AS Level Further Mathematics

Topic: Series

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3 (a) By considering $(2r+1)^3-(2r-1)^3$, use the method of differences to prove that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1).$$ [5]

Let $S_n = 1^2+3\times2^2+3^2+3\times4^2+5^2+3\times6^2+\ldots+\left(2+(-1)^n\right)n^2$.

(b) Show that $S_{2n} = \frac{1}{3}n(2n+1)(an+b)$, where a and b are integers to be determined. [3]

(c) State the value of $\lim\limits_{n\to\infty}\frac{S_{2n}}{n^3}$. [1]

$$(2r+1)^3-(2r-1)^3 = \left[{}^3C_0\,8r^3+{}^3C_1\,4r^2+{}^3C_2\,2r+{}^3C_3(1)\right]$$

$$-\left[{}^3C_0\,8r^3-{}^3C_1\,4r^2+{}^3C_2\,2r-{}^3C_3(1)\right]$$

$$= 24r^2+2$$

$$\sum_{r=1}^{n}(2r+1)^3-(2r-1)^3 = \begin{array}{l} 3^3-1^3 \\ 5^3-3^3 \\ 7^3-5^3 \\ 9^3-7^3 \\ \vdots \\ (2n-3)^3-(2n-5)^3 \\ (2n-1)^3-(2n-3)^3 \\ (2n+1)^3-(2n-1)^3 \end{array}$$

$$\sum_{r=1}^{n} 24r^2+2 = (2n+1)^3-1$$

$$24\sum_{r=1}^{n} r^2+2n = 8n^3+12n^2+6n+1-1$$

$$24\sum_{r=1}^{n} r^2 = 8n^3+12n^2+4n = 4n(2n^2+3n+1)$$

$$= 4n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

(b) Given

$$S_n = 1^2 + 3\times 2^2 + 3^2 + 3\times 4^2 + 5^2 + 3\times 6^2 + \cdots + (2 + (-1)^n)n^2$$

$$S_{2n} = [1^2 + 2^2 + 3^2 + \cdots (2n)^2] + 2[2^2 + 4^2 + \cdots + (2n)^2]$$

$$= \sum_{r=1}^{2n} r^2 + 2\sum_{r=1}^{n} (2r)^2$$

$$= \frac{1}{6}(2n)(2n+1)(4n+1) + 8\times\frac{1}{6}n(n+1)(2n+1)$$

$$= \frac{1}{3}n(2n+1)[4n+1+4n+4]$$

$$= \frac{1}{3}n(2n+1)(8n+5)$$

(c) $\lim\limits_{n\to\infty} \frac{S_{2n}}{n^3}$

$$= \lim_{n\to\infty} \frac{1}{3}(2n^2+n)(8n+5) \div n^3$$

$$= \lim_{n\to\infty} \frac{1}{3}[16n^3 + 18n^2 + 5n] \div n^3$$

$$= \lim_{n\to\infty} \left[\frac{16}{3} + \frac{18}{n} + \frac{5}{n^2}\right]$$

$$= \frac{16}{3}$$

Question No 2 Oct/Nov/P12/2022

1 (a) Use the list of formulae (MF19) to find $\sum_{r=1}^{n} r(r+2)$ in terms of n, simplifying your answer. [2]

(b) Express $\frac{1}{r(r+2)}$ in partial fractions and hence find $\sum_{r=1}^{n} \frac{1}{r(r+2)}$ in terms of n. [4]

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{r(r+2)}$. [1]

(a) $\sum_{r=1}^{n} r(r+2) = \sum_{r=1}^{n} r^2 + 2r$

$= \frac{1}{6}n(n+1)(2n+1) + 2 \cdot \frac{1}{2}n(n+1)$

$= \frac{1}{6}n(n+1)[2n+1+6] = \frac{1}{6}n(n+1)(2n+7)$

(b) $\frac{1}{r(r+2)} = \frac{A}{r} + \frac{B}{r+2}$

$1 = A(r+2) + Br$

put $r=0$

$A = \frac{1}{2}$

put $r=-2$

$B = -\frac{1}{2}$

$\frac{1}{r(r+2)} = \frac{1/2}{r} + \frac{-1/2}{r+2} = \frac{1}{2}\left[\frac{1}{r} - \frac{1}{r+2}\right]$

$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+2}\right)$

$= \frac{1}{2}\Big[$
$1 - \cancel{\frac{1}{3}}$
$\frac{1}{2} - \cancel{\frac{1}{4}}$
$\cancel{\frac{1}{3}} - \frac{1}{5}$
$\cancel{\frac{1}{4}} - \frac{1}{6}$
$\vdots$
$\frac{1}{n-2} - \cancel{\frac{1}{n}}$
$\frac{1}{n-1} - \frac{1}{n+1}$
$\cancel{\frac{1}{n}} - \frac{1}{n+2}$

$$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{1}{2}\left[\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right]$$

$$\sum_{r=1}^{\infty} \frac{1}{r(r+2)} = \lim_{n \to \infty} \frac{1}{2}\left[\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right]$$

$$= \frac{1}{2}\left[\frac{3}{2} - 0 - 0 \right]$$

$$= \frac{3}{4}$$

1 Let a be a positive constant.

(a) Use the method of differences to find $\sum_{r=1}^{n} \frac{1}{(ar+1)(ar+a+1)}$ in terms of n and a. [4]

(b) Find the value of a for which $\sum_{r=1}^{\infty} \frac{1}{(ar+1)(ar+a+1)} = \frac{1}{6}$. [3]

(a) $$\frac{1}{(ar+1)(ar+a+1)} = \frac{A}{ar+1} + \frac{B}{ar+a+1}$$

$$1 = A(ar+a+1) + B(ar+1)$$

put $r = -\frac{1}{a}$

$$A = \frac{1}{a}$$

put $r = \frac{-a-1}{a}$

$$B = -\frac{1}{a}$$

$$\frac{1}{(ar+1)(ar+a+1)} = \frac{1/a}{ar+1} + \frac{-\frac{1}{a}}{ar+a+1} = \frac{1}{a}\left[\frac{1}{ar+1} - \frac{1}{ar+a+1}\right]$$

$$\sum_{r=1}^{n} \frac{1}{(ar+1)(ar+a+1)} = \frac{1}{a}\sum_{r=1}^{n}\left[\frac{1}{ar+1} - \frac{1}{ar+a+1}\right]$$

