AS Level Further Mathematics

Topic:

Matrices

Teacher:

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Q.1 May/June/2022/P11+P12

The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & k & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

(a) Find the set of values of k for which $\mathbf{A}$ is non-singular. [3]

(b) Given that $\mathbf{A}$ is non-singular, find, in terms of k, the entries in the top row of $\mathbf{A}^{-1}$. [4]

(c) Given that $\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, give an example of a matrix $\mathbf{C}$ such that $\mathbf{BAC} = \begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix}$. [4]

(d) Find the set of values of k for which the transformation in the x-y plane represented by $\begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix}$ has two distinct invariant lines through the origin. [6]

(a)

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & k & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1(9k-48) - 2(36-42) + 3(32-7k)$$

$$= 9k - 48 - 72 + 84 + 96 - 21k$$

$$= -12k + 60$$

A is non-singular if

$$|A| \neq 0$$

$$-12k + 60 \neq 0$$

$$k \neq 5$$

(b)

$$\frac{1}{60-12k}\begin{vmatrix} k & 6 \\ 8 & 9 \end{vmatrix} = \frac{9k-48}{60-12k}$$

$$\frac{-1}{60-12k}\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = \frac{-(18-24)}{60-12k} = \frac{6}{60-12k}$$

$$\frac{1}{60-12k}\begin{vmatrix} 2 & 3 \\ k & 6 \end{vmatrix} = \frac{12-3k}{60-12k}$$

$$\text{Top row of } A^{-1} = \begin{bmatrix} \frac{9k-48}{60-12k} & \frac{6}{60-12k} & \frac{12-3k}{60-12k} \\ - & - & - \\ - & - & - \end{bmatrix}$$

(c)

$$BA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & k & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 4 & k & 6 \end{pmatrix}$$

let $C = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & k & 6 \end{pmatrix}\begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix}$$

$$\begin{pmatrix} a+2b+3c & d+2e+3f \\ 4a+kb+6c & 4d+ke+6f \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$$

(d) $y = mx$

$$\begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} 2x+y \\ kx+4y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$x' = 2x+y = 2x+mx$

$y' = kx+4mx$

(x', y') lies on $y = mx$

$kx+4mx = m(2x+mx)$

$m^2x + 2mx - 4mx - kx = 0$

$(m^2 - 2m - k)x = 0$

$m^2 - 2m - k = 0$

Two distinct invariant lines, if $b^2 - 4ac > 0$

$4 + 4k > 0$

$k > -1$ and $k \neq 8$

Let $\mathbf{A} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, where a is a positive constant.

(a) State the type of the geometrical transformation in the x–y plane represented by $\mathbf{A}$. [1]

(b) Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}.$$ [5]

Let $\mathbf{B} = \begin{pmatrix} b & b \\ a^{-1} & a^{-1} \end{pmatrix}$, where b is a positive constant.

(c) Find the equations of the invariant lines, through the origin, of the transformation in the x–y plane represented by $\mathbf{A}^n\mathbf{B}$.
[6]

(a) Shear in the x direction

(b) $$A^n = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}$$

For $n=1$

$$A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad (\text{True})$$

suppose result is true for $n=k$

$$A^k = \begin{pmatrix} 1 & ka \\ 0 & 1 \end{pmatrix}$$

Now for $n=k+1$

$$A^{k+1} = A^k \cdot A$$

$$= \begin{pmatrix} 1 & ka \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & a+ak \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (k+1)a \\ 0 & 1 \end{pmatrix}$$

Hence induction is complete.

(c)

$$A^n B = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b & b \\ \frac{1}{a} & \frac{1}{a} \end{pmatrix}$$

$$= \begin{pmatrix} b+n & b+n \\ \frac{1}{a} & \frac{1}{a} \end{pmatrix}$$

Let $y = mx$ be the invariant line

$$\begin{pmatrix} b+n & b+n \\ \frac{1}{a} & \frac{1}{a} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} (b+n)(x+y) \\ \frac{1}{a}(x+y) \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$x' = (b+n)(x+mx)$$

$$y' = \frac{1}{a}(x+mx)$$

(x', y') lies on $y = mx$

$$\frac{1}{a}(x+mx) = m\left[(b+n)(x+mx)\right]$$

$$\left[\frac{1}{a}(1+m) - m(b+n)(1+m)\right]x = 0$$

$$\frac{1}{a}(1+m) - m(b+n)(1+m) = 0$$

$$(1+m)\left[\frac{1}{a} - (b+n)m\right] = 0$$

$$1+m = 0 \quad \Rightarrow \quad m = -1$$

$$\frac{1}{a} - (b+n)m = 0 \quad \Rightarrow \quad m = \frac{1}{a(b+n)}$$

Invariant lines $y = -x$

$$y = \frac{1}{a(b+n)}x$$

The matrix **M** represents the sequence of two transformations in the x-y plane given by a rotation of 60° anticlockwise about the origin followed by a one-way stretch in the x-direction, scale factor d ($d \neq 0$).

