AS Level
Further
Mathematics

Topic: Rational Function

Teacher:

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Q.1 Oct/Nov/P11+P13/2022

7 The curve C has equation $y = \frac{5x^2}{5x-2}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the coordinates of the stationary points on C. [4]

(c) Sketch C. [3]

(d) Sketch the curve with equation $y = \left|\frac{5x^2}{5x-2}\right|$ and find in exact form the set of values of x for which $\left|\frac{5x^2}{5x-2}\right| < 2$. [6]

(a) Vertical asymptotes

$5x - 2 = 0$

$x = \frac{2}{5}$

oblique asymptotes

$$\begin{array}{r|l} & x + \frac{2}{5} \\ \hline 5x-2 & 5x^2 \\ & \underset{-}{}5x^2 \underset{+}{-} 2x \\ \hline & 2x \\ & \underset{-}{+} x \underset{+}{-} \frac{4}{5} \\ \hline & \frac{4}{5} \end{array}$$

$y = x + \frac{2}{5}$ is oblique asymptote

(b)

$$y = \frac{5x^2}{5x-2}$$

$$\frac{dy}{dx} = \frac{(5x-2)(10x) - (5x^2)(5)}{(5x-2)^2} = \frac{50x^2 - 20x - 25x^2}{(5x-2)^2} = \frac{25x^2 - 20x}{(5x-2)^2}$$

For stationary point put $\frac{dy}{dx} = 0$

$$\frac{25x^2 - 20x}{(5x-2)^2} = 0$$

$5x^2 - 4x = 0$

$x(5x - 4) = 0$

$x = 0$

$x = \frac{4}{5}$

$x = 0$	$x = \frac{4}{5}$
$y = 0$	$y = \frac{8}{5}$
$(0, 0)$	$(\frac{4}{5}, \frac{8}{5})$

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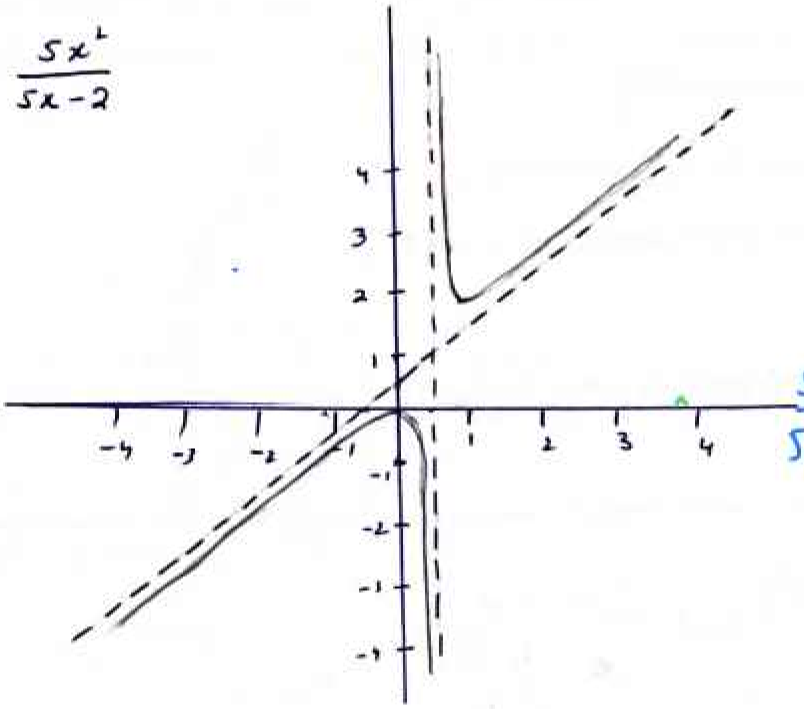
(c) $y = \frac{5x^2}{5x-2}$

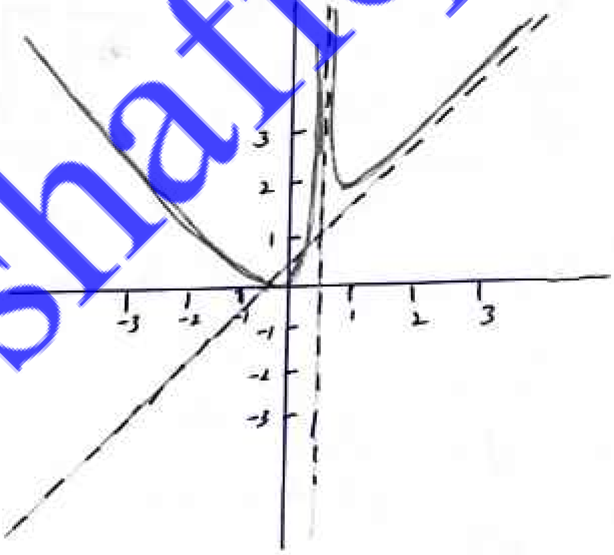


$\left| \frac{5x^2}{5x-2} \right| < 2$

$\left| \frac{5x^2}{5x-2} \right| = 2$

$\frac{5x^2}{5x-2} = 2$ or $\frac{5x^2}{5x-2} = -2$

(d) $y = \left| \frac{5x^2}{5x-2} \right|$



$\left| \frac{5x^2}{5x-2} \right| < 2$

$\frac{5x^2}{5x-2} = 2$ or $\frac{5x^2}{5x-2} = -2$

$5x^2 - 10x + 4 = 0$ or $5x^2 + 10x - 4 = 0$

$x = -1 \pm \frac{3}{5}\sqrt{5}$ or $x = 1 \pm \frac{1}{5}\sqrt{5}$

$\left| \frac{5x^2}{5x-2} \right| < 2$ when

$-1 - \frac{3}{5}\sqrt{5} < x < -1 + \frac{3}{5}\sqrt{5}$, or $1 - \frac{1}{5}\sqrt{5} < x < 1 + \frac{1}{5}\sqrt{5}$

1

1

2

$\frac{5x^2}{5x-2} =$

Q.2 Oct/Nov/P12/2022

7 The curve C has equation $y = \frac{x^2 - x}{x+1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the exact coordinates of the stationary points on C. [4]

(c) Sketch C, stating the coordinates of any intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\frac{x^2 - x}{x+1}\right|$ and find in exact form the set of values of x for which $\left|\frac{x^2 - x}{x+1}\right| < 6$. [5]

(a) Vertical asymptote $x+1=0$ $x=-1$

oblique asymptote

$$\begin{array}{r} x-2 \\ x+1 \overline{\big)\, x^2 - x} \\ \underline{\pm x^2 \pm x} \\ -2x \\ \underline{\mp 2x \mp 2} \\ 2 \end{array}$$

$y = x-2$ is oblique asymptote

(b) $\frac{dy}{dx} = \frac{(x+1)(2x-1)-(x^2-x)(1)}{(x+1)^2} = \frac{2x^2+x-1-x^2+x}{(x+1)^2} = \frac{x^2+2x-1}{(x+1)^2}$

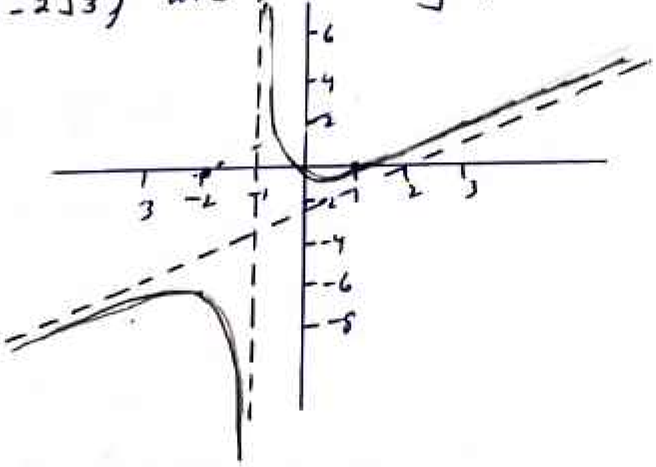
For stationary point put $\frac{dy}{dx}=0$

$x^2+2x=-1$

$x=-1+\sqrt{2}$ $x=-1-\sqrt{2}$

$y=-3+2\sqrt{2}$ $y=-3-2\sqrt{2}$

$(-1+\sqrt{2}, -3+2\sqrt{2})$, $(-1-\sqrt{2}, -3-2\sqrt{3})$ are stationary points



(c) For y-intercept put $x=0$

$y=0$

$(0,0)$

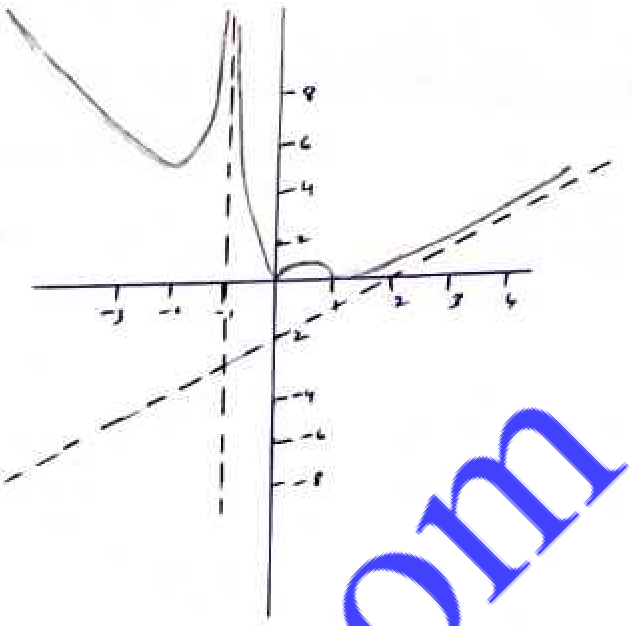
For x-intercept put $y=0$

$x^2-x=0$

$x=0$ $y=0$ $(0,0)$

$x=1$ $y=0$ $(1,0)$

(d) $y = \left| \frac{x^2 - x}{x+1} \right|$



$$\left| \frac{x^2 - x}{x+1} \right| < 6$$

$\frac{x^2 - x}{x+1} = 6$ or $\frac{x^2 - x}{x+1} = -6$

$x^2 - 7x - 6 = 0$ $\quad$ $x^2 + 5x + 6 = 0$

$x = \frac{7}{2} \pm \frac{1}{2}\sqrt{73}$ $\quad$ $x = -3, -2$

$\left| \frac{x^2 - x}{x+1} \right| < 6$ when

$-3 < x < -2$ and $\frac{7}{2} - \frac{1}{2}\sqrt{73} < x < \frac{7}{2} + \frac{1}{2}\sqrt{73}$

Q.3 May/June/p11/2022

The curve C has equation $y = \dfrac{2x^2-x-1}{x^2+x+1}$.