$$= \frac{1}{a}\left[\frac{1}{a+1} - \cancel{\frac{1}{2a+1}}\right.$$

$$\cancel{\frac{1}{2a+1}} - \cancel{\frac{1}{3a+1}}$$

$$\cancel{\frac{1}{3a+1}} - \frac{1}{4a+1}$$

$$\vdots$$

$$\frac{1}{(n-2)a+1} - \cancel{\frac{1}{(n-1)a+1}}$$

$$\cancel{\frac{1}{(n-1)a+1}} - \cancel{\frac{1}{na+1}}$$

$$\cancel{\frac{1}{na+1}} - \frac{1}{na+a+1}$$

$$= \frac{1}{a}\left[\frac{1}{a+1} - \frac{1}{an+a+1}\right]$$

(b) Given

$$\sum_{r=1}^{\infty} \frac{1}{(ar+1)(ar+a+1)} = \frac{1}{6}$$

$$\lim_{n\to\infty} \frac{1}{a}\left[\frac{1}{a+1} - \frac{1}{an+a+1}\right] = \frac{1}{6}$$

$$\frac{1}{a}\left[\frac{1}{a+1} - 0\right] = \frac{1}{6}$$

$$\frac{1}{a^2+a} = \frac{1}{6}$$

$$a^2+a-6=0$$

$$(a+3)(a-2)=0$$

$a=2$ or $a=-3$ (not possible as a is +ve)

Question No 4 May/June/P13/2022

4 Let $u_r = e^{rx}(e^{2x} - 2e^x + 1)$.

(a) Using the method of differences, or otherwise, find $\sum_{r=1}^{n} u_r$ in terms of n and x. [3]

(b) Deduce the set of non-zero values of x for which the infinite series

$$u_1 + u_2 + u_3 + \dots$$

is convergent and give the sum to infinity when this exists. [3]

(c) Using a standard result from the list of formulae (MF19), find $\sum_{r=1}^{n} \ln u_r$ in terms of n and x. [3]

(a)
$$\sum_{r=1}^{n} e^{rx}(e^{2x} - 2e^{x} + 1) = \sum_{r=1}^{n} e^{(r+2)x} - 2e^{(r+1)x} + e^{rx}$$

$$= e^{3x} - 2e^{2x} + e^{x}$$
$$e^{4x} - 2e^{3x} + e^{2x}$$
$$e^{5x} - 2e^{4x} + e^{3x}$$
$$\vdots$$
$$e^{nx} - 2e^{(n-1)x} + e^{(n-2)x}$$
$$e^{(n+1)x} - 2e^{nx} + e^{(n-1)x}$$
$$e^{(n+2)x} - 2e^{(n+1)x} + e^{nx}$$

$$= -2e^{2x} + e^{x} + e^{2x} + e^{(n+1)x} + e^{(n+2)x} - 2e^{(n+1)x}$$

$$= e^{x} - e^{2x} - e^{(n+1)x} + e^{(n+2)x}$$

(b) When $x < 0$, then infinite series is convergent

Then $e^{nx} \to 0$ as $n \to \infty$

So $$u_1 + u_2 + u_3 + \dots = \sum_{r=1}^{\infty} u_r = \lim_{n \to \infty}\left[e^{x} - e^{2x} - e^{(n+1)x} + e^{(n+2)x}\right]$$

$$= e^{x} - e^{2x} - 0 + 0$$

$$= e^{x} - e^{2x}$$

(c) $\sum_{r=1}^{n} \ln u_r = \sum_{r=1}^{n} \ln[e^{rx}(e^{2x} - 2e^{x} + 1)]$

$$= \sum_{r=1}^{n} \ln[e^{rx}(e^{x}-1)^{2}]$$

$$= \sum_{r=1}^{n} [\ln e^{rx} + \ln(e^{x}-1)^{2}]$$

$$= \sum_{r=1}^{n} [rx + \ln(e^{x}-1)^{2}]$$

$$= \frac{1}{2}n(n+1)x + n\ln(e^{x}-1)^{2}$$

Question No 5 May/June/P11+P12/2021

2 (a) Use standard results from the List of formulae (MF19) to find $\sum_{r=1}^{n}(1-r-r^2)$ in terms of n, simplifying your answer. [3]

(b) Show that

$$\frac{1-r-r^2}{(r^2+2r+2)(r^2+1)}=\frac{r+1}{(r+1)^2+1}-\frac{r}{r^2+1}$$

and hence use the method of differences to find $\sum_{r=1}^{n}\frac{1-r-r^2}{(r^2+2r+2)(r^2+1)}$. [5]

(c) Deduce the value of $\sum_{r=1}^{\infty}\frac{1-r-r^2}{(r^2+2r+2)(r^2+1)}$. [1]

(a) $$\sum_{r=1}^{n}(1-r-r^2) = \sum_{r=1}^{n}1-\sum_{r=1}^{n}r-\sum_{r=1}^{n}r^2$$

$$= n-\frac{1}{2}n(n+1)-\frac{1}{6}n(n+1)(2n+1)$$

$$= n\left[1-\frac{1}{2}n-\frac{1}{2}-\frac{1}{6}(2n^2+3n+1)\right]$$

$$= n\left[1-\frac{1}{2}-\frac{1}{6}-\frac{1}{2}n-\frac{1}{2}n-\frac{1}{3}n^2\right]$$

$$= n\left[\frac{1}{3}-n-\frac{1}{3}n^2\right]$$

$$= \frac{1}{3}n-n^2-\frac{1}{3}n^3$$

(b) $$\frac{r+1}{(r+1)^2+1}-\frac{r}{r^2+1}=\frac{(r+1)(r^2+1)-r(r^2+2r+2)}{\left[(r+1)^2+1\right](r^2+1)}$$

$$=\frac{r^3+r^2+r+1-r^3-2r^2-2r}{(r^2+2r+2)(r^2+1)}$$

$$=\frac{1-r-r^2}{(r^2+2r+2)(r^2+1)}$$

→

$$\sum_{r=1}^{n} \frac{1-r-r^2}{(r^2+2r+2)(r^2+1)} = \sum_{r=1}^{n} \left(\frac{r+1}{(r+1)^2+1} - \frac{r}{r^2+1} \right)$$