(a) Find **M** in terms of d. **[4]**

(b) The unit square in the x-y plane is transformed by **M** onto a parallelogram of area $\frac{1}{2}d^2$ units^2.

Show that $d = 2$. **[2]**

The matrix **N** is such that $\mathbf{MN} = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

(c) Find **N**. **[3]**

(d) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **MN**. **[5]**

(a)

$$M = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos 60 & -\sin 60 \\ \sin 60 & \cos 60 \end{pmatrix}$$

$$= \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$M = \begin{pmatrix} \frac{d}{2} & -\frac{d\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

(b)

$$|M| = \begin{vmatrix} \frac{d}{2} & -\frac{d\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{vmatrix}$$

$$= \frac{d}{4} + \frac{3d}{4} = d$$

$$d \times 1 = \frac{1}{2} d^2$$

$$2d - d^2 = 0$$

$$d(2-d) = 0$$

$d = 0$ or $2 - d = 0$

ignore $\quad d = 2$

(c) $|M| = 2$

$$\text{adj}\, M = \begin{pmatrix} \frac{1}{2} & \sqrt{3} \\ -\frac{\sqrt{3}}{2} & 1 \end{pmatrix}$$

$$M^{-1} = \frac{1}{2}\begin{pmatrix} \frac{1}{2} & \sqrt{3} \\ -\frac{\sqrt{3}}{2} & 1 \end{pmatrix}$$

Given

$$MN = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$N = M^{-1}\begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \frac{1}{2} & \sqrt{3} \\ -\frac{\sqrt{3}}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \frac{1}{2} + \frac{\sqrt{3}}{2} & \frac{1}{2} + \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} + \frac{1}{2} & -\frac{\sqrt{3}}{2} + \frac{1}{2} \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1+\sqrt{3} & 1+\sqrt{3} \\ 1-\sqrt{3} & 1-\sqrt{3} \end{pmatrix}$$

(d) Let $y = mx$

$$\begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$x' = x + y = x + mx$

$y' = \frac{1}{2}x + \frac{1}{2}y = \frac{1}{2}x + \frac{1}{2}mx$

(x', y') lies on $y = mx$

$\frac{1}{2}x + \frac{1}{2}mx = m(x + mx)$

$(1+m)x = (2m + 2m^2)x$

$(2m^2 + m - 1)x = 0$

$2m^2 + m - 1 = 0$

$m = \frac{1}{2}$, $m = -1$

Invariant lines are

$y = \frac{1}{2}x$ and

$y = -x$

Q.4 May/June/2021/P13

The matrices **A**, **B** and **C** are given by

$$\mathbf{A}=\begin{pmatrix}2 & k & k\\ 5 & -1 & 3\\ 1 & 0 & 1\end{pmatrix},\quad \mathbf{B}=\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 0\end{pmatrix} \text{ and } \mathbf{C}=\begin{pmatrix}0 & 1 & 1\\ -1 & 2 & 0\end{pmatrix},$$

where k is a real constant.

(a) Find **CAB**. [3]

(b) Given that **A** is singular, find the value of k. [3]

(c) Using the value of k from part **(b)**, find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **CAB**. [5]

(a)

$$CA = \begin{pmatrix}0 & 1 & 1\\ -1 & 2 & 0\end{pmatrix}\begin{pmatrix}2 & k & k\\ 5 & -1 & 3\\ 1 & 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}6 & -1 & 4\\ 8 & -k-2 & -k+6\end{pmatrix}$$

$$CAB = \begin{pmatrix}6 & -1 & 4\\ 8 & -k-2 & -k+6\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 0\end{pmatrix}$$

$$= \begin{pmatrix}10 & -1\\ -k+14 & -k-2\end{pmatrix}$$

(b) Given A is singular, so

$$|A| = 0$$

$$\begin{vmatrix}2 & k & k\\ 5 & -1 & 3\\ 1 & 0 & 1\end{vmatrix} = 0$$

$$2(-1-0) - k(5-3) + k(0+1) = 0$$

$$-2-2k+k=0$$

$$\boxed{k=-2}$$

1c) $CAB = \begin{pmatrix} 10 & -1 \\ 16 & 0 \end{pmatrix}$ as $k = -2$

Let $y = mx$

$$\begin{pmatrix} 10 & -1 \\ 16 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$x' = 10x - y = 10x - mx$

