AS Level Further Mathematics

Topic: Linear Motion

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Q.1 Oct/Nov/P31+P33/2022

4 A particle of mass 0.5 kg moves along a horizontal straight line. Its velocity is $v\,\mathrm{m\,s^{-1}}$ at time t s. The forces acting on the particle are a driving force of magnitude 50 N and a resistance of magnitude $2v^2$ N. The initial velocity of the particle is $3\,\mathrm{m\,s^{-1}}$.

(a) Find an expression for v in terms of t. [7]

(b) Deduce the limiting value of v. [1]

Given $m = 0.5$ kg, $t=0$, $V=3$

$$F = 50 - 2v^2$$

$$F = ma$$

$$50 - 2v^2 = m\frac{dv}{dt}$$

$$2(25 - v^2) = \frac{1}{2}\frac{dv}{dt}$$

$$4\,dt = \frac{1}{25-v^2}\,dv$$

$$\frac{1}{(5-v)(5+v)}\,dv = 4\,dt$$

$$\frac{1}{(5-v)(5+v)} = \frac{A}{5-v} + \frac{B}{5+v}$$

$$1 = A(5+v) + B(5-v)$$

put $v=5 \Rightarrow A = \frac{1}{10}$

put $v=-5 \Rightarrow B = \frac{1}{10}$

$$\int \frac{1}{10(5-v)} + \frac{1}{10(5+v)}\,dv = 4\int dt$$

$$\frac{1}{10}(-\ln(5-v)) + \frac{1}{10}\ln(5+v) = 4t + c$$

put $t=0$, $v=3 \Rightarrow c = \frac{1}{10}\ln 4$

$$\ln\frac{5+v}{5-v} = 40t + \ln 4$$

$$\ln\frac{5+v}{4(5-v)} = 40t$$

$$\frac{5+v}{20-4v} = e^{40t}$$

→

$$5+v = 20e^{40t} - 4ve^{40t}$$

$$v(1+4e^{40t}) = 20e^{40t} - 5$$

$$v = \frac{20e^{40t} - 5}{1+4e^{40t}}$$

(b) $$v = \frac{20 - \frac{5}{e^{40t}}}{\frac{1}{e^{40t}} + 4}$$

$t \to \infty$

$$v = \frac{20-0}{0+4}$$

$$v = 5$$

4 A particle P of mass 5 kg moves along a horizontal straight line. At time t s, the velocity of P is $v\,\mathrm{m\,s^{-1}}$ and its displacement from a fixed point O on the line is x m. The forces acting on P are a force of magnitude $\frac{500}{v}$ N in the direction OP and a resistive force of magnitude $\frac{1}{2}v^2$ N. When $t = 0$, $x = 0$ and $v = 5$.

(a) Find an expression for v in terms of x. [6]

(b) State the value that the speed approaches for large values of x. [1]

Given $m = 5\text{ kg}$ $t=0$, $x=0$, $v=5$

$$F = \frac{500}{v} - \frac{v^2}{2}$$

$$F = ma$$

$$\frac{500}{v} - \frac{v^2}{2} = 5\left[v\frac{dv}{dx}\right]$$ as $a = \frac{dv}{dt}$, $a = \frac{dv}{dx}\cdot\frac{dx}{dt} = v\frac{dv}{dx}$

