AS Level Further Mathematics

Topic: Polar Coordinates 3D

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6 (a) Show that the curve with Cartesian equation

$$(x^2+y^2)^2 = 36(x^2-y^2)$$

has polar equation $r^2 = 36\cos 2\theta$. [3]

The curve C has polar equation $r^2 = 36\cos 2\theta$, for $-\frac{1}{4}\pi \leqslant \theta \leqslant \frac{1}{4}\pi$.

(b) Sketch C and state the maximum distance of a point on C from the pole. [3]

(c) Find the area of the region enclosed by C. [2]

(d) Find the maximum distance of a point on C from the initial line, giving the answer in exact form. [6]

$$(x^2+y^2)^2 = 36(x^2-y^2)$$

put $x = r\cos\theta$, $y = r\sin\theta$

$$[r^2\cos^2\theta + r^2\sin^2\theta]^2 = 36[r^2\cos^2\theta - r^2\sin^2\theta]$$

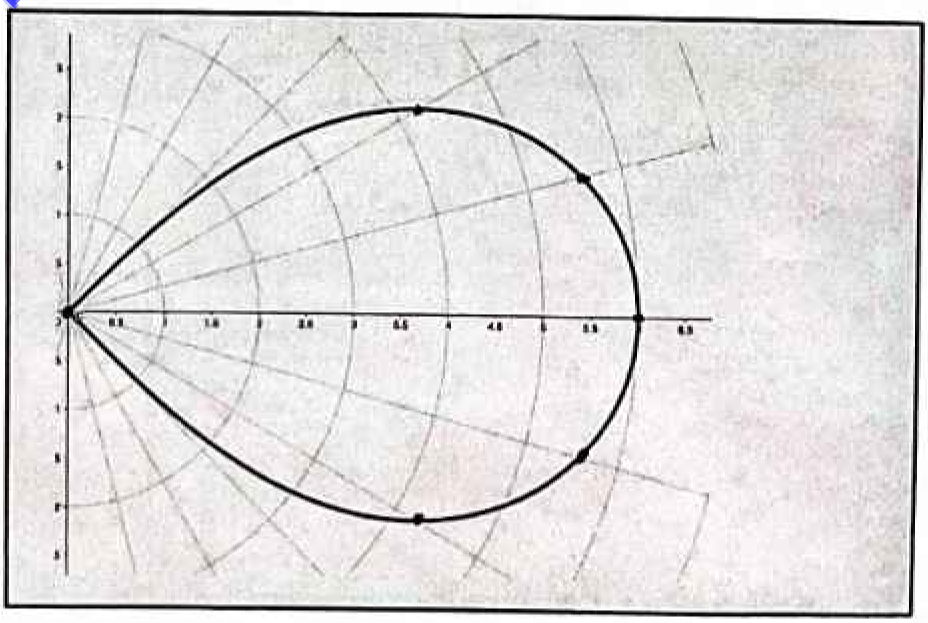
$$[r^2(\cos^2\theta+\sin^2\theta)]^2 = 36r^2[\cos^2\theta - \sin^2\theta]$$

$$r^4 = 36r^2\cos 2\theta$$

as $\cos^2\theta + \sin^2\theta = 1$
$\cos^2\theta - \sin^2\theta = \cos 2\theta$

$$r^2 = 36\cos 2\theta$$

θ	r
$-\pi/4$	0
$-\frac{\pi}{6}$	4.24
$-\frac{\pi}{12}$	5.58
0	6
$\frac{\pi}{12}$	5.58
$\frac{\pi}{6}$	4.24
$\frac{\pi}{4}$	0



(c)

$$\text{Area} = \frac{1}{2}\int_a^b r^2\,d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 36\cos 2\theta\,d\theta$$

$$= 18\left|\frac{\sin 2\theta}{2}\right|_{-\frac{\pi}{4}}^{\pi/4} = 9\left[\sin -\frac{\pi}{2} - \sin\frac{\pi}{2}\right] = 9(2) = 18$$

(d) For maximum distance from initial line, put

$$\frac{dy}{d\theta} = 0$$

$$y = r\sin\theta = \sqrt{36\cos 2\theta}\cdot\sin\theta$$

$$\frac{dy}{d\theta} = 6\sqrt{\cos 2\theta}\cdot\cos\theta + 6\sin\theta\cdot\frac{-2\sin 2\theta}{2\sqrt{\cos 2\theta}}$$

$$= \frac{6}{\sqrt{\cos 2\theta}}\left[\cos 2\theta\cos\theta - \sin 2\theta\sin\theta\right]$$

put $\frac{dy}{d\theta} = 0$

$$\cos 2\theta\cos\theta - \sin 2\theta\sin\theta = 0$$

$$\cos 3\theta = 0 \quad \text{as } \cos A\cos B - \sin A\sin B = \cos(A+B)$$

$$3\theta = \pm\frac{\pi}{2}$$

$$\theta = \pm\frac{\pi}{6}$$

$$\text{Maximum distance} = \sqrt{36\cos\frac{\pi}{2}} \times \sin\left(\frac{\pi}{6}\right)$$

$$= \frac{3\sqrt{2}}{2}$$

5 The curve C has polar equation $r = a\sec^2\theta$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{4}\pi$.

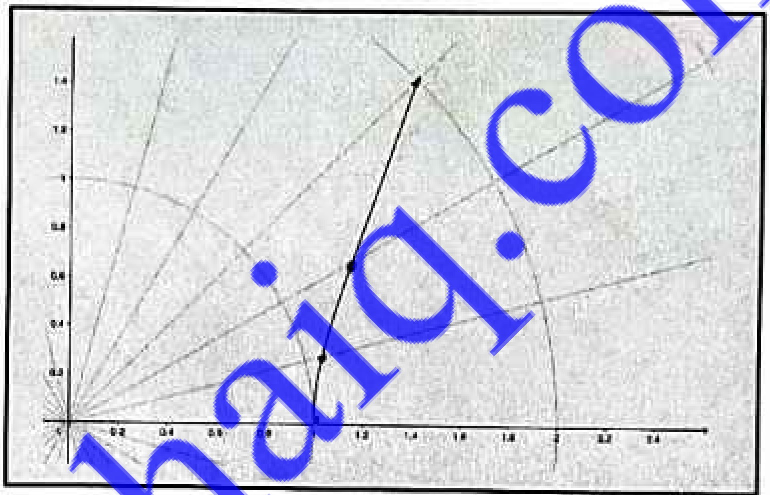
(a) Sketch C, stating the polar coordinates of the point of intersection of C with the initial line and also with the half-line $\theta = \frac{1}{4}\pi$. [3]

(b) Find the maximum distance of a point of C from the initial line. [2]

(c) Find the area of the region enclosed by C, the initial line and the half-line $\theta = \frac{1}{4}\pi$. [4]

(d) Find, in the form $y = f(x)$, the Cartesian equation of C. [3]

θ	r
0	a
$\frac{\pi}{12}$	$1.07a$
$\frac{\pi}{6}$	$1.33a$
$\frac{\pi}{4}$	$2a$



(b) For maximum distance from initial line, put $\frac{dy}{d\theta} = 0$

$$y = r\sin\theta$$
$$= a\sec^2\theta\sin\theta = \frac{a\sin\theta}{\cos^2\theta}$$

$$\frac{dy}{d\theta} = \frac{\cos^2\theta(a\cos\theta) - a\sin\theta(2\cos\theta)(-\sin\theta)}{\cos^4\theta}$$

$$= \frac{a\cos^3\theta + 2a\cos\theta(1-\cos^2\theta)}{\cos^4\theta}$$

$$= \frac{a\cos^3\theta + 2a\cos\theta - 2a\cos^3\theta}{\cos^4\theta} = \frac{2a\cos\theta - a\cos^3\theta}{\cos^4\theta}$$

put $\frac{2a\cos\theta - a\cos^3\theta}{\cos^4\theta} = 0 \Rightarrow a\cos\theta(2-\cos^2\theta) = 0$

$\cos$

6 The curve C has polar equation $r^2 = \tan^{-1}\left(\frac{1}{2}\theta\right)$, where $0 \leqslant \theta \leqslant 2$.

