AS Level Further Mathematics

Topic: Mathematical Induction

Teacher:
Muhammad Shafiq ur Rehman
Aitchison College Lahore

Q.1 Oct/Nov/P11+P13/2022

2 Prove by mathematical induction that, for all positive integers n, $7^{2n} + 97^n - 50$ is divisible by 48. [6]

For $n=1$ $\quad 7^2 + 97 - 50 = 96 = 2 \times 48$ (True)

Suppose result is True for $n = k$

$7^{2k} + 97^k - 50$ is divisible by 48

Now for $n = k+1$

$$7^{2k+2} + 97^{k+1} - 50 = 49 \cdot 7^{2k} + 97 \cdot 97^k - 50$$

$$= 7^{2k} + 97^k - 50 + 48 \cdot 7^{2k} + 96 \cdot 97^k$$

$$= \underbrace{(7^{2k} + 97^k - 50)}_{\text{Divisible by Supposition}} + 48\underbrace{(7^{2k} + 2 \times 97^k)}_{\text{divisible by 48}}$$

So

$7^{2k+2} + 97^{k+1} - 50$ is divisible by 48

Hence induction is complete.

Q.2 Oct/Nov/P12/2022

4 The function f is such that $f''(x) = f(x)$.

Prove by mathematical induction that, for every positive integer n,

$$\frac{d^{2n-1}}{dx^{2n-1}}(xf(x)) = xf'(x) + (2n-1)f(x).$$

[7]

For $n=1$

$$\frac{d}{dx}(xf(x)) = xf'(x) + 1f(x) \quad (\text{True})$$

Suppose result is True for $n=k$

$$\frac{d^{2k-1}}{dx^{2k-1}}(xf(x)) = xf'(x) + (2k-1)f(x)$$

Now for $n=k+1$

$$\frac{d^{2k+1}}{dx^{2k+1}}(xf(x)) = \frac{d^2}{dx^2}\left[xf'(x) + (2k-1)f(x)\right]$$

$$= \frac{d}{dx}\left[xf''(x) + f'(x) + (2k-1)f'(x)\right]$$

$$= \frac{d}{dx}\left[xf(x) + 2kf'(x)\right]$$

$$= xf'(x) + f(x) + 2kf''(x)$$

$$= xf'(x) + f(x) + 2kf(x) \quad \text{Given } f''(x) = f(x)$$

$$= xf'(x) + (2k+1)f(x)$$

Hence induction is complete.

1 Prove by mathematical induction that $2^{4n}+31^n-2$ is divisible by 15 for all positive integers n. [6]

For $n=1$ $\quad 2^4+31-2=45=15\times 3$ (True)

Suppose result is true for $n=k$

$2^{4k}+31^k-2$ is divisible by 15

Now for $n=k+1$

$$2^{4(k+1)}+31^{k+1}-2 = 2^4\times 2^{4k}+31\times 31^k-2$$
$$= 2^{4k}+31^k-2+15\times 2^{4k}+30\times 31^k$$
$$= \underbrace{(2^{4k}+31^k-2)}_{\text{divisible by 15 by supposition}}+15\underbrace{(2^{4k}+2\times 31^k)}_{\text{divisible by 15}}$$

So $\quad 2^{4(k+1)}+31^{k+1}-2$ is divisible by 15

Hence induction is complete.

Q.5 May/June/P13/2021

3 (a) Prove by mathematical induction that, for all positive integers n,

$$\sum_{r=1}^{n}\left(5r^4+r^2\right)=\tfrac{1}{2}n^2(n+1)^2(2n+1).$$ [6]

(b) Use the result given in part (a) together with the List of formulae (MF19) to find $\sum_{r=1}^{n} r^4$ in terms of n, fully factorising your answer. [3]