$$= \cancel{\frac{2}{5}} - \frac{1}{2}$$

$$\cancel{\frac{3}{10}} - \cancel{\frac{2}{5}}$$

$$\frac{4}{17} - \cancel{\frac{3}{10}}$$

$$\vdots$$

$$\cancel{\frac{n-1}{(n-1)^2+1}} - \frac{n-2}{(n-2)^2+1}$$

$$\cancel{\frac{n}{n^2+1}} - \cancel{\frac{n-1}{(n-1)^2+1}}$$

$$\frac{n+1}{(n+1)^2+1} - \cancel{\frac{n}{n^2+1}}$$

$$= -\frac{1}{2} + \frac{n+1}{(n+1)^2+1}$$

(c) $$\sum_{r=1}^{\infty} \frac{1-r-r^2}{(r^2+2r+2)(r^2+1)} = \lim_{n\to\infty} \left[-\frac{1}{2} + \frac{n+1}{n^2+2n+2} \right]$$

$$= \lim_{n\to\infty} \left[-\frac{1}{2} + \frac{1+\frac{1}{n}}{n+2+\frac{2}{n}} \right]$$

$$= -\frac{1}{2} + \frac{1+0}{\infty+2+0}$$

$$= -\frac{1}{2} + 0$$

$$= -\frac{1}{2}$$

Question No 6 May/June/P13/2021

1 (a) Show that

$$\tan(r+1)-\tan r=\frac{\sin 1}{\cos(r+1)\cos r}.$$ [2]

Let $u_r=\dfrac{1}{\cos(r+1)\cos r}$.

(b) Use the method of differences to find $\sum_{r=1}^{n}u_r$. [3]

(c) Explain why the infinite series $u_1+u_2+u_3+\ldots$ does not converge. [1]

(a) $\tan(r+1)-\tan r=\dfrac{\sin(r+1)}{\cos(r+1)}-\dfrac{\sin r}{\cos r}$

$$=\frac{\sin(r+1)\cos r-\sin r\cos(r+1)}{\cos(r+1)\cos r}$$

$$=\frac{\sin(r+1-r)}{\cos(r+1)\cos r}=\frac{\sin 1}{\cos(r+1)\cos r}$$

(b) Given $u_r=\dfrac{1}{\cos(r+1)\cos r}$

$$(\sin 1)u_r=\frac{\sin 1}{\cos(r+1)\cos r}=\tan(r+1)-\tan r$$

$$(\sin 1)\sum_{r=1}^{n}u_r=\sum_{r=1}^{n}\tan(r+1)-\tan r$$

$$=\cancel{\tan 2}-\tan 1$$
$$\cancel{\tan 3}-\cancel{\tan 2}$$
$$\tan 4-\cancel{\tan 3}$$
$$\vdots$$
$$\cancel{\tan(n-1)}-\tan(n-2)$$
$$\cancel{\tan n}-\cancel{\tan(n-1)}$$
$$\tan(n+1)-\cancel{\tan n}$$

$$\sum_{r=1}^{n}u_r=\frac{\tan(n+1)-\tan 1}{\sin 1}$$

(c) as $n \to \infty$

$\tan(n+1)$ oscillates

so $u_1 + u_2 + u_3 + \cdots$ does not converge.

2 (a) Use standard results from the list of formulae (MF19) to find $\sum_{r=1}^{n} r(r+1)(r+2)$ in terms of n, fully factorising your answer. [3]

(b) Express $\frac{1}{r(r+1)(r+2)}$ in partial fractions and hence use the method of differences to find

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}.$$ [5]

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)}$. [1]

(a)

$$\sum_{r=1}^{n} r(r+1)(r+2) = \sum_{r=1}^{n} r(r^2+3r+2)$$

$$= \sum_{r=1}^{n} r^3+3r^2+2r$$

$$= \frac{1}{4}n^2(n+1)^2 + 3 \times \frac{1}{6} n(n+1)(2n+1) + 2 \cdot \frac{1}{2} n(n+1)$$

$$= \frac{1}{4} n(n+1)\left[n^2+n+4n+2+4 \right]$$

$$= \frac{1}{4} n(n+1)(n^2+5n+6)$$

$$= \frac{1}{4} n(n+1)(n+2)(n+3)$$

(b)

$$\frac{1}{r(r+1)(r+2)} = \frac{A}{r} + \frac{B}{r+1} + \frac{C}{r+2}$$

$$1 = A(r+1)(r+2) + Br(r+2) + Cr(r+1)$$

put $r=0$ $A = 1/2$

$r=-1$ $B=-1$

$r=-2$ $C = \frac{1}{2}$

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \sum_{r=1}^{n} \frac{1}{2r} - \frac{1}{r+1} + \frac{1}{2(r+2)}$$

$$= \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{6}} \qquad \frac{1}{2n-4} - \frac{1}{n-1} + \cancel{\tfrac{1}{2n}}$$

$$\frac{1}{4} - \cancel{\tfrac{1}{3}} + \frac{1}{8} \qquad \frac{1}{2n-2} - \cancel{\tfrac{1}{n}} + \frac{1}{2n+2}$$

$$\cancel{\tfrac{1}{6}} - \frac{1}{4} + \frac{1}{10} \qquad \cancel{\tfrac{1}{2n}} - \frac{1}{n+1} + \frac{1}{2n+4}$$

$\vdots$

$$= \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$$

(c) $$\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)} = \lim_{n \to \infty} \left[\frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)} \right]$$

$$= \frac{1}{4} - 0 + 0$$

$$= \frac{1}{4}$$

3 Let $S_n = \sum_{r=1}^{n} \ln \frac{r(r+2)}{(r+1)^2}$.

(a) Using the method of differences, or otherwise, show that $S_n = \ln \frac{n+2}{2(n+1)}$. [4]

Let $S = \sum_{r=1}^{\infty} \ln \frac{r(r+2)}{(r+1)^2}$.

(b) Find the least value of n such that $S_n - S < 0.01$. [3]

(a) $S_n = \sum_{r=1}^{n} \ln \frac{r(r+2)}{(r+1)^2}$

$= \sum_{r=1}^{n} \ln r + \ln(r+2) - 2\ln(r+1)$

$= \ln 1 + \cancel{\ln 3} - 2\ln 2$

$\ln 2 + \cancel{\ln 4} - 2\cancel{\ln 3}$

$\cancel{\ln 3} + \ln 5 - 2\cancel{\ln 4}$

$\cancel{\ln 4} + \ln 6 - 2\ln 5$

$\vdots$

$\ln(n-2) + \cancel{\ln(n)} - 2\ln(n-1)$

$\ln(n-1) + \ln(n+1) - 2\cancel{\ln(n)}$

$\ln(\cancel{n}) + \ln(n+2) - 2\ln(n+1)$

$= \ln 1 - \ln 2 - \ln(n+1) + \ln(n+2)$

$= \ln(n+2) - (\ln 2 + \ln(n+1))$

$= \ln(n+2) - \ln 2(n+1)$

$= \ln \frac{n+2}{2(n+1)}$

$S = \sum_{r=1}^{\infty} \ln \frac{r(r+2)}{(r+1)^2} = \lim_{n \to \infty} \ln \frac{n+2}{2n+2}$