(a) Sketch C and state, in exact form, the greatest distance of a point on C from the pole. [3]

(b) Find the exact value of the area of the region bounded by C and the half-line $\theta = 2$. [5]

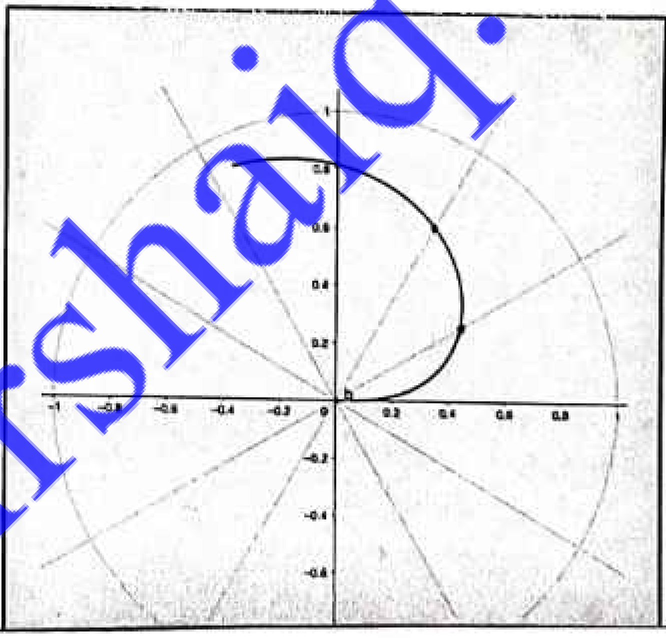
Now consider the part of C where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(c) Show that, at the point furthest from the half-line $\theta = \frac{1}{2}\pi$,

$$(\theta^2+4)\tan^{-1}\left(\tfrac{1}{2}\theta\right)\sin\theta - \cos\theta = 0$$

and verify that this equation has a root between 0.6 and 0.7. [5]

θ	r
0	0
$\frac{\pi}{6}$	0.51
$\frac{\pi}{3}$	0.69
$\frac{\pi}{2}$	0.82
2	0.89



(b) Area $= \frac{1}{2}\int_0^2 1 \cdot \tan^{-1}\left(\frac{\theta}{2}\right) d\theta$ using By parts

$= \frac{1}{2}\left[\left|\theta \tan^{-1}\left(\frac{\theta}{2}\right)\right|_0^2 - \int_0^2 \theta \cdot \frac{2}{\theta^2+4} \right]$ as $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

$= \frac{1}{2}\left[\left|\theta \tan^{-1}\left(\frac{\theta}{2}\right)\right|_0^2 - \left[\ln(\theta^2+4)\right]_0^2 \right]$

$= \frac{1}{2}\left[(2\tan^{-1}(1) - 0) - (\ln 8 - \ln 4) \right]$

$= \frac{1}{2}\left[\frac{\pi}{2} - \ln 2 \right] = \frac{\pi}{4} - \frac{1}{2}\ln 2$

(c) For the point furthest from half line $\theta = \frac{\pi}{2}$

put $\frac{dx}{d\theta} = 0$

$$x = r\cos\theta$$

$$x = \left(\tan^{-1}\left(\frac{\theta}{2}\right)\right)^{1/2}\cos\theta$$

$$\frac{dx}{d\theta} = -\left(\tan^{-1}\left(\frac{\theta}{2}\right)\right)^{1/2}\sin\theta + \cos\theta \cdot \frac{1}{2\sqrt{\tan^{-1}\frac{\theta}{2}}} \cdot \frac{1}{1+\frac{\theta^2}{4}} \cdot \frac{1}{2}$$

$$-\sin\theta\sqrt{\tan^{-1}\left(\frac{\theta}{2}\right)} + \frac{\cos\theta}{(4+\theta^2)\sqrt{\tan^{-1}\left(\frac{\theta}{2}\right)}} = 0$$

$$-(\theta^2+4)\tan^{-1}\left(\frac{\theta}{2}\right)\sin\theta + \cos\theta = 0$$

$$(\theta^2+4)\tan^{-1}\left(\frac{\theta}{2}\right)\sin\theta - \cos\theta = 0$$

Let $f(\theta) = (\theta^2+4)\tan^{-1}\left(\frac{\theta}{2}\right)\sin\theta - \cos\theta$

$f(0.6) = -0.108$

$f(0.7) = 0.209$

This shows that root lie between 0.6 and 0.7

6 The curve C has Cartesian equation $x^2+xy+y^2=a$, where a is a positive constant.

(a) Show that the polar equation of C is $r^2=\dfrac{2a}{2+\sin 2\theta}$. [3]

(b) Sketch the part of C for $0 \leqslant \theta \leqslant \frac{1}{4}\pi$. [2]

The region R is enclosed by this part of C, the initial line and the half-line $\theta=\frac{1}{4}\pi$.

(c) It is given that $\sin 2\theta$ may be expressed as $\dfrac{2\tan\theta}{1+\tan^2\theta}$. Use this result to show that the area of R is

$$\frac{1}{2}a\int_0^{\frac{1}{4}\pi}\frac{1+\tan^2\theta}{1+\tan\theta+\tan^2\theta}\,d\theta$$

and use the substitution $t=\tan\theta$ to find the exact value of this area. [8]

(a)

$$x^2+xy+y^2=a$$

put $x=r\cos\theta$, $y=r\sin\theta$, $x^2+y^2=r^2$

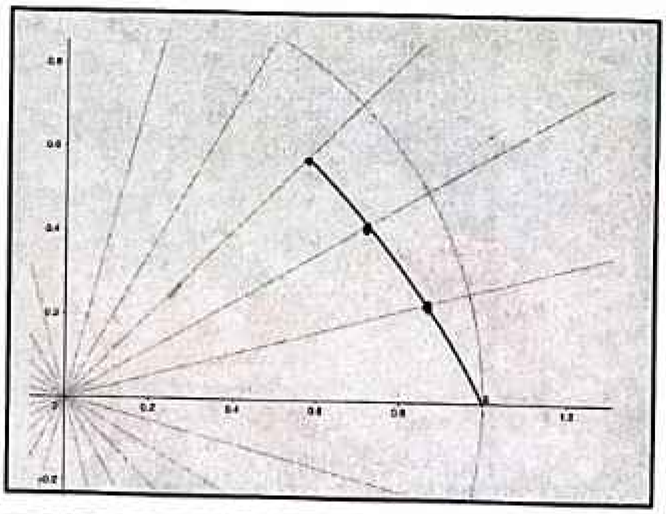
$$x^2+y^2+xy=a$$

$$r^2+r^2\sin\theta\cos\theta=a$$

$$r^2\left(1+\tfrac{1}{2}\sin 2\theta\right)=a$$

$$r^2=\frac{2a}{2+\sin 2\theta}$$

θ	r
0	a
$\frac{\pi}{12}$	$0.9a$
$\frac{\pi}{6}$	$0.84a$
$\frac{\pi}{4}$	$0.82a$



(c) $R = \text{Area} = \frac{1}{2}\int_0^{\pi/4} r^2\,d\theta = \frac{1}{2}\int_0^{\pi/4} \frac{2a}{2+\sin 2\theta}\,d\theta$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{2a}{2+\frac{2\tan\theta}{1+\tan^2\theta}}\,d\theta$$

$$= a\int_0^{\pi/4} \frac{1+\tan^2\theta}{2(1+\tan^2\theta)+2\tan\theta} = \frac{a}{2}\int_0^{\pi/4}\frac{1+\tan^2\theta}{1+\tan\theta+\tan^2\theta}\,d\theta$$

put $t = \tan\theta$

θ	0	$\pi/4$
t	0	1

$dt = \sec^2\theta\,d\theta$

$\frac{dt}{1+\tan^2\theta} \cdot d\theta$

$\frac{dt}{1+t^2} = d\theta$

$$R = \frac{1}{2}a\int_0^1 \frac{1+t^2}{1+t+t^2}\cdot\frac{1}{1+t^2}\,dt$$

$$= \frac{1}{2}a\int_0^1 \frac{1}{1+t+t^2}\,dt = \frac{1}{2}a\int_0^1 \frac{1}{(t+\frac{1}{2})^2+\frac{3}{4}}\,dt$$

$$= \frac{1}{2}a\int_0^1 \frac{1}{(t+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dt$$

$$= \frac{1}{2}a\left|\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t+1/2}{\sqrt{3}/2}\right)\right|_0^1 \quad \text{as } \int\frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{a}{\sqrt{3}}\left|\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right|_0^1 = \frac{a}{\sqrt{3}}\left[\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}\right]$$

$$= \frac{a}{\sqrt{3}}\left[\frac{\pi}{3}-\frac{\pi}{6}\right]$$

$$= \frac{\pi a}{6\sqrt{3}}$$

5 The curve C has polar equation $r = a\cot\left(\frac{1}{3}\pi - \theta\right)$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{6}\pi$.

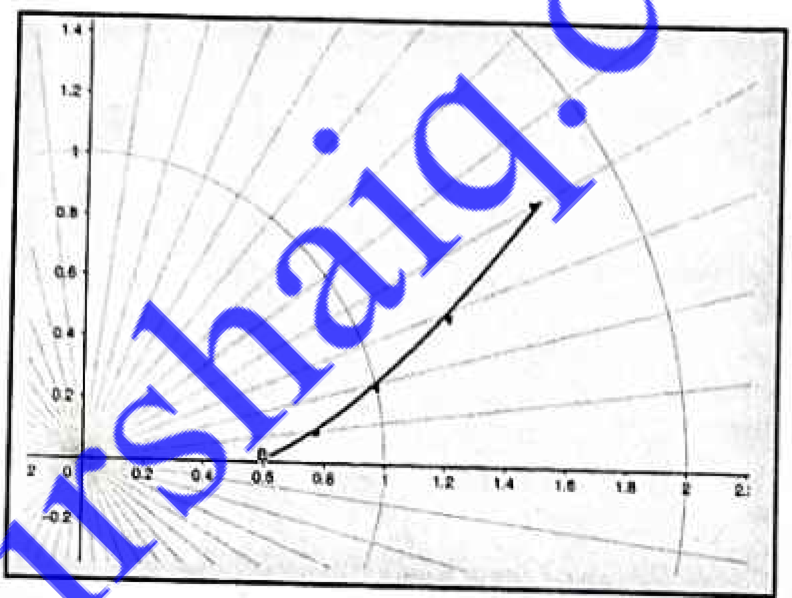
It is given that the greatest distance of a point on C from the pole is $2\sqrt{3}$.

