AS Level Further Mathematics

Topic: Projectile Motion

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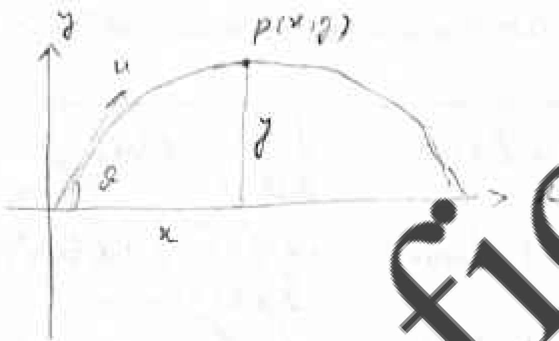
5 A particle P is projected with speed $u\,\mathrm{m\,s^{-1}}$ at an angle of θ above the horizontal from a point O on a horizontal plane and moves freely under gravity. The horizontal and vertical displacements of P from O at a subsequent time t s are denoted by x m and y m respectively.

(a) Show that the equation of the trajectory is given by

$$y = x\tan\theta - \frac{gx^2}{2u^2}(1+\tan^2\theta).$$ [4]

In the subsequent motion P passes through the point with coordinates (30, 20).

(b) Given that one possible value of $\tan\theta$ is $\frac{4}{3}$, find the other possible value of $\tan\theta$. [5]



(a) Let $P(x, y)$ be the point after time t sec

$x = (u\cos\theta)t$ —— ①

$y = (u\sin\theta)t - \frac{1}{2}gt^2$ —— ②

from ① $t = \frac{x}{u\cos\theta}$

Substitute in ②

$y = (u\sin\theta)\cdot\frac{x}{u\cos\theta} - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$

$y = x\tan\theta - \frac{gx^2}{2u^2}\sec^2\theta$

$y = x\tan\theta - \frac{gx^2}{2u^2}(1+\tan^2\theta)$

(b) Given when $x = 30$
$y = 20$

and $\tan\theta = \frac{4}{3}$

by substituting these values in above equation of trajectory

$$y = x\tan\theta - \frac{gx^2}{2u^2}(1+\tan^2\theta)$$

$$20 = 30\left(\frac{4}{3}\right) - \frac{10(30)^2}{2u^2}\left(1+\frac{16}{9}\right)$$

$$20 = 40 - \frac{9000}{2u^2}\left(\frac{25}{9}\right)$$

$$-20 = -\frac{25000}{2u^2}$$

$$40u^2 = 25000$$

$$u^2 = 625$$

So

$$y = x\tan\theta - \frac{gx^2}{2u^2}(1+\tan^2\theta)$$

$$20 = 30\tan\theta - \frac{10\times 30^2}{2\times 625}(1+\tan^2\theta)$$

$$\frac{36}{5}(1+\tan^2\theta) - 30\tan\theta + 20 = 0$$

$$36\tan^2\theta - 150\tan\theta + 100 + 36 = 0$$

$$18\tan^2\theta - 75\tan\theta + 68 = 0$$

$$\Rightarrow \tan\theta = \frac{4}{3}, \quad \tan\theta = \frac{17}{6}$$

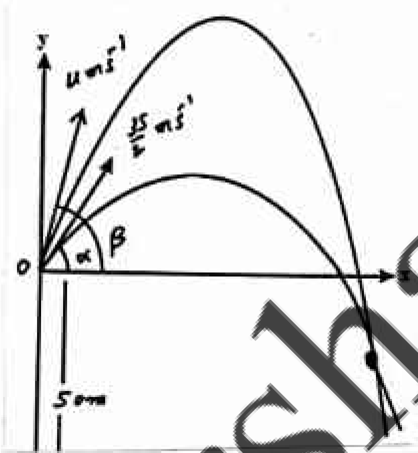
Q.2 M/J/2022/P31+P32

7 Particles P and Q are projected in the same vertical plane from a point O at the top of a cliff. The height of the cliff exceeds 50 m. Both particles move freely under gravity. Particle P is projected with speed $\frac{35}{2}\,\mathrm{m\,s^{-1}}$ at an angle α above the horizontal, where $\tan\alpha = \frac{4}{3}$. Particle Q is projected with speed $u\,\mathrm{m\,s^{-1}}$ at an angle β above the horizontal, where $\tan\beta = \frac{1}{2}$. Particle Q is projected one second after the projection of particle P. The particles collide T s after the projection of particle Q.