(a) Show that C has no vertical asymptotes and state the equation of the horizontal asymptote of C. [3]

(b) Find the coordinates of the stationary points on C. [4]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{2x^2-x-1}{x^2+x+1}\right|$ and state the set of values of k for which $\left|\dfrac{2x^2-x-1}{x^2+x+1}\right| = k$ has 4 distinct real solutions. [2]

(a) vertical asymptote

put $x^2+x+1=0$

$b^2-4ac = 1-4(1)(1) = -3 < 0$

This shows that C has no vertical asymptote

Horizontal asymptote

$y=2$

(b) $y = \dfrac{2x^2-x-1}{x^2+x+1}$

$$\frac{dy}{dx} = \frac{(x^2+x+1)(4x-1)-(2x^2-x-1)(2x+1)}{(x^2+x+1)^2}$$

$$= \frac{3x^2+6x}{(x^2+x+1)^2}$$

For stationary point put $\frac{dy}{dx}=0$

$\dfrac{3x^2+6x}{(x^2+x+1)^2} = 0 \Rightarrow 3x^2+6x=0$

$x=0$ $\quad$ $x=-2$

$y=-1$ $\quad$ $y=3$

$(0,-1)$ and $(-2,3)$ are the stationary points

(c) $y = \dfrac{2x^2 - x + 1}{x^2 + x + 1}$

For x-intercept put $y=0$

$\dfrac{2x^2 - x + 1}{x^2+x+1} = 0$

$2x^2 - x + 1 = 0$

$x = 1$ $\quad x = -\frac{1}{2}$

$y = 0$ $\quad y = 0$

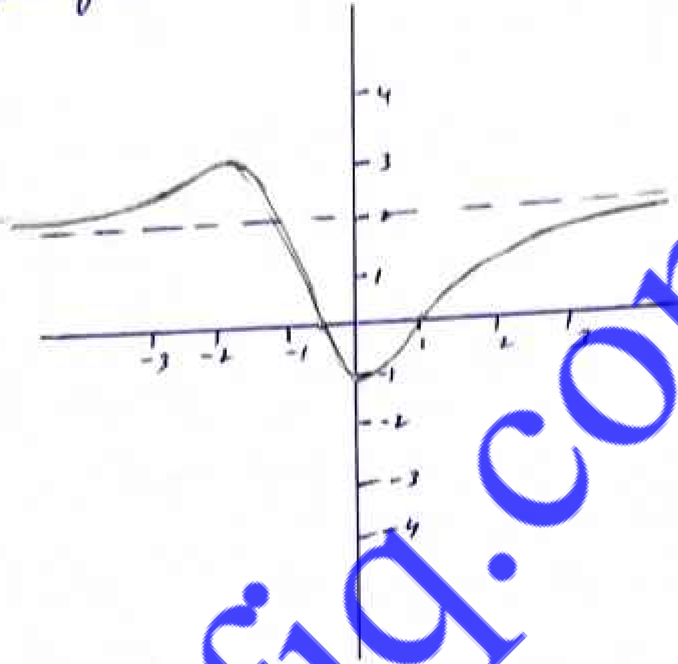
$(1, 0)$, $(-\frac{1}{2}, 0)$

are x-intercepts

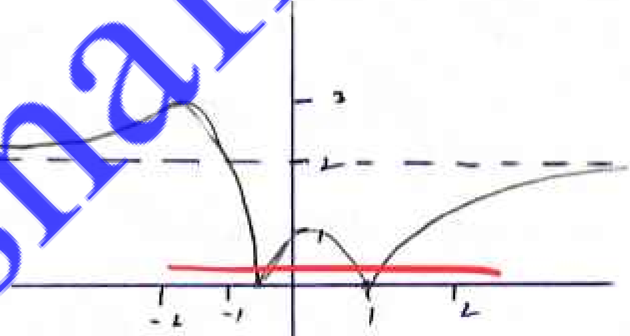
For y-intercept put $x=0$

$y = \dfrac{0 - 0 + 1}{0 + 0 + 1} = 1$

$(0, -1)$ is y-intercept



(d) $y = \left| \dfrac{2x^2 - x + 1}{x^2 + x + 1} \right|$



when $0 < K < 1$ Then

$\left| \dfrac{2x^2 - x - 1}{x^2 + x + 1} \right| = K$ has 4 distinct real solutions

Q.4 May/June/p13/2022

(a) Sketch the curve with equation $y = \frac{x+1}{x-1}$. **[2]**

(b) Sketch the curve with equation $y = \frac{|x|+1}{|x|-1}$ and find the set of values of x for which $\frac{|x|+1}{|x|-1} < -2$. **[4]**

(a) Vertical asymptote $x - 1 = 0$ $x = 1$

Horizontal asymptote $y = 1$

x-intercept put $y=0$

$$\frac{x+1}{x-1} = 0$$

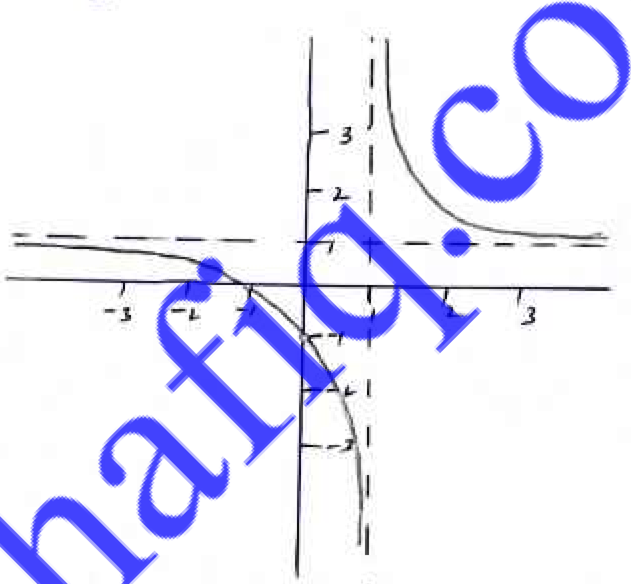
$x = -1$

$(-1, 0)$

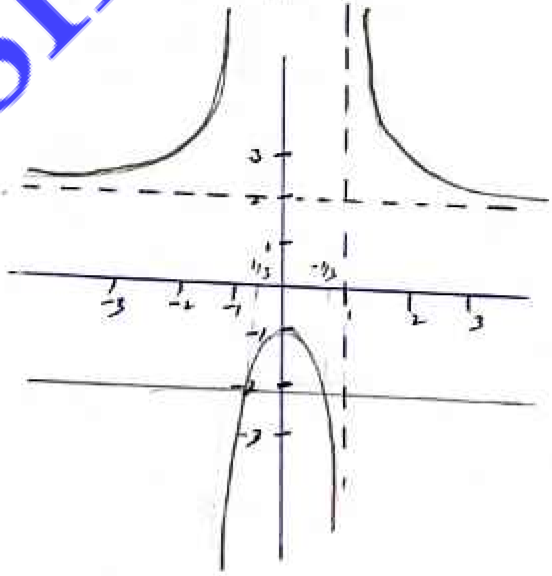
y-intercept put $x=0$

$y = -1$

$(0, -1)$



(b) $y = \frac{|x|+1}{|x|-1}$



$$\frac{|x|+1}{|x|-1} = -2$$

$|x| + 1 = -2|x| + 2$

$3|x| = 1$

$|x| = \frac{1}{3}$

So $\frac{|x|+1}{|x|-1} < -2$ when

$-1 < x < -\frac{1}{3}$, $\frac{1}{3} < x < 1$

Q.5 May/June/P11+P12/2021

The curve C has equation $y = \dfrac{x^2+x+9}{x+1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the coordinates of the stationary points on C. [4]

(c) Sketch C, stating the coordinates of any intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{x^2+x+9}{x+1}\right|$ and find the set of values of x for which $2|x^2+x+9| > 13|x+1|$. [5]

(a) vertical asymptote $x+1=0$ $\boxed{x=-1}$

oblique asymptote

$$\begin{array}{r|l} & x \\ x+1 & x^2+x+9 \\ & -\,x^2+x \\ \hline & 9 \end{array}$$

$\boxed{y=x}$

(b) $y = \dfrac{x^2+x+9}{x+1}$

$$\frac{dy}{dx} = \frac{(x+1)(2x+1)-(x^2+x+9)(1)}{(x+1)^2} = \frac{2x^2+3x+1-x^2-x-9}{(x+1)^2} = \frac{x^2+2x-8}{(x+1)^2}$$

For stationary points put $\frac{dy}{dx}=0$

$\dfrac{x^2+2x-8}{(x+1)^2}=0$ $x^2+2x-8=0 \Rightarrow (x+1)^2=9$

$x=2$ $x=-4$

$y=5$ $y=-7$

$(2,5)$ and $(-4,-7)$ are the stationary points

(c) For x-intercept put $y=0$

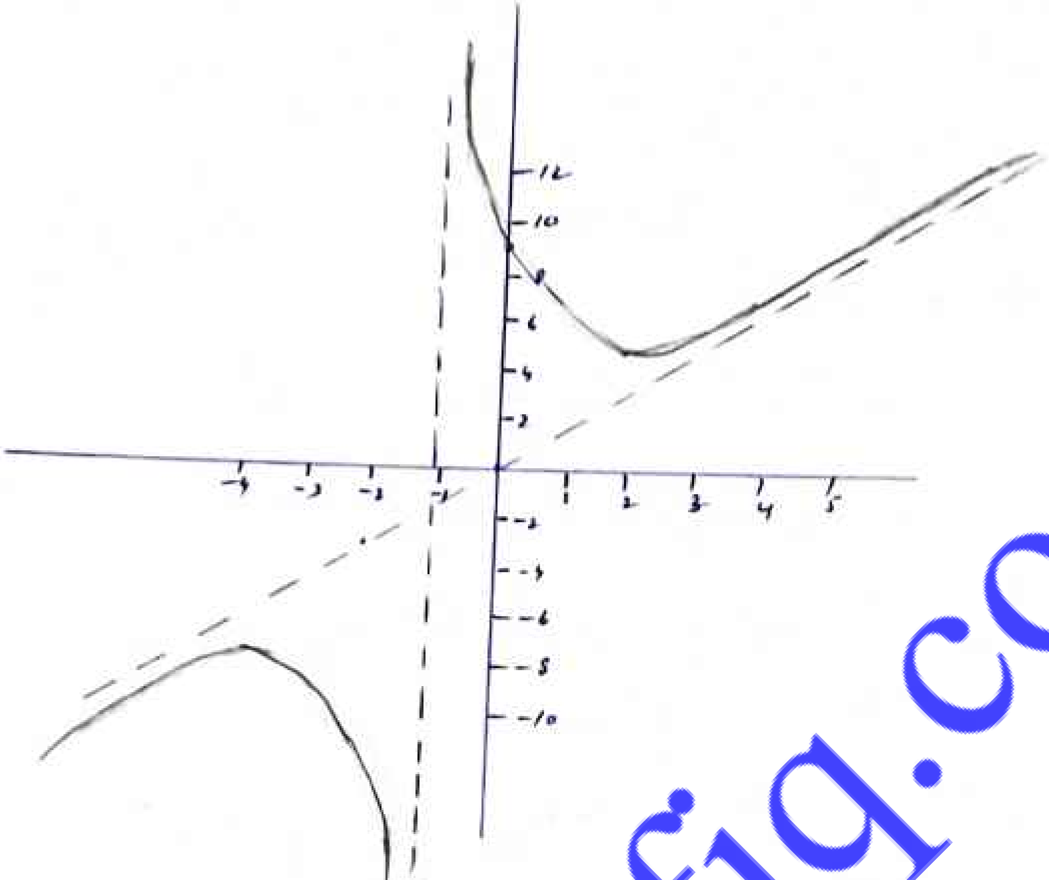
$\dfrac{x^2+x+9}{x+1}=0$ $x^2+x+9=0$

$b^2-4ac = 1-36 = -35<0$ (No x-intercept)