$= \lim_{n \to \infty} \ln \frac{1 + \frac{2}{n}}{2 + \frac{2}{n}}$

$= \ln \frac{1}{2} = -\ln 2$

$$S_n - S = \ln \frac{(n+2)}{2(n+1)} + \ln 2$$

$$= \ln \frac{n+2}{n+1}$$

Given

$$S_n - S < 0.01$$

$$\ln \frac{n+2}{n+1} < 0.01$$

$$n+2 < e^{0.01}(n+1)$$

$$n(1 - e^{0.01}) < e^{0.01} - 2$$

$$n < \frac{e^{0.01} - 2}{1 - e^{0.01}}$$

Question No 9 May/June/P11+P12/2020

4 (a) By first expressing $\frac{1}{r^2-1}$ in partial fractions, show that

$$\sum_{r=2}^{n} \frac{1}{r^2-1} = \frac{3}{4} - \frac{an+b}{2n(n+1)},$$

where a and b are integers to be found. [5]

(b) Deduce the value of $\sum_{r=2}^{\infty} \frac{1}{r^2-1}$. [1]

(c) Find the limit, as $n \to \infty$, of $\sum_{r=n+1}^{2n} \frac{n}{r^2-1}$. [4]

(a) $\frac{1}{r^2-1} = \frac{1}{(r-1)(r+1)} = \frac{A}{r-1} + \frac{B}{r+1}$

$1 = A(r+1) + B(r-1)$

put $r=1$ $A = 1/2$

$r=-1$ $B = -1/2$

$\frac{1}{r^2-1} = \frac{1/2}{r-1} + \frac{-1/2}{r+1}$

$\sum_{r=2}^{n} \frac{1}{r^2-1} = \frac{1}{2}\sum_{r=2}^{n} \frac{1}{r-1} - \frac{1}{r+1}$

$= \frac{1}{2}\Big[1 - \cancel{\frac{1}{3}}$

$\frac{1}{2} - \cancel{\frac{1}{4}}$

$\cancel{\frac{1}{3}} - \frac{1}{5}$

$\cancel{\frac{1}{4}} - \frac{1}{6}$

$\vdots$

$\frac{1}{n-3} - \cancel{\frac{1}{n-1}}$

$\frac{1}{n-2} - \frac{1}{n}$

$\cancel{\frac{1}{n-1}} - \frac{1}{n+1}$

$= \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right]$

$= \frac{1}{2}\left[\frac{3}{2} - \frac{2n+1}{n(n+1)}\right] = \frac{3}{4} - \frac{2n+1}{2n(n+1)}$

(b) $$\sum_{n=2}^{\infty} \frac{1}{r^2-1} = \lim_{n\to\infty} \left[\frac{3}{4} - \frac{2n+1}{2n^2+2n}\right]$$

$$= \lim_{n\to\infty} \left[\frac{3}{4} - \frac{2+1/n}{2n+2}\right]$$

$$= \frac{3}{4} - \frac{2+0}{\infty+2}$$

$$= \frac{3}{4} - 0 = \frac{3}{4}$$

(c) $$\sum_{r=2}^{2n} \frac{n}{r^2-1} = n\sum_{r=2}^{2n} \frac{1}{2}\left[\frac{1}{r-1} - \frac{1}{r+1}\right]$$

$$\sum_{r=n+1}^{2n} \frac{n}{r^2-1} = \sum_{r=2}^{n} \frac{n}{r^2-1} - \sum_{r=2}^{n} \frac{n}{r^2-1}$$

$$= n\left[\frac{3}{4} - \frac{4n+1}{4n(2n+1)}\right] - n\left[\frac{3}{4} - \frac{2n+1}{2n(n+1)}\right]$$

$$= \frac{n(2n+1)}{2n(n+1)} - \frac{n(4n+1)}{4n(2n+1)}$$

$$= \frac{2n+1}{2n+2} - \frac{4n+1}{8n+4}$$

as $n \to \infty$

$$= \lim_{n\to\infty}\left[\frac{2+\frac{1}{n}}{2+\frac{2}{n}} - \frac{4+\frac{1}{n}}{8+\frac{4}{n}}\right]$$

$$= 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

$$= \frac{n}{2}\left[1 - \frac{1}{3}\right.$$

$\frac{1}{2} - \frac{1}{4}$

$\frac{1}{3} - \frac{1}{5}$

$\frac{1}{4} - \frac{1}{6}$

$\vdots$

$\frac{1}{2n-3} - \frac{1}{2n-1}$

$\frac{1}{2n-2} - \frac{1}{2n}$

$\frac{1}{2n-1} - \frac{1}{2n+1}$

$$= \frac{n}{2}\left[1 + \frac{1}{2} - \frac{1}{2n} - \frac{1}{2n+1}\right]$$

$$= \frac{n}{2}\left[\frac{3}{2} - \frac{4n+1}{2n(2n+1)}\right]$$

$$= \frac{3n}{4} - \frac{4n^2+n}{4n(2n+1)}$$

$$= \frac{3n(n)(2n+1) - 4n^2 - n}{4n(2n+1)}$$

$$= \frac{6n^3+3n^2-4n^2-n}{8n^3+4n}$$

3 Let $S_n = 2^2 + 6^2 + 10^2 + \ldots + (4n-2)^2$.

(a) Use standard results from the List of Formulae (MF19) to show that $S_n = \frac{4}{3}n(4n^2-1)$. [4]

(b) Express $\frac{n}{S_n}$ in partial fractions and find $\sum_{n=1}^{N} \frac{n}{S_n}$ in terms of N. [4]

(c) Deduce the value of $\sum_{n=1}^{\infty} \frac{n}{S_n}$. [1]

