AS Level Further Mathematics

Topic: Equilibrium of rigid body

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Q.1 Oct/Nov/P32/2022

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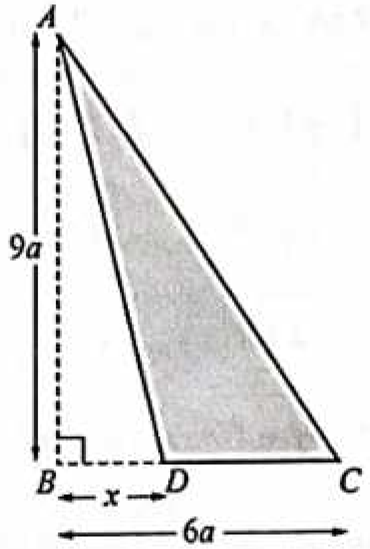


A uniform lamina is in the form of a triangle ABC in which angle B is a right angle, $AB = 9a$ and $BC = 6a$. The point D is on BC such that $BD = x$ (see diagram). The region ABD is removed from the lamina. The resulting shape ADC is placed with the edge DC on a horizontal surface and the plane ADC is vertical.

Find the set of values of x, in terms of a, for which the shape is in equilibrium. [6]

Let $B(0,0)$, then $A(0,9a)$, $C(6a,0)$, $D(x,0)$

Centre of mass of $ABC = \left(\frac{0+0+6a}{3}, \frac{0+9a+0}{3}\right) = (2a, 3a)$

Centre of mass of $ABD = \left(\frac{0+0+x}{3}, \frac{9a+0+0}{3}\right) = \left(\frac{x}{3}, 3a\right)$

Area of $ABC = \frac{1}{2} \times 6a \times 9a = 27a^2$

Area of $ABD = \frac{1}{2}(x)(9a) = \frac{9}{2}ax$

Area of $ADC = 27a^2 - \frac{9}{2}ax$

	Area	Distance from AB
ABC	$27a^2$	$2a$
ABD	$\frac{9}{2}ax$	$\frac{1}{3}x$
Shape ADC	$27a^2 - \frac{9}{2}ax$	$\bar{x}$

→

Momentum about AB $= \bar{x}$

$$\bar{x} = \frac{27a^2 \times 2a - \frac{9}{2}ax \times \frac{x}{3}}{\left(27a^2 - \frac{9}{2}ax\right)}$$

$$\bar{x} = \frac{54a^3 - \frac{3}{2}ax^2}{27a^2 - \frac{9}{2}ax}$$

Shape will be in equilibrium if

$$x \leq \bar{x}$$

$$x \leq \frac{54a^3 - \frac{3}{2}ax^2}{27a^2 - \frac{9}{2}ax}$$

$$27a^2x - \frac{9}{2}ax^2 \leq 54a^3 - \frac{3}{2}ax^2$$

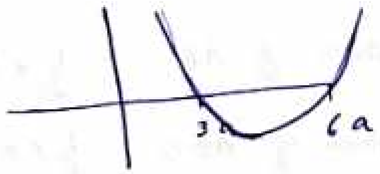
$$3x^2 - 27ax + 54a^2 \geq 0$$

$$3x^2 - 27ax + 54a^2 = 0$$

$$x = 3a \quad , \quad x = 6a$$

$$x \leq 3a \quad \text{or} \quad x \geq 6a$$

(Not possible)



Then $x \leq \bar{x}$

$$\boxed{x \leq 3a}$$

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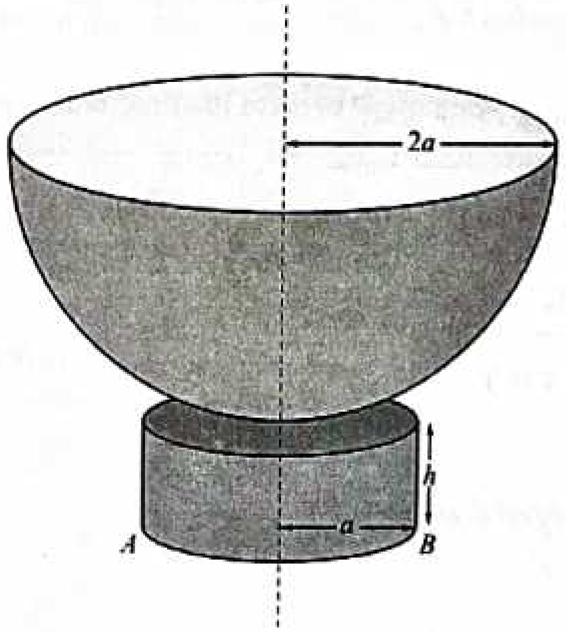


An object is composed of a hemispherical shell of radius $2a$ attached to a closed hollow circular cylinder of height h and base radius a. The hemispherical shell and the hollow cylinder are made of the same uniform material. The axes of symmetry of the shell and the cylinder coincide. AB is a diameter of the lower end of the cylinder (see diagram).

(a) Find, in terms of a and h, an expression for the distance of the centre of mass of the object from AB. [4]

The object is placed on a rough plane which is inclined to the horizontal at an angle θ, where $\tan\theta = \frac{2}{3}$. The object is in equilibrium with AB in contact with the plane and lying along a line of greatest slope of the plane.

(b) Find the set of possible values of h, in terms of a. [4]

(a)

Surface area of cylinder $= 2\pi a h + 2\pi a^2$

centre of mass of cylinder from AB $= \frac{1}{2}h$

Surface area of hemispherical Shell $= 2\pi(2a)^2$

centre of mass of the Shell from AB $= h+a$

	Area	Centre of mass from AB
Cylinder	$2\pi ah + 2\pi a^2$	$\frac{1}{2}h$
Shell	$2\pi(2a)^2$	$h+a$

Let $\bar{x}$ = Momentum about AB

$$\bar{x} = \frac{2\pi(2a)^2(h+a) + (2\pi ah + 2\pi a^2)(\frac{1}{2}h)}{2\pi ah + 2\pi a^2 + 2\pi(2a)^2}$$

$$\bar{x} = \frac{8\pi a^2 h + 8\pi a^3 + \pi a h^2 + \pi a^2 h}{2\pi a h + 10\pi a^2}$$

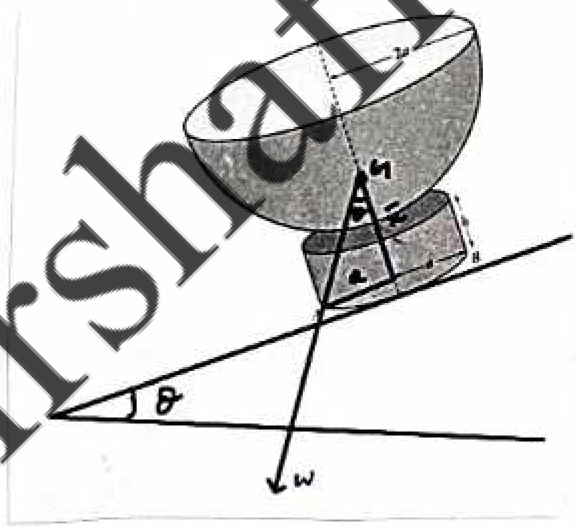
$$= \frac{\pi a(h^2 + 9ah + 8a^2)}{2\pi a(h + 5a)}$$

$$\bar{x} = \frac{h^2 + 9ah + 8a^2}{2(h + 5a)}$$

(b)

The object will be in equilibrium if

$$\tan\theta \le \frac{a}{\bar{x}}$$



$$\frac{2}{3}\bar{x} \le a$$

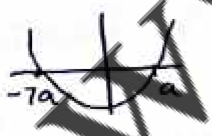
$$\bar{x} \le \frac{3}{2}a$$

$$\frac{h^2 + 9ah + 8a^2}{2(h+5a)} \le \frac{3}{2}a$$

$$h^2 + 9ah + 8a^2 \le 3ah + 15a^2$$

$$h^2 + 6ah - 7a^2 \le 0$$

$$(h-a)(h+7a) \le 0$$



$$-7a \le h \le a$$

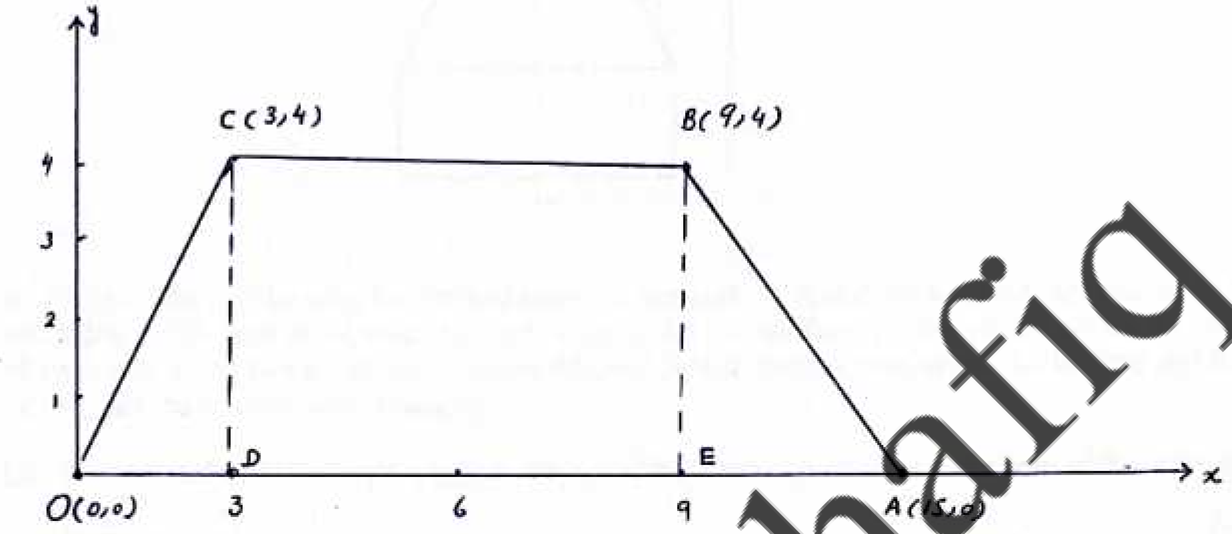
So object will be in equilibrium if

$$h \le a$$

Q.3 May/June/P33/2022

1 A uniform lamina $OABC$ is a trapezium whose vertices can be represented by coordinates in the x-y plane. The coordinates of the vertices are $O(0,0)$, $A(15,0)$, $B(9,4)$ and $C(3,4)$.