(a) For $n=1$

$$5(1)^4+(1)^2=\frac{1}{2}(1)^2(1+1)^2(2+1)$$

$$5+1=\frac{1}{2}\times 4\times 3$$

$$6=6 \quad \text{(True)}$$

Suppose result is True for $n=k$

$$\sum_{r=1}^{k}(5r^4+r^2)=\frac{1}{2}k^2(k+1)^2(2k+1)$$

Now for $n=k+1$

$$\sum_{r=1}^{k+1}(5r^4+r^2)=\frac{1}{2}k^2(k+1)^2(2k+1)+5(k+1)^4+(k+1)^2$$

$$=\frac{1}{2}(k+1)^2\left[k^2(2k+1)+10(k+1)^2+2\right]$$

$$=\frac{1}{2}(k+1)^2\left[2k^3+k^2+10k^2+20k+10+2\right]$$

$$=\frac{1}{2}(k+1)^2\left[2k^3+11k^2+20k+12\right]$$

$$\begin{array}{r|rrrr}
-2 & 2 & 11 & 20 & 12 \\
 & 0 & -4 & -14 & -12 \\ \hline
-2 & 2 & 7 & 6 & 0 \\
 & 0 & -4 & -6 & \\ \hline
 & 2 & 3 & 0 &
\end{array}$$

$$2k^3+11k^2+20k+12=(k+2)^2(2k+3)$$

$$\sum_{r=1}^{k+1}(5r^4+r^2)=\frac{1}{2}(k+1)^2(k+2)^2(2k+3)$$

Hence induction is complete.

Q.3 2022/May/June/P11+P12

3 The sequence of positive numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 > 4$ and, for $n \geqslant 1$,

$$u_{n+1} = \frac{u_n^2 + u_n + 12}{2u_n}.$$

(a) By considering $u_{n+1} - 4$, or otherwise, prove by mathematical induction that $u_n > 4$ for all positive integers n. [5]

(b) Show that $u_{n+1} < u_n$ for $n \geqslant 1$. [3]

(a) For $n=1$ $u_1 > 4$ True by given

Suppose result is True for $n = k$

$$u_k > 4$$

Now for $n = k+1$

$$u_{k+1} - 4 = \frac{u_k^2 + u_k + 12}{2u_k} - 4$$

$$u_{k+1} - 4 = \frac{u_k^2 + u_k + 12 - 8u_k}{2u_k}$$

$$= \frac{u_k^2 - 7u_k + 12}{2u_k}$$

$$= \frac{u_k^2 - 4u_k - 3u_k + 12}{2u_k}$$

$$= \frac{(u_k - 4)(u_k - 3)}{2u_k}$$

$$= \frac{(+ve)(+ve)}{+ve} > 0 \quad \text{as } u_k > 4$$

$$u_{k+1} - 4 > 0$$

$\Rightarrow$ $u_{k+1} > 4$ Hence induction is complete.

(b) $$u_{n+1} - u_n = \frac{u_n^2 + u_n + 12}{2u_n} - u_n$$

$$= \frac{u_n^2 + u_n + 12 - 2u_n^2}{2u_n}$$

$$= \frac{-(u_n^2 - u_n - 12)}{2u_n}$$

$$= -\frac{[(u_n - 4)(u_n + 3)]}{2u_n}$$

$$= -\frac{(+ve)(+ve)}{+ve} \quad \because u_n > 4$$

$$= -ve < 0$$

$$u_{n+1} - u_n < 0$$

$$u_{n+1} < u_n \quad \text{for } n \geq 1$$

proved.

3 The sequence of real numbers $a_1, a_2, a_3, \dots$ is such that $a_1 = 1$ and

$$a_{n+1} = \left(a_n + \frac{1}{a_n}\right)^3.$$

(a) Prove by mathematical induction that $\ln a_n \geqslant 3^{n-1} \ln 2$ for all integers $n \geqslant 2$. [6]

[You may use the fact that $\ln\left(x + \frac{1}{x}\right) > \ln x$ for $x > 0$.]