(a) Sketch C and show that $a = 2$. [3]

(b) Find the exact value of the area of the region bounded by C, the initial line and the half-line $\theta = \frac{1}{6}\pi$. [4]

(c) Show that C has Cartesian equation $2\left(x + y\sqrt{3}\right) = \left(x\sqrt{3} - y\right)\sqrt{x^2 + y^2}$. [3]

θ	r
0	0.58a
$\frac{\pi}{24}$	0.77a
$\frac{\pi}{12}$	a
$\frac{\pi}{8}$	1.3 a
$\frac{\pi}{6}$	1.73a



(b)

$$\text{Area} = \frac{1}{2}\int_0^{\frac{\pi}{6}} 4\cot^2\left(\frac{\pi}{3} - \theta\right) d\theta$$

$$= 2\int_0^{\pi/6} \left[\operatorname{cosec}^2\left(\frac{\pi}{3} - \theta\right) - 1\right] d\theta \quad \text{as} \quad 1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

$$= 2\left|\cot\left(\frac{\pi}{3} - \theta\right) - \theta\right|_0^{\pi/6}$$

$$= 2\left[\left(\sqrt{3} - \frac{\pi}{6}\right) - \frac{1}{\sqrt{3}}\right]$$

$$= 2\sqrt{3} - \frac{2}{\sqrt{3}} - \frac{\pi}{3} = \frac{4}{\sqrt{3}} - \frac{\pi}{3}$$

(c) $r = \dfrac{2}{\tan\left(\frac{\pi}{3}-\theta\right)}$

$$= 2\,\frac{\cos(\pi/3-\theta)}{\sin\left(\frac{\pi}{3}-\theta\right)}$$

$$= 2\,\frac{\cos\pi/3\cos\theta + \sin\frac{\pi}{3}\sin\theta}{\sin\frac{\pi}{3}\cos\theta - \cos\frac{\pi}{3}\sin\theta}$$

$$= 2\,\frac{\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta}{\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta}$$

$$r = 2\,\frac{r\cos\theta + \sqrt{3}\,r\sin\theta}{\sqrt{3}\,r\cos\theta - r\sin\theta}$$

$$\sqrt{x^2+y^2} = \frac{2(x+y\sqrt{3})}{(x\sqrt{3}-y)}$$

$$2(x+y\sqrt{3}) = (x\sqrt{3}-y)\sqrt{x^2+y^2}$$

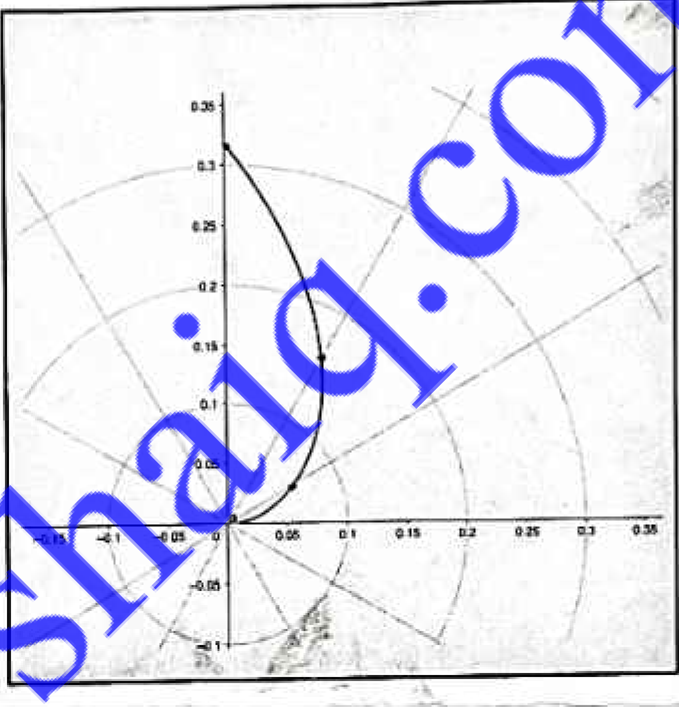
Q.6 May/June/P13/2021

5 The curve C has polar equation $r = \frac{1}{\pi - \theta} - \frac{1}{\pi}$, where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Sketch C. [3]

(b) Show that the area of the region bounded by the half-line $\theta = \frac{1}{2}\pi$ and C is $\frac{3 - 4\ln 2}{4\pi}$. [6]

θ	r
0	0
$\frac{\pi}{6}$	0.063
$\frac{\pi}{3}$	0.159
$\frac{\pi}{2}$	0.318



(b)

$$\text{Area} = \frac{1}{2}\int_0^{\pi/2}\left(\frac{1}{\pi-\theta} - \frac{1}{\pi}\right)^2 d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{1}{(\pi-\theta)^2} - \frac{2}{\pi(\pi-\theta)} + \frac{1}{\pi^2}\, d\theta$$

$$= \frac{1}{2}\left| \frac{1}{\pi-\theta} + \frac{2}{\pi}\ln(\pi-\theta) + \frac{\theta}{\pi^2}\right|_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{2}{\pi} + \frac{2}{\pi}\ln\frac{\pi}{2} + \frac{1}{2\pi}\right) - \left(\frac{1}{\pi} + \frac{2}{\pi}\ln\pi\right)\right]$$

$$= \frac{1}{2}\left[\frac{2}{\pi} + \frac{1}{2\pi} - \frac{1}{\pi} + \frac{2}{\pi}\left(\ln\frac{\pi}{2} - \ln\pi\right)\right]$$

$$= \frac{1}{2}\left[\frac{3}{2\pi} + \frac{2}{\pi}\ln\frac{1}{2}\right] = \frac{3}{4\pi} - \frac{1}{\pi}\ln 2 = \frac{3 - 4\ln 2}{4\pi}$$

Q.7 Oct/Nov/P11+P13/2021

6 The curve C has polar equation $r = 2\cos\theta(1+\sin\theta)$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Find the polar coordinates of the point on C that is furthest from the pole. [5]

(b) Sketch C. [2]

(c) Find the area of the region bounded by C and the initial line, giving your answer in exact form. [6]

(a) For the point that is furthest from pole, put $\frac{dr}{d\theta} = 0$

$$r = 2\cos\theta(1+\sin\theta)$$

$$\frac{dr}{d\theta} = 2\cos\theta(\cos\theta) + 2(1+\sin\theta)(-\sin\theta)$$

$$= 2\cos^2\theta - 2\sin^2\theta - 2\sin\theta$$

$$= 2[1-\sin^2\theta - \sin^2\theta - \sin\theta]$$

$$= 2[1-\sin\theta - 2\sin^2\theta]$$

put $\frac{dr}{d\theta} = 0$

$$1 - \sin\theta - 2\sin^2\theta = 0$$

$$\sin\theta = -1, \quad \sin\theta = \frac{1}{2}$$

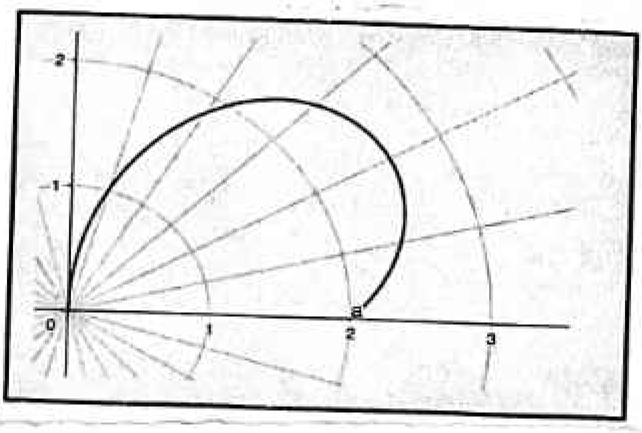
$$\theta = \frac{2\pi}{3} \quad \theta = \frac{\pi}{6}$$

(ignore)

$$r = 2\cos\left(\frac{\pi}{6}\right)\left(1+\sin\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2}$$

$$\left(\frac{3\sqrt{2}}{2}, \frac{\pi}{6}\right)$$

θ	r
0	2
$\frac{\pi}{12}$	2.43
$\frac{\pi}{6}$	2.6
$\frac{\pi}{4}$	2.41
$\frac{\pi}{3}$	1.87
$\frac{5\pi}{12}$	1.02
$\frac{\pi}{2}$	0



(c)

$$\text{Area} = \frac{1}{2}\int_0^{\pi/2} r^2\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 4\cos^2\theta\,(1+\sin\theta)^2\,d\theta$$

$$= 2\int_0^{\pi/2} \cos^2\theta\,(1+2\sin\theta+\sin^2\theta)\,d\theta$$

$$= 2\int_0^{\pi/2} \left[\cos^2\theta + 2\cos^2\theta\sin\theta + \sin^2\theta\cos^2\theta\right] d\theta$$

$$= 2\int_0^{\pi/2} \frac{1}{2}(\cos 2\theta+1) - 2(\cos\theta)^2(-\sin\theta) + \frac{1}{4}\sin^2 2\theta$$

$$= \int_0^{\pi/2} \cos 2\theta + 1 - 4(\cos\theta)^2(-\sin\theta) + \frac{1}{4}(1-\cos 4\theta)\,d\theta$$

$$= \int_0^{\pi/2} \left(\frac{5}{4} + \cos 2\theta - 4(\cos\theta)^2(-\sin\theta) - \frac{1}{4}\cos 4\theta\right) d\theta$$