$$\frac{1000 - v^3}{2v} = 5v\frac{dv}{dx}$$

$$\frac{10v^2}{1000 - v^3}\,dv = dx$$

$$-\frac{10}{3}\int\frac{-3v^2}{1000-v^3}\,dv = \int dx$$

$$-\frac{10}{3}\ln(1000 - v^3) = x + c$$

put $x = 0$, $v = 5$

$$c = -\frac{10}{3}\ln 875$$

$$-\frac{10}{3}\ln(1000 - v^3) = x - \frac{10}{3}\ln 875$$

$$x = \frac{10}{3}\ln 875 - \frac{10}{3}\ln(1000 - v^3)$$

$$= \frac{10}{3}\ln\frac{875}{1000 - v^3}$$

$$\frac{3}{10}x = \ln\frac{875}{1000 - v^3}$$

$$e^{0.3x} = \frac{875}{1000 - v^3}$$

$$1000 - v^3 = 875e^{-0.3x}$$

$$v^3 = 1000 - 875e^{-0.3x}$$

$$v = \left[1000 - 875e^{-0.3x}\right]^{1/3}$$

as $x \to \infty$

$$v \to [1000 - 0]^{1/3} = 10$$

3 A particle P is moving in a horizontal straight line. Initially P is at the point O on the line and is moving with velocity $25\,\mathrm{m\,s^{-1}}$. At time t s after passing through O, the acceleration of P is $\frac{4000}{(5t+4)^3}\,\mathrm{m\,s^{-2}}$ in the direction PO. The displacement of P from O at time t is x m.

Find an expression for x in terms of t. [5]

Given $t=0$ $v = 25\ ms^{-1}$ and

$a = -\frac{4000}{(5t+4)^3}$ as t is increases a is decrease

$\frac{dv}{dt} = -\frac{4000}{(5t+5)^3}$

$dv = -4000(5t+4)^{-3}\,dt$

$\int dv = -4000\int (5t+4)^{-3}\,dt$

$v = -4000\frac{(5t+4)^{-2}}{-2(5)} + c$

$v = \frac{400}{(5t+4)^2} + c$

put $t=0$, $v=25 \Rightarrow c=0$

$v = \frac{400}{(5t+4)^2}$

$\frac{dx}{dt} = \frac{400}{(5t+4)^2}$

$dx = 400(5t+4)^{-2}\,dt$

$\int dx = 400\int (5t+4)^{-2}\,dt$

$x = 400\frac{(5t+4)^{-1}}{5(-1)} + c'$

$x = \frac{-80}{5t+4} + c'$

put $t=0$, $x=0 \Rightarrow c' = 20$

$x = \frac{-80}{5t+4} + 20 = \frac{100t}{5t+4}$

5 A particle P of mass 4 kg is moving in a horizontal straight line. At time t s the velocity of P is $v\,\mathrm{m\,s^{-1}}$ and the displacement of P from a fixed point O on the line is x m. The only force acting on P is a resistive force of magnitude $(4e^{-x}+12)e^{-x}$ N. When $t=0$, $x=0$ and $v=4$.

(a) Show by integration that $v = \dfrac{1+3e^{x}}{e^{x}}$. [4]

(b) Find an expression for x in terms of t. [4]

(a) Given $F = -(4e^{-x}+12)e^{-x}$, $t=0$, $x=0$, $v=4$

$m = 4$ kg

$$F = mv\frac{dv}{dx}$$

$$-(4e^{-x}+12)e^{-x} = 4v\frac{dv}{dx}$$

$$4v\,dv = (-4e^{-2x} - 12e^{-x})\,dx$$

$$\int 4v\,dv = \int(-4e^{-2x} - 12e^{-x})\,dx$$

$$2v^2 = -4\frac{e^{-2x}}{-2} - \frac{12e^{-x}}{-1} + c$$

$$v^2 = e^{-2x} + 6e^{-x} + c$$

put $x=0$, $v=4$ $\Rightarrow c = 9$

$$v^2 = e^{-2x} + 6e^{-x} + 9$$

$$v^2 = (e^{-x}+3)^2$$

$$v = e^{-x} + 3 = \frac{1}{e^x} + 3 = \frac{1+3e^x}{e^x}$$

(b)