Find the x-coordinate of the centre of mass of the lamina. [4]



Area of Triangle $OCD = \frac{1}{2} \times 3 \times 4 = 6$

Centre of mass $= \left(\frac{0+3+3}{3}, \frac{0+0+4}{3} \right) = \left(2, \frac{4}{3} \right)$

Area of Rectangle $DEBC = 6 \times 4 = 24$

Centre of mass $= (6, 2)$

Area of Triangle $BAE = \frac{1}{2} \times 6 \times 4 = 12$

Centre of mass $= \left(\frac{9+15+9}{3}, \frac{0+0+4}{3} \right) = \left(11, \frac{4}{3} \right)$

$\bar{x}$ = centre of mass of the lamina from oy

$$= \frac{6 \times 2 + 24 \times 6 + 12 \times 11}{6 + 24 + 12}$$

$$= \frac{288}{42} = 6.86$$

Q.4 May/June/P31+P32/2021

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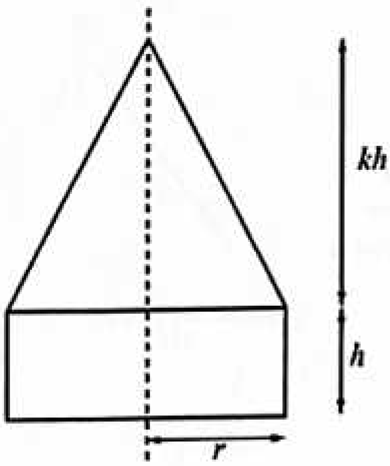


A uniform solid circular cone has vertical height kh and radius r. A uniform solid cylinder has height h and radius r. The base of the cone is joined to one of the circular faces of the cylinder so that the axes of symmetry of the two solids coincide (see diagram, which shows a cross-section). The cone and the cylinder are made of the same material.

(a) Show that the distance of the centre of mass of the combined solid from the base of the cylinder is $\dfrac{h(k^2+4k+6)}{4(3+k)}$. [4]

The solid is placed on a plane that is inclined to the horizontal at an angle θ. The base of the cylinder is in contact with the plane. The plane is sufficiently rough to prevent sliding. It is given that $3h = 2r$ and that the solid is on the point of toppling when $\tan\theta = \frac{4}{3}$.

(b) Find the value of k. [3]

(a) Volume of the cone $= \frac{1}{3}\pi r^2 kh$

Centre of mass of the cone from base of cylinder $= h + \frac{1}{4}kh$

Volume of the cylinder $= \pi r^2 h$

Centre of the mass from base $= \frac{h}{2}$

	Volume	Centre of mass from base of cylinder
Cone	$\frac{1}{3}\pi r^2 kh$	$h+\frac{kh}{4}$
Cylinder	$\pi r^2 h$	$\frac{h}{2}$
Combined	$\pi r^2 h\left(\frac{1}{3}k+1\right)$	$\bar{x}$

$\bar{x}$ = centre of the mass of the solid from base of cylinder

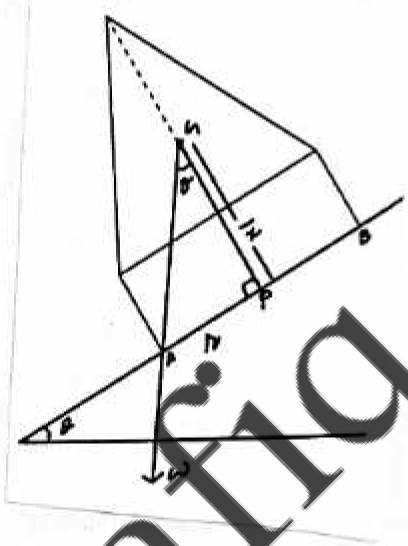
$$= \frac{\frac{1}{3}\pi r^2 kh\left(h+\frac{kh}{4}\right)+\pi r^2 h\left(\frac{h}{2}\right)}{\frac{1}{3}\pi r^2 kh+\pi r^2 h}$$

$$= \frac{\frac{1}{3}\pi r^2 h^2 k+\frac{1}{12}\pi r^2 h^2 k^2+\frac{\pi r^2 h^2}{2}}{\frac{1}{3}\pi r^2 h(k+3)} = \frac{h(k^2+4k+6)}{4(3+k)}$$

$\rightarrow$

(b) Given that the solid is on the point of toppling, so

$$\tan\theta = \frac{r}{\bar{x}}$$



$$\frac{4}{3} = \frac{3}{2}h \times \frac{4(3+k)}{h(k^2+4k+6)}$$

$$\frac{4}{3} = \frac{6(3+k)}{k^2+4k+6}$$

$$4k^2+16k+24 = 54+18k$$

$$4k^2-2k-30=0$$

$$2k^2-k-15=0$$

$k=3$ $\quad\quad$ $k=-\frac{5}{2}$ (not possible)

Q.5 May/June/P33/2021

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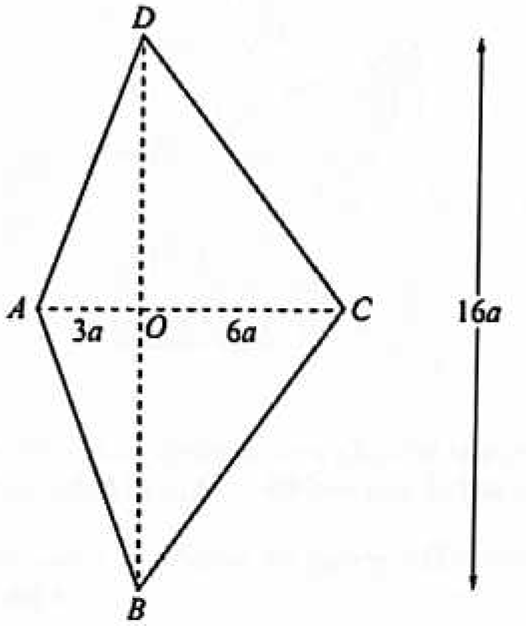

A uniform lamina *ABCD* consists of two isosceles triangles *ABD* and *BCD*. The diagonals of *ABCD* meet at the point *O*. The length of *AO* is $3a$, the length of *OC* is $6a$ and the length of *BD* is $16a$ (see diagram).

Find the distance of the centre of mass of the lamina from *DB*. **[3]**

Area of $ABD = \frac{1}{2} \times 16a \times 3a = 24a^2$

centre of mass from $BD = -a$

Area of $BCD = \frac{1}{2}(16a)(6a) = 48a^2$

centre of mass from $BD = 2a$

	Area	Centre of mass from *DB*
ABD	$24a^2$	$-a$
BCD	$48a^2$	$2a$
Combined	$72a^2$	$\bar{x}$

$\bar{x}$ = centre of mass of lamina from BD

$$= \frac{(24a^2)(-a) + (48a^2)(2a)}{24a^2 + 48a^2}$$

$$= \frac{72a^3}{72a^2} = a$$

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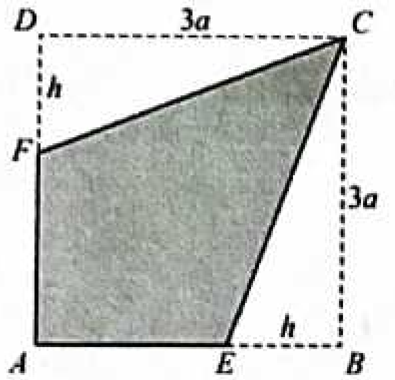

A uniform lamina *AECF* is formed by removing two identical triangles *BCE* and *CDF* from a square lamina *ABCD*. The square has side $3a$ and $EB = DF = h$ (see diagram).

(a) Find the distance of the centre of mass of the lamina *AECF* from *AD* and from *AB*, giving your answers in terms of a and h. **[5]**

The lamina *AECF* is placed vertically on its edge *AE* on a horizontal plane.