(a)

$$S_n = \sum_{r=1}^{n} (4r-2)^2$$

$$= \sum_{r=1}^{n} (16r^2 - 16r + 4)$$

$$= 16\sum_{r=1}^{n} r^2 - 16\sum_{r=1}^{n} r + 4n$$

$$= 16 \times \frac{1}{6} n(n+1)(2n+1) - 16 \times \frac{1}{2} n(n+1) + 4n$$

$$= \frac{8}{3} n(n+1)(2n+1) - 8n(n+1) + 4n$$

$$= \frac{4}{3} n\left[(2)(n+1)(2n+1) - 6(n+1) + 3\right]$$

$$= \frac{4}{3} n\left[4n^2 + 6n + 2 - 6n - 6 + 3\right]$$

$$= \frac{4}{3} n(4n^2 - 1)$$

(b)

$$\frac{n}{S_n} = \frac{n}{\frac{4}{3}n(2n-1)(2n+1)} = \frac{3}{4(2n-1)(2n+1)}$$

$$= \frac{3}{4}\left[\frac{1/2}{2n-1} - \frac{1/2}{2n+1}\right]$$

$$= \frac{3}{8}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$$

$$\sum_{n=1}^{N} \frac{n}{S_n} = \frac{3}{8}\left[1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{5}} - \frac{1}{7} + \cdots \frac{1}{2N-2} - \cancel{\frac{1}{2N-1}} + \cancel{\frac{1}{2N-1}} - \frac{1}{2N+1}\right]$$

$$= \frac{3}{8}\left[1 - \frac{1}{2N+1}\right]$$

(c) as $N \to \infty$

$$\sum_{n=1}^{\infty} \frac{n}{S_n} = \lim_{N \to \infty} \frac{3}{8}\left[1 - \frac{1}{2N+1}\right]$$

$$= \frac{3}{8}\left[1 - 0\right]$$

$$= \frac{3}{8}$$

2 (a) Use standard results from the List of Formulae (MF19) to show that

$$\sum_{r=1}^{n}(7r+1)(7r+8) = an^3+bn^2+cn,$$

where a, b and c are constants to be determined. [3]

(b) Use the method of differences to find $\sum_{r=1}^{n}\frac{1}{(7r+1)(7r+8)}$ in terms of n. [4]

(c) Deduce the value of $\sum_{r=1}^{\infty}\frac{1}{(7r+1)(7r+8)}$. [1]

(a) $\sum_{r=1}^{n}(7r+1)(7r+8) = \sum_{r=1}^{n} 49r^2+63r+8$

$= 49 \times \frac{1}{6}n(n+1)(2n+1) + \frac{63}{2}n(n+1) + 8n$

$= \frac{49}{6}[n(2n^2+3n+1)] + \frac{63}{2}n^2 + \frac{63}{2}n + 8n$

$= \frac{49}{6} \times 2n^3 + \frac{49}{6} \times 3n^2 + \frac{49}{6}n + \frac{63}{2}n^2 + \frac{63}{2}n + 8n$

$= \frac{49}{3}n^3 + 56n^2 + \frac{143}{3}n$

(b) $\frac{1}{(7r+1)(7r+8)} = \frac{A}{7r+1} + \frac{B}{7r+8}$

$1 = A(7r+8) + B(7r+1)$

put $r = -\frac{1}{7}$, $A = \frac{1}{7}$

$r = \frac{-8}{7}$, $B = -\frac{1}{7}$

$\sum_{r=1}^{n}\frac{1}{(7r+1)(7r+8)} = \frac{1}{7}\sum_{r=1}^{n}\left(\frac{1}{7r+1} - \frac{1}{7r+8}\right)$

$= \frac{1}{7}\Big[\ \frac{1}{8} - \cancel{\frac{1}{15}}$

$\cancel{\frac{1}{15}} - \frac{1}{22}$

$\vdots$

$\frac{1}{7n-6} - \cancel{\frac{1}{7n+1}}$

$\cancel{\frac{1}{7n+1}} - \frac{1}{7n+8}\ \Big]$

$$= \frac{1}{7}\left[\frac{1}{8} - \frac{1}{7n+8}\right]$$

(c) $$\sum_{r=1}^{\infty} \frac{1}{(7r+1)(7r+8)} = \lim_{n\to\infty} \frac{1}{7}\left[\frac{1}{8} - \frac{1}{7n+8}\right]$$

$$= \frac{1}{7}\left[\frac{1}{8} - 0\right]$$

$$= \frac{1}{56}$$

3 (a) By simplifying $\left(x^n-\sqrt{x^{2n}+1}\right)\left(x^n+\sqrt{x^{2n}+1}\right)$, show that $\dfrac{1}{x^n-\sqrt{x^{2n}+1}}=-x^n-\sqrt{x^{2n}+1}$. [1]

Let $u_n=x^{n+1}+\sqrt{x^{2n+2}+1}+\dfrac{1}{x^n-\sqrt{x^{2n}+1}}$.

(b) Use the method of differences to find $\sum\limits_{n=1}^{N}u_n$ in terms of N and x. [3]

(c) Deduce the set of values of x for which the infinite series

$$u_1+u_2+u_3+\ldots$$

is convergent and give the sum to infinity when this exists. [3]

(a) $\left(x^n-\sqrt{x^{2n}+1}\right)\left(x^n+\sqrt{x^{2n}+1}\right)=\left(x^n\right)^2-\left(\sqrt{x^{2n}+1}\right)^2$

$=x^{2n}-x^{2n}-1$

$=-1$

$x^n+\sqrt{x^{2n}+1}=\dfrac{-1}{x^n-\sqrt{x^{2n}+1}}$

$\dfrac{1}{x^n-\sqrt{x^{2n}+1}}=-x^n-\sqrt{x^{2n}+1}$

$u_n=x^{n+1}+\sqrt{x^{2n+2}+1}-x^n-\sqrt{x^{2n}+1}$

$$\begin{aligned}\sum_{n=1}^{N}u_n=\;&\cancel{x^2}+\cancel{\sqrt{x^4+1}}-x-\sqrt{x^2+1}\\&\cancel{x^3}+\cancel{\sqrt{x^6+1}}-\cancel{x^2}-\cancel{\sqrt{x^4+1}}\\&x^4+\sqrt{x^8+1}-\cancel{x^3}-\cancel{\sqrt{x^6+1}}\\&\vdots\\&\cancel{x^{N-1}}+\cancel{\sqrt{x^{2N-2}+1}}-x^{N-2}-\sqrt{x^{2N-4}+1}\\&\cancel{x^N}+\cancel{\sqrt{x^{2N}+1}}-\cancel{x^{N-1}}-\cancel{\sqrt{x^{2N-2}+1}}\\&x^{N+1}+\sqrt{x^{2N+2}+1}-\cancel{x^N}-\cancel{\sqrt{x^{2N}+1}}\end{aligned}$$

$$\sum_{n=1}^{N} u_n = x^{N+1} + \sqrt{x^{2N+2}+1} - x - \sqrt{x^2+1}$$

(c) when $-1 < x < 1$

Then $u_1, u_2, u_3, \dots$ is convergent

as $n \to \infty$, then

$x^{N+1} \to 0$ and

$x^{2N+2} \to 0$

$$\sum_{n=1}^{\infty} u_n = 0 + \sqrt{0+1} - x - \sqrt{x^2+1}$$

$$= 1 - x - \sqrt{x^2+1}$$

Question No 13 May/June/P11+P12/2019

2 Let $u_n = \dfrac{4\sin(n-\frac{1}{2})\sin\frac{1}{2}}{\cos(2n-1)+\cos 1}$.

(i) Using the formulae for $\cos P \pm \cos Q$ given in the List of Formulae MF10, show that

$$u_n = \frac{1}{\cos n} - \frac{1}{\cos(n-1)}.$$ [2]