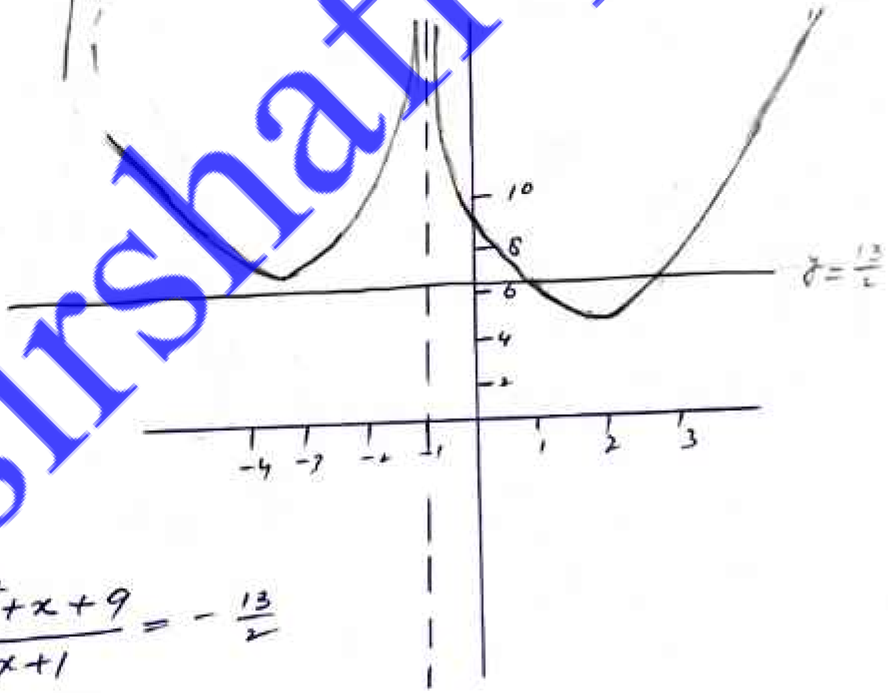
For y-intercept put $x=0$

$y = \dfrac{0+0+9}{0+1} = 9$

$(0,9)$ is the y-intercept



(d)



$$\frac{|x^2+x+9|}{|x+1|} < \frac{13}{2}$$

$$\frac{x^2+x+9}{x+1} = \frac{13}{2} \quad \text{or} \quad \frac{x^2+x+9}{x+1} = -\frac{13}{2}$$

$$x^2 - \frac{11}{2}x + \frac{5}{2} = 0 \quad \text{or} \quad x^2 + \frac{15}{2}x + \frac{31}{2} = 0$$

$$x = \frac{1}{2}, 5$$

No solution

$2|x^2+x+9| > 13|x+1|$ when

$x < \frac{1}{2}$ and $x > 5$

Q.6 May/June/P13/2021

The curve C has equation $y = \dfrac{x^2 - x - 3}{1 + x - x^2}$.

(a) Find the equations of the asymptotes of C. [2]

(b) Find the coordinates of any stationary points on C. [3]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{x^2 - x - 3}{1 + x - x^2}\right|$ and find in exact form the set of values of x for which $\left|\dfrac{x^2 - x - 3}{1 + x - x^2}\right| < 3$. [6]

(a) Vertical asymptote $1 + x - x^2 = 0$

$x = \frac{1-\sqrt{5}}{2}$, $x = \frac{1+\sqrt{5}}{2}$ are Horizontal asymptote

Horizontal asymptote $y = -1$

(b) $y = \dfrac{x^2 - x - 3}{1 + x - x^2}$

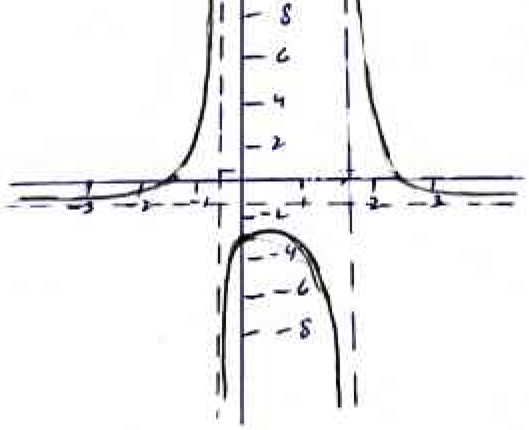
$$\frac{dy}{dx} = \frac{(1+x-x^2)(2x-1) - (x^2-x-3)(1-2x)}{(1+x-x^2)^2} = \frac{-2(2x-1)}{(1+x-x)^2}$$

For stationary points put $\frac{dy}{dx} = 0$

$-2(2x-1) = 0$

$x = \frac{1}{2}$

$y = -\frac{13}{5}$, So $\left(\frac{1}{2}, -\frac{13}{5}\right)$ is stationary point



(c) x-intercept

$\frac{x^2 - x - 3}{1 + x - x^2} = 0$

$x^2 - x - 3 = 0$

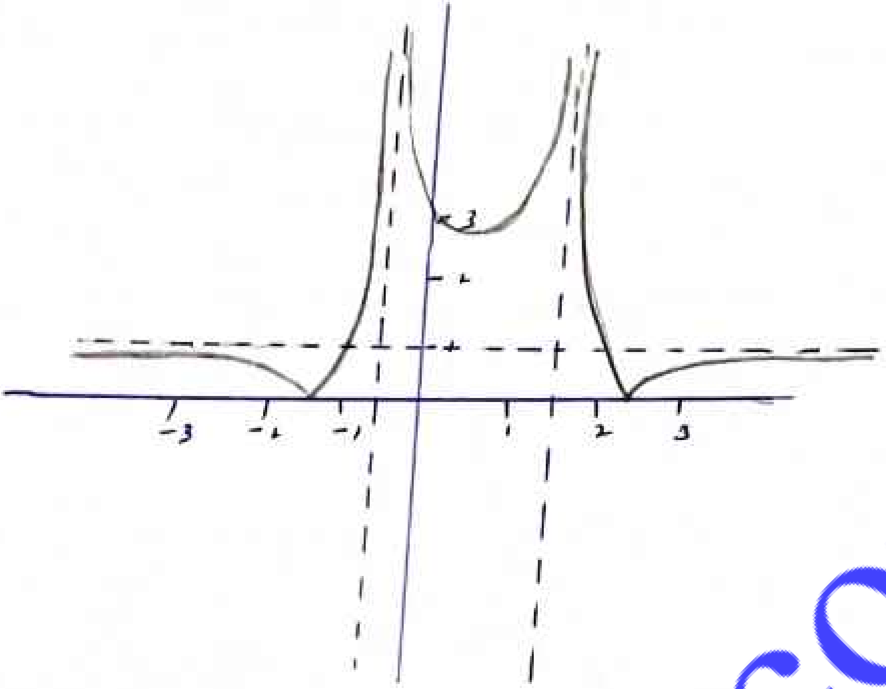
$x = \frac{1}{2} \pm \frac{1}{2}\sqrt{13}$

$\left(\frac{1}{2} + \frac{1}{2}\sqrt{13}, 0\right), \left(\frac{1}{2} - \frac{1}{2}\sqrt{13}, 0\right)$

y-intercept

$(0, -3)$

(d)



$$\left|\frac{x^2-x-3}{1+x-x^2}\right|<3$$

$\frac{x^2-x-3}{1+x-x^2}=3$ or $\frac{x^2-x-3}{1+x-x^2}=-3$

$4x^2-4x-6=0$ or $-2x^2+2x=0$

$x=\frac{1}{2}\pm\frac{1}{2}\sqrt{7}$ $x=0, x=1$

$\left|\frac{x^2-x-3}{1+x-x^2}\right|<3$ when

$x<\frac{1}{2}-\frac{1}{2}\sqrt{7}$, $0<x<1$, $x>\frac{1}{2}+\frac{1}{2}\sqrt{7}$

Q.7 Oct/Nov/P11+P13/2021

The curve C has equation $y = \dfrac{4x+5}{4-4x^2}$.

(a) Find the equations of the asymptotes of C. [2]

(b) Find the coordinates of any stationary points on C. [4]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{4x+5}{4-4x^2}\right|$ and find in exact form the set of values of x for which $4|4x+5| > 5|4-4x^2|$. [6]

(a) Vertical Asymptote

$4-4x^2=0$

$x^2=1$

$x=\pm 1$ are vertical asymptotes

Horizontal Asymptote

$y=0$

(b) $y = \dfrac{4x+5}{4-4x^2}$

$\dfrac{dy}{dx} = \dfrac{(4-4x^2)(4)-(4x+5)(-8x)}{(4-4x^2)^2} = \dfrac{16x^2+40x+16}{(4-4x^2)^2}$