$$\frac{dx}{dt} = \frac{1+3e^x}{e^x}$$

$$\frac{e^x}{1+3e^x}\,dx = dt$$

$$\frac{1}{3}\int \frac{3e^x}{3e^x+1}\,dx = \int dt$$

$$\frac{1}{3}\ln(3e^x+1) = t + c'$$

put $t=0$, $x=0$ $\Rightarrow c' = \frac{1}{3}\ln 4$ $\rightarrow$

$$\frac{1}{3}\ln(3e^x+1) = t + \frac{1}{3}\ln 4$$

$$\ln(3e^x+1) = 3t + \ln 4$$

$$\ln\frac{3e^x+1}{4} = 3t$$

$$\frac{3e^x+1}{4} = e^{3t}$$

$$3e^x = 4e^{3t} - 1$$

$$e^x = \frac{4}{3}e^{3t} - \frac{1}{3}$$

$$x = \ln\left[\frac{4}{3}e^{3t} - \frac{1}{3}\right]$$

1 A particle P of mass 1 kg is moving along a straight line against a resistive force of magnitude $\frac{10\sqrt{v}}{(t+1)^2}$ N, where $v\,\mathrm{m\,s^{-1}}$ is the speed of P at time t s. When $t = 0$, $v = 25$.

Find an expression for v in terms of t. [5]

Given $m = 1\text{kg}$ $t=0,\ v=25$

$$F = -\frac{10\sqrt{v}}{(t+1)^2}$$

use $F = ma$

$$\frac{-10\sqrt{v}}{(t+1)^2} = 1 \cdot \frac{dv}{dt}$$

$$\frac{-10}{(t+1)^2}\,dt = \frac{1}{\sqrt{v}}\,dv$$

$$\int v^{-1/2}\,dv = -10\int (t+1)^{-2}\,dt$$

$$\frac{v^{1/2}}{1/2} = -10\frac{(t+1)^{-1}}{-1} + c$$

$$2\sqrt{v} = \frac{10}{t+1} + c$$

put $t=0,\ v=25 \Rightarrow c=0$

$$2\sqrt{v} = \frac{10}{t+1}$$

$$\sqrt{v} = \frac{5}{t+1}$$

$$v = \frac{25}{(t+1)^2}$$

Q.6 May/June/P33/2021

5 A particle P of mass m kg is projected vertically upwards from a point O, with speed $20\,\text{m s}^{-1}$, and moves under gravity. There is a resistive force of magnitude $2mv$ N, where $v\,\text{m s}^{-1}$ is the speed of P at time t s after projection.

(a) Find an expression for v in terms of t, while P is moving upwards. [6]

The displacement of P from O is x m at time t s.

(b) Find an expression for x in terms of t, while P is moving upwards. [2]

(a) Given mass $= m$ $t = 0$, $v = 20$

$$F = -2mv - mg$$

use $F = ma$

$$-2mv - mg = m\frac{dv}{dt}$$

$$-2(v+5) = \frac{dv}{dt}$$

$$-2\,dt = \frac{1}{v+5}\,dv$$

$$\int \frac{1}{v+5}\,dv = \int -2\,dt$$

$$\ln(v+5) = -2t + c$$

put $t = 0$, $v = 20 \Rightarrow c = \ln 25$

$$\ln(v+5) = -2t + \ln 25$$

$$\ln 25 - \ln(v+5) = 2t$$

$$\ln\frac{25}{v+5} = 2t$$

$$\frac{25}{v+5} = e^{2t}$$

$$e^{2t}v = 25 - 5e^{2t}$$

$$v = 25e^{-2t} - 5$$

(b)

$$\frac{dx}{dt} = 25e^{-2t} - 5$$

$$dx = (25e^{-2t} - 5)\,dt$$

$$\int dx = \int (25e^{-2t} - 5)\,dt$$

$$x = \frac{25}{-2}e^{-2t} - 5t + c'$$

put $t = 0$, $x = 0 \Rightarrow c' = \frac{25}{2}$

$$x = -\frac{25}{2}e^{-2t} - 5t + \frac{25}{2}$$

$$x = \frac{25}{2}(1 - e^{-2t}) - 5t$$

2 A particle P of mass m kg moves along a horizontal straight line with acceleration $a\,\mathrm{ms}^{-2}$ given by

$$a = \frac{v(1-2t^2)}{t},$$

where $v\,\mathrm{ms}^{-1}$ is the velocity of P at time t s.