(ii) Use the method of differences to find $\sum_{n=1}^{N} u_n$. [2]

(iii) Explain why the infinite series $u_1 + u_2 + u_3 + \dots$ does not converge. [1]

(i) $u_n = \dfrac{4\sin(n-\frac{1}{2})\sin\frac{1}{2}}{\cos(2n-1)+\cos 1}$

as

$\cos P + \cos Q = 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$

$\cos P - \cos Q = -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$

$= \dfrac{2[\cos(n-1) - \cos n]}{2\cos(n)\cos(n-1)}$

$= \dfrac{1}{\cos(n)} - \dfrac{1}{\cos(n-1)}$

(ii) $\sum_{n=1}^{N} u_n = \sum_{n=1}^{N}\left[\dfrac{1}{\cos(n)} - \dfrac{1}{\cos(n-1)}\right]$

$= \cancel{\dfrac{1}{\cos 1}} - \dfrac{1}{}$

$\cancel{\dfrac{1}{\cos 2}} - \cancel{\dfrac{1}{\cos 1}}$

$\dfrac{1}{\cos 3} - \cancel{\dfrac{1}{\cos 2}}$

$\vdots$

$\cancel{\dfrac{1}{\cos(N-2)}} - \dfrac{1}{\cos(N-3)}$

$\dfrac{1}{\cos(N-1)} - \cancel{\dfrac{1}{\cos(N-2)}}$

$\dfrac{1}{\cos N} - \cancel{\dfrac{1}{\cos(N-1)}}$

$$\sum_{n=1}^{N} u_n = -1 + \frac{1}{\cos N}$$

(iii) $\cos N$ oscillates

as $n \longrightarrow \infty$

so $u_1, u_2, u_3, \cdots$ does not converge.

4 (i) Use the method of differences to show that $\sum^{N} \frac{1}{(3r+1)(3r-2)} = \frac{1}{3} - \frac{1}{3(3N+1)}$. [4]

(ii) Find the limit, as $N \to \infty$, of $\sum_{r=N+1}^{N^2} \frac{N}{(3r+1)(3r-2)}$. [4]

(i) $\frac{1}{(3r+1)(3r-2)} = \frac{A}{3r+1} + \frac{B}{3r-2}$

$1 = A(3r-2) + B(3r+1)$

put $r = -\frac{1}{3}$, $A = -\frac{1}{3}$

$r = \frac{2}{3}$, $B = +\frac{1}{3}$

$$\sum_{r=1}^{N} \frac{1}{(3r+1)(3r-2)} = \sum_{r=1}^{N} \frac{1}{3}\left(-\frac{1}{3r+1} + \frac{1}{3r-2}\right)$$

$$= \frac{1}{3}\Big[\; 1 - \frac{1}{4}$$

$$\frac{1}{4} - \frac{1}{7}$$

$$\frac{1}{7} - \frac{1}{10}$$

$$\vdots$$

$$\frac{1}{3N-5} - \frac{1}{3N-2}$$

$$\frac{1}{3N-2} - \frac{1}{}$$

$$\frac{1}{3N-2} - \frac{1}{3N+1}\Big]$$

$$= \frac{1}{3}\left[1 - \frac{1}{3N+1}\right]$$

(ii) $$\sum_{r=N+1}^{N^2} \frac{N}{(3r+1)(3r-2)} = \sum_{r=1}^{N^2} \frac{N}{(3r+1)(3r-2)} - \sum_{r=1}^{N} \frac{N}{(3r+1)(3r-2)}$$

$$= \left(\frac{N}{3} - \frac{N}{3(3N^2+1)}\right) - \left(\frac{N}{3} - \frac{N}{3(3N+1)}\right)$$

$$= \frac{N}{3(3N+1)} - \frac{N}{3(3N^2+1)} = \frac{N^3 - N^2}{(3N+1)(3N^2+1)}$$

as $N \to \infty$

$$\lim_{N \to \infty} \frac{N^3 - N^2}{9N^3 + 3N + 3N^2 + 1}$$

$$= \lim_{N \to \infty} \frac{1 - \frac{1}{N}}{9 + \frac{3}{N^2} + \frac{3}{N} + \frac{1}{N^3}}$$

$$= \frac{1 - 0}{9 + 0 + 0 + 0} = \frac{1}{9}$$

5 Let $S_N = \sum_{r=1}^{N}(5r+1)(5r+6)$ and $T_N = \sum_{r=1}^{N}\frac{1}{(5r+1)(5r+6)}$.

(i) Use standard results from the List of Formulae (MF10) to show that

$$S_N = \tfrac{1}{3}N(25N^2+90N+83).$$ [3]

(ii) Use the method of differences to express T_N in terms of N. [4]

(iii) Find $\lim_{N\to\infty}(N^{-3}S_NT_N)$. [2]

(i) $S_N = \sum_{r=1}^{N}(5r+1)(5r+6)$

$= \sum_{r=1}^{N}(25r^2+35r+6)$

$= 25\left[\frac{1}{6}N(N+1)(2N+1)\right] + 35\left[\frac{1}{2}N(N+1)\right] + 6N$

$= \frac{25}{6}N(2N^2+3N+1) + \frac{35}{2}N(N+1) + 6N$

$= \frac{1}{3}N\left[25N^2+90N+83\right]$

(ii) $T_N = \sum_{r=1}^{N}\frac{1}{(5r+1)(5r+6)}$

$\frac{1}{(5r+1)(5r+6)} = \frac{A}{(5r+1)} + \frac{B}{(5r+6)}$

$1 = A(5r+6) + B(5r+1)$

put $r = -\frac{1}{5} \Rightarrow A = \frac{1}{5}$

$r = \frac{-6}{5} \Rightarrow B = -\frac{1}{5}$

$T_N = \sum_{r=1}^{N}\frac{1}{5}\left(\frac{1}{5r+1} - \frac{1}{5r+6}\right)$

$= \frac{1}{5}\left[\frac{1}{6} - \frac{1}{11} + \frac{1}{11} - \frac{1}{16} + \cdots + \frac{1}{5N-4} - \frac{1}{5N+1} + \frac{1}{5N+1} - \frac{1}{5N+6}\right]$

