1) = 0$

$2c^3 - 4 = 0$

$c = \sqrt[3]{2}$

Q.12 Oct/Nov/P12/20

The cubic equation $x^3+bx^2+cx+d=0$, where b, c and d are constants, has roots α, β, γ. It is given that $\alpha\beta\gamma=-1$.

(a) State the value of d. [1]

(b) Find a cubic equation, with coefficients in terms of b and c, whose roots are $\alpha+1$, $\beta+1$, $\gamma+1$. [3]

(c) Given also that $\gamma+1=-\alpha-1$, deduce that $(c-2b+3)(b-3)=b-c$. [4]

(a) $x^3+bx^2+cx+d=0$ has roots α,β,γ

$\alpha\beta\gamma=-d$

$(-1)=-d$ Given $\alpha\beta\gamma=-1$

$d=1$

(b) Let $y=x+1$

$x=y-1$

$(y-1)^3+b(y-1)^2+c(y-1)+1=0$

$y^3-3y^2+3y-1+by^2-2by+b+cy-c+1=0$

$y^3+(b-3)y^2+(c-2b+3)y+(b-c)=0$

(c) Given $\gamma+1=-\alpha-1 \Rightarrow (\alpha+1)+(\gamma+1)=0$

$=-(\alpha+1)$

$(\alpha+1)+(\beta+1)+(\gamma+1)=-(b-3)$

$\beta+1=-(b-3)$

$(\alpha+1)(\beta+1)+(\alpha+1)(\gamma+1)+(\beta+1)(\gamma+1)=c-2b+3$

$(\alpha+1)(\beta+1)+(\alpha+1)[-(\alpha+1)]+(\beta+1)[-(\alpha+1)]=c-2b+3$

$-(\alpha+1)(\alpha+1)=c-2b+3$

$(\alpha+1)(\beta+1)(\gamma+1)=-(b-c)$

$(\alpha+1)(\gamma+1)(\beta+1)=-(b-c)$

$[-(\alpha+1)(\alpha+1)](\beta+1)=-(b-c)$

$-(c-2b+3)(b-3)=-(b-c) \Rightarrow (c-2b+3)(b-3)=b-c$

Q.13 May/June/P11+P12/19

The equation

$$x^3 - x + 1 = 0$$

has roots α, β, γ.

(i) Use the relation $x = y^{\frac{1}{3}}$ to show that the equation

$$y^3 + 3y^2 + 2y + 1 = 0$$

has roots $\alpha^3, \beta^3, \gamma^3$. Hence write down the value of $\alpha^3 + \beta^3 + \gamma^3$. [3]

Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(ii) Find the value of S_{-3}. [2]

(iii) Show that $S_6 = 5$ and find the value of S_9. [4]

(i) $x^3 - x + 1 = 0$ has roots α, β, γ

Given $x = y^{1/3}$

$$y - y^{1/3} + 1 = 0$$

$$y^{1/3} = y + 1$$

$$y = (y+1)^3$$

$$y = y^3 + 3y^2 + 3y + 1$$

$$y^3 + 3y^2 + 2y + 1 = 0 \quad \text{—— (1)}$$

This equation has roots $\alpha^3, \beta^3, \gamma^3$

$$\alpha^3 + \beta^3 + \gamma^3 = -3$$

(ii) $S_{-3} = \frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} = \frac{\alpha^3\beta^3 + \beta^3\gamma^3 + \alpha^3\gamma^3}{(\alpha\beta\gamma)^3}$

$$= \frac{2}{-1} = -2$$

(iii) $S_6 = (\alpha^3)^2 + (\beta^3)^2 + (\gamma^3)^2$

$$= (\Sigma \alpha^3)^2 - 2 \Sigma \alpha^3\beta^3$$

$$= (-3)^2 - 2(2) = 5$$

$S_3 = -3S_2 - 2S_1 - 3$ from (1)

$\alpha^9 + \beta^9 + \gamma^9 = -3(5) - 2(-3) - 3$

$= -12$

Q.14 May/June/P13/19

A cubic equation $x^3 + bx^2 + cx + d = 0$ has real roots α, β and γ such that

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = -\frac{5}{12},$$

$$\alpha\beta\gamma = -12,$$

$$\alpha^3 + \beta^3 + \gamma^3 = 90.$$

(i) Find the values of c and d. [3]

(ii) Express $\alpha^2 + \beta^2 + \gamma^2$ in terms of b. [2]

(iii) Show that $b^3 - 15b + 126 = 0$. [4]

(iv) Given that $3 + i\sqrt{(12)}$ is a root of $y^3 - 15y + 126 = 0$, deduce the value of b. [2]

(i)

$$x^3 + bx^2 + cx + d = 0$$

$$\Sigma\alpha = -b$$

$$\Sigma\alpha\beta = c$$

$$\Sigma\alpha\beta\gamma = -d$$

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$$

$$\frac{-5}{12} = \frac{c}{-d}$$

$$5d = 12c \quad — ①$$

$$-d = -12$$

$$d = 12$$

put in ①

$$5 \times 12 = 12c$$

$$c = 5$$

(iii)

$$\alpha^2 + \beta^2 + \gamma^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$$

$$= b^2 - 2c$$

$$= b^2 - 2(5)$$

$$= b^2 - 10$$

(iii)

$$S_3 + bS_2 + 5S_1 + 36 = 0$$

$$90 + b(b^2-10) - 5b + 36 = 0$$

$$b^3 - 15b + 126 = 0$$

(iv) Given

$$b = 3 + i\sqrt{12} = 3 + i\sqrt{1}$$

Q.15 Oct/Nov/p11+P12+P13/19

The equation $x^3 + 2x^2 + x + 7 = 0$ has roots α, β, γ.