(b) Show that $\ln a_{n+1} - \ln a_n > 3^{n-1} \ln 4$ for $n \geqslant 2$. [2]

(a) For $n=2$ $\quad \ln a_2 \geqslant 3^{2-1} \ln 2$

$$\ln\left(1+\frac{1}{1}\right)^3 \geqslant 3\ln 2$$

$$3\ln 2 \geqslant 3\ln 2 \quad \text{(True)}$$

Suppose result is true for $n = k$

$$\ln a_k \geqslant 3^{k-1}\ln 2$$

Now for $n = k+1$

$$a_{k+1} = \left(a_k + \frac{1}{a_k}\right)^3$$

$$\ln a_{k+1} = 3\ln\left(a_k + \frac{1}{a_k}\right)$$

$$> 3\ln a_k$$

$$> 3 \cdot 3^{k-1}\ln 2 \quad \text{by supposition}$$

$$= 3^k \ln 2$$

$$\ln a_{k+1} > 3^k \ln 2$$

Hence induction is complete.

(b)

$$\ln a_{n+1} - \ln a_n = \ln\left(a_n + \frac{1}{a_n}\right)^3 - \ln a_n$$

$$= 3\ln\left(a_n + \frac{1}{a_n}\right) - \ln a_n$$

$$> 3\ln a_n - \ln a_n$$

$$= 2\ln a_n$$

$$> 2 \cdot 3^{n-1}\ln 2$$

$$= 3^{n-1}\ln 4$$

Proved

Q.7 Oct/Nov/P12/2021

2 It is given that $y = xe^{ax}$, where a is a constant.

Prove by mathematical induction that, for all positive integers n,

$$\frac{d^n y}{dx^n} = (a^n x + na^{n-1})e^{ax}.$$ [6]

For $n = 1$

$$\frac{d}{dx} y = \frac{d}{dx}(x e^{ax})$$

$$= x e^{ax}(a) + e^{ax}(1)$$

$$= (ax + 1) e^{ax} \quad \text{True}$$

Suppose result is true for $n = k$

$$\frac{d^k y}{dx^k} = [a^k x + k a^{k-1}] e^{ax}$$

Now for $n = k+1$

$$\frac{d^{k+1} y}{dx^{k+1}} = \frac{d}{dx}\left[\frac{d^k y}{dx^k}\right]$$

$$= \frac{d}{dx}[a^k x + k a^{k-1}] e^{ax}$$

$$= [a^k x + k a^{k-1}] e^{ax}(a) + e^{ax} + a^k$$

$$= [a^{k+1} x + k a^k + a^k] e^{ax}$$

$$= [a^{k+1} x + (k+1) a^k] e^{ax}$$

Hence induction is complete.

Q.8 May/June/P13/2020

2 The sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 1$ and $u_{n+1} = 2u_n + 1$ for $n \geqslant 1$.

(a) Prove by induction that $u_n = 2^n - 1$ for all positive integers n. [5]

(b) Deduce that u_{2n} is divisible by u_n for $n \geqslant 1$. [2]

(a) For $n=1$

$$u_1 = 2^1 - 1$$
$$1 = 2 - 1$$
$$1 = 1 \quad \text{(True)}$$

Suppose result is true for $n = k$

$$u_k = 2^k - 1$$

Now for $n = k+1$

$$u_{k+1} = 2u_k + 1 \quad \text{by given}$$
$$= 2(2^k - 1) + 1$$
$$= 2^{k+1} - 2 + 1$$
$$= 2^{k+1} - 1 \quad \text{Hence induction is complete.}$$

(b)

$$u_{2n} = 2^{2n} - 1$$
$$= (u_n + 1)^2 - 1 \quad \text{as } u_n = 2^n - 1$$
$$= u_n^2 + 2u_n + 1 - 1$$
$$= u_n^2 + 2u_n$$
$$= u_n(u_n + 2)$$