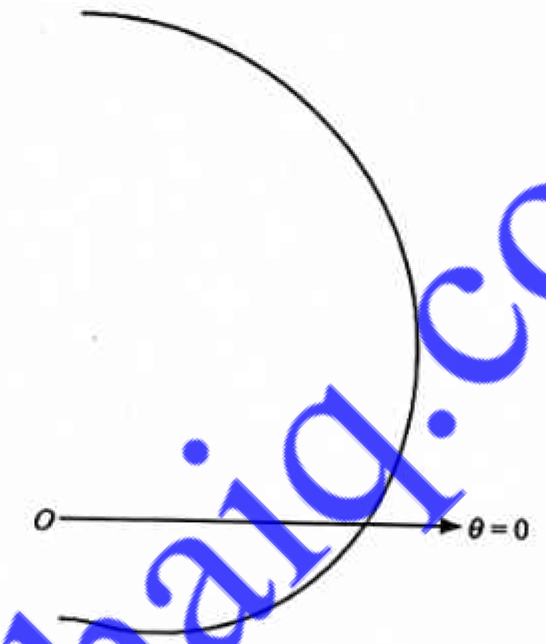
$$= \left|\frac{5}{4}\theta + \frac{1}{2}\sin 2\theta - \frac{4}{3}\cos^3\theta - \frac{1}{16}\sin 4\theta\right|_0^{\pi/2}$$

$$= \left(\frac{5}{8}\pi + 0 - 0 - 0\right) - \left(0 + 0 - \frac{4}{3} - 0\right)$$

$$= \frac{5}{8}\pi + \frac{4}{3}$$

5 The curve C has polar equation $r = 3 + 2\sin\theta$, for $-\pi < \theta \leqslant \pi$.

(a) The diagram shows part of C. Sketch the rest of C on the diagram. [1]

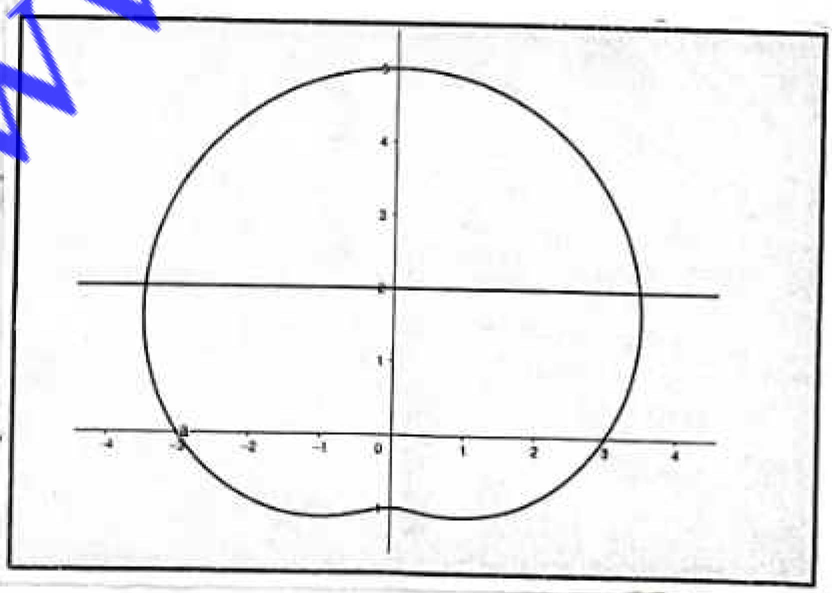


The straight line l has polar equation $r\sin\theta = 2$.

(b) Add l to the diagram in part (a) and find the polar coordinates of the points of intersection of C and l. [5]

(c) The region R is enclosed by C and l, and contains the pole.

Find the area of R, giving your answer in exact form. [6]



(b) $r = 3 + 2\sin\theta$

$r\sin\theta = 2$

$\frac{2}{\sin\theta} = 3 + 2\sin\theta$

$2\sin^2\theta + 3\sin\theta - 2 = 0$

$\sin\theta = \frac{1}{2}$ or $\sin\theta = -2$ (ignore)

$\theta = \frac{\pi}{6}$

$r = 3 + 2\sin(\frac{\pi}{6}) = 4$

$(4, \frac{\pi}{6})$, $(4, \frac{5}{6}\pi)$

(c)

$R = 2 \times \frac{1}{2}\int_{-\pi/2}^{\pi/6} (3 + 2\sin\theta)^2\, d\theta$ + area of the triangle

$= \int_{-\pi/2}^{\pi/6} (9 + 12\sin\theta + 4\sin^2\theta)\, d\theta + 2 \times (4\cos\frac{\pi}{6})$

$= \int_{-\pi/2}^{\pi/6} 9 + 12\sin\theta + 2(1 - \cos 2\theta)\, d\theta + 8\cos\frac{\pi}{6}$

$= \int_{-\pi/2}^{\pi/6} (11 + 12\sin\theta - 2\cos 2\theta)\, d\theta + 8 \times \frac{\sqrt{3}}{2}$

$= \left[11\theta - 12\cos\theta - 2\frac{\sin 2\theta}{2} \right]_{-\pi/2}^{\pi/6} + 4\sqrt{3}$

$= \left(11\frac{\pi}{6} - 12\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2}\right) - \left(-11\frac{\pi}{2} - 12(0) - 0\right) + 4\sqrt{3}$

$= \frac{22}{3}\pi - \frac{5}{2}\sqrt{3}$

Q.9 May/June/P11+P12/2020

7 The curve C_1 has polar equation $r = \theta\cos\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) The point on C_1 furthest from the line $\theta = \frac{1}{2}\pi$ is denoted by P. Show that, at P,

$$2\theta\tan\theta - 1 = 0$$

and verify that this equation has a root between 0.6 and 0.7. [5]

The curve C_2 has polar equation $r = \theta\sin\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. The curves C_1 and C_2 intersect at the pole, denoted by O, and at another point Q.

(b) Find the polar coordinates of Q, giving your answers in exact form. [2]

(c) Sketch C_1 and C_2 on the same diagram. [3]

(d) Find, in terms of π, the area of the region bounded by the arc OQ of C_1 and the arc OQ of C_2. [7]

(a) Furthest point from the line $\theta = \frac{\pi}{2}$, put $\frac{dx}{d\theta} = 0$

$x = r\cos\theta$

$x = \theta\cos^2\theta$

$\frac{dx}{d\theta} = \cos^2\theta - 2\theta\sin\theta\cos\theta$

$\cos^2\theta - 2\theta\sin\theta\cos\theta = 0$

$\cos\theta - 2\theta\sin\theta = 0$

$1 - 2\theta\tan\theta = 0$

$2\theta\tan\theta - 1 = 0$

Let $f(\theta) = 2\theta\tan\theta - 1$

$\left.\begin{aligned} f(0.6) &= -0.179 \\ f(0.7) &= 0.179 \end{aligned}\right\}$ This shows that root lie between 0.6 and 0.7

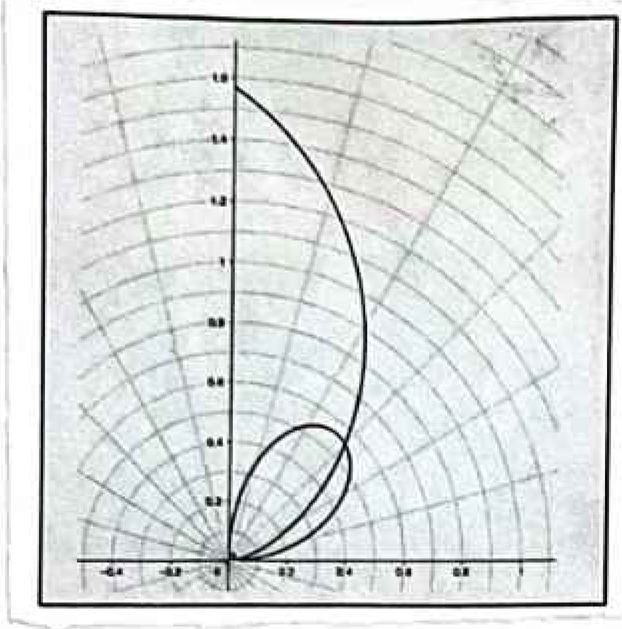
(b) C_1 $r = \theta\cos\theta$

C_2 $r = \theta\sin\theta$

$\theta\sin\theta = \theta\cos\theta$

$\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

$r = \frac{\pi}{4}\cos\left(\frac{\pi}{4}\right) = \frac{\pi}{4\sqrt{2}}$ $\left(\frac{\pi}{4\sqrt{2}}, \frac{\pi}{4}\right)$



(d) $\text{Area} = \frac{1}{2}\int_0^{\pi/4} \theta^2\cos^2\theta\,d\theta - \frac{1}{2}\int_0^{\pi/4} \theta^2\sin^2\theta\,d\theta$

$$= \frac{1}{2}\int_0^{\pi/4} \theta^2(\cos^2\theta - \sin^2\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \theta^2\cos 2\theta\,d\theta$$

$$= \frac{1}{2}\left[\left|\theta^2\frac{\sin 2\theta}{2}\right|_0^{\pi/4} - \int_0^{\pi/4} \frac{\sin 2\theta}{2}(2\theta)\,d\theta \right]$$

$$= \frac{1}{2}\left[\left(\frac{\pi^2}{16}\cdot\frac{1}{2} - 0\right) - \left[\left|\theta\cdot\frac{-\cos 2\theta}{2}\right|_0^{\pi/4} - \int_0^{\pi/4} \frac{-\cos 2\theta}{2}\,d\theta \right] \right]$$

$$= \frac{1}{2}\left[\frac{\pi^2}{32} + (0 - 0) - \frac{1}{2}\left|\frac{\sin 2\theta}{2}\right|_0^{\pi/4} \right]$$

$$= \frac{\pi^2}{64} - \frac{1}{8}$$

Q.12 May/June/P11+P12/2019

11 The curve C_1 has polar equation $r^2 = 2\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(i) The point on C_1 furthest from the line $\theta = \frac{1}{2}\pi$ is denoted by P. Show that, at P,