For stationary points put $\dfrac{dy}{dx}=0$

$\dfrac{16x^2+40x+16}{(4-4x^2)^2}=0$

$16x^2+40x+16=0$

$x=-2$ $\qquad$ $x=-\frac{1}{2}$

$y=\frac{1}{4}$ $\qquad$ $y=1$

$(-2,\frac{1}{4})$, $(-\frac{1}{2},1)$ are stationary points

(c) x-intercepts put $y=0$

$$\frac{4x+5}{4-4x^2}=0$$

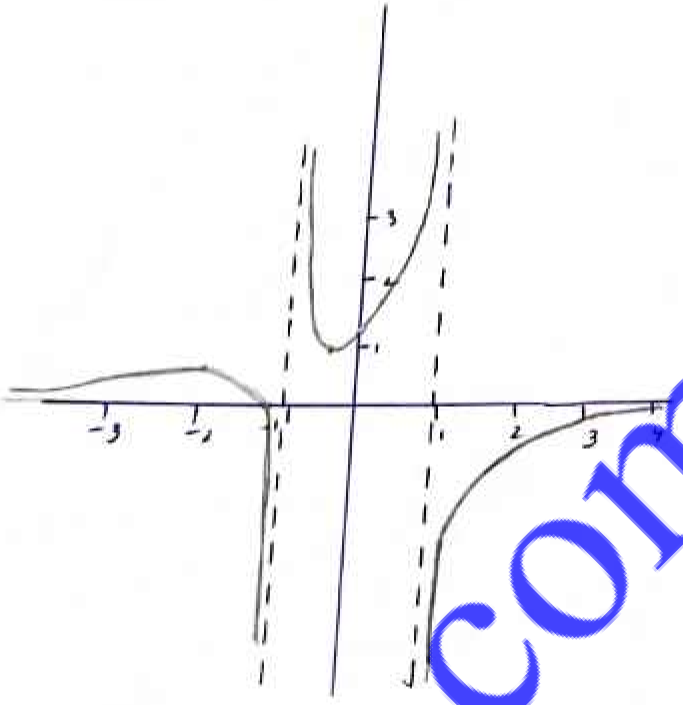
$$x=-\frac{5}{4}$$

$$\left(-\frac{5}{4},0\right)$$

y-intercept put $x=0$

$$y=\frac{5}{4}$$

$$\left(0,\frac{5}{4}\right)$$



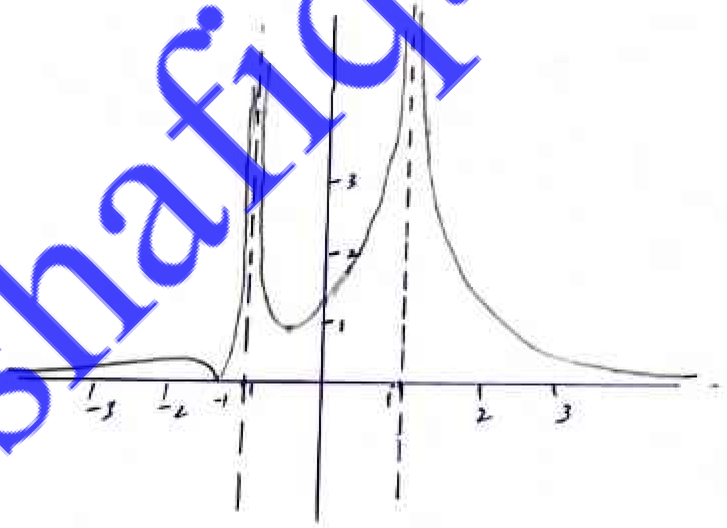
(d) $4|4x+5|>5|4-4x^2|$

$$\left|\frac{4x+5}{4-4x^2}\right|>\frac{5}{4}$$

$$\frac{4x+5}{4-4x^2}=\frac{5}{4} \quad \text{or} \quad \frac{4x+5}{4-4x^2}=-\frac{5}{4}$$

$$5x^2+4x=0 \quad \text{or} \quad 5x^2-4x-10=0$$

$$x=-\frac{4}{5},\ x=0 \quad \text{or} \quad x=\frac{2}{5}\pm\frac{3}{5}\sqrt{6}$$



$4|4x+5|>5|4-4x^2|$ when

$$\frac{2}{5}-\frac{3}{5}\sqrt{6}<x<-\frac{4}{5}\ , \quad 0<x<\frac{2}{5}+\frac{3}{5}\sqrt{6}$$

Q.8 Oct/Nov/P12/2021

The curve C has equation $y = \frac{x^2}{x-3}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that there is no point on C for which $0 < y < 12$. [4]

(c) Sketch C. [2]

(d) (i) Sketch the graphs of $y = \left|\frac{x^2}{x-3}\right|$ and $y = |x| - 3$ on a single diagram, stating the coordinates of the intersections with the axes. [4]

(a) vertical Asymptote

$x - 3 = 0$ $\boxed{x = 3}$

oblique Asymptote

$$\begin{array}{r|l} & x+3 \\ \hline x-3 & x^2 \\ & -x^2 \overset{+}{-} 3x \\ \hline & 3x \\ & 3x - 9 \\ \hline & 9 \end{array}$$

$\boxed{y = 3 + x}$ is oblique Asymptote

(b) $y = \frac{x^2}{x-3}$

$x^2 = y(x-3)$

$x^2 - yx + 3y = 0$

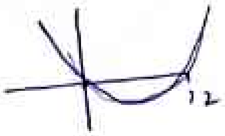
$b^2 - 4ac < 0$

$y^2 - 4(1)(3y) < 0$

$y^2 - 12y < 0$

$y^2 - 12y = 0$

$y = 0$, $y = 12$



$\Rightarrow$ $0 < y < 12$

(c) x - intercept put y = 0

$\frac{x^2}{x-3} = 0$

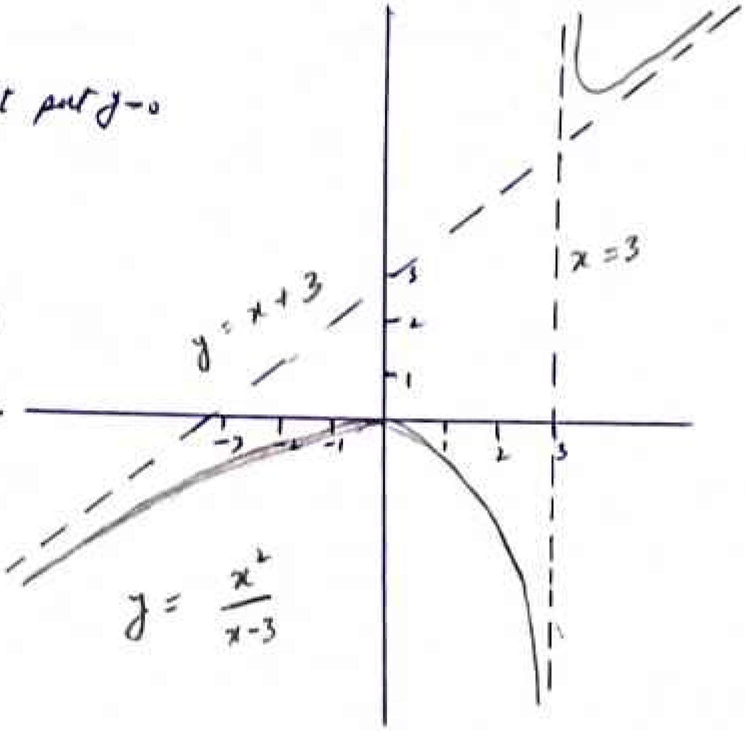
$x^2 - x + 3 = 0$

No x - intercept

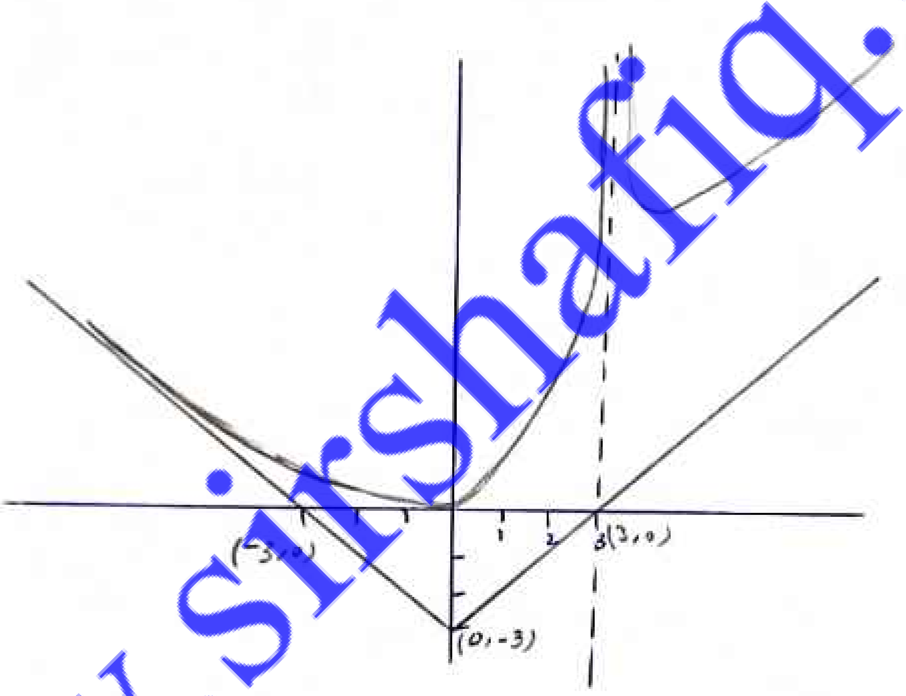
y = intercept put x = 0

$y = \frac{0}{0-3} = 0$

(0, 0)



(d)



Q.9 May/June/P11+P12/2020

Let a be a positive constant.

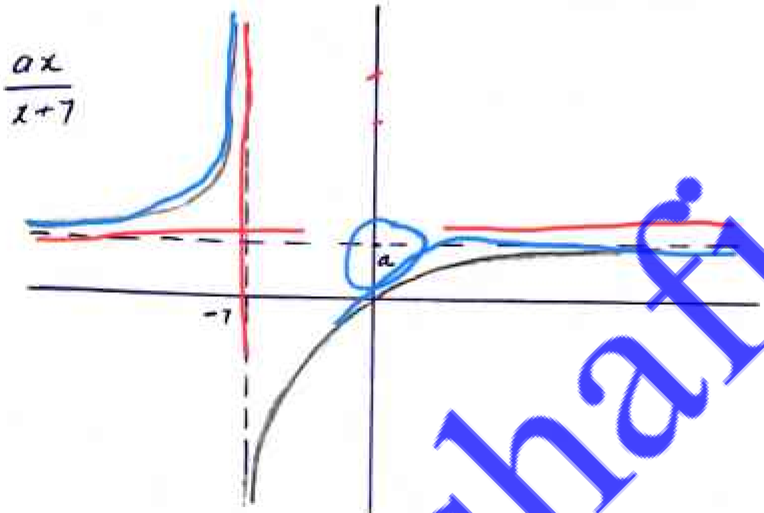
(a) Sketch the curve with equation $y = \frac{ax}{x+7}$. [2]

(b) Sketch the curve with equation $y = \left|\frac{ax}{x+7}\right|$ and find the set of values of x for which $\left|\frac{ax}{x+7}\right| > \frac{a}{2}$. [4]

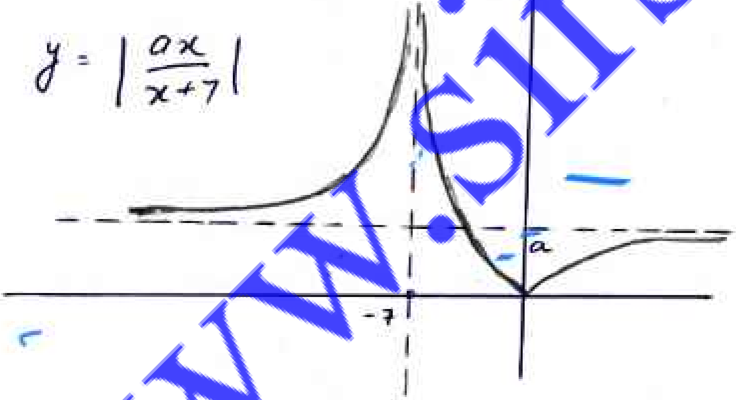
(a) vertical Asymptote $x + 7 = 0$ $\boxed{x = -7}$

Horizontal Asymptote $\boxed{y = a}$

$y = \frac{ax}{x+7}$



$y = \left|\frac{ax}{x+7}\right|$



$$\left|\frac{ax}{x+7}\right| > \frac{a}{2}$$

$$\frac{ax}{x+7} = \frac{a}{2} \quad \text{or} \quad \frac{ax}{x+7} = -\frac{a}{2}$$

$$x = 7 \quad \text{or} \quad x = -\frac{7}{3}$$

$\left|\frac{ax}{x+7}\right| > \frac{a}{2}$ when

$x < -7$, $-7 < x < -\frac{7}{3}$, $x > 7$

Q.10 May/June/P13/2020

The curve C has equation $y = \dfrac{10+x-2x^2}{2x-3}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that C has no turning points. [3]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