(b) Find, in terms of a, the set of values of h for which the lamina remains in equilibrium. **[3]**

Area of Square ABCD $= 9a^2$

Centre of mass from AD $= \frac{3a}{2}$

Area of Triangle CDF $= \frac{1}{2}(3a)(h) = \frac{3}{2}ah$

Centre of mass from AD $= a$

Area of Triangle BEC $= \frac{3}{2}ah$

Centre of mass from AD $= \frac{3a + 3a + 3a - h}{3} = 3a - \frac{h}{3}$

$\bar{x}$ = centre of mass of the lamina AECF from AD

$$= \frac{(9a^2)(\frac{3a}{2}) - (\frac{3}{2}ah)(a) - \frac{3}{2}ah\,(3a - \frac{h}{3})}{9a^2 - \frac{3}{2}ah - \frac{3}{2}ah}$$

$$= \frac{\frac{27}{2}a^3 h - \frac{3}{2}a^2 h - \frac{9}{2}a^2 h - \frac{1}{2}ah^2}{9a^2 - 3ah} = \frac{27a^2 - 12ah + h^2}{6(3a - h)}$$

$$= \frac{(9a - h)(3a - h)}{6(3a - h)} = \frac{9a - h}{6}$$

Centre of mass of the lamina AB $= \bar{x} = \dfrac{9a-h}{6}$

(b) lamina will be in equilibrium if

$$\bar{x} \leq 3a-h$$

$$\frac{27a^2-12ah+h^2}{6(3a-h)} \leq (3a-h)$$

or

$$\frac{9a-h}{6} \leq 3a-h$$

$$9a-h \leq 18a-6h$$

$$5h \leq 9a$$

$$h \leq \frac{9}{5}a$$

Q.7 Oct/Nov/P32/2021

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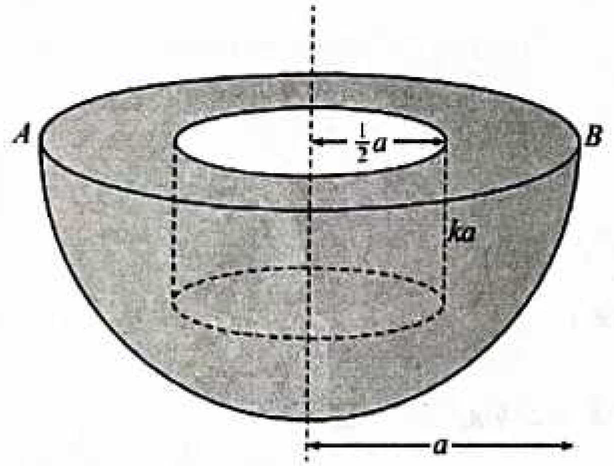


An object is formed by removing a solid cylinder, of height ka and radius $\frac{1}{2}a$, from a uniform solid hemisphere of radius a. The axes of symmetry of the hemisphere and the cylinder coincide and one circular face of the cylinder coincides with the plane face of the hemisphere. AB is a diameter of the circular face of the hemisphere (see diagram).

(a) Show that the distance of the centre of mass of the object from AB is $\dfrac{3a(2-k^2)}{2(8-3k)}$. **[4]**

When the object is freely suspended from the point A, the line AB makes an angle θ with the downward vertical, where $\tan\theta = \frac{7}{18}$.

(b) Find the possible values of k. **[3]**

	Volume	Centre of mass from AB
Hemisphere	$\frac{2}{3}\pi a^3$	$\frac{3}{8}a$
Cylinder	$\pi ka\left(\frac{a}{2}\right)^2$	$\frac{ka}{2}$
Remainder	$\frac{2}{3}\pi a^3 - \pi ka\left(\frac{a}{2}\right)^2$	$\bar{x}$

(a) Volume of hemisphere $= \frac{2}{3}\pi a^3$

Centre of mass from AB $= \frac{3}{8}a$

Volume of cylinder $= \pi\left(\frac{a}{2}\right)^2 ka = \frac{\pi k a^3}{4}$

Centre of mass from AB $= \frac{ka}{2}$

Let $\bar{x}$ = centre of mass of the object from AB

$$\bar{x} = \frac{\left(\frac{2}{3}\pi a^3\right)\left(\frac{3}{8}a\right) - \left(\frac{\pi k a^3}{4}\right)\left(\frac{ka}{2}\right)}{\frac{2}{3}\pi a^3 - \frac{\pi k a^3}{4}}$$

$$\bar{x} = \frac{\frac{\pi a^4}{4} - \frac{\pi k^2 a^4}{8}}{\frac{2}{3}\pi a^3 - \frac{\pi k a^3}{4}} = \frac{\frac{\pi a^4}{8}(2-k^2)}{\frac{\pi a^3}{12}(8-3k)} = \frac{3a(2-k^2)}{2(8-3k)}$$

$\rightarrow$

(b)

$$\tan\theta = \frac{\bar{x}}{a}$$

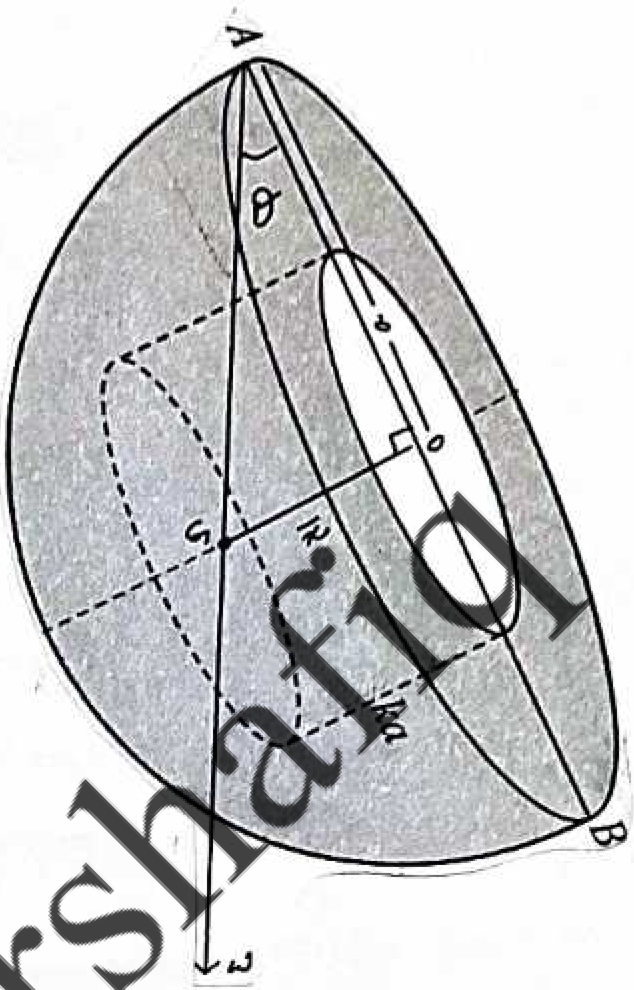
$$\frac{7}{18} = \frac{3a(2-k^2)}{2(8-3k)} \cdot \frac{1}{a}$$

$$112 - 42k = 108 - 54k^2$$

$$54k^2 - 42k + 4 = 0$$

$$27k^2 - 21k + 2 = 0$$

$$k = \frac{2}{3} \quad \text{and} \quad k = \frac{1}{9}$$



Q.8 May/June/P31+32/2020

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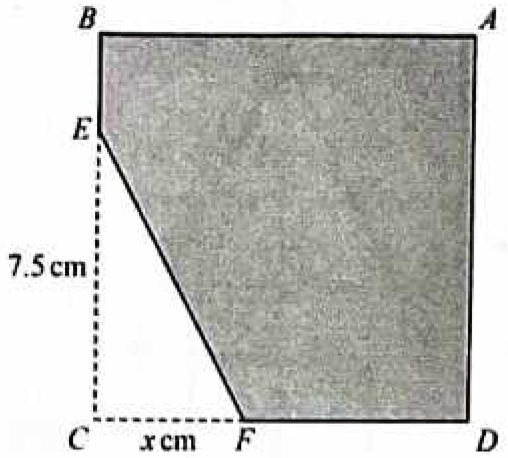

A uniform square lamina $ABCD$ has sides of length 10 cm. The point E is on BC with $EC = 7.5$ cm, and the point F is on DC with $CF = x$ cm. The triangle EFC is removed from $ABCD$ (see diagram). The centre of mass of the resulting shape $ABEFD$ is a distance $\bar{x}$ cm from CB and a distance $\bar{y}$ cm from CD.

(a) Show that $\bar{x} = \dfrac{400 - x^2}{80 - 3x}$ and find a corresponding expression for $\bar{y}$. [4]

The shape $ABEFD$ is in equilibrium in a vertical plane with the edge DF resting on a smooth horizontal surface.

(b) Find the greatest possible value of x, giving your answer in the form $a + b\sqrt{2}$, where a and b are constants to be determined. [3]

(a) Area of Triangle $CEF = \frac{1}{2}(x)(7\cdot5) = \frac{15}{4}x$

Centre of mass from BC $= \dfrac{0+0+x}{3} = \dfrac{x}{3}$

Centre of mass from CD $= \dfrac{0+7\cdot5+0}{3} = \dfrac{5}{2}$

Area of Square ABCD $= 100$

Centre of mass from BC $= 5$

Centre of mass from CD $= 5$

	Area	Centre of mass from BC	Centre of mass from DC
Square	100	5	5
Triangle	½ x. 15/2	$\frac{1}{3}x$	$\frac{5}{2}$
Shape $ABEFD$	$100 - \frac{15}{4}x$	$\bar{x}$	$\bar{y}$

$$\bar{x} = \frac{100 \times 5 - \frac{15}{4}x \cdot \frac{x}{3}}{100 - \frac{15}{4}x} = \frac{\frac{15}{12}(400 - x^2)}{\frac{5}{4}(80 - 3x)} = \frac{400 - x^2}{80 - 3x}$$

$$\bar{y} = \frac{100 \times 5 - \frac{15}{4}x \cdot \frac{5}{2}}{100 - \frac{15}{4}x} = \frac{500 - \frac{75}{8}x}{\frac{400 - 15x}{4}} = \frac{4000 - 75x}{800 - 30x} = \frac{800 - 15x}{160 - 6x}$$

$\rightarrow$

(b) For greatest value of x

$$\bar{x} \geq x$$

$$\frac{400 - x^2}{80 - 3x} \geq x$$

$$400 - x^2 \geq 80x - 3x^2$$

$$2x^2 - 80x + 400 \geq 0$$

$x = 20 - 10\sqrt{2}$ only $x = 20 + 10\sqrt{2}$ (Not possible)

Q.9 May/June/P33/2020

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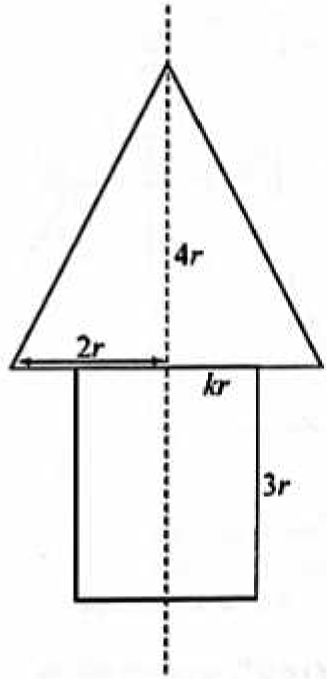


A uniform solid circular cone, of vertical height $4r$ and radius $2r$, is attached to a uniform solid cylinder, of height $3r$ and radius kr, where k is a constant less than 2. The base of the cone is joined to one of the circular faces of the cylinder so that the axes of symmetry of the two solids coincide (see diagram). The cone and the cylinder are made of the same material.