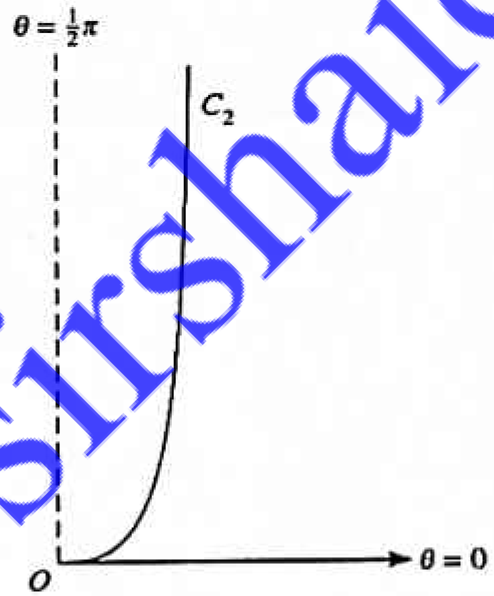
$$2\theta \tan\theta = 1$$

and verify that this equation has a root between 0.6 and 0.7. [5]

The curve C_2 has polar equation $r^2 = \theta\sec^2\theta$, for $0 \leqslant \theta < \frac{1}{2}\pi$. The curves C_1 and C_2 intersect at the pole, denoted by O, and at another point Q.

(ii) Find the exact value of θ at Q. [2]

(iii) The diagram below shows the curve C_2. Sketch C_1 on this diagram. [2]



(iv) Find, in exact form, the area of the region OPQ enclosed by C_1 and C_2. [5]

(i) For furthest from the line $\theta = \frac{\pi}{2}$, put $\frac{dx}{d\theta} = 0$

$x = r\cos\theta$

$x = \sqrt{2}\,\theta^{1/2}\cos\theta$

$\frac{dx}{d\theta} = \sqrt{2}\left[(-\sin\theta)\sqrt{\theta} + \cos\theta\frac{1}{2\sqrt{\theta}}\right] = \sqrt{2}\left[\frac{-2\theta\sin\theta + \cos\theta}{2\sqrt{\theta}}\right]$

put $\frac{dx}{d\theta} = 0$

$-2\theta\sin\theta + \cos\theta = 0 \Rightarrow -2\theta\sin\theta = -\cos\theta \Rightarrow 2\theta\tan\theta = 1$

let $f(\theta) = 2\theta\tan\theta - 1$

$f(0.6) = -0.179$

$f(0.7) = 0.179$

This shows that root lies between 0.6 and 0.7

(ii) C_1 $\quad r^2 = 2\theta$

C_2 $\quad r^2 = \theta\sec^2\theta$

$2\theta = \theta\sec^2\theta$

$\theta(2 - \sec^2\theta) = 0$

$\theta = 0 \qquad \sec^2\theta = 2$

$\Rightarrow \cos^2\theta = \frac{1}{2}$

$\cos\theta = \frac{1}{\sqrt{2}}$ as $0 \le \theta < \frac{\pi}{2}$

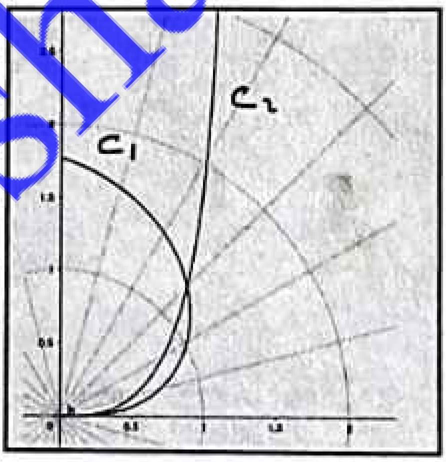
$\theta = \frac{\pi}{4}$

C_1

θ	r
0	0
$\frac{\pi}{12}$	0.72
$\frac{\pi}{6}$	1.02
$\frac{\pi}{4}$	1.25
$\frac{\pi}{3}$	1.45
$\frac{5\pi}{12}$	1.62
$\frac{\pi}{2}$	1.77

C_2

θ	r
0	0
$\frac{\pi}{12}$	0.53
$\frac{\pi}{6}$	0.84
$\frac{\pi}{4}$	1.25
$\frac{\pi}{3}$	2.05
$\frac{5\pi}{12}$	4.42
$\frac{\pi}{2}$	∞



(iv) Area $= \frac{1}{2}\int_0^{\pi/4} 2\theta\, d\theta - \frac{1}{2}\int_0^{\pi/4} \theta\sec^2\theta\, d\theta$

$= \frac{1}{2}\int_0^{\pi/4} \theta(2 - \sec^2\theta)\, d\theta$

$= \frac{1}{2}\left[\left|\theta(2\theta - \tan\theta)\right|_0^{\pi/4} - \int_0^{\pi/4} (2\theta - \tan\theta)\, d\theta \right]$

$= \frac{1}{2}\left[\left|\theta(2\theta - \tan\theta)\right|_0^{\pi/4} - \left|\theta^2 + \ln\cos\theta\right|_0^{\pi/4} \right]$

$= \frac{1}{2}\left[\frac{\pi}{4}\left(\frac{\pi}{2} - 1\right) - \left[\left(\frac{\pi^2}{16} + \ln\frac{1}{\sqrt{2}}\right) - 0\right]\right]$

$= \frac{1}{2}\left[\frac{\pi^2}{8} - \frac{\pi}{4} - \frac{\pi^2}{16} + \frac{1}{2}\ln 2 \right] = \frac{1}{4}\ln 2 + \frac{\pi}{8}\left(\frac{\pi}{4} - 1\right)$

Q.12 May/June/P11+P12/2019

5 The curve C has polar equation $r = \ln(1 + \pi - \theta)$, for $0 \leqslant \theta \leqslant \pi$.

(a) Sketch C and state the polar coordinates of the point of C furthest from the pole. [3]

(b) Using the substitution $u = 1 + \pi - \theta$, or otherwise, show that the area of the region enclosed by C and the initial line is

$$\tfrac{1}{2}(1+\pi)\ln(1+\pi)\big(\ln(1+\pi)-2\big)+\pi.$$ [6]

(c) Show that, at the point of C furthest from the initial line,

$$(1+\pi-\theta)\ln(1+\pi-\theta)-\tan\theta = 0$$

and verify that this equation has a root between 1.2 and 1.3. [5]

Q.11 Oct/Nov/P11+P12+P13/2020

7 (a) Show that the curve with Cartesian equation

$$(x^2+y^2)^{\frac{5}{2}} = 4xy(x^2-y^2)$$

has polar equation $r = \sin 4\theta$. [4]

The curve C has polar equation $r = \sin 4\theta$, for $0 \leqslant \theta \leqslant \frac{1}{4}\pi$.

(b) Sketch C and state the equation of the line of symmetry. [3]

(c) Find the exact value of the area of the region enclosed by C. [4]

(d) Using the identity $\sin 4\theta \equiv 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta$, find the maximum distance of C from the line $\theta = \frac{1}{2}\pi$. Give your answer correct to 2 decimal places. [6]

(a) $(x^2+y^2)^{\frac{5}{2}} = 4xy(x^2-y^2)$ put $x^2+y^2 = r^2$, $x = r\cos\theta$, $y = r\sin\theta$

$(r^2)^{5/2} = 4(r\cos\theta)(r\sin\theta)[r^2\cos^2\theta - r^2\sin^2\theta]$

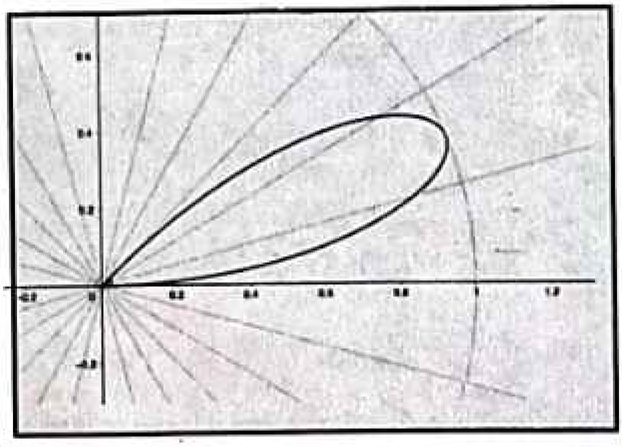
$r^5 = 4r^4\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)$

$r = 2(2\sin\theta\cos\theta)(\cos 2\theta)$

$= 2\sin 2\theta\cos 2\theta = \sin 4\theta$

$r = \sin 4\theta$

θ	r
0	0
$\frac{\pi}{12}$	0.87
$\frac{\pi}{6}$	0.87
$\frac{\pi}{4}$	0



(c)

$$\text{Area} = \frac{1}{2}\int_0^{\pi/4} r^2\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \sin^2 4\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \left(\frac{1-\cos 8\theta}{2}\right) d\theta$$

$$= \frac{1}{4}\left|\theta - \frac{\sin 8\theta}{8}\right|_0^{\pi/4}$$

$$= \frac{1}{4}\left[\left(\frac{\pi}{4} - 0\right) - 0\right] = \frac{\pi}{16}$$