(a) Vertical Asymptote $2x-3=0$ $\boxed{x=\frac{3}{2}}$

oblique Asymptote

$$\begin{array}{r|l} & -x-1 \\ 2x-3 & -2x^2+x+10 \\ & \underset{+}{-}2x^2\underset{-}{+}3x \\ \hline & -2x+10 \\ & \underset{+}{-}2x\underset{-}{+}3 \\ \hline & 7 \end{array}$$

$\boxed{y=-x-1}$ is oblique Asymptote

(b) $$y = \frac{-2x^2+x+10}{2x-3}$$

$$\frac{dy}{dx} = \frac{(2x-3)(-4x+1)-(-2x^2+x+10)(2)}{(2x-3)^2} = \frac{4x^2-12x+23}{(2x-3)^2}$$

For stationary point put $\frac{dy}{dx}=0$

$$\frac{4x^2-12x+23}{(2x-3)^2}=0$$

$$4x^2-12x+23=0$$

$$b^2-4ac=(-12)^2-4(4)(23)=-224<0$$

This shows that there is no Turning point

(c) y-intercept put $x=0$

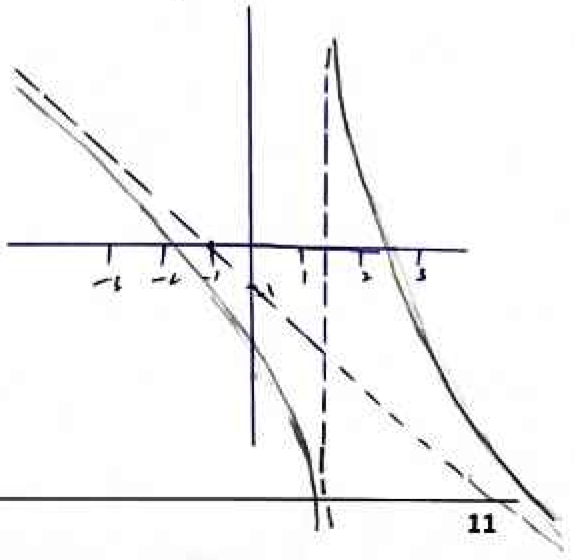
$y=-\frac{10}{3}$ $\left(0,-\frac{10}{3}\right)$

x-intercept put $y=0$

$-2x^2+x+10=0$

$x=-2$ $x=\frac{5}{2}$ $(-2,0),\left(\frac{5}{2},0\right)$

$y=0$ $y=0$



Q.11 Oct/Nov/P11/2020

The curve C has equation $y = \dfrac{x^2+x-1}{x-1}$.

(a) Find the equations of the asymptotes of C. **[3]**

(b) Show that there is no point on C for which $1 < y < 5$. **[4]**

(c) Find the coordinates of the intersections of C with the axes, and sketch C. **[3]**

(a) vertical Asymptote $x-1=0$ $\boxed{x=1}$

oblique Asymptote

$$\begin{array}{r|l} & x+2 \\ \hline x-1 & x^2+x-1 \\ & x^2-x \\ \hline & 2x-1 \\ & 2x-2 \\ \hline & 3 \end{array}$$

$\boxed{y = x+2}$ is oblique Asymptote

(b) $y(x-1) = x^2+x-1$

$x^2+(1-y)x+(y-1)=0$

$b^2-4ac<0$

$(1-y)^2-4(1)(y-1)<0$

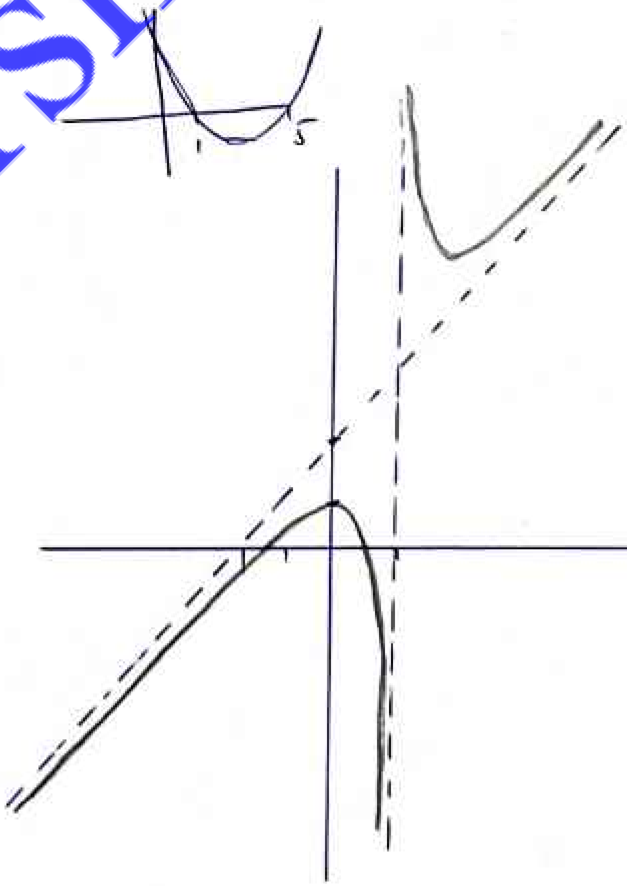
$y^2-6y+5<0$

$y^2-6y+5=0$

$y=1 \ , \ y=5$

$y^2-6y+5<0$ is true for

$1<y<5$



(c) y-intercept

$x=0$
$y=1$ $(0,1)$

x-intercept

$x^2+x-1=0$

$x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$

$(-\frac{1}{2}+\frac{1}{2}\sqrt{5},0)$, $(-\frac{1}{2}-\frac{1}{2}\sqrt{5},0)$

Q.11 Oct/Nov/P12/2020

Let a be a positive constant.

(a) The curve C_1 has equation $y = \frac{x-a}{x-2a}$. [2]

Sketch C_1.

The curve C_2 has equation $y = \left(\frac{x-a}{x-2a}\right)^2$. The curve C_3 has equation $y = \left|\frac{x-a}{x-2a}\right|$.

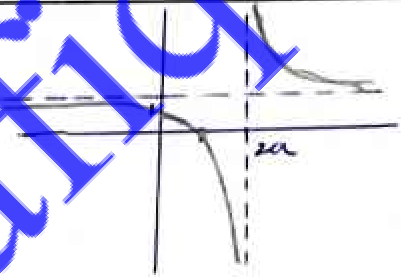
(b) (i) Find the coordinates of any stationary points of C_2. [3]

(ii) Find also the coordinates of any points of intersection of C_2 and C_3. [3]

(c) Sketch C_2 and C_3 on a single diagram, clearly identifying each curve. Hence find the set of values of x for which $\left(\frac{x-a}{x-2a}\right)^2 \leqslant \left|\frac{x-a}{x-2a}\right|$. [5]

(a) vertical Asymptote $x - 2a = 0$ $\boxed{x = 2a}$

Horizontal Asymptote $\boxed{y = 1}$



(b) (i) $y = \left(\frac{x-a}{x-2a}\right)^2$

$$\frac{dy}{dx} = 2\left(\frac{x-a}{x-2a}\right) \cdot \frac{x-2a-x+a}{(x-2a)^2} = -\frac{2a(x-a)}{(x-2a)^3}$$

put $\frac{dy}{dx} = 0$

$$\frac{-2a(x-a)}{x-2a} = 0$$

$x = a$

$y = 0$ $(a, 0)$ is Stationary point

(ii) $\left(\frac{x-a}{x-2a}\right)^2 = \left|\frac{x-a}{x-2a}\right|$

$\left(\frac{x-a}{x-2a}\right)^2 = \frac{x-a}{x-2a}$ or $\left(\frac{x-a}{x-2a}\right)^2 = -\frac{x-a}{x-2a}$

$(x-a)^2 = (x-a)(x-2a)$ or $(x-a)^2 = -(x-a)(x-2a)$

$(x-a)[\cancel{x}-a-\cancel{x}+2a] = 0$ $(x-a)[x-a+x-2a)] = 0$

$x = a$ $x = a$, $x = \frac{3}{2}a$

$y = 0$

when $x = a$ $\qquad$ $x = \frac{3}{2}a$

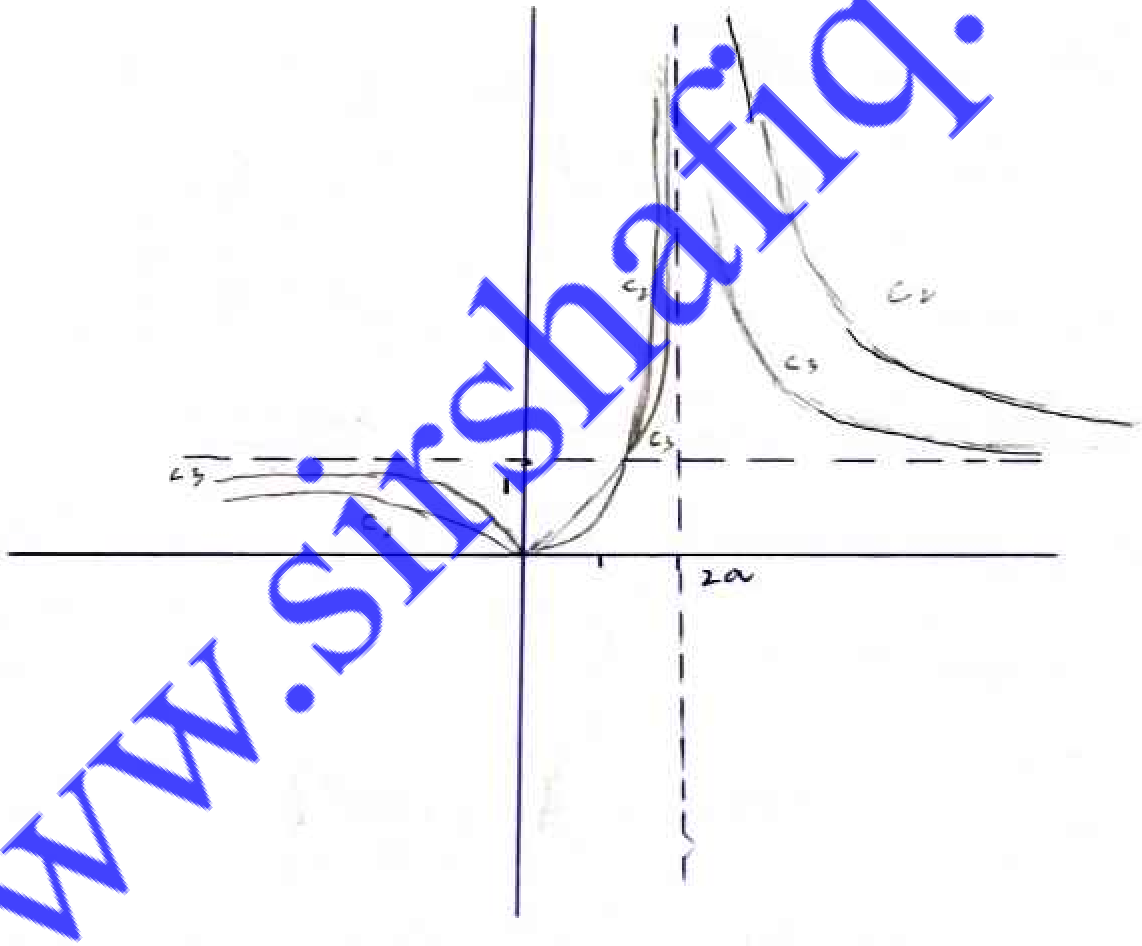
$y = 0$ $\qquad$ $y = 1$

$(a, 0)$ and $(\frac{3}{2}a, 1)$

(c) C_2 $\quad y = \left(\frac{x-a}{x-2a}\right)^2$

Vertical Asymptote $x = 2a$

Horizontal Asymptote $y = 1$



Q.13 Oct/Nov/P13/2020

The curve C has equation $y = \dfrac{x^2+x-1}{x-1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that there is no point on C for which $1 < y < 5$. [4]