This shows that

u_{2n} is divisible by u_n

Q.9 Oct/Nov/P11+P13/2020

5 Prove by mathematical induction that, for every positive integer n,

$$\frac{d^{2n-1}}{dx^{2n-1}}(x\sin x) = (-1)^{n-1}\left(x\cos x + (2n-1)\sin x\right).$$

[7]

For $n=1$

$$\frac{d}{dx}(x\sin x) = x\cos x + \sin x \quad (\text{True})$$

Suppose result is true for $n=k$

$$\frac{d^{2k-1}}{dx^{2k-1}}(x\sin x) = (-1)^{k-1}\left[x\cos x + (2k-1)\sin x\right]$$

Now for $n=k+1$

$$\frac{d^{2(k+1)-1}}{dx^{2(k+1)-1}}(x\sin x)$$

$$= \frac{d^{2k+1}}{dx^{2k+1}}(x\sin x)$$

$$= \frac{d^2}{dx^2}\left[\frac{d^{2k-1}}{dx^{2k-1}}(x\sin x)\right]$$

$$= \frac{d^2}{dx^2}\left[(-1)^{k-1}\left(x\cos x + (2k-1)\sin x\right)\right]$$

$$= \frac{d}{dx}\left[(-1)^{k-1}\left(x(-\sin x) + \cos x + (2k-1)\cos x\right)\right]$$

$$= \frac{d}{dx}\left[(-1)^{k-1}\left(-x\sin x + 2k\cos x\right)\right]$$

$$= (-1)^{k-1}\left[-x\cos x - \sin x - 2k\sin x\right]$$

$$= (-1)^{k}\left[x\cos x + (2k+1)\sin x\right]$$

Hence induction is complete.

Q.10 Oct/Nov/P12/2020

2 Prove by mathematical induction that $7^{2n}-1$ is divisible by 12 for every positive integer n. [5]

For $n=1$

$$7^2-1 = 49-1$$
$$= 48 = 12\times 4 \quad \text{(True)}$$

Suppose result is true for $n=k$

$7^{2k}-1$ is divisible by 12

Now for $n=k+1$

$$7^{2(k+1)}-1 = 7^{2k+2}-1$$
$$= 49\times 7^{2k}-1$$
$$= \underbrace{7^{2k}-1}_{\substack{\text{Divisible by 12} \\ \text{by supposition}}} + \underbrace{48\times 7^{2k}}_{\text{divisible by 12}}$$

so

$7^{2k+2}-1$ is divisible by 12

Hence induction is complete.

Q.11 May/June/P11+P12/2019

8 (i) Prove by mathematical induction that, for $z \neq 1$ and all positive integers n,

$$1 + z + z^2 + \ldots + z^{n-1} = \frac{z^n - 1}{z - 1}.$$ [5]

For $n = 1$

$$1 = \frac{z^1 - 1}{z - 1} = 1 \quad (\text{True})$$

Suppose result is true for $n = k$

$$1 + z + z^2 + \cdots + z^{k-1} = \frac{z^k - 1}{z - 1}$$

Now for $n = k+1$

Adding on both sides z^k

$$1 + z + z^2 + \cdots + z^{k-1} + z^k = \frac{z^k - 1}{z - 1} + z^k$$

$$= \frac{z^k - 1 + z^{k+1} - z^k}{z - 1}$$

$$= \frac{z^{k+1} - 1}{z - 1}$$

Hence induction is complete.

Q.12 May/June/P13/2019

1 Prove by mathematical induction that $3^{3n} - 1$ is divisible by 13 for every positive integer n. [5]

For $n=1$ $3^3 - 1 = 27-1$

$= 26 = 13 \times 2$ (True)

Suppose result is true for $n = k$

$3^{3k} - 1$ is divisible by 13

Now for $n = k+1$

$$3^{3(k+1)} - 1 = 3^{3k+3} - 1$$

$$= 27 \times 3^{3k} - 1$$

$$= \underbrace{3^{3k} - 1}_{\text{By supposition divisible by 13}} + \underbrace{26 \times 3^{3k}}_{\text{divisible by 13}}$$

So $3^{3(k+1)} - 1$ is divisible by 13

Hence induction is complete.