$y' = 16x$

(x', y') lies on $y = mx$

$16x = m(10x - mx)$

$(m^2 - 10m + 16)x = 0$

$m^2 - 10m + 16 = 0$

$m = 2, \quad m = 8$

Invariant lines are

$y = 2x$ and $y = 8x$

The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) The matrix $\mathbf{M}$ represents a sequence of two geometrical transformations.

State the type of each transformation, and make clear the order in which they are applied. [2]

(b) Find the values of θ, for $0 \leqslant \theta \leqslant \pi$, for which the transformation represented by $\mathbf{M}$ has exactly one invariant line through the origin, giving your answers in terms of π. [9]

(a) Stretch with scale factor 3 parallel to the x-axis followed by a anticlockwise rotation about the origin at angle θ

(b) $M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 3\cos\theta & -\sin\theta \\ 3\sin\theta & \cos\theta \end{pmatrix}$

Let $y = mx$

$\begin{pmatrix} 3\cos\theta & -\sin\theta \\ 3\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$

$x' = (3\cos\theta)x - (\sin\theta)y = (3\cos\theta)x - mx\sin\theta$

$y' = (3\sin\theta)x + (\cos\theta)y = (3\sin\theta)x + (m\cos\theta)x$

(x', y') lies on $y = mx$

$(3\sin\theta)x + (m\cos\theta)x = m\left[(3\cos\theta)x - (m\sin\theta)x\right]$

$(\sin\theta)m^2 - (2\cos\theta)m + 3\sin\theta = 0$

Given M has exactly one ~~line~~ invariant line, so

$b^2 - 4ac = 0$

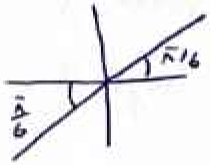
$4\cos^2\theta - 12\sin^2\theta = 0$

$$\tan^2\theta = \frac{1}{3}$$

$$\tan\theta = \pm\frac{1}{\sqrt{3}}$$

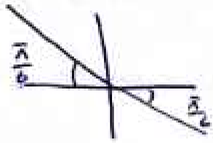
$$\tan\theta = \frac{1}{\sqrt{3}} \quad , \quad \tan\theta = -\frac{1}{\sqrt{3}}$$

$\alpha = \frac{\pi}{6}$



$\theta = \frac{\pi}{6}$

$\alpha = \frac{\pi}{6}$



$\theta = \pi - \frac{\pi}{6}$

$\theta = \frac{5\pi}{6}$

(a) Give full details of the geometrical transformation in the x-y plane represented by the matrix $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$. **[1]**

Let $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix}$.

(b) The triangle DEF in the x-y plane is transformed by $\mathbf{A}$ onto triangle PQR.

Given that the area of triangle DEF is $13\,\text{cm}^2$, find the area of triangle PQR. **[2]**

(c) Find the matrix $\mathbf{B}$ such that $\mathbf{AB} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$. **[2]**

(d) Show that the origin is the only invariant point of the transformation in the x-y plane represented by $\mathbf{A}$. **[4]**

(a) Enlargement about (0,0) with scale factor 6

(b) $A = \begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix}$

$|A| = 6 - 8 = -2$

Area of $\triangle PQR$ = det(A) × Area of $\triangle DEF$

$= 2 \times 13 = 26\ \text{cm}^2$

(c) $B = A^{-1}\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$

$A^{-1} = \frac{1}{-2}\begin{pmatrix} 2 & -4 \\ -2 & 3 \end{pmatrix}$

$B = \frac{1}{-2}\begin{pmatrix} 2 & -4 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$

$= -\frac{1}{2}\begin{pmatrix} 12 & -24 \\ -12 & 18 \end{pmatrix} = \begin{pmatrix} -6 & 12 \\ 6 & -9 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} 3x + 4y \\ 2x + 2y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$3x + 4y = x$
$2x + 2y = y$

$2x + 4y = 0 \quad -(1)$
$2x + y = 0 \quad -(2)$

By solving

$x = 0$
$y = 0$

Q.7 May/June/2020/P11+P12

Let $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$.