(c) Find the coordinates of the intersections of C with the axes, and sketch C. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{x^2+x-1}{x-1}\right|$. [2]

(a)

$$y = \frac{x^2+x-1}{x-1}$$

Vertical Asymptote $x-1=0$ $\boxed{x=1}$

Oblique Asymptote

$$x-1\,\overline{\smash{)}\,x^2+x-1}\quad \text{quotient } x+2$$

x^2-x

$2x-1$

$2x-2$

1

$\boxed{y=x+2}$ is oblique Asymptote

(b)

$$y = \frac{x^2+x-1}{x-1}$$

$$(x-1)y = x^2+x-1$$

$$x^2+(1-y)x+(y-1)=0$$

This equation has no solution if

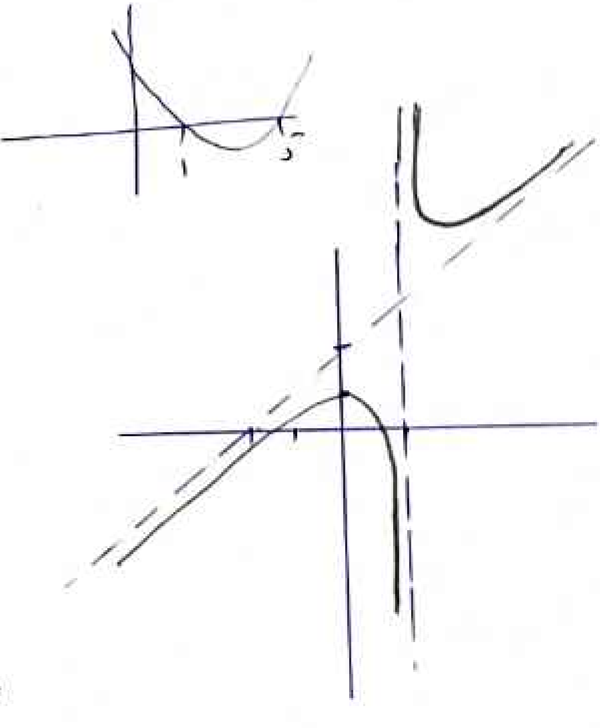
$$b^2-4ac<0$$

$$(1-y)^2-4(1)(y-1)<0$$

$$y^2-6y+5<0$$

$$y^2-6y+5=0 \Rightarrow y=1,\ y=5$$

$$1<y<5$$



(c)

x-intercept

$$\frac{x^2+x-1}{x-1}=0$$

$$x^2+x-1=0$$

$$x=-\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{5}$$

y-intercept

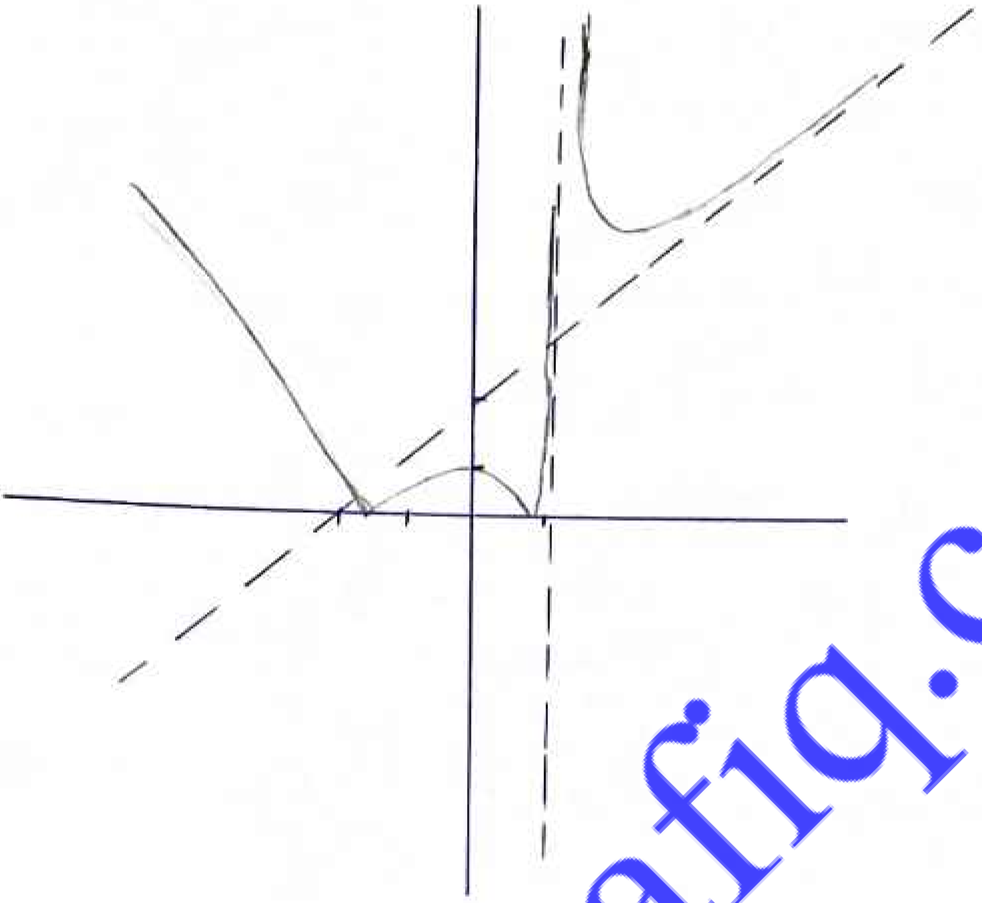
$x=0$

$y=1$

$(0,1)$

$(-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{5},0),\ (-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{5},0)$

ⓒ



Q.14 May/June/P11+P12/2019

The curves C_1 and C_2 have equations

$$y = \frac{ax}{x+5} \quad \text{and} \quad y = \frac{x^2 + (a+10)x + 5a + 26}{x+5}$$

respectively, where a is a constant and $a > 2$.

(i) Find the equations of the asymptotes of C_1. [2]

(ii) Find the equation of the oblique asymptote of C_2. [2]

(iii) Show that C_1 and C_2 do not intersect. [2]

(iv) Find the coordinates of the stationary points of C_2. [3]

(v) Sketch C_1 and C_2 on a single diagram. [You do not need to calculate the coordinates of any points where C_2 crosses the axes.] [3]

(i) vertical Asymptote $x = -5$

Horizontal Asymptote $y = a$

(ii)

$$\begin{array}{r|l} & x + a + 5 \\ \hline x+5 & x^2 + (a+10)x + 5a + 26 \\ & -x^2 - 5x \\ \hline & (a+5)x + 5a + 26 \\ & (a+5)x + 5a + 25 \\ \hline & 1 \end{array}$$

$y = x + a + 5$ oblique Asymptote of C_2

(iii) $\frac{ax}{x+5} = \frac{x^2 + (a+10)x + 5a + 26}{x+5}$

$x^2 + 10x + 5a + 26 = 0$

$b^2 - 4ac = (10)^2 - 4(1)(5a+26)$

$= -4 - 20a < 0$ as $a > 2$

This shows that c_1 and c_2 do not intersect

(iv) $$y = \frac{x^2 + (a+10)x + 5a + 26}{(x+5)}$$

$$\frac{dy}{dx} = \frac{(x+5)(2x + a + 10) - (x^2 + (a+10)x + 5a + 26)(1)}{(x+5)^2}$$

$$= \frac{x^2 + 10x + 24}{(x+5)^2}$$

For stationary points put $\frac{dy}{dx} = 0$

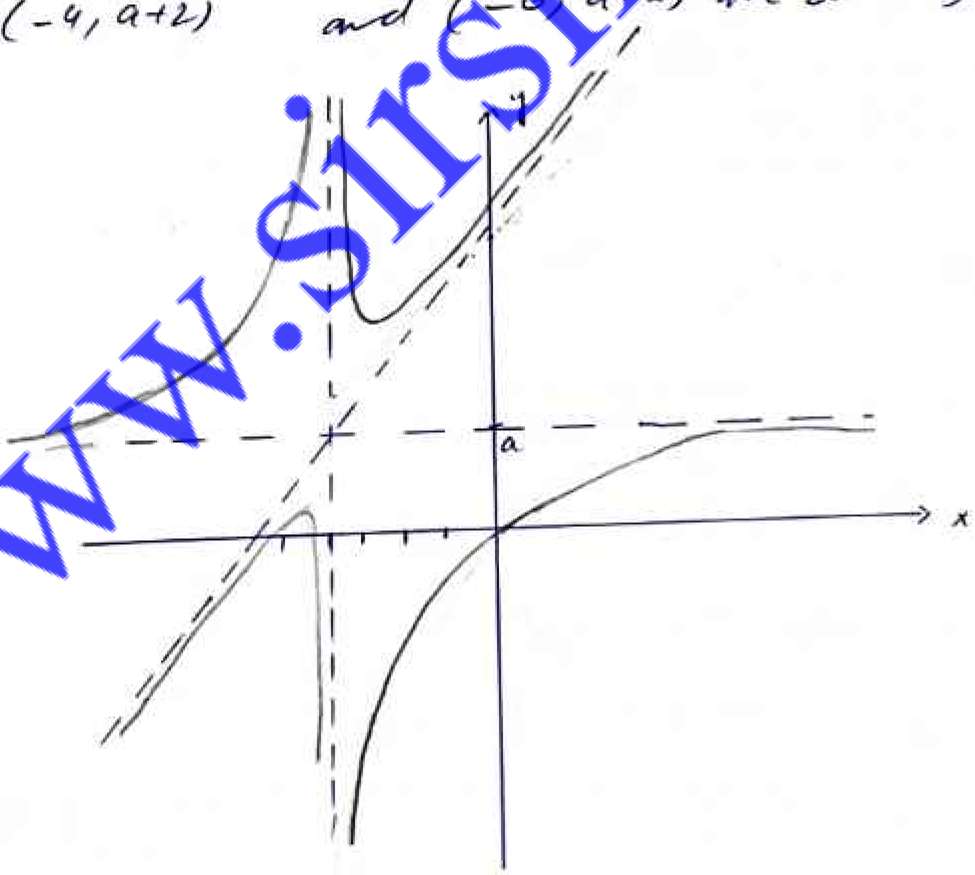
$$\frac{x^2 + 10x + 24}{(x+5)^2} = 0$$

$$x^2 + 10x + 24 = 0$$

$x = -4$ $\quad$ $x = -6$

$y = a+2$ $\quad$ $y = a-2$

$(-4, a+2)$ and $(-6, a-2)$ are stationary points



Q.15 May/June/P13/2019

The curve C has equation

$$y = \frac{x^2}{kx-1},$$

where k is a positive constant.