(a) Write down expressions, in terms of T, for the horizontal displacements of P and Q from O when they collide and hence show that $4uT = 21\sqrt{5}(T+1)$. [4]

(b) Find the value of T. [4]

(c) Find the horizontal and vertical displacements of the particles from O when they collide. [3]



(a) let T be the time of particle Q, then

$T+1$ be the time of particle P when both collide

Let $P(x,y)$ be the collision position of particle P and Q

For particle P

$x = \left(\frac{35}{2}\cos\alpha\right)(T+1)$ —① as given initial velocity of $P = \frac{35}{2}$ and angle of projection α

For particle Q

$x = (u\cos\beta)(T)$ —②

Given $\tan\alpha = \frac{4}{3}$ and $\tan\beta = \frac{1}{2}$

$\sin\alpha = \frac{4}{5}$ $\sin\beta = \frac{1}{\sqrt{5}}$

$\cos\alpha = \frac{3}{5}$ $\cos\beta = \frac{2}{\sqrt{5}}$

From ① and ② we can write

$$\frac{35}{2} \times \frac{3}{5} (T+1) = u \cdot \frac{2}{\sqrt{5}} T$$

$$\frac{21}{2} (T+1) = \frac{2u}{\sqrt{5}} T$$

$$4uT = 21\sqrt{5} (T+1) \quad \text{——— ③}$$

(b) For P

$$y = (u \sin\alpha)(T+1) - \frac{1}{2} g (T+1)^2 \quad \text{——— ④}$$

For Q

$$y = (u \sin\beta) T - \frac{1}{2} g T^2 \quad \text{——— ⑤}$$

From ④ and ⑤

$$\frac{35}{2} \times \frac{4}{5} (T+1) - \frac{g}{2} (T+1)^2 = u \cdot \frac{1}{\sqrt{5}} T - \frac{g}{2} T^2$$

$$14(T+1) - 5(T+1)^2 = \frac{21\sqrt{5}(T+1)}{4T} \cdot \frac{T}{\sqrt{5}} - 5T^2 \quad \text{from ③}$$

$$14T + 14 - 5T^2 - 10T - 5 = \frac{21}{4} T + \frac{21}{4} - 5T^2$$

$$56T + 56 - 40T - 20 = 21T + 21$$

$$16T + 36 = 21T + 21$$

$$5T = 15 \Rightarrow \boxed{T = 3}$$

(c)

$$\text{Horizontal displacement} = \left(\frac{35}{2} \cos\alpha\right)(T+1) = \frac{35}{2} \times \frac{3}{5} \times 4 = 42$$

$$\text{vertical displacement} = \left(\frac{35}{2} \sin\alpha\right)(T+1) - \frac{g}{2} (T+1)^2$$

$$= \frac{35}{2} \times \frac{4}{5} \times 4 - 5(4)^2$$

$$= 56 - 80$$

$$= -24 \text{ (24m below from O)}$$

3 A particle P is projected with speed $25\,\mathrm{m\,s^{-1}}$ at an angle θ above the horizontal from a point O on a horizontal plane and moves freely under gravity. After 2 s the speed of P is $15\,\mathrm{m\,s^{-1}}$.

(a) Find the value of $\sin\theta$. [5]

(b) Find the range of the flight. [3]

(a) Given $u = 25\ m\,s^{-1}$

When $T = 2$ sec

$V = 15\ m\,s^{-1}$

x - component of $v = u\cos\theta = 25\cos\theta$

y - component of $v = u\sin\theta - gT = 25\sin\theta - 20$

$$V = \sqrt{(25\cos\theta)^2 + (25\sin\theta - 20)^2}$$

$$15^2 = 25^2\cos^2\theta + 25^2\sin^2\theta - 1000\sin\theta + 400$$

$$1000\sin\theta = 800$$

$$\sin\theta = \frac{800}{1000} = \boxed{\frac{4}{5}}$$

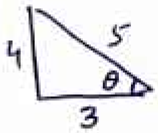
(b)
$$y = (u\sin\theta)T - \frac{1}{2}gT^2$$