(a) Find an expression for v in terms of t and an arbitrary constant. [3]

(b) Given that $a = 5$ when $t = 1$, find an expression, in terms of m and t, for the horizontal force acting on P at time t. [3]

(a) Given

$$a = \frac{v(1-2t^2)}{t}$$

$$\frac{dv}{dt} = \frac{v(1-2t^2)}{t}$$

$$\frac{dv}{v} = \left(\frac{1}{t} - 2t\right) dt$$

$$\int \frac{dv}{v} = \int \left(\frac{1}{t} - 2t\right) dt$$

$$\ln|v| = \ln t - t^2 + c$$

$$\ln|v| = \ln t - t^2 + \ln A \qquad \text{let } c = \ln A$$

$$\ln|v| = \ln At - t^2$$

$$\ln \frac{|v|}{At} = -t^2$$

$$|v| = Ate^{-t^2} \Rightarrow v = -Ate^{-t^2}$$

(b)

$$a = \frac{-Ate^{-t^2}(1-2t^2)}{t}$$

put $a = 5,\ t = 1 \Rightarrow A = 5e$

$$a = -5e \cdot e^{-t^2}(1-2t^2) = 5e^{1-t^2}(2t^2-1)$$

$$\text{Force} = ma$$

$$= m \cdot 5e^{1-t^2}(2t^2-1)$$

$$= 5me^{1-t^2}(2t^2-1)$$

6 A particle P of mass 2 kg moves along a horizontal straight line. The point O is a fixed point on this line. At time t s the velocity of P is $v\,\mathrm{m\,s^{-1}}$ and the displacement of P from O is x m.

A force of magnitude $\left(8x - \frac{128}{x^3}\right)$ N acts on P in the direction OP. When $t = 0$, $x = 8$ and $v = -15$.

(a) Show that $v = -\frac{2}{x}(x^2 - 4)$. [5]

(b) Find an expression for x in terms of t. [4]

(a) Given $m = 2$ kg

$$F = 8x - \frac{128}{x^3} \qquad t = 0,\ x = 8,\ v = -15$$

use $F = ma$

$$8x - \frac{128}{x^3} = 2 \cdot v\frac{dv}{dx}$$

$$2v\,dv = (8x - 128x^{-3})\,dx$$

$$\int 2v\,dv = \int (8x - 128x^{-3})\,dx$$

$$v^2 = 4x^2 + 64x^{-2} + c$$

put $x = 8$, $v = -15 \Rightarrow c = -32$

$$v^2 = 4x^2 + \frac{64}{x^2} - 32$$

$$= \frac{4}{x^2}(x^4 - 8x^2 + 16)$$

$$v^2 = \frac{4}{x^2}(x^2 - 4)^2$$

$$v = -\frac{2}{x}(x^2 - 4)$$

(b)

$$\frac{dx}{dt} = -\frac{2}{x}(x^2 - 4)$$

$$\frac{x}{x^2 - 4}\,dx = -2\,dt$$

$$\int \frac{x}{x^2 - 4}\,dx = \int -2\,dt$$

$$\frac{1}{2}\ln(x^2 - 4) = -2t + c'$$

put $t = 0$, $x = 8 \Rightarrow c' = \frac{1}{2}\ln 60$

$$\frac{1}{2}\ln(x^2 - 4) = -2t + \frac{1}{2}\ln 60$$

$$\frac{1}{2}\left[\ln(x^2 - 4) - \ln 60\right] = -2t$$

$$\ln\frac{x^2 - 4}{60} = -4t$$

$$\frac{x^2 - 4}{60} = e^{-4t}$$

$$x = \sqrt{60e^{-4t} + 4}$$

5 A particle P is moving along a straight line with acceleration $3ku - kv$ where v is its velocity at time t, u is its initial velocity and k is a constant. The velocity and acceleration of P are both in the direction of increasing displacement from the initial position.