(a) Show that the distance of the centre of mass of the combined solid from the vertex of the cone is $\dfrac{(99k^2+96)r}{18k^2+32}$. **[4]**

The point C is on the circumference of the base of the cone. When the combined solid is freely suspended from C and hanging in equilibrium, the diameter through C makes an angle α with the downward vertical, where $\tan\alpha = \frac{1}{8}$.

(b) Given that the centre of mass of the combined solid is within the cylinder, find the value of k. **[4]**

Volume of the cone $= \frac{1}{3}\pi(2r)^2(4r) = \frac{16}{3}\pi r^3$

Centre of the mass from vertex $= \frac{3}{4}(4r) = 3r$

volume of the cylinder $= \pi(kr)^2(3r) = 3\pi k^2 r^3$

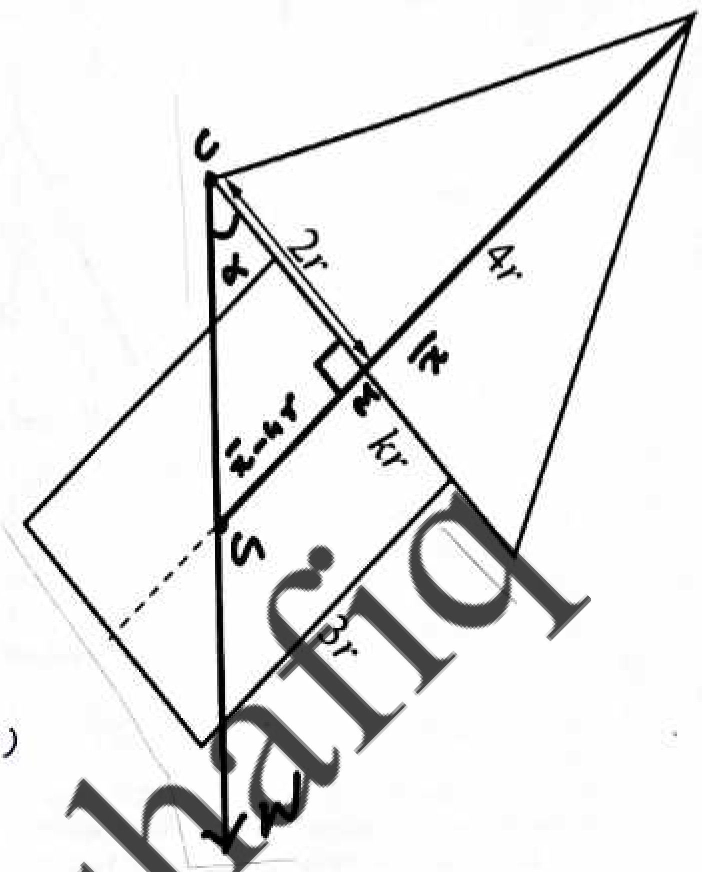
Centre of the mass from vertex $= \frac{3r}{2} + 4r = \frac{11}{2}r$

$\bar{x}$ = centre of the mass of the solid from vertex

$$= \frac{\frac{16}{3}\pi r^3 \cdot 3r + 3\pi k^2 r^3 \cdot \frac{11}{2}r}{\frac{16}{3}\pi r^3 + 3\pi k^2 r^3} = \frac{16\pi r^4 + \frac{33}{2}\pi r^4 k^2}{\pi r^3\left(\frac{16+9k^2}{3}\right)}$$

$$= \frac{\pi r^4(32+33k^2)}{\pi r^3(16+9k^2)} \times \frac{3}{2}$$

$$\bar{x} = \frac{(99k^2+96)r}{18k^2+32}$$



(b) Solid will be in equilibrium if

$$\tan\alpha = \frac{\bar{x}-4r}{2r}$$

$$\frac{1}{8} = \frac{\frac{(99k^2+96)r}{18k^2+32} - 4r}{2r}$$

$$\frac{1}{8} = \frac{(99k^2+96) - 4(18k^2+32)}{2(18k^2+32)}$$

$$\frac{1}{8} = \frac{27k^2-32}{36k^2+64}$$

$$216k^2 - 256 = 36k^2 + 64$$

$$180k^2 = 320$$

$$k^2 = \frac{16}{9}$$

$$k = \frac{4}{3}$$

Q.10 Oct/Nov/P31/2020

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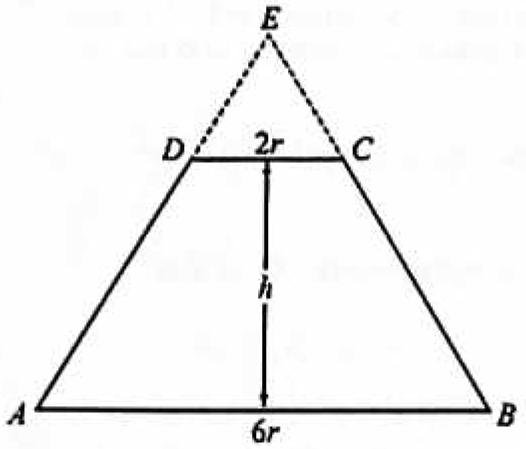


The diagram shows the cross-section $ABCD$ of a uniform solid object which is formed by removing a cone with cross-section DCE from the top of a larger cone with cross-section ABE. The perpendicular distance between AB and DC is h, the diameter AB is $6r$ and the diameter DC is $2r$.

(a) Find an expression, in terms of h, for the distance of the centre of mass of the solid object from AB. **[4]**

The object is freely suspended from the point B and hangs in equilibrium. The angle between AB and the downward vertical through B is θ.

(b) Given that $h = \frac{13}{4}r$, find the value of $\tan\theta$. **[2]**

Volume of small cone $= \frac{1}{3}\pi r^2 \frac{h}{2} = \frac{\pi r^2 h}{6}$

Centre of mass of small cone from AB $= h + \frac{1}{4}\left(\frac{h}{2}\right) = \frac{9h}{8}$

Volume of Larger Cone $= \frac{\pi}{3}(3r)^2 \cdot \frac{3}{2}h = \frac{9}{2}\pi r^2 h$

Centre of mass of Larger Cone from AB $= \frac{1}{4}\left(\frac{3}{2}h\right) = \frac{3h}{8}$

$\bar{x}$ = centre of mass of the solid from AB

	Volume	Centre of mass from AB
Small cone	$\frac{1}{3}\pi r^2 \cdot \frac{h}{2}$	$h + \frac{1}{4}\cdot\frac{h}{2}\left(= \frac{9h}{8}\right)$
Large cone	$\frac{1}{3}\pi(3r)^2 \cdot \frac{3h}{2}$	$\frac{1}{4}\cdot\frac{3h}{2}\left(= \frac{3h}{8}\right)$
Object	$\frac{26}{6}\pi(r)^2 h$	$\bar{x}$

$$= \frac{\frac{9}{2}\pi r^2 h \cdot \frac{3h}{8} - \frac{1}{6}\pi r^2 h \cdot \frac{9h}{8}}{\frac{9}{2}\pi r^2 h - \frac{1}{6}\pi r^2 h}$$

$$= \frac{\frac{3}{2}\pi r^2 h^2}{\frac{13}{3}\pi r^2 h} = \frac{9h}{26}$$

→

(b) Solid will be in equilibrium if

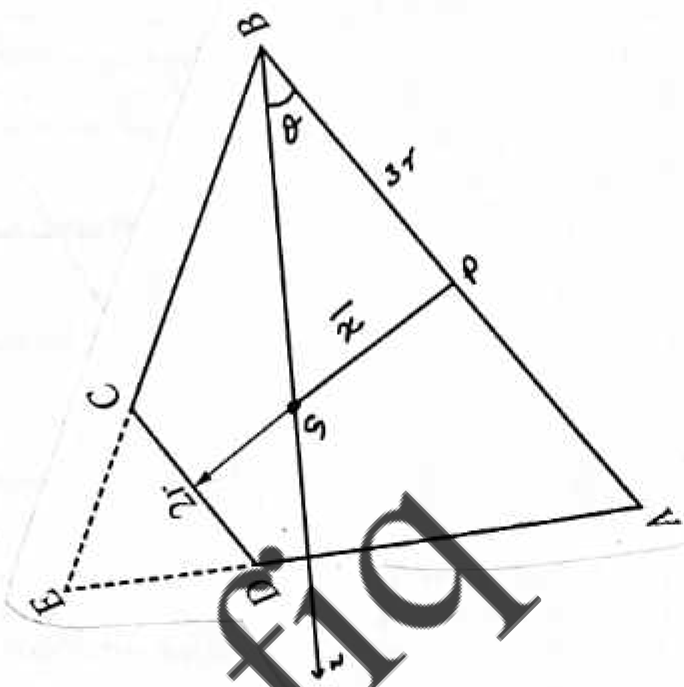
$$\tan\theta = \frac{\bar{x}}{3r}$$

$$\tan\theta = \frac{9h}{26} \cdot \frac{1}{3r}$$

$$= \frac{3h}{26r}$$

$$= \frac{3}{26r} \cdot \frac{13}{4}r \quad \text{by given } h = \frac{13}{4}r$$

$$\tan\theta = \frac{3}{8}$$



Q.11 Oct/Nov/P32/2020

3 An object consists of a uniform solid circular cone, of vertical height $4r$ and radius $3r$, and a uniform solid cylinder, of height $4r$ and radius $3r$. The circular base of the cone and one of the circular faces of the cylinder are joined together so that they coincide. The cone and the cylinder are made of the same material.