$$0 = 25 \times \frac{4}{5}\,T - 5T^2$$

$$5T(4 - T) = 0$$

$$T = 0\ ,\ T = 4$$

$$y = x\tan\theta - \frac{gx^2}{2u^2}\sec^2\theta$$



$$0 = \frac{4}{3}x - \frac{10x^2}{2\times 25^2}\left(\frac{5}{3}\right)^2$$

$$0 = \frac{4x}{3} - \frac{x^2}{45}$$

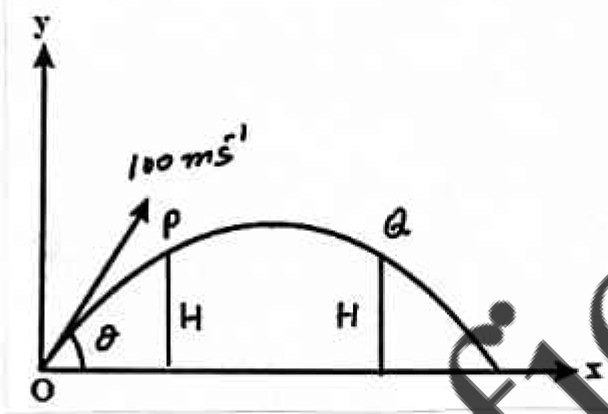
$$0 = 60x - x^2 \Rightarrow x(60 - x) = 0 \Rightarrow x = 0 \text{ or } \boxed{x = 60}$$

Q.4 May/June/P31+32/2021

7 A particle P is projected from a point O on a horizontal plane and moves freely under gravity. The initial velocity of P is $100\,\mathrm{m\,s^{-1}}$ at an angle θ above the horizontal, where $\tan\theta = \frac{4}{3}$. The two times at which P's height above the plane is H m differ by 10 s.

(a) Find the value of H. [5]

(b) Find the magnitude and direction of the velocity of P one second before it strikes the plane. [4]

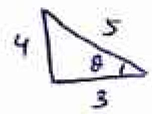


Let P and Q be two points on the path such that vertical hight is H

Let the particle reaches at P after t sec, then particle will reach at Q after (t+10) sec

Given $u = 100\ \mathrm{ms^{-1}}$

$\tan\theta = \frac{4}{3}$



$\sin\theta = \frac{4}{5}$

$\cos\theta = \frac{3}{5}$

$H = (u\sin\theta)t - \frac{1}{2}gt^2$

$H = 100\times\frac{4}{5}t - 5t^2$ —— ①

Also

$H = (u\sin\theta)(t+10) - \frac{1}{2}g(t+10)^2$

$= 100\times\frac{4}{5}(t+10) - 5(t^2+20t+100)$

$= 80t + 800 - 5t^2 - 100t - 500$

$= -5t^2 + 20t + 300$ —— ②

From ① and ②

$-5t^2 + 20t + 300 = 80t - 5t^2$

$100t = 300 \Rightarrow t = 3$

put the value of $t = 3$ in ①

$$H = 100 \times \frac{4}{5} \times 3 - 5 \times 9 = \boxed{195}$$

(b)

Time of flight $T = \dfrac{2u\sin\theta}{g}$

$$T = \frac{2 \times 100 \times \frac{4}{5}}{10} = 16 \text{ sec}$$

$\uparrow$ $V = u\sin\theta - gt$

$$V = 100 \times \frac{4}{5} - 10 \times 15 = -70$$

$\rightarrow$ $V = u\cos\theta = 100 \times \frac{3}{5} = 60$

magnitude $= \sqrt{(60)^2 + (-70)^2} = 92.2 \text{ ms}^{-1}$

Direction $= \tan^{-1}\left(\frac{70}{60}\right) = 49.4^\circ$

Q.5 May/June/P33/2021

7 A particle P is projected with speed u at an angle θ above the horizontal from a point O on a horizontal plane and moves freely under gravity. The horizontal and vertical displacements of P from O at a subsequent time t are denoted by x and y respectively.