Q.13 Oct/Nov/P11+P12+P13/2019

2 It is given that $y = \ln(ax+1)$, where a is a positive constant. Prove by mathematical induction that, for every positive integer n,

$$\frac{d^n y}{dx^n} = (-1)^{n-1}\frac{(n-1)!a^n}{(ax+1)^n}.$$ [6]

For $n=1$

$$\frac{d^1 y}{dx^1} = \frac{d}{dx}\ln(ax+1)$$

$$= (-1)^0 \frac{a}{ax+1} = \frac{0!\,a^1}{(ax+1)^1} \quad (\text{True})$$

Suppose result is true for $n=k$

$$\frac{d^k y}{dx^k} = (-1)^{k-1}\frac{(k-1)!\,a^k}{(ax+1)^k}$$

Now for $n=k+1$

$$\frac{d^{k+1} y}{dx^{k+1}} = \frac{d}{dx}\left[\frac{d^k y}{dx^k}\right]$$

$$= \frac{d}{dx}\left[(-1)^{k-1}\frac{(k-1)!\,a^k}{(ax+1)^k}\right]$$

$$= (-1)^{k-1}(k-1)!\,a^k\left[-k(ax+1)^{-k-1}(a)\right]$$

$$= \frac{(-1)^k\,k(k-1)!\,a^{k+1}}{(ax+1)^{k+1}}$$

$$= (-1)^k\frac{k!\,a^{k+1}}{(ax+1)^{k+1}}$$

Hence induction is complete.

Q.14 May/June/P11+P12/2018

2 It is given that $f(n) = 2^{3n} + 8^{n-1}$. By simplifying $f(k) + f(k+1)$, or otherwise, prove by mathematical induction that $f(n)$ is divisible by 9 for every positive integer n. [6]

For $n = 1$

$$f(1) = 2^3 + 8^0$$
$$= 8 + 1 = 9 \quad (\text{True})$$

Suppose result is True for $n = k$

$$f(k) = 2^{3k} + 8^{k-1} \text{ is divisible by } 9$$

Now for $n = k+1$

$$f(k+1) = 2^{3(k+1)} + 8^{k+1-1}$$
$$= 2^{3k+3} + 8^{k}$$
$$= 8 \times 2^{3k} + 8 \times 8^{k-1}$$
$$= \underbrace{(2^{3k} + 8^{k-1})}_{\text{By supposition divisible by } 9} 8$$

So $f(k+1)$ is divisible by 9

Hence induction is complete.

Q.15 May/June/P13/2018

9 For the sequence $u_1, u_2, u_3, \ldots$, it is given that $u_1 = 8$ and

$$u_{r+1} = \frac{5u_r - 3}{4}$$

for all r.

(i) Prove by mathematical induction that

$$u_n = 4\left(\tfrac{5}{4}\right)^n + 3,$$

for all positive integers n. [5]

For $n=1$

$$u_1 = 4\left(\frac{5}{4}\right)^1 + 3$$
$$= 5+3$$
$$= 8 \quad \text{(True) by given}$$

Suppose result is true for $n=k$

$$u_k = 4\left(\frac{5}{4}\right)^k + 3$$

Now for $n = k+1$

$$u_{k+1} = \frac{5u_k - 3}{4}$$
$$= \frac{5\left[4\left(\frac{5}{4}\right)^k + 3\right] - 3}{4} \quad \text{by supposition}$$
$$= \frac{20\left(\frac{5}{4}\right)^k + 12}{4}$$
$$= 5\left(\frac{5}{4}\right)^k + 3$$
$$= 4\left(\frac{5}{4}\right)\left(\frac{5}{4}\right)^k + 3$$
$$= 4\left(\frac{5}{4}\right)^{k+1} + 3$$

Hence induction is complete.