(a) Find the distance of the centre of mass of the object from the end of the cylinder that is not attached to the cone. [4]

(b) Show that the object can rest in equilibrium with the curved surface of the cone in contact with a horizontal surface. [3]

(a) Volume of the cone $= \frac{1}{3}\pi(3r)^2 \cdot 4r$

$= 12\pi r^3$

Centre of mass of cone from base $= 4r + r$

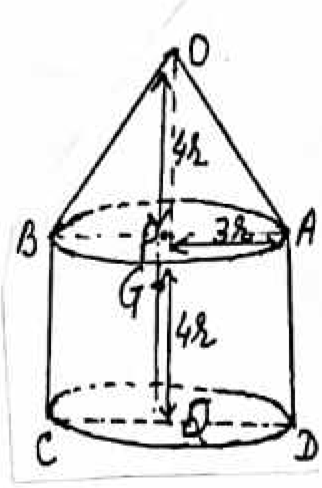
$= 5r$

	Volume	Centre of mass from base
Cone	$\frac{1}{3}\pi(3r)^2 \cdot 4r$	$4r+r$
Cylinder	$\pi(3r)^2 \cdot 4r$	$2r$
Combined	$\frac{4}{3}\pi(3r)^2 \cdot 4r$	$\bar{x}$

Volume of cylinder $= \pi(3r)^2 \cdot 4r$

$= 36\pi r^3$

Centre of mass of cylinder from base $= 2r$



$\bar{x}$ = centre of mass of solid from base

$$\bar{x} = \frac{(12\pi r^3)(5r) + (36\pi r^3)(2r)}{12\pi r^3 + 36\pi r^3}$$

$$= \frac{132\pi r^4}{48\pi r^3} = \frac{11}{4}r$$

(b)

object will be in equilibrium if

$OG\cos\theta < OA$

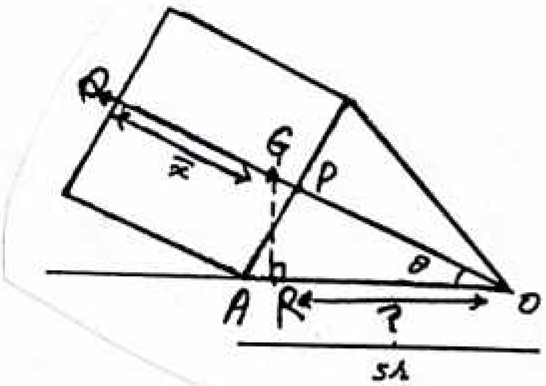
or $OR < OA$

$\frac{21}{4}r\cos\theta < 5r$

$\frac{21}{4}r \times \frac{4}{5} < 5r$

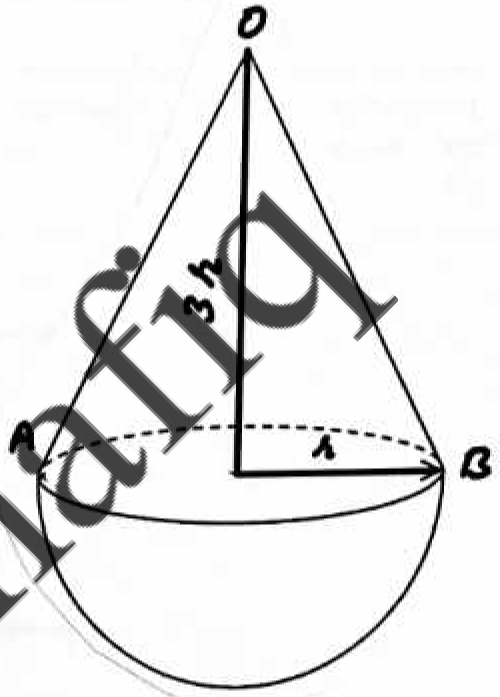
$21 < 25$

True



1 A child's toy consists of a uniform solid circular cone, of vertical height $3r$ and radius r, and a uniform solid hemisphere of radius r. The circular bases of the cone and the hemisphere are joined together so that they coincide. The cone and the hemisphere are made of the same material.

Show that the centre of mass of the toy is at a distance $\frac{27}{10}r$ from the vertex of the cone. [4]



Volume of the cone $= \frac{1}{3}\pi r^2 \times 3r$

$= \pi r^3$

Centre of mass of cone from vertex $= \frac{3}{4}(3r)$

$= \frac{9}{4}r$

Volume of hemisphere $= \frac{2}{3}\pi r^3$

Centre of mass from vertex $= 3r + \frac{3}{8}r$

$= \frac{27}{8}r$

$\bar{x}$ = Centre of mass of solid from vertex

$$\bar{x} = \frac{(\pi r^3)(\frac{9}{4}r) + (\frac{2}{3}\pi r^3)(\frac{27}{8}r)}{\pi r^3 + \frac{2}{3}\pi r^3}$$

$$= \frac{\frac{9}{2}\pi r^4}{\frac{5}{3}\pi r^3}$$

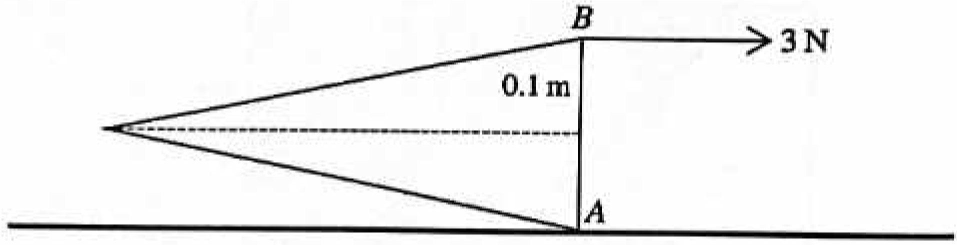
$$= \frac{27}{10}r$$

Density ρ

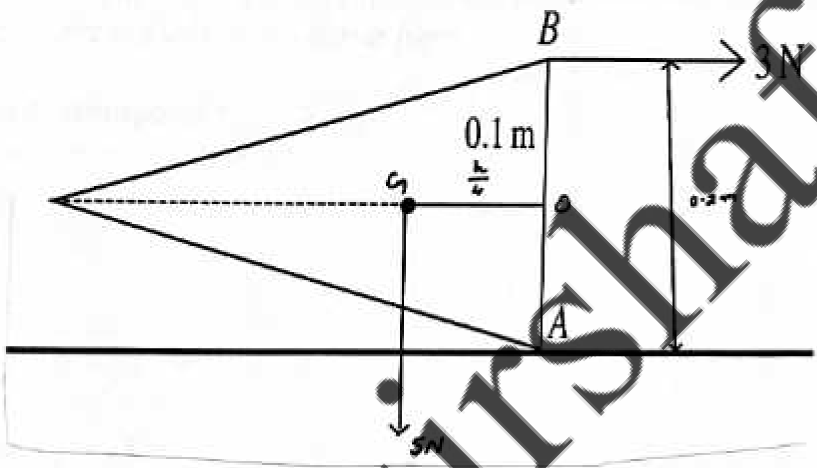
	Volume	COM from vertex
Cone	$\frac{1}{3}\pi r^2 \times 3r$	$\frac{3}{4} \times 3r$
Hemisphere	$\frac{2}{3}\pi r^3$	$3r + \frac{3}{8}r$
Combined	$\frac{5}{3}\pi r^3$	$\bar{x}$

Q.13 9709/51/O/N/19

1



A uniform solid cone has weight 5 N and base radius 0.1 m. AB is a diameter of the base of the cone. The cone is held in equilibrium, with A in contact with a rough horizontal surface and AB vertical, by a force applied at B. This force has magnitude 3 N and acts parallel to the axis of the cone (see diagram). Calculate the height of the cone. [3]



Let h = be the hight of the cone, then

$\frac{h}{4}$ = centre of mass from vertex

Taking moments about A

$$5 \times \frac{h}{4} = 3 \times 0.2$$

$$h = 0.6 \times \frac{4}{5}$$

$$h = 0.48 \text{ m}$$

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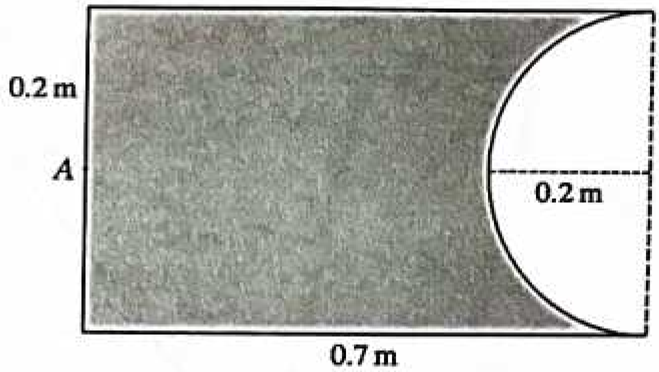
3



The diagram shows the cross-section through the centre of mass of a uniform solid object. The object is a cylinder of radius 0.2 m and length 0.7 m, from which a hemisphere of radius 0.2 m has been removed at one end. The point A is the centre of the plane face at the other end of the object. Find the distance of the centre of mass of the object from A. [5]

[The volume of a hemisphere is $\frac{2}{3}\pi r^3$.]