(d) For the maximum distance from the line $\theta = \frac{\pi}{2}$

put $\frac{dx}{d\theta} = 0$

$$x = r\cos\theta$$

$$x = (\sin 4\theta)\cos\theta$$

$$x = [4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta]\cos\theta \quad \text{by given Identity}$$

$$x = 4\sin\theta\cos^4\theta - 4\sin^3\theta\cos^2\theta$$

put $\frac{dx}{d\theta} = 0$

$$4\left[\left(\cos^5\theta + \sin\theta(4\cos^3\theta)(-\sin\theta)\right) - \left(3\sin^2\theta\cos^3\theta + \sin^3\theta(2\cos\theta)(-\sin\theta)\right)\right] = 0$$

$$\cos^5\theta - 4\sin^2\theta\cos^3\theta - 3\sin^2\theta\cos^3\theta + 2\sin^4\theta\cos\theta = 0$$

$$\cos\theta\left[\cos^4\theta - 7\sin^2\theta\cos^2\theta + 2\sin^4\theta\right] = 0$$

$\cos\theta = 0$ or $2\sin^4\theta - 7\sin^2\theta\cos^2\theta + \cos^4\theta = 0$

$$2\tan^4\theta - 7\tan^2\theta + 1 = 0$$

$$\tan^2\theta = \frac{7 \pm \sqrt{41}}{4}$$

$$\tan^2\theta = 3.35, \quad \tan^2\theta = 0.15$$

$$\tan\theta = \pm 1.83, \quad \tan\theta = \pm 0.39$$

$\theta = \frac{\pi}{2}$ $\quad \theta = \pm 1.071, \quad \theta = \pm 0.369$

x is maximum at $\theta = 0.369$

$$x = 0.93$$

Q.10 May/June/P13/2020

5 The curve C has polar equation $r = a\tan\theta$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{4}\pi$.

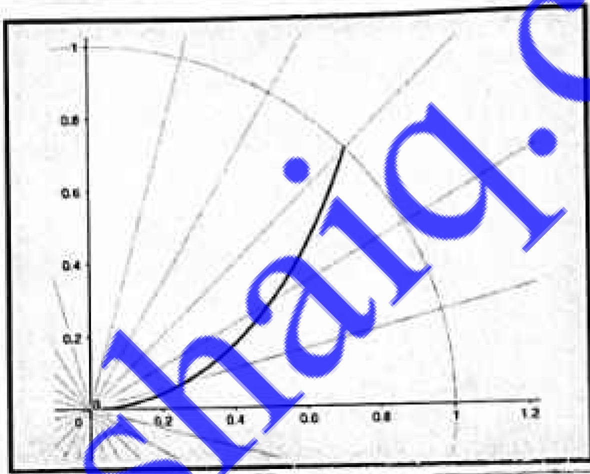
(a) Sketch C and state the greatest distance of a point on C from the pole. [2]

(b) Find the exact value of the area of the region bounded by C and the half-line $\theta = \frac{1}{4}\pi$. [4]

(c) Show that C has Cartesian equation $y = \dfrac{x^2}{\sqrt{a^2 - x^2}}$. [3]

(d) Using your answer to part (b), deduce the exact value of $\displaystyle\int_0^{\frac{1}{2}a\sqrt{2}} \frac{x^2}{\sqrt{a^2-x^2}}\,dx$. [2]

θ	r
0	0
$\frac{\pi}{12}$	$0.27a$
$\frac{\pi}{6}$	$0.58a$
$\frac{\pi}{4}$	a



(b) $\text{Area} = \frac{1}{2}\int_0^{\pi/4} a^2 \tan^2\theta \, d\theta$

$= \frac{a^2}{2}\int_0^{\pi/4} (\sec^2\theta - 1)\, d\theta$

$= \frac{a^2}{2}\left|\tan\theta - \theta\right|_0^{\pi/4}$

$= \frac{a^2}{2}\left[\left(1 - \frac{\pi}{4}\right) - 0\right] = \frac{a^2}{2}\left(1 - \frac{\pi}{4}\right)$

(c) $r = a\,\frac{\sin\theta}{\cos\theta}$

$r = a\,\frac{r\sin\theta}{r\cos\theta}$

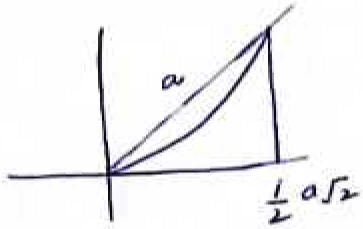
$\sqrt{x^2+y^2} = a\,\frac{y}{x}$

$x^2(x^2+y^2) = a^2y^2$

$x^4 = (a^2 - x^2)y^2 \Rightarrow y = \frac{x^2}{\sqrt{a^2-x^2}}$

①

$$\int_0^{\frac{1}{2}a\sqrt{2}} \frac{x^2}{\sqrt{a^2-x^2}}\,dx$$



$$= \frac{1}{2}\left(a\cos\frac{\pi}{4}\right)\left(a\sin\frac{\pi}{4}\right) - \frac{1}{2}a^2\left(1-\frac{\pi}{4}\right)$$

$$= \frac{1}{2}\cdot\left(\frac{1}{\sqrt{2}}a\right)\cdot\left(a\frac{1}{\sqrt{2}}\right) - \frac{a^2}{2} + \frac{\pi}{8}a^2$$

$$= \frac{a^2}{4} - \frac{a^2}{2} + \frac{\pi}{8}a^2$$

$$= -\frac{a^2}{4} + \frac{\pi}{8}a^2$$

$$= \frac{a^2}{4}\left(\frac{\pi}{2}-1\right)$$

Q.13 May/June/P13/2019

2 The curve C has polar equation $r^2 = \ln(1+\theta)$, for $0 \leqslant \theta \leqslant 2\pi$.

(i) Sketch C. [2]

(ii) Using the substitution $u = 1 + \theta$, or otherwise, find the area of the region bounded by C and the initial line, leaving your answer in an exact form. [5]

θ	r
0	0
$\frac{\pi}{2}$	0.97
π	1.19
$\frac{3\pi}{2}$	1.32
2π	1.41

(ii) $\text{Area} = \frac{1}{2}\int_0^{2\pi} \ln(1+\theta)\,d\theta$

put $u = 1+\theta$

$du = d\theta$

θ	0	2π
u	1	$2\pi+1$

$$= \frac{1}{2}\int_1^{2\pi+1} \ln u \, du$$

$$= \frac{1}{2}\left[\left| u \ln u \right|_1^{2\pi+1} - \int_1^{2\pi+1} u \cdot \frac{1}{u}\, du \right]$$

$$= \frac{1}{2}\left[\left| u \ln u \right|_1^{2\pi+1} - \left| u \right|_1^{2\pi+1} \right]$$

$$= \frac{1}{2}\left[\left((2\pi+1)\ln(2\pi+1) - 0 \right) - (2\pi+1-1) \right]$$

$$= \frac{1}{2}(2\pi+1)\ln(2\pi+1) - \pi$$

The curves C_1 and C_2 have polar equations, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$, as follows:

$$C_1 : r = 2(e^{\theta} + e^{-\theta}),$$
$$C_2 : r = e^{2\theta} - e^{-2\theta}.$$

The curves intersect at the point P where $\theta = \alpha$.

(i) Show that $e^{2\alpha} - 2e^{\alpha} - 1 = 0$. Hence find the exact value of α and show that the value of r at P is $4\sqrt{2}$. [6]

(ii) Sketch C_1 and C_2 on the same diagram. [3]

(iii) Find the area of the region enclosed by C_1, C_2 and the initial line, giving your answer correct to 3 significant figures. [5]

C_1: $r = 2(e^{\theta} + e^{-\theta})$

C_2: $r = e^{2\theta} - e^{-2\theta}$

$$e^{2\alpha} - e^{-2\alpha} = 2(e^{\alpha} + e^{-\alpha})$$
$$(e^{\alpha})^2 - (e^{-\alpha})^2 = 2(e^{\alpha} + e^{-\alpha})$$
$$(e^{\alpha} + e^{-\alpha})(e^{\alpha} - e^{-\alpha}) = 2(e^{\alpha} + e^{-\alpha})$$
$$e^{\alpha} - e^{-\alpha} = 2$$
$$e^{2\alpha} - 1 = 2e^{\alpha}$$
$$e^{2\alpha} - 2e^{\alpha} - 1 = 0$$

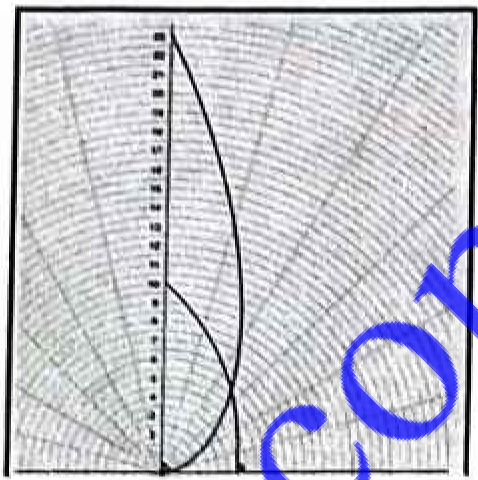
$$e^{\alpha} = 1 + \sqrt{2} \quad \text{or} \quad e^{\alpha} = 1 - \sqrt{2} \text{ (Ignore)}$$

$$\alpha = \ln(1 + \sqrt{2})$$

$$r = 2\left[e^{\ln(1+\sqrt{2})} + \frac{1}{e^{\ln(1+\sqrt{2})}}\right]$$
$$= 2\left[1 + \sqrt{2} + \frac{1}{1+\sqrt{2}}\right]$$
$$= 2\left[1 + \sqrt{2} + \frac{1-\sqrt{2}}{-1}\right] = 2\left[1 + \sqrt{2} - 1 + \sqrt{2}\right] = 4\sqrt{2}$$