Q.16 Oct/Nov/P11+P13/2018

3 The sequence of positive numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 < 3$ and, for $n \geqslant 1$,

$$u_{n+1} = \frac{4u_n + 9}{u_n + 4}.$$

(i) By considering $3 - u_{n+1}$, or otherwise, prove by mathematical induction that $u_n < 3$ for all positive integers n. [5]

(ii) Show that $u_{n+1} > u_n$ for $n \geqslant 1$. [3]

For $n = 1$ $u_1 < 3$ given (True)

suppose result is true for $n = k$

$$u_k < 3$$

Now for $n = k+1$

$$3 - u_{k+1} = 3 - \frac{4u_k + 9}{u_k + 4}$$

$$= \frac{3(u_k + 4) - 4u_k - 9}{u_k + 4}$$

$$= \frac{-u_k + 3}{u_k + 4}$$

given u_k is +ve and

$u_k < 3$ by supposition, so

$$= \frac{+ve}{+ve} > 0$$

$$3 - u_{k+1} > 0$$

$$3 > u_{k+1}$$

$$\Rightarrow u_{k+1} < 3$$

Hence induction is complete.

Q.17 Oct/Nov/P12/2018

6 It is given that $y = e^x u$, where u is a function of x. The rth derivatives $\frac{d^r y}{dx^r}$ and $\frac{d^r u}{dx^r}$ are denoted by $y^{(r)}$ and $u^{(r)}$ respectively. Prove by mathematical induction that, for all positive integers n,

$$y^{(n)} = e^x\left(\binom{n}{0}u + \binom{n}{1}u^{(1)} + \binom{n}{2}u^{(2)} + \ldots + \binom{n}{r}u^{(r)} + \ldots + \binom{n}{n}u^{(n)}\right).$$ [8]

[You may use without proof the result $\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$.]

For $n=1$ $\quad y' = \frac{d}{dx}(e^x u)$

$$= e^x u + e^x u'$$

$$= e^x\left[\binom{1}{0}u + \binom{1}{1}u'\right] \quad \text{(True)}$$

Suppose result is true for $n=k$

$$y^k = e^x\left[\binom{k}{0}u + \binom{k}{1}u' + \binom{k}{2}u^2 + \cdots + \binom{k}{k}u^k\right]$$

Now for $n=k+1$

$$y^{k+1} = \frac{d}{dx}y^k$$

$$= \frac{d}{dx}e^x\left[\binom{k}{0}u + \binom{k}{1}u' + \binom{k}{2}u^2 + \cdots + \binom{k}{k}u^k\right]$$

$$= e^x\left[\binom{k}{0}u + \binom{k}{1}u' + \binom{k}{2}u^2 + \cdots + \binom{k}{k}u^k\right]$$

$$+ e^x\left[\binom{k}{0}u' + \binom{k}{1}u^2 + \binom{k}{2}u^3 + \cdots + \binom{k}{k}u^{k+1}\right]$$

$$= e^x\left[\binom{k}{0}u + \left[\binom{k}{1} + \binom{k}{0}\right]u' + \left[\binom{k}{2} + \binom{k}{1}\right]u^2 + \cdots + \binom{k}{k}u^{k+1}\right]$$

$$= e^x\left[\binom{k+1}{0}u + \binom{k+1}{1}u' + \binom{k+1}{2}u^2 + \cdots + \binom{k+1}{k+1}u^{k+1}\right]$$

as $\binom{k}{0} = \binom{k+1}{0}$ and $\binom{k}{k} = \binom{k+1}{k+1}$

Hence induction is complete.