(i) Use the relation $x^2 = -7y$ to show that the equation

$$49y^3 + 14y^2 - 27y + 7 = 0$$

has roots $\frac{\alpha}{\beta\gamma}, \frac{\beta}{\gamma\alpha}, \frac{\gamma}{\alpha\beta}$. [4]

(ii) Show that $\frac{\alpha^2}{\beta^2\gamma^2} + \frac{\beta^2}{\gamma^2\alpha^2} + \frac{\gamma^2}{\alpha^2\beta^2} = \frac{58}{49}$. [3]

(iii) Find the exact value of $\frac{\alpha^3}{\beta^3\gamma^3} + \frac{\beta^3}{\gamma^3\alpha^3} + \frac{\gamma^3}{\alpha^3\beta^3}$. [2]

(i) $x^3 + 2x^2 + x + 7 = 0$

Given $x^2 = -7y$

$x = \sqrt{-7y}$

$\sqrt{-7y}(-7y) + 2(-7y) + \sqrt{-7y} + 7 = 0$

$\sqrt{-7y}(-7y+1) = 14y - 7$

$-7y(-7y+1)^2 = (14y-7)^2$

$-7y(49y^2 - 14y + 1) = 196y^2 - 196y + 49$

$-343y^3 + 98y^2 - 7y = 196y^2 - 196y + 49$

$49y^3 - 14y^2 + y = -28y^2 + 28y - 7$

$49y^3 + 14y^2 - 27y + 7 = 0$

has root $\frac{\alpha}{\beta\gamma}, \frac{\beta}{\alpha\gamma}, \frac{\gamma}{\alpha\beta}$

$\sum \frac{\alpha}{\beta\gamma} = \frac{\alpha}{\beta\gamma} + \frac{\beta}{\alpha\gamma} + \frac{\gamma}{\alpha\beta} = \frac{-14}{49} = -\frac{2}{7}$

$\sum \left(\frac{\alpha}{\beta\gamma}\right)\left(\frac{\beta}{\alpha\gamma}\right) = \frac{-27}{49}$

$$\frac{\alpha^2}{\beta^2\gamma^2} + \frac{\beta^2}{\alpha^2\gamma^2} + \frac{\gamma^2}{\alpha^2\beta^2} = \left(\Sigma \frac{\alpha}{\beta\gamma}\right)^2 - 2\,\Sigma\left(\frac{\alpha}{\beta\gamma}\right)\left(\frac{\beta}{\alpha\gamma}\right)$$

$$= \left(-\frac{2}{7}\right)^2 - 2\left(-\frac{27}{49}\right) = \frac{58}{49}$$

(iii) $49 S_3 + 14 S_2 - 27 S_1 + 21 = 0$

$$49 S_3 = -14\left(\frac{58}{49}\right) + 27\left(-\frac{2}{7}\right) - 21$$

$$S_3 = -\frac{317}{343}$$

$$\frac{\alpha^3}{\beta^3\gamma^3} + \frac{\beta^3}{\alpha^3\gamma^3} + \frac{\gamma^3}{\alpha^3\beta^3} = -\frac{317}{343}$$

Q.16 May/June/P13/18

The equation

$$9x^3 - 9x^2 + x - 2 = 0$$

has roots α, β, γ.

(i) Use the substitution $y = 3x - 1$ to show that $3\alpha - 1, 3\beta - 1, 3\gamma - 1$ are the roots of the equation

$$y^3 - 2y - 7 = 0.$$ [2]

The sum $(3\alpha - 1)^n + (3\beta - 1)^n + (3\gamma - 1)^n$ is denoted by S_n.

(ii) Find the value of S_3. [2]

(iii) Find the value of S_{-2}. [4]

(i) Given $y = 3x - 1$

$$x = \frac{y+1}{3}$$

$9x^3 - 9x^2 + x - 2 = 0$ has roots α, β, γ

$$9\left(\frac{y+1}{3}\right)^3 - 9\left(\frac{y+1}{3}\right)^2 + \frac{y+1}{3} - 2 = 0$$

$$\frac{y^3 + 3y^2 + 3y + 1}{3} - (y^2 + 2y + 1) + \frac{y+1}{3} - 2 = 0$$

$$y^3 + 3y^2 + 3y + 1 - 3y^2 - 6y - 3 + y + 1 - 6 = 0$$

$$y^3 - 2y - 7 = 0$$

This equation has roots $3\alpha - 1, 3\beta - 1, 3\gamma - 1$

(ii) $S_3 = 2S_1 + 21$

$S_3 = 2(0) + 21 = 21$

(iii) $$S_{-2} = \frac{\left[\sum (3\alpha-1)(3\beta-1)\right]^2 - 2(3\alpha-1)(3\beta-1)(3\gamma-1)\sum(3\alpha-1)}{\left[(3\alpha-1)(3\beta-1)(3\gamma-1)\right]^2}$$

$$= \frac{(2)^2 - 2(7)(0)}{(7)^2} = \frac{4}{49}$$