θ	r
0	4
$\frac{\pi}{12}$	4.14
$\frac{\pi}{6}$	4.56
$\frac{\pi}{4}$	5.3
$\frac{\pi}{3}$	6.4
$\frac{5\pi}{12}$	7.95

θ	r
0	0
$\frac{\pi}{12}$	1.1
$\frac{\pi}{6}$	2.5
$\frac{\pi}{4}$	4.6
$\frac{\pi}{3}$	8
$\frac{5\pi}{12}$	13.6



(iii) Area $= \frac{1}{2}\int_0^{\ln(1+\sqrt{2})} 4\left(e^{\theta}+e^{-\theta}\right)^2 - \frac{1}{2}\int_0^{\ln(1+\sqrt{2})} \left(e^{2\theta}-e^{-2\theta}\right)^2 d\theta$

$$= \int_0^{\ln(1+\sqrt{2})} 2\left(e^{2\theta}+e^{-2\theta}+2\right) - \frac{1}{2}\left(e^{4\theta}+e^{-4\theta}-2\right) d\theta$$

$$= \int_0^{\ln(1+\sqrt{2})} \left(5+2e^{2\theta}+2e^{-2\theta}-\frac{1}{2}e^{4\theta}-\frac{1}{2}e^{-4\theta}\right) d\theta$$

$$= \left| 5\theta + e^{2\theta} - e^{-2\theta} - \frac{1}{8}e^{4\theta} + \frac{1}{8}e^{-4\theta} \right|_0^{\ln(1+\sqrt{2})}$$

$$= \left(5\ln(1+\sqrt{2}) + (1+\sqrt{2})^2 - (1+\sqrt{2})^2 - \frac{1}{8}(1+\sqrt{2})^4 + \frac{1}{8}(1+\sqrt{2})^{-4}\right) - \left(0+1-1-\frac{1}{8}+\frac{1}{8}\right)$$

$$= 5.82$$

Q.15 May/June/P11+P12/2018

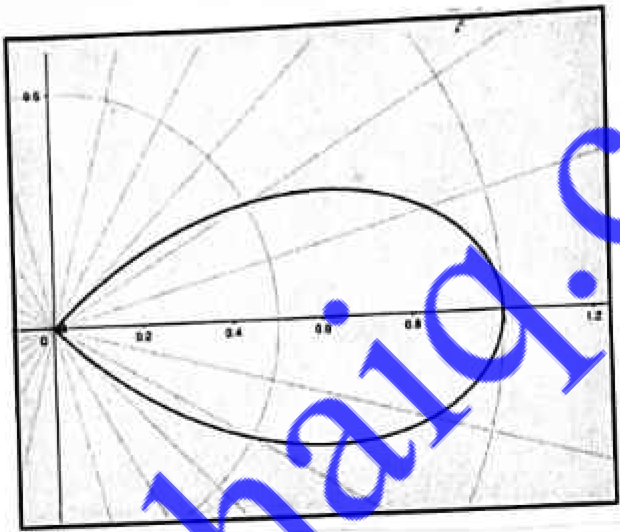
3 The curve C has polar equation $r = \cos 2\theta$, for $-\frac{1}{4}\pi \leqslant \theta \leqslant \frac{1}{4}\pi$.

(i) Sketch C. [2]

(ii) Find the area of the region enclosed by C, showing full working. [3]

(iii) Find a cartesian equation of C. [3]

θ	r
$-\pi/4$	0
$-\pi/6$	0.5
$-\pi/12$	0.87
0	1
$\frac{\pi}{12}$	0.87
$\frac{\pi}{6}$	0.5
$\frac{\pi}{4}$	0



(ii)

$$\text{Area} = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta \, d\theta$$

$$= \frac{1}{2}\int_{-\pi/4}^{\pi/4} \left(\frac{1+\cos 4\theta}{2}\right) d\theta$$

$$= \frac{1}{4}\left|\theta + \frac{\sin 4\theta}{4}\right|_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{4}\left[\left(\frac{\pi}{4}+0\right) - \left(-\frac{\pi}{4}-0\right)\right] = \frac{1}{4}\left(\frac{\pi}{4}+\frac{\pi}{4}\right) = \frac{\pi}{8}$$

(iii)

$$r = \cos 2\theta$$

$$r = \cos^2\theta - \sin^2\theta$$

$$r^3 = r^2\cos^2\theta - r^2\sin^2\theta$$

$$(x^2+y^2)^{3/2} = x^2 - y^2$$

Q.16 May/June/P13/2018

8 The curves C_1 and C_2 have polar equations, for $0 \leqslant \theta \leqslant \pi$, as follows:

$$C_1: r = a,$$
$$C_2: r = 2a|\cos\theta|,$$

where a is a positive constant. The curves intersect at the points P_1 and P_2.

(i) Find the polar coordinates of P_1 and P_2. [2]

(ii) In a single diagram, sketch C_1, C_2 and their line of symmetry. [3]

(iii) The region R enclosed by C_1 and C_2 is bounded by the arcs OP_1, P_1P_2 and P_2O, where O is the pole. Find the area of R, giving your answer in exact form. [5]

(i) $C_1: \quad r = a$

$C_2: \quad r = 2a|\cos\theta|$

$$2a|\cos\theta| = a$$
$$|\cos\theta| = \frac{1}{2}$$

$\cos\theta = \frac{1}{2}$ $\qquad$ $\cos\theta = -\frac{1}{2}$

$\theta = \frac{\pi}{3}$ $\qquad$ $\theta = \frac{2\pi}{3}$

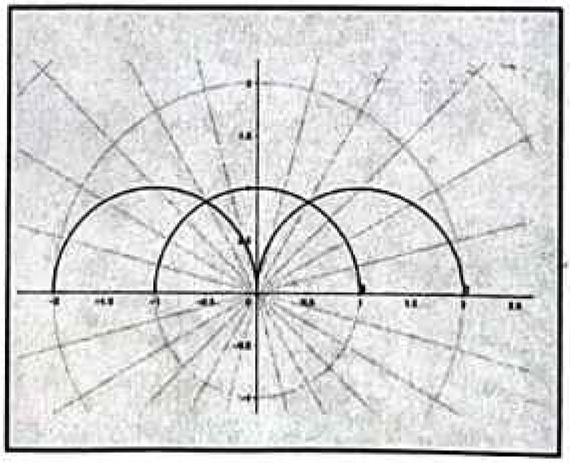
$r = a$ $\qquad$ $r = a$

$\left(a, \frac{\pi}{3}\right)$ $\qquad$ $\left(a, \frac{2\pi}{3}\right)$

(ii)

θ	r
0	1
$\frac{\pi}{6}$	1
$\frac{\pi}{3}$	1
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	1
$\frac{5\pi}{6}$	1
π	1

θ	r
0	2
$\frac{\pi}{6}$	1.73
$\frac{\pi}{3}$	1
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	1
$\frac{5\pi}{6}$	1.73
π	2



(iii) $\text{Area} = 2 \times \frac{1}{2}\int_{\pi/3}^{\pi/2} (2a|\cos\theta|)^2 \, d\theta$

$$= 4a^2 \int_{\pi/3}^{\pi/2} \cos^2\theta \, d\theta$$

$$= 4a^2 \int_{\pi/3}^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= 2a^2 \left| \theta + \frac{\sin 2\theta}{2} \right|_{\pi/3}^{\pi/2}$$

$$= 2a^2 \left[\left(\frac{\pi}{2} + 0\right) - \left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right) \right]$$

$$= a^2 \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} \right)$$

Required Area $= \frac{\pi a^2}{6} - a^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$

$$= -\frac{\pi a^2}{6} + \frac{a^2\sqrt{3}}{2}$$

9 The curve C has polar equation

$$r = 5\sqrt{(\cot\theta)},$$

where $0.01 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(i) Find the area of the finite region bounded by C and the line $\theta = 0.01$, showing full working. Give your answer correct to 1 decimal place. [3]

Let P be the point on C where $\theta = 0.01$.