$= \frac{1}{5}\left[\frac{1}{6} - \frac{1}{5N+6}\right] = \frac{1}{30} - \frac{1}{5(5N+6)}$

(iii)

$$\lim_{N\to\infty} N^{-3} S_N T_N = \lim_{N\to\infty} \frac{1}{N^3}\left[\frac{25}{3}N^3 + 30N^2 + \frac{83}{3}N\right]\left[\frac{1}{30} - \frac{1}{25N+30}\right]$$

$$= \lim_{N\to\infty}\left[\frac{25}{3} + \frac{30}{N} + \frac{83}{3N^2}\right]\left[\frac{1}{30} - \frac{1}{25N+30}\right]$$

$$= \left(\frac{25}{3} + 0 + 0\right)\left(\frac{1}{30} - 0\right)$$

$$= \frac{25}{3} \times \frac{1}{30}$$

$$= \frac{5}{18}$$

5 Let $S_n = \sum_{r=1}^{n}(-1)^{r-1}r^2$.

(i) Use the standard result for $\sum_{r=1}^{n} r^2$ given in the List of Formulae (MF10) to show that

$$S_{2n} = -n(2n+1).$$ [4]

(ii) State the value of $\lim_{n\to\infty} \frac{S_{2n}}{n^2}$ and find $\lim_{n\to\infty} \frac{S_{2n+1}}{n^2}$. [4]

(i)
$$S_{2n} = 1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \cdots + (-1)^{2n-1}(2n)^2$$
$$= \sum_{r=1}^{2n} r^2 - 2\sum_{r=1}^{n}(2r)^2$$
$$= \frac{1}{6}(2n)(2n+1)(4n+1) - 8 \times \frac{1}{6} n(n+1)(2n+1)$$
$$= \frac{1}{3} n(2n+1)(4n+1-4n-4)$$
$$= \frac{1}{3} n(2n+1)(-3)$$
$$= -n(2n+1)$$

(ii)
$$\lim_{n\to\infty} \frac{S_{2n}}{n^2} = \lim_{n\to\infty} \frac{-2n^2+n}{n^2}$$
$$= \lim_{n\to\infty} \left(-2+\frac{1}{n}\right) = -2+0 = -2$$

$$S_{2n+1} = S_{2n} + (-1)^{2n}(2n+1)^2$$
$$= -n(2n+1) + (2n+1)^2$$
$$= (2n+1)(-n+2n+1)$$
$$= (2n+1)(n+1)$$

$$\lim_{n\to\infty} \frac{S_{2n+1}}{n^2} = \lim_{n\to\infty} \frac{(2n+1)(n+1)}{n^2} = \lim_{n\to\infty} \frac{n^2\left(2+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)}{n^2}$$
$$= (2+0)(1+0) = 2$$

Question No 17 May/June/P13/2018

2 (i) Verify that

$$\frac{n(e-1)+e}{n(n+1)e^{n+1}} = \frac{1}{ne^n} - \frac{1}{(n+1)e^{n+1}}.$$ [1]

Let $S_N = \sum_{n=1}^{N} \frac{n(e-1)+e}{n(n+1)e^{n+1}}$.

(ii) Express S_N in terms of N and e. [2]

Let $S = \lim_{N\to\infty} S_N$.

(iii) Find the least value of N such that $(N+1)(S - S_N) < 10^{-3}$. [3]

(i) $$\frac{1}{ne^n} - \frac{1}{(n+1)e^{n+1}} = \frac{(n+1)e - n}{n(n+1)e^{n+1}}$$

$$= \frac{ne - n + e}{n(n+1)e^{n+1}} = \frac{n(e-1)+e}{n(n+1)e^{n+1}}$$

(ii) $$S_N = \sum_{n=1}^{N} \frac{n(e-1)+e}{n(n+1)e^{n+1}}$$

$$= \sum_{n=1}^{N} \left[\frac{1}{ne^n} - \frac{1}{(n+1)e^{n+1}}\right]$$

$$= \frac{1}{e} - \cancel{\frac{1}{2e^2}} + \cancel{\frac{1}{2e^2}} - \frac{1}{3e^3} + \cdots + \frac{1}{N-1\,e^{N-1}} - \cancel{\frac{1}{Ne^N}} + \cancel{\frac{1}{Ne^N}} - \frac{1}{(N+1)e^{N+1}}$$

$$= \frac{1}{e} - \frac{1}{(N+1)e^{N+1}}$$

Also $$S = \lim_{N\to\infty} S_N = \frac{1}{e} - 0$$

$$S = \frac{1}{e}$$

$$(N+1)(S - S_N) < 10^{-3}$$

$$(N+1)\left[\frac{1}{e} - \frac{1}{e} + \frac{1}{N+1\,e^{N+1}}\right] < 10^{-3}$$

$$\frac{1}{e^{N+1}} < 10^{-3}$$

$$e^{N+1} > 10^{3}$$

$$N+1 > \ln 10^{3}$$

$$N+1 > 6.91$$

$$N > 5.91$$

$\Rightarrow$ least value of N is 6

Question No 18 Oct/Nov/P11/2018

(i) By considering $(2r+1)^2-(2r-1)^2$, use the method of differences to prove that

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1).$$ [3]

(ii) By considering $(2r+1)^4-(2r-1)^4$, use the method of differences and the result given in part (i) to prove that

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2.$$ [5]

The sums S and T are defined as follows:

$$S = 1^3+2^3+3^3+4^3+\ldots+(2N)^3+(2N+1)^3,$$
$$T = 1^3+3^3+5^3+7^3+\ldots+(2N-1)^3+(2N+1)^3.$$

(iii) Use the result given in part (ii) to show that $S=(2N+1)^2(N+1)^2$. [1]

(iv) Hence, or otherwise, find an expression in terms of N for T, factorising your answer as far as possible. [2]