(a) Use the equation of the trajectory given in the List of formulae (MF19), together with the condition $y = 0$, to establish an expression for the range R in terms of u, θ and g. [2]

(b) Deduce an expression for the maximum height H, in terms of u, θ and g. [2]

It is given that $R = \frac{4H}{\sqrt{3}}$.

(c) Show that $\theta = 60°$. [1]

It is given also that $u = \sqrt{40}\,\text{m s}^{-1}$.

(d) Find, by differentiating the equation of the trajectory or otherwise, the set of values of x for which the direction of motion makes an angle of less than 45° with the horizontal. [4]

(a) $y = x\tan\theta - \frac{gx^2}{2u^2}(1+\tan^2\theta)$

put $y = 0$

$x\left[\tan\theta - \frac{gx}{2u^2\cos^2\theta}\right] = 0$

$x = 0$ or $\frac{gx}{2u^2\cos^2\theta} = \tan\theta$

$x = \frac{\sin\theta}{\cos\theta} \times \frac{2u^2\cos^2\theta}{g}$

$R = \text{Range} = x = \frac{2u^2\sin\theta\cos\theta}{g}$

(b) Maximum hight H will be at $x = \frac{u^2\sin\theta\cos\theta}{g}$

$H = \frac{u^2\sin\theta\cos\theta}{g}\cdot\frac{\sin\theta}{\cos\theta} - \frac{g}{2u^2\cos^2\theta} \times \frac{u^4\sin^2\theta\cos^2\theta}{g^2}$

$= \frac{u^2\sin^2\theta}{g} - \frac{u^2\sin^2\theta}{2g}$

$= \frac{u^2\sin^2\theta}{2g}$

(c) given

$$R = \frac{4}{\sqrt{3}} H$$

$$\frac{2u^2 \sin\theta \cos\theta}{g} = \frac{4}{\sqrt{3}} \times \frac{u^2 \sin^2\theta}{2g}$$

$$\Rightarrow \tan\theta = \sqrt{3}$$

$$\theta = 60^\circ$$

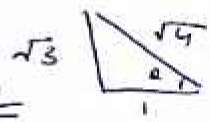
(d)

$$y = x\tan\theta - \frac{g x^2}{2u^2\cos^2\theta}$$

$$\frac{dy}{dx} = \tan\theta - \frac{g x}{u^2 \cos^2\theta}$$

$$\tan\theta - \frac{g x}{u^2\cos^2\theta} = \pm 1$$

$\tan\theta = \sqrt{3}$



$\cos\theta = \frac{1}{\sqrt{4}}$

$$\tan\theta - \frac{g x}{u^2\cos^2\theta} = 1 \quad \text{or} \quad \tan\theta - \frac{g x}{u^2\cos^2\theta} = -1$$

as $u = \sqrt{40}\ ms^{-1}$

$$\sqrt{3} - \frac{10x}{40 \cdot \frac{1}{4}} = 1 \qquad \sqrt{3} - \frac{10x}{40 \cdot \frac{1}{4}} = -1$$

$$x = \sqrt{3} + 1 \qquad x = \sqrt{3} - 1$$

so when

$$\sqrt{3} - 1 < x < \sqrt{3} + 1$$

Then angle must be less 45°

5 A particle P is projected from a point O on a horizontal plane and moves freely under gravity. Its initial speed is $u\,\mathrm{m\,s^{-1}}$ and its angle of projection is $\sin^{-1}(\frac{4}{5})$ above the horizontal. At time 8 s after projection, P is at the point A. At time 32 s after projection, P is at the point B. The direction of motion of P at B is perpendicular to its direction of motion at A.

Find the value of u. [7]

Given $u = u$ $\quad \theta = \sin^{-1}\frac{4}{5} \Rightarrow \sin\theta = \frac{4}{5}$

At A $\quad t = 8$ sec

$\rightarrow$ component of velocity $= u\cos\theta$

$\uparrow$ vertical component $= u\sin\theta - gt = u\sin\theta - 8g$

Let α be the direction of motion at A

$$\tan\alpha = \frac{u\sin\theta - 8g}{u\cos\theta}$$

At B $\quad t = 32$ sec

Horizontal component of velocity $= u\cos\theta$

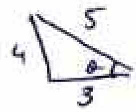
Vertical component of velocity $= u\sin\theta - 32g$