(ii) Find the distance of P from the initial line, giving your answer correct to 1 decimal place. [2]

(iii) Find the maximum distance of C from the initial line. [3]

(iv) Sketch C. [2]

(i) $$\text{Area} = \frac{1}{2}\int_{0.01}^{\pi/2} r^2\, d\theta$$

$$= \frac{1}{2}\int_{0.01}^{\pi/2} 25\cot\theta\, d\theta$$

$$= \frac{25}{2}\int_{0.01}^{\pi/2} \frac{\cos\theta}{\sin\theta}\, d\theta$$

$$= \frac{25}{2}\Big|\ln(\sin\theta)\Big|_{0.01}^{\pi/2}$$

$$= 0 - \frac{25}{2}\ln\sin 0.01 \approx 57.6$$

(ii) $$y = r\sin\theta = 5\frac{\cos^{1/2}\theta}{\sin^{1/2}\theta}\cdot\sin\theta$$

$$= 5\cos^{1/2}\theta\sin^{1/2}\theta$$

$$= \frac{5}{\sqrt{2}}\sqrt{\sin 2\theta}$$

$$= \frac{5}{\sqrt{2}}\sqrt{\sin(0.02)}$$

$$\approx 0.5$$

(iii) For maximum distance from initial line

put $\frac{dy}{d\theta} = 0$

$$y = \frac{5}{\sqrt{2}} \sqrt{\sin 2\theta}$$

$$\frac{dy}{d\theta} = \frac{5}{\sqrt{2}} \cdot \frac{2\cos 2\theta}{2\sqrt{\sin 2\theta}}$$

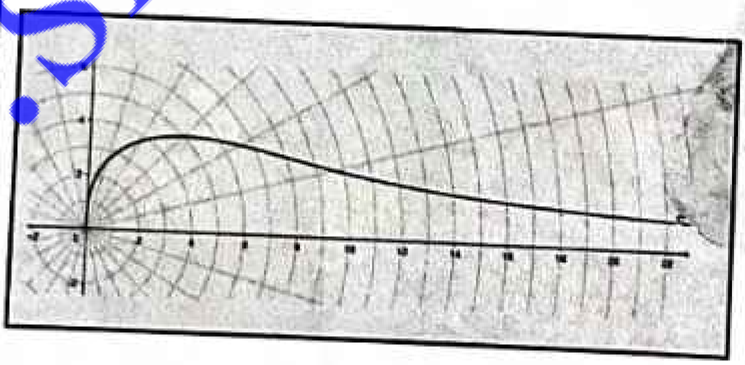
$$\frac{5}{\sqrt{2}} \frac{\cos 2\theta}{\sqrt{\sin 2\theta}} = 0$$

$$\cos 2\theta = 0$$

$$\theta = \frac{\pi}{4}$$

$$y = \frac{5}{\sqrt{2}} \sqrt{\sin \frac{\pi}{2}} = \frac{5}{\sqrt{2}} = 3.54$$

θ	r
0.01	50
$\frac{\pi}{12}$	9.5
$\frac{\pi}{6}$	6.5
$\frac{\pi}{4}$	4.9
$\frac{\pi}{3}$	3.8
$\frac{5\pi}{12}$	2.5
$\frac{\pi}{2}$	0



3 The curve C has polar equation $r = a\cos 3\theta$, for $-\frac{1}{6}\pi \leqslant \theta \leqslant \frac{1}{6}\pi$, where a is a positive constant.

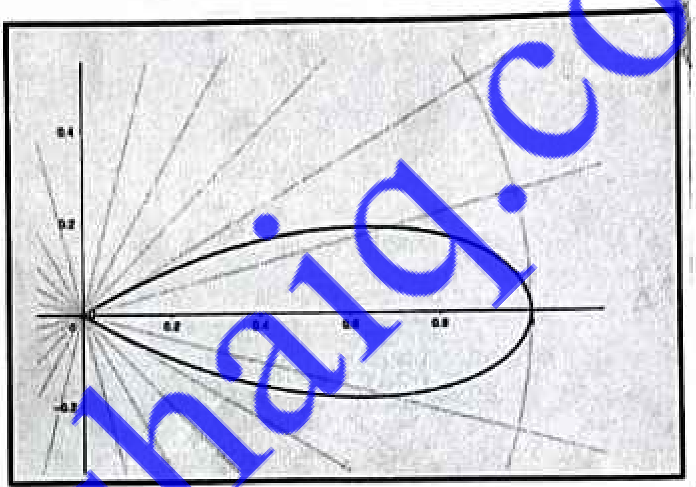
(i) Sketch C. [2]

(ii) Find the area of the region enclosed by C, showing full working. [3]

(iii) Using the identity $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$, find a cartesian equation of C. [3]

(i)

θ	r
0	1
$\frac{\pi}{12}$	0.7
$\frac{\pi}{6}$	0
$-\frac{\pi}{12}$	0.7
$-\frac{\pi}{6}$	0



(ii)

$$\text{Area} = \frac{1}{2}\int_{-\pi/6}^{\pi/6} a^2\cos^2 3\theta \, d\theta$$

$$= \frac{a^2}{2}\int_{-\pi/6}^{\pi/6}\left(\frac{1+\cos 6\theta}{2}\right)d\theta$$

$$= \frac{a^2}{4}\left|\theta + \frac{\sin 6\theta}{6}\right|_{-\pi/6}^{\pi/6}$$

$$= \frac{a^2}{4}\left[\left(\frac{\pi}{6}+0\right)-\left(-\frac{\pi}{6}+0\right)\right]$$

$$= \frac{a^2}{4}\times\frac{2\pi}{6}$$

$$= \frac{\pi a^2}{12}$$

(iii)

$$r = a\cos 3\theta$$

$$= a\left[4\cos^3\theta - 3\cos\theta\right]$$

$$r = a\cos\theta\left[4\cos^2\theta - 3\right]$$

$$r = a\left(\frac{r\cos\theta}{r}\right)\left[4\,\frac{r^2\cos^2\theta}{r^2} - 3\right]$$

$$r = a\,\frac{x}{r}\left(4\,\frac{x^2}{r^2} - 3\right)$$

$$= \frac{ax}{r}\left[\frac{4x^2 - 3r^2}{r^2}\right]$$

$$r^4 = ax\,(4x^2 - 3r^2)$$

$$(x^2+y^2)^2 = ax\left[4x^2 - 3(x^2+y^2)\right]$$

$$(x^2+y^2)^2 = ax\,(x^2 - 3y^2)$$

11 The curve C has polar equation $r = a(1 + \sin\theta)$ for $-\pi < \theta \leqslant \pi$, where a is a positive constant.

(i) Sketch C. [2]

(ii) Find the area of the region enclosed by C. [4]

(iii) Show that the length of the arc of C from the pole to the point furthest from the pole is given by

$$s = (\sqrt{2})a \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sqrt{(1 + \sin\theta)}\,d\theta.$$ [3]

(iv) Show that the substitution $u = 1 + \sin\theta$ reduces this integral for s to $(\sqrt{2})a \int_0^2 \frac{1}{\sqrt{(2-u)}}\,du$. Hence evaluate s. [4]

θ	r
$-\pi$	1
$-\frac{3\pi}{4}$	0.3
$-\frac{\pi}{2}$	0
$-\frac{\pi}{4}$	0.3
0	1
$\frac{\pi}{4}$	1.71
$\frac{\pi}{2}$	2
$\frac{3\pi}{4}$	1.71
π	1

(ii) Area $= \frac{1}{2}\int_{-\pi}^{\pi} a^2(1+\sin\theta)^2\,d\theta$

$= \frac{a^2}{2}\int_{-\pi}^{\pi} 1 + 2\sin\theta + \left(\frac{1-\cos 2\theta}{2}\right) d\theta$

$= \frac{a^2}{2}\left| \theta - 2\cos\theta + \frac{\theta}{2} - \frac{\sin 2\theta}{4} \right|_{-\pi}^{\pi}$

$= \frac{a^2}{2}\left[\frac{3\theta}{2} - 2\cos\theta - \frac{\sin 2\theta}{4} \right]_{-\pi}^{\pi}$

$= \frac{a^2}{2}\left[\left(\frac{3\pi}{2} - 2(-1) - 0\right) - \left(-\frac{3\pi}{2} - 2(-1) - 0\right) \right]$

$= \frac{a^2}{2}\left[\frac{3\pi}{2} + 2 + \frac{3\pi}{2} - 2 \right] = \frac{3\pi a^2}{2}$

(ii) ~~Arc~~ Arc length $= \int_a^b \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} \sqrt{a^2(1+2\sin\theta+\sin^2\theta)+a^2\cos^2\theta}\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{a^2[1+2\sin\theta+\sin^2\theta+\cos^2\theta]}\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{2a^2(1+\sin\theta)}\, d\theta$$

$$= \sqrt{2}\, a \int_{-\pi/2}^{\pi/2} \sqrt{1+\sin\theta}\, d\theta$$

(iv) put $u = 1+\sin\theta$

θ	$-\pi/2$	$\pi/2$
u	0	2

$du = \cos\theta\, d\theta$

$du = \sqrt{1-\sin^2\theta}\, d\theta$

$= \sqrt{1-(u-1)^2}\, d\theta$

$= \sqrt{1-(u^2-2u+1)}\, d\theta$

$du = \sqrt{2u-u^2}\, d\theta$

$\frac{du}{\sqrt{2u-u^2}} = d\theta$

Arc length $= \sqrt{2}\, a \int_0^2 \sqrt{u} \cdot \frac{du}{\sqrt{2u-u^2}}$

$$= \sqrt{2}\, a \int_0^2 \frac{du}{\sqrt{2-u}}$$

$$= \sqrt{2}\, a \int_0^2 (2-u)^{-1/2}\, du$$

$$= \sqrt{2}\, a \left| \frac{(2-u)^{1/2}}{-1/2} \right|_0^2$$

$$= 4a$$

Q.21 Oct/Nov/P11+P12+P13/2017

The polar equation of a curve C is $r = a(1 + \cos\theta)$ for $0 \leqslant \theta < 2\pi$, where a is a positive constant.

(i) Sketch C. [2]

(ii) Show that the cartesian equation of C is

$$x^2 + y^2 = a\left(x + \sqrt{(x^2 + y^2)}\right).$$ [2]

(iii) Find the area of the sector of C between $\theta = 0$ and $\theta = \frac{1}{3}\pi$. [4]

(iv) Find the arc length of C between the point where $\theta = 0$ and the point where $\theta = \frac{1}{3}\pi$. [5]