Volume of the cylinder $= \pi(0.2)^2(0.7) = 0.028\pi$

Centre of mass from A $= \frac{0.7}{2} = 0.35$

Volume of hemisphere $= \frac{2}{3}\pi(0.2)^3 = \frac{2}{375}\pi$

Centre of mass from A $= 0.7 - \frac{3}{8} \times 0.2 = \frac{5}{8}$

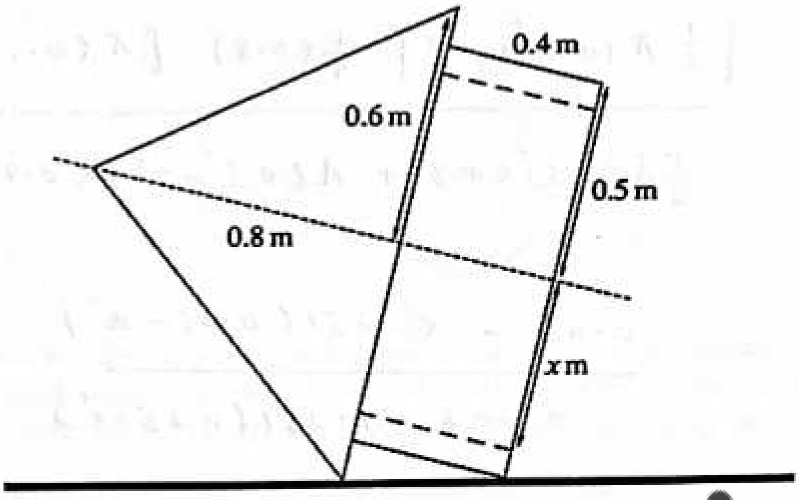
$\bar{x}$ = centre of mass of the object from A

$$= \frac{(0.028\pi)(0.35) - \frac{2}{375}\pi \times \frac{5}{8}}{0.028\pi - \frac{2}{375}\pi}$$

$\bar{x} = 0.285$

Q.14 9709/52/M/J/18

5



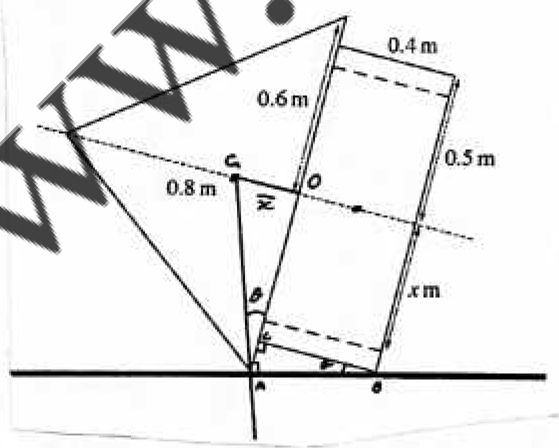
A uniform object is made by joining a solid cone of height 0.8 m and base radius 0.6 m and a cylinder. The cylinder has length 0.4 m and radius 0.5 m. The cylinder has a cylindrical hole of length 0.4 m and radius x m drilled through it along the axis of symmetry. A plane face of the cylinder is attached to the base of the cone so that the object has an axis of symmetry perpendicular to its base and passing through the vertex of the cone. The object is placed with points on the base of the cone and the base of the cylinder in contact with a horizontal surface (see diagram). The object is on the point of toppling.

(i) Show that the centre of mass of the object is 0.15 m from the base of the cone. [3]

(ii) Find x. [4]

[The volume of a cone is $\frac{1}{3}\pi r^2 h$.]

(a)



In $\triangle ABC$ $\quad \tan\theta = \frac{AC}{BC} = \frac{0.6 - 0.5}{0.4} = \frac{1}{4}$

In rt $\triangle AOG$ $\quad \tan\theta = \frac{OG}{AO} = \frac{\bar{x}}{0.6}$

$\frac{1}{4} = \frac{\bar{x}}{0.6} \Rightarrow \bar{x} = 0.15\text{ m}$

(b)

$$\bar{x} = \frac{\left[\frac{1}{3}\pi(0.6)^2 \times 0.8\right] \times \frac{1}{4}(0.8) - \left[\pi(0.5^2 - x^2) \times 0.4\right]\frac{0.4}{2}}{\frac{1}{3}\pi(0.6)^2 \times 0.8 + \pi(0.5^2 - x^2) \times 0.4}$$

$$0.15 = \frac{0.06 - 0.251(0.25 - x^2)}{0.302 + 1.257(0.25 - x^2)}$$

$$0.045 + 0.047 - 0.189x^2 = 0.06 - 0.063 + 0.251x^2$$

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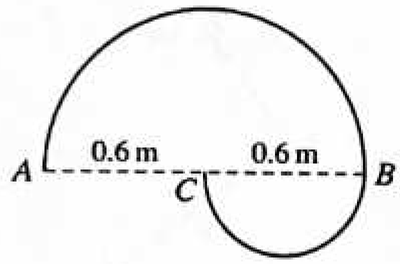
2



A uniform wire is bent to form an object which has a semicircular arc with diameter AB of length 1.2 m, with a smaller semicircular arc with diameter BC of length 0.6 m. The end C of the smaller arc is at the centre of the larger arc (see diagram). The two semicircular arcs of the wire are in the same plane.

(i) Show that the distance of the centre of mass of the object from the line ACB is 0.191 m, correct to 3 significant figures. [3]

The object is freely suspended at A and hangs in equilibrium.

(ii) Find the angle between ACB and the vertical. [4]

(i) Let D = distance of com from the line ACB

Length of wire	Bigger circle	Smaller circle	object
	0.6π	0.3π	$0.6\pi + 0.3\pi = 0.9\pi$
com from ACB	$\frac{0.6\sin\frac{\pi}{2}}{\frac{\pi}{2}} = \frac{1.2}{\pi}$	$-\frac{0.3\sin\frac{\pi}{2}}{\pi/2} = -\frac{0.6}{\pi}$	D

Moments about ACB

$$D = \frac{0.6\pi \times \frac{1.2}{\pi} + 0.3\pi \times \left(-\frac{0.6}{\pi}\right)}{0.6\pi + 0.3\pi} = \frac{0.72 - 0.18}{0.9\pi} = 0.191\text{ m}$$

(ii) Moments about A

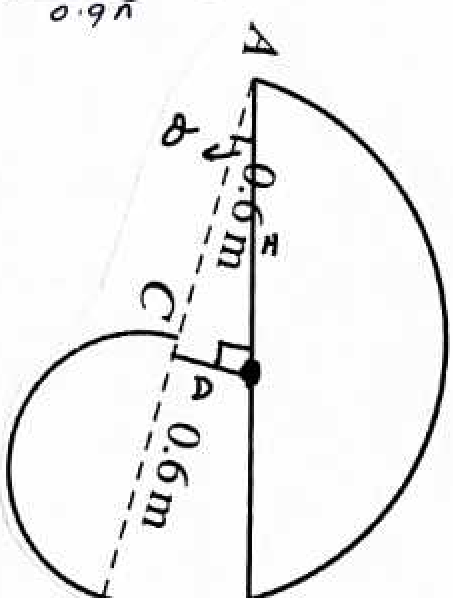
$$(0.6\pi + 0.3\pi)H = 0.6\pi \times 0.6 + 0.3\pi \times 0.9$$

$$H = 0.7$$

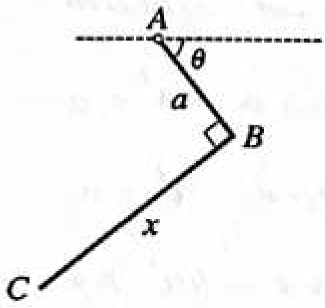
Let θ be the angle b/w ACB and vertical

$$\tan\theta = \frac{0.191}{0.7}$$

$$\theta = 15.3^\circ$$



3



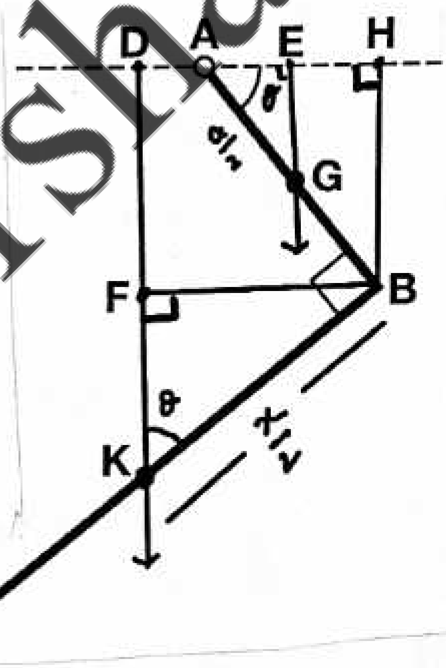
ABC is an object made from a uniform wire consisting of two straight portions AB and BC, in which $AB = a$, $BC = x$ and angle $ABC = 90°$. When the object is freely suspended from A and in equilibrium, the angle between AB and the horizontal is θ (see diagram).

(i) Show that $x^2 \tan\theta - 2ax - a^2 = 0$. [3]

(ii) Given that $\tan\theta = 1.25$, calculate the length of the wire in terms of a. [2]

(i)

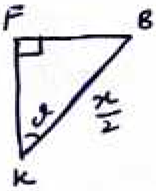
Taking moments about A



$x(AD) = a(AE)$ —(1)

$AD = FB - AH$

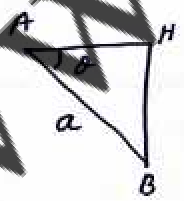
In rt △ BFK



$FB = BK \sin\theta$

$F = \frac{x}{2}\sin\theta$

In rt △ AHB

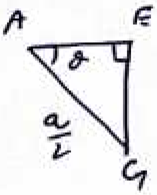


$AH = AB\cos\theta$

$AH = a\cos\theta$

$AD = \frac{x}{2}\sin\theta - a\cos\theta$

In rt△ AEG



$AE = AG\cos\theta$

$= \frac{a}{2}\cos\theta$

Putting AD and AE in (1)

$x\left(\frac{x}{2}\sin\theta - a\cos\theta\right) = a\left(\frac{a}{2}\cos\theta\right)$

$\frac{x^2}{2}\sin\theta - ax\cos\theta - \frac{a^2}{2}\cos\theta = 0$

$x^2\sin\theta - 2ax\cos\theta - a^2\cos\theta = 0$

$x^2\tan\theta - 2ax - a^2 = 0$

(ii) Given $\tan\theta = 1.25$

put in above required equation

$$1.25x^2 - 2ax - a^2 = 0$$

$$\frac{5}{4}x^2 - 2ax - a^2 = 0$$

$$5x^2 - 8ax - 4a^2 = 0$$

$$(x-2a)(5x+2a) = 0$$

$x = 2a$ or $x = -\frac{2a}{5}$ (Not possible)

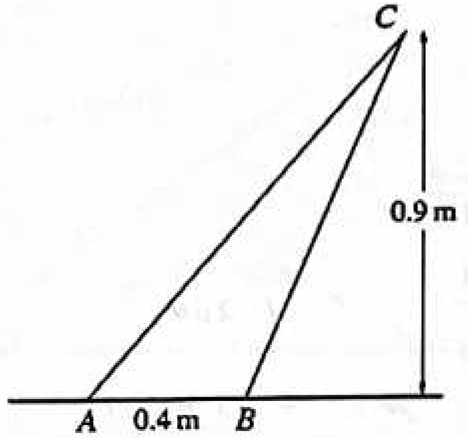
Now the length of wire $= x + a$

$= 2a + a$

$= 3a$

Q.17 9709/51/O/N/19

7



ABC is the cross-section through the centre of mass of a uniform prism which rests with AB on a rough horizontal surface. $AB = 0.4$ m and C is 0.9 m above the surface (see diagram). The prism is on the point of toppling about its edge through B.