Q.18 May/June/P11+P12/2017

2 Prove, by mathematical induction, that $5^n + 3$ is divisible by 4 for all non-negative integers n. [5]

For $n=1$

$$5^1+3 = 8 = 4\times 2 \quad (\text{True})$$

Suppose result is true for $n=k$

5^k+3 is divisible by 4

Now for $n=k+1$

$$5^{k+1}+3 = 5\cdot 5^k+3$$

$$= \underbrace{(5^k+3)}_{\text{by supposition divisible by } 4} + \underbrace{4\cdot 5^k}_{\text{divisible by } 4}$$

So $5^{k+1}+3$ is divisible by 4

Q.19 May/June/P13/2017

3 Prove, by mathematical induction, that $\sum_{r=1}^{n} r\ln\left(\frac{r+1}{r}\right) = \ln\left(\frac{(n+1)^n}{n!}\right)$ for all positive integers n. [6]

For $n=1$

$$1\ln\left(\frac{1+1}{1}\right) = \ln\left(\frac{(1+1)^1}{1!}\right)$$

$$\ln 2 = \ln 2 \quad \text{(True)}$$

Suppose result is True for $n=k$

$$\sum_{r=1}^{k} r\ln\left(\frac{r+1}{r}\right) = \ln\left[\frac{(k+1)^k}{k!}\right]$$

Now for $n=k+1$

$$\sum_{r=1}^{k+1} r\ln\left(\frac{r+1}{r}\right) = \sum_{r=1}^{k} r\ln\left(\frac{r+1}{r}\right) + (k+1)\ln\left(\frac{k+2}{k+1}\right)$$

$$= \ln\left(\frac{(k+1)^k}{k!}\right) + \ln\frac{(k+2)^{k+1}}{(k+1)^{k+1}}$$

$$= \ln\left[\frac{(k+1)^k}{k!} \times \frac{(k+2)^{k+1}}{(k+1)^{k+1}}\right]$$

$$= \ln\left(\frac{(k+2)^{k+1}}{(k+1)k!}\right)$$

$$= \ln\left(\frac{(k+2)^{k+1}}{(k+1)!}\right)$$

Hence induction is complete.

Q.20 Oct/Nov/P11+P12+P13/2017

3 (i) Show that $\frac{d^{n+1}}{dx^{n+1}}(x^{n+1}\ln x) = \frac{d^n}{dx^n}(x^n + (n+1)x^n \ln x)$. [2]

(ii) Prove by mathematical induction that, for all positive integers n,

$$\frac{d^n}{dx^n}(x^n \ln x) = n!\left(\ln x + 1 + \frac{1}{2} + \ldots + \frac{1}{n}\right).$$ [5]

(i)

$$\frac{d^{n+1}}{dx^{n+1}}(x^{n+1}\ln x) = \frac{d^n}{dx^n}\cdot\frac{d}{dx}(x^{n+1}\ln x)$$

$$= \frac{d^n}{dx^n}\left[x^{n+1}\cdot\frac{1}{x} + (n+1)x^n \ln x\right]$$

$$= \frac{d^n}{dx^n}\left(x^n + (n+1)x^n \ln x\right)$$

proved

(ii) For $n=1$

$$\frac{d}{dx}(x\ln x) = \left[\ln x + x\cdot\frac{1}{x}\right]$$

$$= 1!(\ln x + 1) \quad \text{(True)}$$

Suppose result is True for $n=k$

$$\frac{d^k}{dx^k}(x^k \ln x) = k!\left(\ln x + 1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)$$

Now for $n = k+1$

$$\frac{d^{k+1}}{dx^{k+1}}(x^{k+1}\ln x) = \frac{d^k}{dx^k}\cdot\frac{d}{dx}(x^{k+1}\ln x)$$

$$= \frac{d^k}{dx^k}\left[(k+1)x^k \ln x + x^{k+1}\cdot\frac{1}{x}\right]$$

$$= \frac{d^k}{dx^k}\left[(k+1)x^k \ln x + x^k\right]$$

$$= (k+1)\frac{d^k}{dx^k}(x^k \ln x) + k!$$

$$= (k+1)\left[k!\left(\ln x + 1 + \frac{1}{2} + \cdots + \frac{1}{k}\right]\right) + k!$$

$$= (k+1)\,k!\left(\ln x + 1 + \frac{1}{2} + \cdots + \frac{1}{k}\right) + \frac{(k+1)\,k!}{(k+1)}$$

$$= (k+1)!\left(\ln x + 1 + \frac{1}{2} + \cdots + \frac{1}{k}\right) + \frac{(k+1)!}{k+1}$$

$$= (k+1)!\left[\ln x + 1 + \frac{1}{2} + \cdots + \frac{1}{k} + \frac{1}{k+1}\right]$$

Hence induction is complete.