(i) Obtain the equations of the asymptotes of C. [3]

(ii) Find the coordinates of the stationary points of C. [3]

(iii) Sketch C. [3]

(i) Vertical Asymptote $x = \frac{1}{K}$

oblique Asymptote

$$\begin{array}{r l}
 & \frac{1}{K}x + \frac{1}{K^2} \\
Kx-1 \Big) & x^2 \\
 & x^2 - \frac{1}{K}x \\
\hline
 & \frac{1}{K}x \\
 & \frac{1}{K}x - \frac{1}{K^2} \\
\hline
 & \frac{1}{K^2}
\end{array}$$

$y = K^{-1}x + K^{-2}$ is oblique asymptote

(ii)

$$y = \frac{x^2}{(Kx-1)}$$

$$\frac{dy}{dx} = \frac{(Kx-1)(2x) - x^2(K)}{(Kx-1)^2} = \frac{Kx^2-2x}{(Kx-1)^2}$$

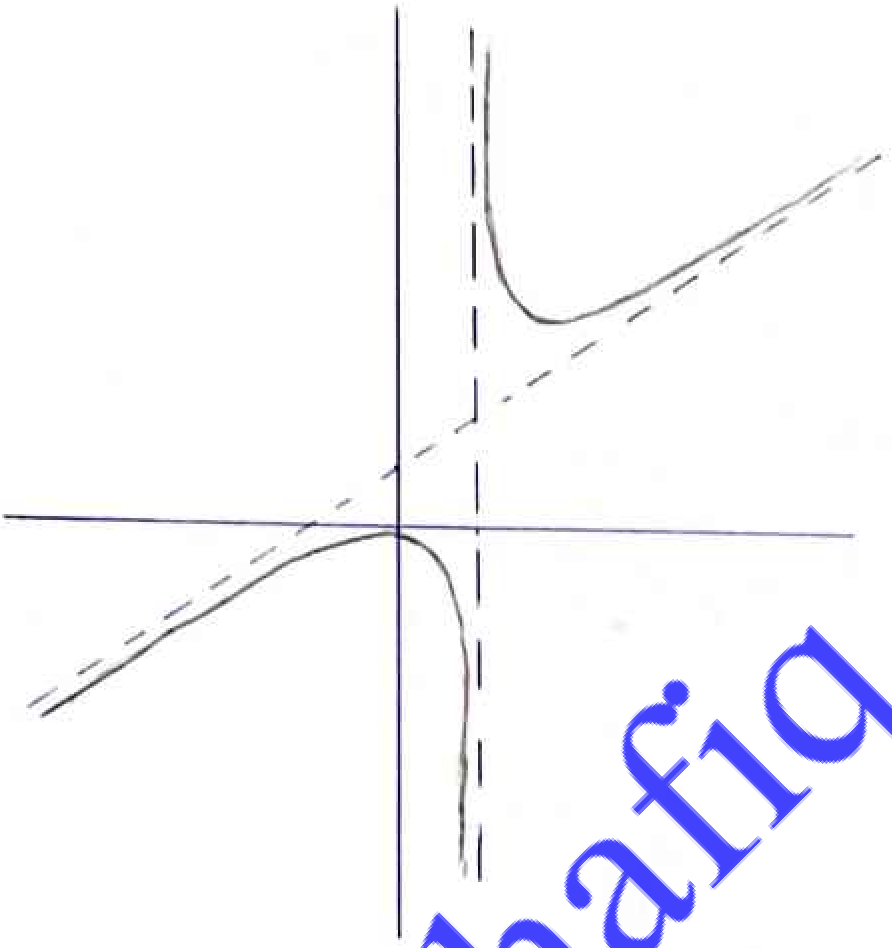
put $\frac{dy}{dx} = 0$

$Kx^2 - 2x = 0$

$x = 0$ $\quad$ $x = 2K^{-1}$

$y = 0$ $\quad$ $y = 4K^{-2}$

$(0,0)$, $(2K^{-1}, 4K^{-2})$ are stationary points



Q.16 Oct/Nov/P11+P12+P13/2019

The line $y = 2x + 1$ is an asymptote of the curve C with equation

$$y = \frac{x^2 + 1}{ax + b}.$$

(i) Find the values of the constants a and b. [3]

(ii) State the equation of the other asymptote of C. [1]

(iii) Sketch C. [Your sketch should indicate the coordinates of any points of intersection with the y-axis. You do not need to find the coordinates of any stationary points.] [3]

(i)

$$\begin{array}{r|l} & a^{-1}x - a^{-2}b \\ \hline ax+b & x^2 + 1 \\ & x^2 + a^{-1}bx \\ \hline & -a^{-1}bx + 1 \\ & -a^{-1}bx - a^{-2}b \\ \hline & a^{2}b + 1 \end{array}$$

$y = a^{-1}x - a^{-2}b$ is oblique Asymptote

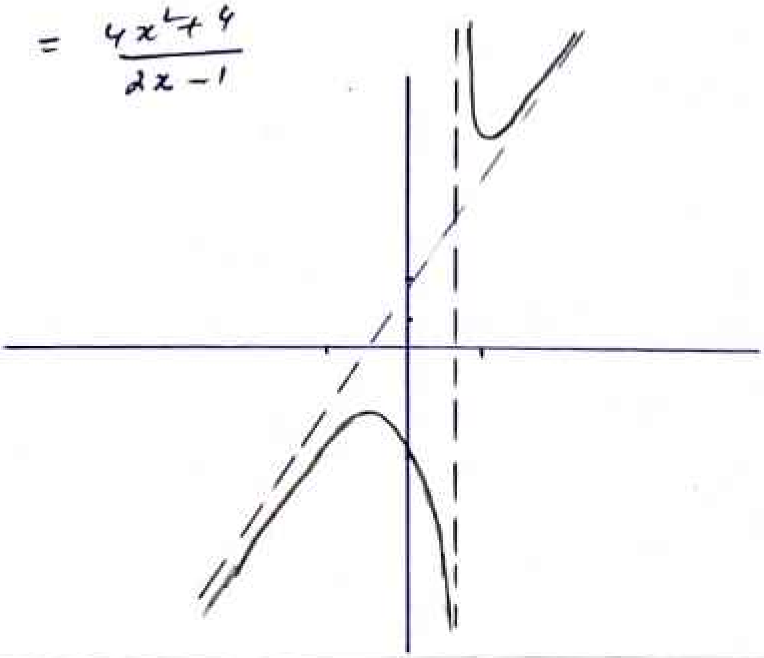
$a^{-1} = 2$

$\frac{1}{a} = 2 \Rightarrow a = \frac{1}{2}$

$-\frac{b}{a^2} = 1 \Rightarrow b = -\frac{1}{4}$

(ii) vertical Asymptote $x = -\frac{b}{a} = \frac{1}{2}$ $\boxed{x = \frac{1}{2}}$

$y = \frac{x^2+1}{\frac{1}{2}x - \frac{1}{4}} = \frac{4x^2+4}{2x-1}$



x-intercept

$\frac{4x^2+4}{2x-1} = 0$

$4x^2+4=0$ no x-intercept

y-intercept

$x=0$

$y=-4$

$(0,-4)$

Q.17 Oct/Nov/P11+P12+P13/2019

The line $y = 2x + 1$ is an asymptote of the curve C with equation

$$y = \frac{x^2 + 1}{ax + b}.$$

(i) Find the values of the constants a and b. [3]

(ii) State the equation of the other asymptote of C. [1]

(iii) Sketch C. [Your sketch should indicate the coordinates of any points of intersection with the y-axis. You do not need to find the coordinates of any stationary points.] [3]

Q.18 May/June/P11+P12/2018

The curve C has equation

$$y = \frac{x^2 + b}{x + b},$$

where b is a positive constant.

(i) Find the equations of the asymptotes of C. [3]

(ii) Show that C does not intersect the x-axis. [1]

(iii) Justifying your answer, find the number of stationary points on C. [2]

(iv) Sketch C. Your sketch should indicate the coordinates of any points of intersection with the y-axis. You do not need to find the coordinates of any stationary points. [3]

(i) vertical Asymptote $x + b = 0$ $\boxed{x = -b}$

oblique Asymptote

$$\begin{array}{r|l} & x - b \\ \hline x+b & x^2 + b \\ & -x^2 \quad -bx \\ \hline & -bx + b \\ & -bx - b \\ \hline & 2b \end{array}$$

$y = x - b$ is oblique Asymptote

(ii) For x-intercept

$$\frac{x^2+b}{x+b} = 0$$

$$x^2 + b = 0$$

$b^2 - 4ac = 0 - 4(1)(b) < 0$ as b is +ve

This show that there is no x-intercept

(iii) $y = \dfrac{x^2+b}{x+b}$

$$\frac{dy}{dx} = \frac{(x+b)(2x) - (x^2+b)(1)}{(x+b)^2} = \frac{x^2 + 2bx - b}{(x+b)^2}$$

put $\dfrac{dy}{dx} = 0$

$$x^2 + 2bx - b = 0$$

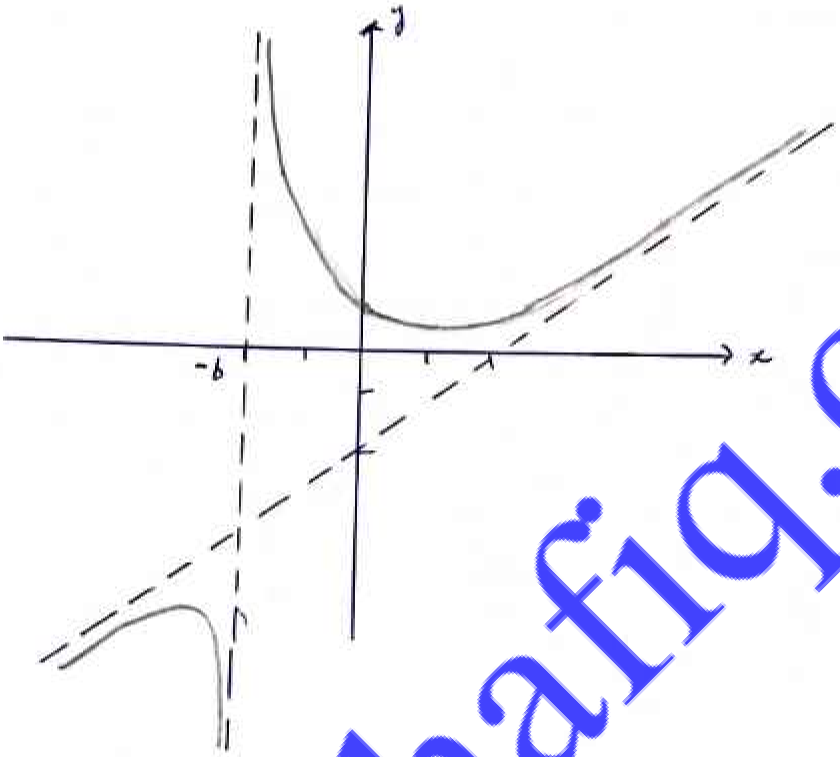
$b^2 - 4ac = 4b^2 + 4b > 0$ as b is +ve

This shows that there are two stationary points on C

(iv) For y-intercept

$x = 0$

$y = 1$ $\quad (0,1)$



Q.19 May/June/P13/2018

The curve C has equation

$$y = \frac{x^2 + 7x + 6}{x - 2}.$$

(i) Find the coordinates of the points of intersection of C with the axes. [2]