(i) Show that angle $BAC = 48.4°$, correct to 3 significant figures. [3]

A force of magnitude 18 N acting in the plane of the cross-section and perpendicular to AC is now applied to the prism at C. The prism is on the point of rotating about its edge through A.

(ii) Calculate the weight of the prism. [3]

(iii) Given also that the prism is on the point of slipping, calculate the coefficient of friction between the prism and the surface. [4]

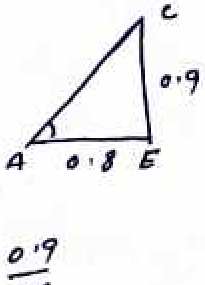
(i) Since $\triangle MGB$ and $\triangle MCE$ are similar, so

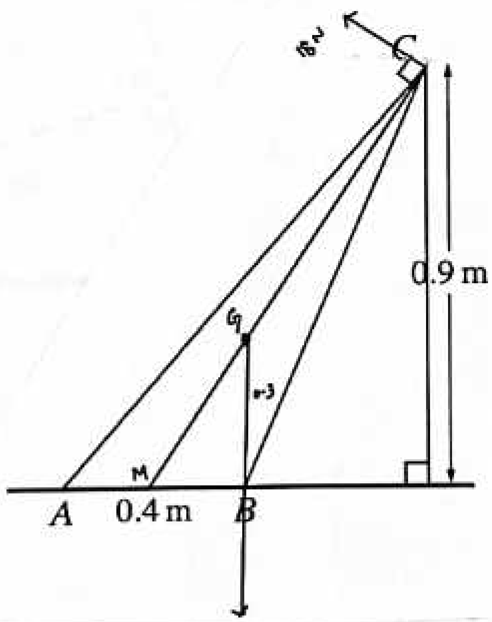
$ME = 3 \times 0.2 = 0.6$

$BG = \dfrac{0+0+0.9}{3} = 0.3$

given prism on the point of toppling at B

In rt $\triangle$ AEC





$\tan EAC = \tan BAC$

$\tan BAE = \dfrac{CE}{AB} = \dfrac{0.9}{0.8}$

$\angle B\hat{A}C = \tan^{-1}\left(\dfrac{0.9}{0.8}\right)$

$= 48.4°$

→

(ii) Taking Moments about A

$W(0\cdot4) = 18(AC)$

In rt $\triangle$ ACE

$$\sin 48\cdot4^\circ = \frac{CE}{AC}$$

$$AC = \frac{0\cdot9}{\sin 48\cdot4^\circ} = 1\cdot2041$$

$\Rightarrow \quad W = (18 \times 1\cdot2041)/0\cdot4 = 54\cdot2\,N$

(iii) Resolve forces horizontally

$18 \sin A = 18 \sin 48\cdot4 = 13\cdot46$

Resolve forces ~~varticall~~ vertically

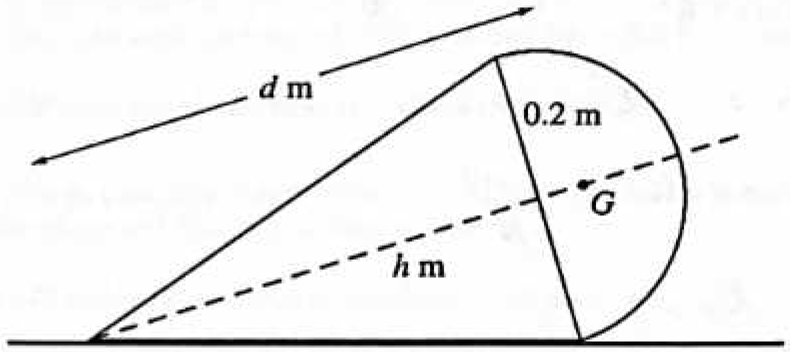
$54\cdot2 - 18 \cos 48\cdot4 = 42\cdot25$

$$\mu = \frac{F}{R} = \frac{13\cdot46}{42\cdot25}$$

$\mu = 0\cdot319$

Q.18 9709/53/O/N/15

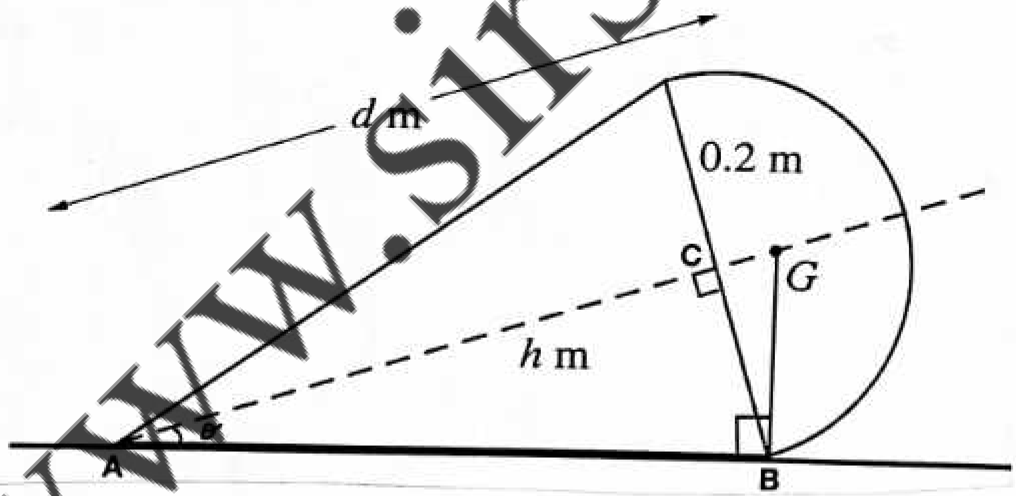
6



An object is formed by joining a hemispherical shell of radius 0.2 m and a solid cone with base radius 0.2 m and height h m along their circumferences. The centre of mass, G, of the object is d m from the vertex of the cone on the axis of symmetry of the object. The object rests in equilibrium on a horizontal plane, with the curved surface of the cone in contact with the plane (see diagram). The object is on the point of toppling.

(i) Show that $d = h + \dfrac{0.04}{h}$. [3]

(ii) It is given that the cone is uniform and of weight 4 N, and that the hemispherical shell is uniform and of weight W N. Given also that $h = 0.8$, find W. [6]



(i) In rt △ ACB

$$\frac{AC}{AB} = \cos\theta$$

$$\frac{h}{AB} = \cos\theta$$

$$\cos\theta = \frac{h}{\sqrt{(0.2)^2 + h^2}} \quad \text{—(1)}$$

as $(AB)^2 = (0.2)^2 + h^2$

In rt △ ACB

In rt △ ABG

$$\frac{AB}{AG} = \cos\theta$$

$$\frac{\sqrt{(0.2)^2 + h^2}}{d} = \cos\theta \quad \text{—(2)}$$

→

From ① and ②

$$\frac{h}{\sqrt{(0.2)^2+h^2}} = \frac{\sqrt{(0.2)^2+h^2}}{d}$$

$$dh = h^2 + 0.04$$

$$d = h + \frac{0.04}{h}$$

(ii) Given $h = 0.8$

$$d = 0.8 + \frac{0.04}{0.8} = 0.85$$

Taking moments about A

$$(4+w)d = 4\times0.6 + w\times0.9$$

as COM from vertex of the cone $= \frac{4}{3}h$

$$(4+W)(0.85) = 4\times0.6 + w\times0.9$$

COM from centre of hemispherical shell $= \frac{1}{2}R$

$$(0.9-0.85)w = 4\times0.85 - 4\times0.6$$

$$0.05\,w = 1$$

$$w = 20$$

Q.19 9709/52/F/M/17

2 A cylindrical container is open at the top. The curved surface and the circular base of the container are both made from the same thin uniform material. The container has radius 0.2 m and height 0.9 m.

(i) Show that the centre of mass of the container is 0.405 m from the base. [3]

The container is placed with its base on a rough inclined plane. The container is in equilibrium on the point of slipping down the plane and also on the point of toppling.