(a) The transformation in the x-y plane represented by $\mathbf{A}^{-1}$ transforms a triangle of area $30\,\text{cm}^2$ into a triangle of area $d\,\text{cm}^2$.

Find the value of d. **[3]**

(b) Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}.$$ **[5]**

(c) The line $y = 2x$ is invariant under the transformation in the x-y plane represented by $\mathbf{A}^n\mathbf{B}$, where $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 33 & 0 \end{pmatrix}$.

Find the value of n. **[5]**

(a) $A = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$

$\det(A) = 2 - 0 = 2$

We know

$\det(A^{-1}) = (\det(A))^{-1} = 2^{-1} = \frac{1}{2}$

Area of Image of object $= \det(A^{-1}) \times$ Area of the object

$d = \frac{1}{2} \times 30 = 15$

(b) $A^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}$

For $n = 1$

$A = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$ (True)

Suppose the result is true for $n = k$

$A^k = \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix}$

Now for $n = k+1$

$$A^{k+1} = A^k \cdot A$$

$$= \begin{pmatrix} 2^k & 0 \\ 2^k-1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & 0 \\ 2^{k+1}-2+1 & 1 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & 0 \\ 2^{k+1}-1 & 1 \end{pmatrix}$$

Hence induction is complete

(c)

$$A^nB = \begin{pmatrix} 2^n & 0 \\ 2^n-1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 33 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2^n & 0 \\ 2^n+32 & 0 \end{pmatrix}$$

Given $y = 2x$ is an invariant line

$$\begin{pmatrix} 2^n & 0 \\ 2^n+32 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} 2^n x \\ (2^n+32)x \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$x' = 2^n x$$
$$y' = (2^n+32)x$$

(x', y') lies on $y = 2x$

$$(2^n+32)x = 2(2^n x)$$
$$2^n+32 = 2 \cdot 2^n$$
$$2^n = 32 = 2^5$$
$$\boxed{n=5}$$

4 The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} k & 0 & 2 \\ 0 & -1 & -1 \\ 1 & 1 & -k \end{pmatrix},$$

where k is a real constant.

(a) Show that **A** is non-singular. [3]

The matrices **B** and **C** are given by

$$\mathbf{B} = \begin{pmatrix} 0 & -3 \\ -1 & 3 \\ 0 & 0 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} -3 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

It is given that $\mathbf{CAB} = \begin{pmatrix} -2 & -\frac{3}{2} \\ -1 & -\frac{3}{2} \end{pmatrix}$.

(b) Find the value of k. [3]

(c) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **CAB**. [5]

(a)

$$A = \begin{pmatrix} k & 0 & 2 \\ 0 & -1 & -1 \\ 1 & 1 & -k \end{pmatrix}$$

$$|A| = k(k+1) + 0 + 2(0+1)$$

$$|A| = k^2 + k + 2$$

$$b^2 - 4ac = (1)^2 - 4(1)(2) = -7$$

This shows that $k^2 + k + 2 > 0$ for all values of k

$|A| \neq 0$ for all values of k

Hence is A is non-singular

(b) Given

$$CAB = \begin{pmatrix} -2 & -\frac{3}{2} \\ -1 & -\frac{3}{2} \end{pmatrix}$$

$$CAB = \begin{pmatrix} -3 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} k & 0 & 2 \\ 0 & -1 & -1 \\ 1 & 1 & -k \end{pmatrix}\begin{pmatrix} 0 & -3 \\ -1 & 3 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & -3k \\ 1 & -3 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 9k+3 \\ -1 & -3k-3 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 9k+3 \\ -1 & -3k-3 \end{pmatrix} = \begin{pmatrix} -2 & -\frac{3}{2} \\ -1 & -\frac{3}{2} \end{pmatrix}$$

$$9k+3 = -\frac{3}{2}$$

$$18k+6 = -3$$

$$18k = -9$$

$$\boxed{k = -\frac{1}{2}}$$

(c) Let $y = mx$ be the invariant line

$$\begin{pmatrix} -2 & -\frac{3}{2} \\ -1 & -\frac{3}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$x' = -2x - \frac{3}{2}y = -2x - \frac{3}{2}mx$$

$$y' = -x - \frac{3}{2}y = -x - \frac{3}{2}mx$$

(x', y') lies on $y = mx$

$$\left(-1 - \frac{3}{2}m\right)x = m\left[\left(-2 - \frac{3}{2}m\right)x\right]$$

$$\left(1 + \frac{3}{2}m\right)x = \left(2m + \frac{3}{2}m^2\right)x$$

$$(2+3m)x = (4m+3m^2)x$$

$$3m^2 + 4m = 2 + 3m$$

$$3m^2 + m - 2 = 0$$

$$3m^2 + 3m - 2m - 2 = 0$$

$$(3m-2)(m+1) = 0$$

$$m = -1, \quad m = \frac{2}{3}$$

Invariant lines are

$y = -x$ and $y = \frac{2}{3}x$

1 The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, where a and b are positive constants.

(a) The matrix $\mathbf{M}$ represents a sequence of two geometrical transformations.