(a) Find the time taken for P to achieve a velocity of $2u$. [3]

(b) Find an expression for the displacement of P from its initial position when its velocity is $2u$. [5]

(a) Given $a = 3ku - kv$

$$\frac{dv}{dt} = k(3u - v)$$

$$\frac{dv}{3u-v} = k\,dt$$

$$\int \frac{dv}{3u-v} = \int k\,dt$$

$$-\ln(3u-v) = kt + c$$

put $t=0$, $v=u$

$$-\ln 2u = c$$

$$-\ln(3u-v) = kt - \ln 2u$$

$$\ln \frac{2u}{3u-v} = kt$$

put $v = 2u$

$$\ln 2 = kt \Rightarrow t = \frac{1}{k}\ln 2$$

(b)

$$a = 3ku - kv$$

$$v\frac{dv}{dx} = k(3u - v)$$

$$\frac{v}{3u-v}\,dv = k\,dx$$

$$\int \frac{v}{3u-v}\,dv = \int k\,dx$$

$$\int -1 + \frac{3u}{3u-v}\,dv = \int k\,dx$$

$$-v - 3u\ln(3u-v) = kx + c'$$

put $x=0$, $v=u \Rightarrow c' = -u - 3u\ln 2u$

$$-v - 3u\ln(3u-v) = kx - u - 3u\ln 2u$$

put $v = 2u$

$$-2u - 3u\ln u = kx - u - 3u\ln 2u$$

$$kx = -u + 3u\ln 2$$

$$x = \frac{u}{k}(3\ln 2 - 1)$$

Q.10 May/June/P33/2020

2 A particle Q of mass m kg falls from rest under gravity. The motion of Q is resisted by a force of magnitude mkv N, where $v\,\text{ms}^{-1}$ is the speed of Q at time t s and k is a positive constant.

Find an expression for v in terms of g, k and t. [6]

Given $F = mg - mkv$

use $F = ma$

$$m(g - kv) = m\frac{dv}{dt}$$

$$\frac{dv}{g - kv} = dt$$

$$\int \frac{dv}{g - kv} = \int dt$$

$$-\frac{1}{k}\ln(g - kv) = t + c$$

put $t = 0,\ v = 0 \Rightarrow c = -\frac{1}{k}\ln g$

$$-\frac{1}{k}\ln(g - kv) = t - \frac{1}{k}\ln g$$

$$t = \frac{1}{k}\ln g - \frac{1}{k}\ln(g - kv)$$

$$t = -\frac{1}{k}[-\ln g + \ln(g - kv)]$$

$$t = -\frac{1}{k}\ln\frac{g - kv}{g}$$

$$\ln\frac{g - kv}{g} = -kt$$

$$\frac{g - kv}{g} = e^{-kt}$$

$$g - kv = ge^{-kt}$$

$$g - ge^{-kt} = kv$$

$$v = \frac{g}{k}(1 - e^{-kt})$$

Q.11 Oct/Nov/P31+P33/2020

7 A particle P moving in a straight line has displacement x m from a fixed point O on the line at time t s. The acceleration of P, in m s^{-2}, is given by $\frac{200}{x^2} - \frac{100}{x^3}$ for $x > 0$. When $t = 0$, $x = 1$ and P has velocity 10 m s^{-1} directed towards O.

(a) Show that the velocity v m s^{-1} of P is given by $v = \frac{10(1-2x)}{x}$. [5]

(b) Show that x and t are related by the equation $e^{-40t} = (2x-1)e^{2x-2}$ and deduce what happens to x as t becomes large. [5]

(a) Given $a = \frac{200}{x^2} - \frac{100}{x^3}$

$$v\frac{dv}{dx} = \frac{200}{x^2} - \frac{100}{x^3}$$

$$v\,dv = \left(\frac{200}{x^2} - \frac{100}{x^3}\right)dx$$

$$\int v\,dv = \int\left(\frac{200}{x^2} - \frac{100}{x^3}\right)dx$$

$$\frac{v^2}{2} = \frac{200x^{-1}}{-1} - \frac{100x^{-2}}{-2} + c$$

$$\frac{v^2}{2} = \frac{50}{x^2} - \frac{200}{x} + c$$

put $x = 1$, $v = -10 \Rightarrow c = 200$

$$\frac{v^2}{2} = \frac{50}{x^2} - \frac{200}{x} + 200$$

$$v^2 = \frac{100 - 400x + 200x^2}{x^2} = \frac{100(4x^2 - 4x + 1)}{x^2}$$

$$v^2 = \frac{100(2x-1)^2}{x^2}$$

$$v = -\frac{100(2x-1)}{x}$$

$$v = \frac{10(1-2x)}{x}$$

(b)