Let β be the direction of motion at B

$$\tan\beta = \frac{u\sin\theta - 32g}{u\cos\theta}$$

given $\tan\alpha \cdot \tan\beta = -1$

$$\frac{u\sin\theta - 8g}{u\cos\theta} \cdot \frac{u\sin\theta - 32g}{u\cos\theta} = -1$$



$$\left[u\left(\tfrac{4}{5}\right) - 80\right]\left[u\left(\tfrac{4}{5}\right) - 320\right] = -u^2\left(\tfrac{9}{25}\right)$$

$$(4u - 400)(4u - 1600) = -9u^2$$

$$16u^2 - 6400u - 1600u + 640000 = -9u^2$$

$$25u^2 - 8000u + 640000 = 0$$

$$u^2 - 320u + 25600 = 0 \Rightarrow \boxed{u = 160}$$

Q.7 Oct/Nov/P32/2021

1 A particle is projected with speed u at an angle α above the horizontal from a point O on a horizontal plane. The particle moves freely under gravity.

(a) Write down the horizontal and vertical components of the velocity of the particle at time T after projection. [2]

At time T after projection, the direction of motion of the particle is perpendicular to the direction of projection.

(b) Express T in terms of u, g and α. [2]

(c) Deduce that $T > \frac{u}{g}$. [1]

(a) Horizontal component of velocity $= u\cos\alpha$

Vertical component of velocity $= u\sin\alpha - gT$

(b)
$$\frac{u\sin\alpha - gT}{u\cos\alpha} \cdot \frac{u\sin\alpha}{u\cos\alpha} = -1$$

$$u^2\sin^2\alpha - ug\sin\alpha\, T = -u^2\cos^2\alpha$$

$$u^2 - ug\sin\alpha\, T = 0$$

$$T = \frac{u^2}{ug\sin\alpha}$$

$$T = \frac{u}{g\sin\alpha}$$

(c) Since $\sin\alpha < 1$

$$\frac{u}{Tg} < 1$$

$$\frac{u}{g} < T$$

$$\Rightarrow \quad T > \frac{u}{g}$$

1 A particle P is projected with speed u at an angle of 30° above the horizontal from a point O on a horizontal plane and moves freely under gravity. The particle reaches its greatest height at time T after projection.

Find, in terms of u, the speed of P at time $\frac{2}{3}T$ after projection. [5]

Given $u = u$ $\theta = 30°$

V_x = Horizontal component of speed at $T = \frac{2}{3}T$ $= u\cos\theta$

$= \frac{u\sqrt{3}}{2}$

V_y = Vertical component of speed at $\frac{2}{3}T$ $= u\sin\theta - gT$

$= u\sin 30 - g\left(\frac{2}{3}T\right)$

$= \frac{u}{2} - \frac{2Tg}{3}$

At highest point $V = 0$ and given time is T

$V = u\sin 30 - gT$

$0 = \frac{u}{2} - gT$

$T = \frac{u}{2g}$

Speed $= \sqrt{(V_x)^2 + (V_y)^2}$

$= \sqrt{\frac{3u^2}{4} + \left(\frac{u}{2} - \frac{2g}{3}\cdot\frac{u}{2g}\right)^2}$

$= \sqrt{\frac{3u^2}{4} + \frac{u^2}{36}}$

$= \frac{\sqrt{7}}{3}u$

6 A particle P is projected with speed u at an angle θ above the horizontal from a point O on a horizontal plane and moves freely under gravity. The direction of motion of P makes an angle α above the horizontal when P first reaches three-quarters of its greatest height.