State the type of each transformation, and make clear the order in which they are applied. [2]

The unit square in the x-y plane is transformed by $\mathbf{M}$ onto parallelogram $OPQR$.

(b) Find, in terms of a and b, the matrix which transforms parallelogram $OPQR$ onto the unit square. [2]

It is given that the area of $OPQR$ is $2\,\text{cm}^2$ and that the line $x + 3y = 0$ is invariant under the transformation represented by $\mathbf{M}$.

(c) Find the values of a and b. [5]

(a) Stretch along x direction with scale factor a followed by a shear along x direction with scale factor b

(b) Given M transform unit square onto parallelogram OPQR

M^{-1} will transform parallelogram OPQR onto unit square

$$M = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

$$|M| = a$$

$$\text{adj}M = \begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix}$$

$$M^{-1} = \frac{1}{a}\begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix}$$

(c) $\det(M) \times 1 = 2$

$a \times 1 = 2 \Rightarrow \boxed{a = 2}$

Also given $x + 3y = 0$ is invariant line under M

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} ax+by \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$x' = ax + by = ax + b(-\frac{1}{3}x) = (a - \frac{b}{3})x$

$y' = -\frac{1}{3}x$

(x', y') lies on $x + 3y = 0$

$(a - \frac{b}{3})x - 1x = 0$

$(a - \frac{b}{3} - 1)x = 0$

$a - \frac{b}{3} - 1 = 0$

$2 - \frac{b}{3} - 1 = 0$

$-\frac{b}{3} = -1$

$\boxed{b = 3}$

4 The matrices **A** and **B** are given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}.$$

(a) Give full details of the geometrical transformation in the x-y plane represented by **A**. [1]

(b) Give full details of the geometrical transformation in the x-y plane represented by **B**. [2]

The triangle DEF in the x-y plane is transformed by **AB** onto triangle PQR.

(c) Show that the triangles DEF and PQR have the same area. [3]

(d) Find the matrix which transforms triangle PQR onto triangle DEF. [2]

(e) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **AB**. [5]

(a) Reflection about the line $y = x$

(b) $\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \cos 60 & -\sin 60 \\ \sin 60 & \cos 60 \end{pmatrix}$

Anti clockwise rotation about $(0,0)$ at an angle $60°$

(c) $AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

$= \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$

$\det(AB) = -\frac{3}{4} - \frac{1}{4} = -1$

Area of $\triangle PQR$ = $|\det(AB)|$ x Area of $\triangle DEF$

Area of $\triangle PQR$ = $|-1|$ x Area of $\triangle DEF$

Area of $\triangle PQR$ = Area of $\triangle DEF$

(d) $(AB)^{-1}$ matrix transform $\triangle PQR$ to $\triangle DEF$

$$(AB)^{-1} = \frac{1}{-1}\begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

(e) Let $y = mx$ be the invariant line

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$x' = \frac{\sqrt{3}}{2}x + \frac{1}{2}y = \left(\frac{\sqrt{3}}{2} + \frac{m}{2}\right)x$$

$$y' = \frac{1}{2}x - \frac{\sqrt{3}}{2}y = \left(\frac{1}{2} - \frac{\sqrt{3}}{2}m\right)x$$

(x', y') lies on $y = mx$

$$\left(\frac{1}{2} - \frac{\sqrt{3}}{2}m\right)x = \left(\frac{\sqrt{3}}{2}m + \frac{m^2}{2}\right)x$$

$$\frac{1}{2} - \frac{\sqrt{3}}{2}m = \frac{\sqrt{3}}{2}m + \frac{m^2}{2}$$

$$m^2 + 2\sqrt{3}m - 1 = 0$$

$$m = 2 - \sqrt{3}, \quad m = -2 - \sqrt{3}$$

Invariant lines are

$$y = (2 - \sqrt{3})x$$

$$y = -(2 + \sqrt{3})x$$