(ii) Find the coefficient of friction between the container and the plane. [3]

(i)

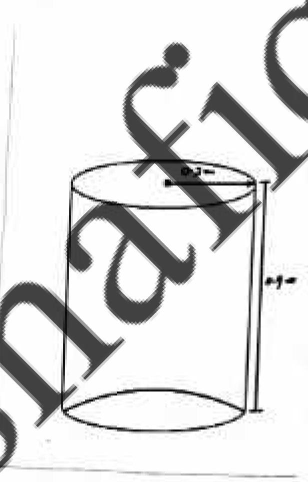
Total mass $= 2\pi r h + \pi r^2$

$= 2\pi \times 0.2 \times 0.9 + \pi(0.2)^2$

$\bar{x}$ = centre of mass from base

$$\bar{x} = \frac{(2\pi \times 0.2 \times 0.9)\left(\frac{0.9}{2}\right)}{2\pi \times 0.2 \times 0.9 + \pi(0.2)^2}$$

$\bar{x} = 0.405$ m



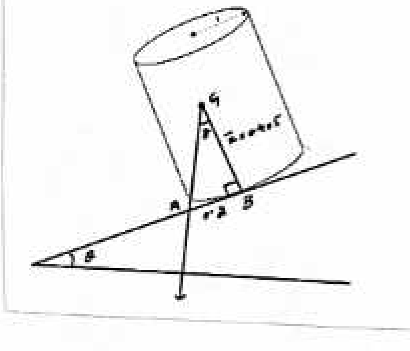
(ii) In rt $\triangle$ ABG

$\tan\theta = \frac{0.2}{0.405} = 0.494$

we know that

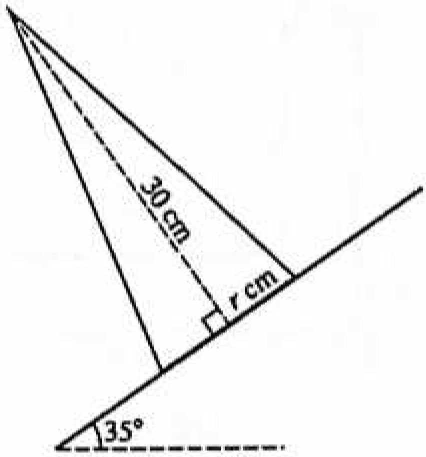
$\mu = \tan\theta$

$\mu = 0.494$



Q.19 9709/51/M/J/10

2



A uniform solid cone has height 30 cm and base radius r cm. The cone is placed with its axis vertical on a rough horizontal plane. The plane is slowly tilted and the cone remains in equilibrium until the angle of inclination of the plane reaches 35°, when the cone topples. The diagram shows a cross-section of the cone.

(i) Find the value of r. [3]

(ii) Show that the coefficient of friction between the cone and the plane is greater than 0.7. [2]

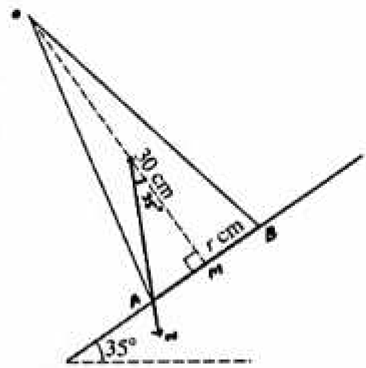
(i) centre of mass of the cone from $O = \frac{3}{4}h$

$= \frac{3}{4} \times 30$

$= \frac{90}{4}$

$GM = \frac{h}{4} = 30 - \frac{90}{4}$

$= \frac{30}{4} = 7.5$ cm



As object topples when $\theta = 35°$

In rt $\triangle$ GMA

$\tan 35 = \frac{AM}{GM}$

$\tan 35 = \frac{r}{7.5}$

$r = 7.5 \times \tan 35°$

$= 5.25$ cm

(ii) when object topples but does not slide, then

$\mu > \tan\theta$

$\Rightarrow \mu > \tan 35°$

$\mu > 0.7$

Q.20 9709/05/O/N/07

7

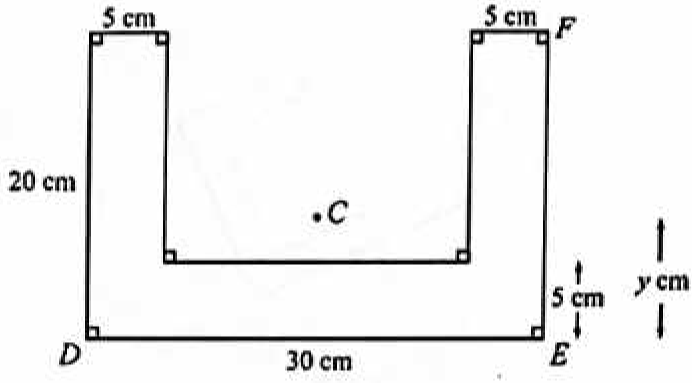
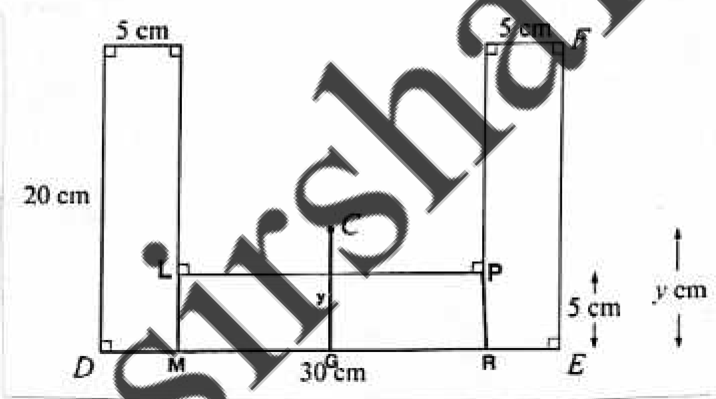


Fig. 1

Fig. 1 shows the cross-section of a uniform solid. The cross-section has the shape and dimensions shown. The centre of mass C of the solid lies in the plane of this cross-section. The distance of C from DE is y cm.

(i) Find the value of y. [3]



(i)

Area of ABMD = 20×5

Area of LMRP = 20×5

Area of QREL = 20×5

Given C is the COM from DE

$\therefore$ $y(20\times5+20\times5+20\times5)=(20\times5)\times10+(20\times5)\times2.5+(20\times5)\times10$

$300y=2250$

$y=\dfrac{2250}{300}=7.5\text{ cm}$

The solid is placed on a rough plane. The coefficient of friction between the solid and the plane is μ. The plane is tilted so that EF lies along a line of greatest slope.

(ii)

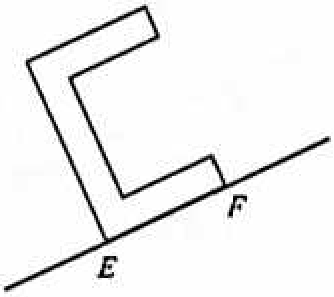
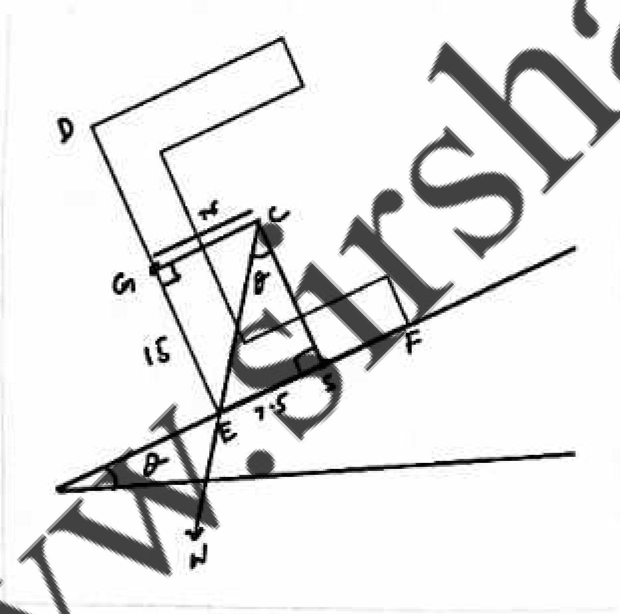


Fig. 2

The solid is placed so that F is higher up the plane than E (see Fig. 2). When the angle of inclination is sufficiently great the solid starts to topple (without sliding). Show that $\mu > \frac{1}{2}$. [3]



Let θ be the angle such that the object is at the point of toppling at E

we know that C is com and

$CG = 7.5 \Rightarrow ES = 7.5$

$CS = GE = 15$

In rt $\triangle CSE$ $\tan\theta = \frac{ES}{CS} = \frac{7.5}{15} = \frac{1}{2}$

Since object is at the point of toppling but not sliding

$\Rightarrow \mu > \frac{1}{2}$

(iii)

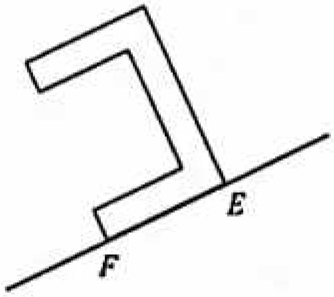
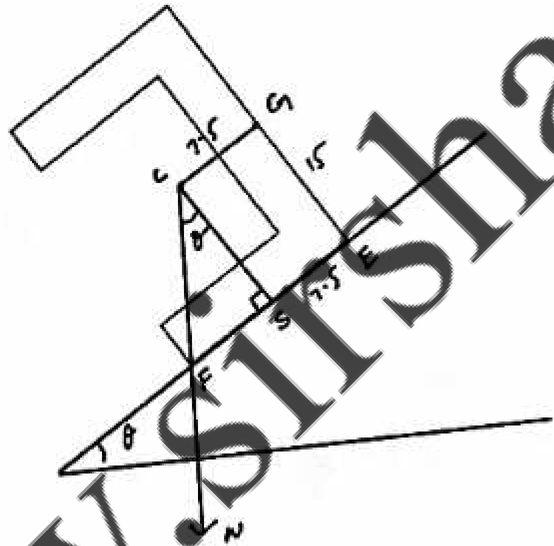


Fig. 3

The solid is now placed so that E is higher up the plane than F (see Fig. 3). When the angle of inclination is sufficiently great the solid starts to slide (without toppling). Show that $\mu < \frac{5}{6}$. [3]



In rt Δ FSC

$$\tan\theta = \frac{FS}{CS} = \frac{12.75}{7.5}$$

$$\tan\theta = \frac{5}{6}$$

given that object sides without toppling

$\Rightarrow \quad \mu < \tan\theta$

$\Rightarrow \quad \mu < \frac{5}{6}$