(a) Show that $\tan\alpha = \frac{1}{2}\tan\theta$. [6]

(b) Given that $\tan\theta = \frac{4}{3}$, find the horizontal distance travelled by P when it first reaches three-quarters of its greatest height. Give your answer in terms of u and g. [4]

(a) Greatest hight $= H = \dfrac{u^2\sin^2\theta}{2g}$

Let v be the velocity of the particle at $\frac{3}{4}H$

Horizontal component of v at $\frac{3}{4}H$

$$v\cos\alpha = u\cos\theta \quad \text{—①}$$

Vertical component of v at $\frac{3}{4}H$

$$(v\sin\alpha)^2 = (u\sin\theta)^2 - 2g\left(\frac{3}{4}\left(\frac{u^2\sin^2\theta}{2g}\right)\right) \quad \text{—②}$$

By using Formula $v_f^2 = v_i^2 - 2aS$

$$(v\sin\alpha)^2 = u^2\sin^2\theta - \frac{3}{4}u^2\sin^2\theta$$

$$(v\sin\alpha)^2 = \frac{u^2\sin^2\theta}{4}$$

$$v\sin\alpha = \frac{1}{2}u\sin\theta \quad \text{—②}$$

from ① and ②

$$\frac{v\sin\alpha}{v\cos\alpha} = \frac{\frac{1}{2}u\sin\theta}{u\cos\theta}$$

$$\tan\alpha = \frac{1}{2}\tan\theta$$

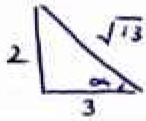
(b) Given $\tan\theta = \frac{4}{3}$

$\sin\theta = 4/5$

$\cos\theta = \frac{3}{5}$

$$\tan\alpha = \frac{1}{2}\tan\theta$$

$$\tan\alpha = \frac{2}{3}$$



At $\frac{3}{4}H$, vertical component of velocity

$$v\sin\alpha = u\sin\theta - gt$$

$$\frac{1}{2}u\sin\theta = u\sin\theta - gt \qquad \text{from } ②$$

$$gt = \frac{1}{2}u\sin\theta$$

$$= \frac{1}{2}u\cdot\frac{4}{5}$$

$$t = \frac{2u}{5g}$$

Horizontal distance at $\frac{3}{4}H = (u\cos\theta)t$

$$= u\left(\frac{3}{5}\right)\left(\frac{2u}{5g}\right)$$

$$= \frac{6u^2}{25g}$$

5 A particle P is projected with speed u at an angle α above the horizontal from a point O on a horizontal plane and moves freely under gravity. The horizontal and vertical displacements of P from O at a subsequent time t are denoted by x and y respectively.

(a) Derive the equation of the trajectory of P in the form

$$y = x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha.$$ [3]

The point Q is the highest point on the trajectory of P in the case where $\alpha = 45°$.

(b) Show that the x-coordinate of Q is $\frac{u^2}{2g}$. [3]

(c) Find the other value of α for which P would pass through the point Q. [4]

(a)
Let $P(x, y)$ be the point after time t

$x = (u\cos\alpha)t$ — ①

$y = (u\sin\alpha)t - \frac{1}{2}gt^2$ — ②

from ① $t = \frac{x}{u\cos\alpha}$

put in ②

$y = u\sin\alpha \cdot \frac{x}{u\cos\alpha} - \frac{1}{2}g \cdot \frac{x^2}{u^2\cos^2\alpha}$

$y = x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha$

(b)
Greatest hight $= \frac{u^2\sin^2\alpha}{2g}$ (given $\alpha = 45°$)

$= \frac{u^2 \cdot \frac{1}{2}}{2g} = \frac{u^2}{2g}$

At greatest hight $v = 0$

$\uparrow$ $0 = u\sin\alpha - gt$

$gt = u\sin\alpha$

$t = \frac{u\sin\alpha}{g}$

x - coordinate $= (u\cos\alpha)t$

$$= u\cos\alpha \cdot \frac{u\sin\alpha}{g}$$

$$= \frac{u^2}{2g} \quad \text{as } \alpha = 45^\circ$$

(c) $\left(\frac{u^2}{2g}, \frac{u^2}{4g}\right)$ be the greatest (stationary) point

$$y = x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha$$

$$\frac{u^2}{4g} = \frac{u^2}{2g}\tan\alpha - \frac{g}{2u^2} \cdot \frac{u^4}{4g^2}(1+\tan^2\alpha)$$

$$2 = 4\tan\alpha - 1 - \tan^2\alpha$$

$$\tan^2\alpha - 4\tan\alpha + 3 = 0$$

$\tan\alpha = 1$ $\qquad$ $\tan\alpha = 3$

$\alpha = 45^\circ$ $\qquad$ $\alpha = 71.6^\circ$