(i) $(2r+1)^2-(2r-1)^2 = 4r^2+4r+1-4r^2+4r-1$

$= 8r$

$$\sum_{r=1}^{n} 8r = \sum_{r=1}^{n} (2r+1)^2-(2r-1)^2$$

$= 3^2-1^2$
5^2-3^2
7^2-5^2
$\vdots$
$(2n-1)^2-(2n-3)^2$
$(2n+1)^2-(2n-1)^2$

$= -1^2+(2n+1)^2$

$$\sum_{r=1}^{n} r = \frac{1}{8}\left[-1+4n^2+4n+1\right] = \frac{4n^2+4n}{8} = \frac{4n(n+1)}{8} = \frac{1}{2}n(n+1)$$

(ii) $(2r+1)^4-(2r-1)^4 = \left[(2r+1)^2+(2r-1)^2\right]\left[(2r+1)^2-(2r-1)^2\right]$

$= \left[4r^2+4r+1+4r^2-4r+1\right]\left[8r\right]$

$= 8r(4r^2+2) = 16(r^3+\ldots)$

$$16\left[4\sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r\right] = \sum_{r=1}^{n} (2r+1)^4 - (2r-1)^4$$

$$= 3^4 - 1^4$$

$$5^4 - 3^4$$

$$7^4 - 5^4$$

$$\vdots$$

$$(2n+1)^4 - (2n-1)^4$$

$$= -1^4 + (2n+1)^4$$

$$64\sum_{r=1}^{n} r^3 = -1 + (2n+1)^4 - 16\cdot\frac{1}{2}(n)(n+1)$$

$$= -1 + 16n^4 + 32n^3 + 24n^2 + 8n + 1 - 8n^2 - 8n$$

$$= 16n^4 + 32n^3 + 16n^2$$

$$= 16n^2(n^2 + 2n + 1)$$

$$64\sum_{r=1}^{n} r^3 = 16n^2(n+1)^2$$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

(iii) $$S = \frac{1}{4}(2N+1)^2(2N+2)^2$$

$$= (2N+1)^2(N+1)^2$$

(iv)

$$S - T = 2^3 + 4^3 + \cdots (2N)^3$$

$$S - T = \sum_{r=1}^{N} (2r)^3$$

$$T = S - \sum_{r=1}^{N}(2r)^3 = (2N+1)^2(N+1)^2 - 2N^2(N+1)^2$$

$$= (N+1)^2(2N^2 + 4N + 1)$$

7 Let

$$S_N = \sum_{r=1}^{N}(3r+1)(3r+4) \quad \text{and} \quad T_N = \sum_{r=1}^{N}\frac{1}{(3r+1)(3r+4)}.$$

(i) Use standard results from the List of Formulae (MF10) to show that

$$S_N = N(3N^2 + 12N + 13).$$ [3]

(ii) Use the method of differences to show that

$$T_N = \frac{1}{12} - \frac{1}{3(3N+4)}.$$ [3]

(iii) Deduce that $\frac{S_N}{T_N}$ is an integer. [2]

(iv) Find $\lim_{N\to\infty} \frac{S_N}{N^3 T_N}$. [2]

$$S_N = \sum_{r=1}^{N}(3r+1)(3r+4)$$

$$= \sum_{r=1}^{N}(9r^2+15r+4)$$

$$= 9\sum_{r=1}^{N} r^2 + 15\sum_{r=1}^{N} r + 4N$$

$$= 9\left(\frac{1}{6}N(N+1)(2N+1)\right) + 15\left(\frac{1}{2}N(N+1)\right) + 4N$$

$$= N\left[\frac{9}{6}(2N^2+3N+1) + \frac{15}{2}N + \frac{15}{2} + 4\right]$$

$$= N(3N^2 + 12N + 13)$$

$$\frac{1}{(3r+1)(3r+4)} = \frac{A}{3r+1} + \frac{B}{3r+4}$$

$$1 = A(3r+4) + B(3r+1)$$

put $r = -1/3 \Rightarrow A = \frac{1}{3}$

$r = -4/3 \Rightarrow B = -\frac{1}{3}$

$$T_N = \sum_{r=1}^{N}\frac{1}{(3r+1)(3r+4)} = \sum_{r=1}^{N}\frac{1}{3}\left(\frac{1}{3r+1} - \frac{1}{3r+4}\right)$$

$$= \frac{1}{3}\left[\frac{1}{4} - \frac{1}{7} + \frac{1}{7} - \frac{1}{10} + \cdots + \frac{1}{3(N-1)+1} - \frac{1}{3N+4}\right]$$

$$= \frac{1}{3}\left[\frac{1}{4} - \frac{1}{3N+4}\right] = \frac{1}{12} - \frac{1}{3(3N+4)}$$

(iii) $\frac{S_N}{T_n} = N(3N^2+12N+13) \div \left[\frac{1}{12} - \frac{1}{3(3N+4)}\right]$

$= N(3N^2+12N+13) \div \left[\frac{3N+4-4}{12(3N+4)}\right]$

$= N(3N^2+12N+13) \times \frac{{}^4\cancel{12}(3N+4)}{\cancel{3}\cancel{N}}$

$= 4(3N+4)(3N^2+12N+13)$

= This is an integer as N is an +ve integer

(iv) $\lim_{N\to\infty} \frac{S_N}{N^3 T_N} = \lim_{N\to\infty} \frac{4(3N+4)(3N^2+12N+13)}{N^3}$

$= \lim_{N\to\infty} \frac{36N^3+192N^2+348N+208}{N^3}$

$= \lim_{N\to\infty} \left[36 + \frac{192}{N} + \frac{348}{N^2} + \frac{208}{N^3}\right]$

$= 36+0+0+0$

$= 36$

1 It is given that $\sum_{r=1}^{n} u_r = n^2(2n+3)$, where n is a positive integer.

(i) Find $\sum_{r=n+1}^{2n} u_r$. [2]

(ii) Find u_r. [3]

(i) Given $\sum_{r=1}^{n} u_r = n^2(2n+3)$

$$\sum_{r=n+1}^{2n} u_r = \sum_{r=1}^{2n} u_r - \sum_{r=1}^{n} u_r$$

$$= (2n)^2(4n+3) - n^2(2n+3)$$

$$= 14n^3 + 9n^2$$

(ii)

$$S_r - S_{r-1} = r^2(2r+3) - (r-1)^2(2r+1)$$

$$= (2r^3+3r^2) - (r^2-2r+1)(2r+1)$$

$$= (2r^3+3r^2) - (2r^3+r^2-4r^2-\cancel{2r}+\cancel{2r}+1)$$

$$= 3r^3+3r^2-2r^3+3r^2-1$$

$$= 6r^2-1$$