(ii) Find the equation of each of the asymptotes of C. [3]

(iii) Sketch C. [3]

(i) x - intercept put $y=0$

$\frac{x^2+7x+6}{x-2} = 0 \Rightarrow x^2+7x+6=0 \Rightarrow \begin{matrix} x=-1 \\ y=0 \end{matrix} \quad \begin{matrix} x=-6 \\ y=0 \end{matrix}$

$(-1,0), (-6,0)$ are x-intercepts

y- intercept put $x=0$

$y=-3$

$(0,-3)$

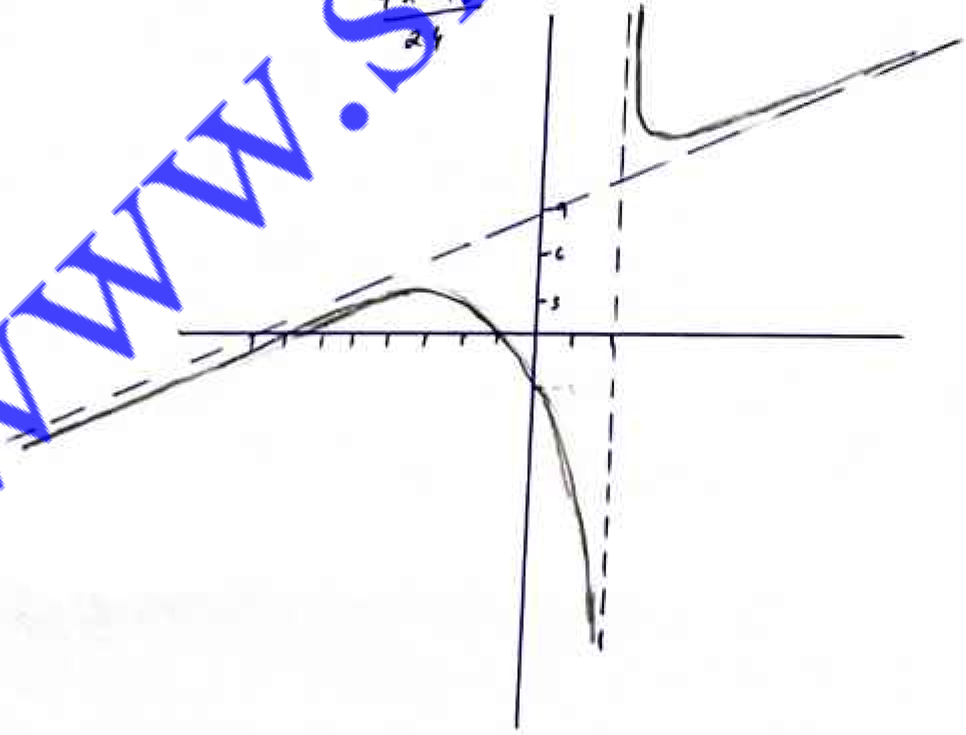
(ii) vertical Asymptote

$x-2=0 \quad \boxed{x=2}$

oblique Asymptote

$$\begin{array}{r|l} & x+9 \\ \hline x-2 & x^2+7x+6 \\ & x^2-2x \\ \hline & 9x+6 \\ & 9x-18 \\ \hline & 24 \end{array}$$

$\boxed{y=x+9}$ is oblique Asymptote



Q.20 Oct/Nov/P11+P13/2018

The curve C has equation

$$y = \frac{x^2 + ax - 1}{x + 1},$$

where a is constant and $a > 1$.

(i) Find the equations of the asymptotes of C. [3]

(ii) Show that C intersects the x-axis twice. [1]

(iii) Justifying your answer, find the number of stationary points on C. [2]

(iv) Sketch C, stating the coordinates of its point of intersection with the y-axis. [3]

(i) Vertical Asymptote $x+1=0$ $\boxed{x=-1}$

oblique Asymptote

$$\begin{array}{r|l} & x+(a-1) \\ \hline x+1 & x^2+ax-1 \\ & x^2+x \\ \hline & (a-1)x-1 \\ & (a-1)x+a-1 \\ \hline & -a \end{array}$$

$y = x + a - 1$ is oblique asymptote

(ii) For x-intercept put $y=0$

$$\frac{x^2+ax-1}{x+1} = 0$$

$$x^2+ax-1=0$$

$$b^2-4ac = a^2+4 > 0 \quad \text{as } a>1$$

This is shown That There two x-intercepts

(iii)

$$\frac{dy}{dx} = \frac{(x+1)(2x+a)-(x^2+ax-1)(1)}{(x+1)^2} = \frac{x^2+2x+a+1}{(x+1)^2}$$

put $\frac{dy}{dx}=0$

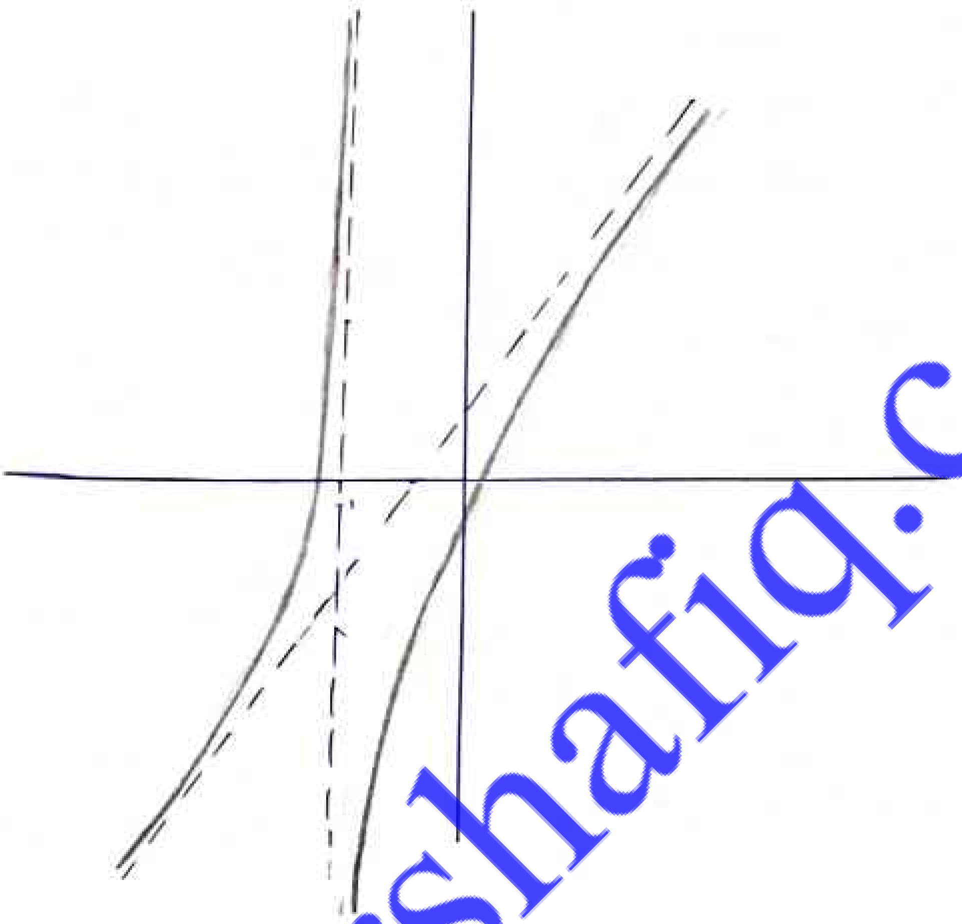
$$x^2+2x+a+1=0$$

$$b^2-4ac = 4-4(a+1) = -4a < 0 \quad \text{as } a>1$$

This shows That There are no Stationary point

(iv) For y-intercept put $x = 0$

$y = -1$ $(0, -1)$



Q.21 Oct/Nov/P12/2018

The curve C has equation

$$y = \frac{5x^2 + 5x + 1}{x^2 + x + 1}.$$

(i) Find the equation of the asymptote of C. [2]

(ii) Show that, for all real values of x, $-\frac{1}{3} \leqslant y < 5$. [4]

(iii) Find the coordinates of any stationary points of C. [2]

(iv) Sketch C, stating the coordinates of any intersections with the y-axis. [2]

(i) vertical Asymptote $x^2+x+1=0$

$b^2-4ac = 1-4 = -3 <$

There is no vertical asymptote

Horizontal Asymptote $y = 5$

(ii) $y = \frac{5x^2+5x+1}{x^2+x+1}$

$5x^2+5x+1 = yx^2+yx+y$

$(y-5)x^2+(y-5)x+(y-1)=0$

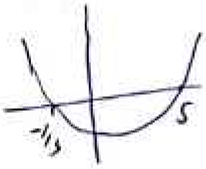
$b^2-4ac \geqslant 0$

$(y-5)^2-4(y-5)(y-1) \geqslant 0$

$(y-5)(y-5-4y+20) \geqslant 0$

$(y-5)(-3y+15 \geqslant 0$

$(y-5)(3y+1) \leq 0$



$-\frac{1}{3} \leq y < 5$ because $y=5$ is an asymptote

(iii) $\frac{dy}{dx} = \frac{(x^2+x+1)(10x+5)-(5x^2+5x+1)(2x+1)}{(x^2+x+1)^2}$

$= (\cancel{10x^3}+\cancel{5x^2}+\cancel{10x^2}+\cancel{5x}+10x+5-\cancel{10x^3}-\cancel{5x^2}-\cancel{10x^2}-\cancel{5x}-2x-1)/(x^2+x+1)^2$

$= \frac{8x+4}{(x^2+x+1)^2}$

put $\frac{dy}{dx} = 0$

$8x + 4 = 0$

$x = -\frac{1}{2}$

$y = -\frac{1}{3}$

$(-\frac{1}{2}, -\frac{1}{3})$ is stationary point