$$v = \frac{10(1-2x)}{x}$$

$$\frac{dx}{dt} = \frac{10(1-2x)}{x}$$

$$\frac{x}{1-2x}dx = 10\,dt$$

$$\frac{1}{2}\left(\frac{1}{1-2x} - 1\right)dx = 10\,dt$$

$$\frac{1}{2}\int \frac{1}{1-2x} - 1\,dx = \int 10\,dt$$

$$-\frac{1}{4}\ln|1-2x| - \frac{x}{2} = 10t + c'$$

put $t = 0$, $x = 1 \Rightarrow c' = -\frac{1}{2}$

$$-\frac{1}{4}\ln|1-2x| - \frac{x}{2} = 10t - \frac{1}{2}$$

$$-\ln|1-2x| - 2x = 40t - 2$$

$$\ln|1-2x| = -40t - 2x + 2$$

$$|1-2x| = e^{-40t} \cdot e^{2-2x}$$

$$e^{-40t} = (2x-1)e^{2x-2}$$

Q.12 Oct/Nov/P32/2020

7 A particle P of mass m kg moves in a horizontal straight line against a resistive force of magnitude mkv^2 N, where $v\,\mathrm{m\,s^{-1}}$ is the speed of P after it has moved a distance x m and k is a positive constant. The initial speed of P is $u\,\mathrm{m\,s^{-1}}$.

(a) Show that $x = \frac{1}{k}\ln 2$ when $v = \frac{1}{2}u$. [4]

Beginning at the instant when the speed of P is $\frac{1}{2}u$, an additional force acts on P. This force has magnitude $\frac{5m}{v}$ N and acts in the direction of increasing x.

(b) Show that when the speed of P has increased again to $u\,\mathrm{m\,s^{-1}}$, the total distance travelled by P is given by an expression of the form

$$\frac{1}{3k}\ln\left(\frac{A-ku^3}{B-ku^3}\right),$$

stating the values of the constants A and B. [7]

(a) Given $F = -mkv^2$

use $F = ma$

$$-mkv^2 = mv\frac{dv}{dx}$$

$$-k\,dx = \frac{1}{v}\,dv$$

$$\int \frac{1}{v}\,dv = \int -k\,dx$$

$$\ln v = -kx + c$$

put $x = 0,\ v = u \Rightarrow c = \ln u$

$$\ln v = -kx + \ln u$$

$$kx = \ln u - \ln v$$

$$x = \frac{1}{k}\left[\ln u - \ln\frac{u}{2}\right]$$

$$x = \frac{1}{k}\ln 2$$

(b) Now $F = -mkv^2 + \frac{5m}{v}$

$$mv\frac{dv}{dx} = -mkv^2 + \frac{5m}{v}$$

$$v\frac{dv}{dx} = \frac{5-kv^3}{v}$$

$$\frac{v^2}{5-kv^3}\,dv = dx$$

→

$$-\frac{1}{3k}\int \frac{-3kv^2}{5-kv^3}\,dv = \int dx$$

$$-\frac{1}{3k}\ln(5-kv^3) = x + c'$$

put $v = \frac{u}{2}$, $x = \frac{1}{k}\ln 2$

$$c' = -\frac{1}{3k}\ln\left(5-\frac{ku^3}{8}\right) - \frac{1}{k}\ln 2$$

$$= -\frac{1}{3k}\ln\left(\frac{40-ku^3}{8}\right) - \frac{1}{3k}\ln 8$$

$$c' = -\frac{1}{3k}\ln(40-ku^3)$$

$$-\frac{1}{3k}\ln(5-kv^3) = x - \frac{1}{3k}\ln(40-ku^3)$$

$$x = \frac{1}{3k}\left[\ln(40-ku^3) - \ln(5-kv^3)\right]$$

$$x = \frac{1}{3k}\ln\left(\frac{40-ku^3}{5-ku^3}\right) \quad \text{use } v = u$$