Q.21 May/June/P11+P12/2016

3 Prove by mathematical induction that, for all positive integers n, $10^n + 3 \times 4^{n+2} + 5$ is divisible by 9. [6]

For $n=1$ $10^1 + 3 \times 4^3 + 5 = 10 + 192 + 5 = 207 = 9 \times 23$ (True)

Suppose result is true for $n=k$

$10^k + 3 \times 4^{k+2} + 5$ is divisible by 9

Now for $n = k+1$

$$10^{k+1} + 3 \times 4^{k+3} + 5$$

$$= 10 \times 10^k + 3 \times 4 \times 4^{k+2} + 5$$

$$= 10^k + 3 \times 4^{k+2} + 5 + 9 \times 10^k + 9 \times 4^{k+2}$$

$$= \underbrace{(10^k + 3 \times 4^{k+2} + 5)}_{\text{by supposition divisible by 9}} + \underbrace{9(10^k + 4^{k+2})}_{\text{Divisible by 9}}$$

So $10^{k+1} + 3 \times 4^{k+3} + 5$ is divisible by 9

Hence induction is complete.

Q.22 May/June/P13/2016

2 It is given that a diagonal of a polygon is a line joining two non-adjacent vertices. Prove, by mathematical induction, that an n-sided polygon has $\frac{1}{2}n(n-3)$ diagonals, where $n \geqslant 3$. [6]

For $n = 3$ (Triangle)

$$\frac{1}{2}n(n-3) = \frac{1}{2}(3)(0)$$

$$= 0 \quad \text{(True)}$$

Suppose result is True for $n = k$

k-sided polygon has $\frac{1}{2}k(k-3)$ diagonals

For $n = k+1$

$$\frac{1}{2}k(k-3) + (k-1) \quad \text{diagonals}$$

$$= \frac{1}{2}\left[k^2 - 3k + 2k - 2\right]$$

$$= \frac{1}{2}\left[k^2 - k - 2\right]$$

$$= \frac{1}{2}\left[k^2 - 2k + k - 2\right]$$

$$= \frac{1}{2}\left[(k+1)(k-2)\right]$$

$$= \frac{1}{2}(k+1)(k-2)$$

Hence induction is complete.

Q.23 Oct/Nov/P11+P12+P13/2016

4 Using factorials, show that $\binom{n}{r-1}+\binom{n}{r}=\binom{n+1}{r}$. [2]

Hence prove by mathematical induction that

$$(a+x)^n=\binom{n}{0}a^n+\binom{n}{1}a^{n-1}x+\ldots+\binom{n}{r}a^{n-r}x^r+\ldots+\binom{n}{n}x^n$$

for every positive integer n. [4]

$$LHS=\binom{n}{r-1}+\binom{n}{r}$$

$$=\frac{n!}{(n-r+1)!\,(r-1)!}+\frac{n!}{(n-r)!\,r!}$$

$$=\frac{n!}{(n-r+1)(n-r)!\,(r-1)!}+\frac{n!}{(n-r)!\,r(r-1)!}$$

$$=n!\left[\frac{r+n-r+1}{(n-r+1)(n-r)!\,r(r-1)!}\right]$$

$$=\frac{(n+1)\,n!}{(n+1-r)!\,r!}=\frac{(n+1)!}{(n+1-r)!\,r!}=\binom{n+1}{r}$$

proved

For Induction

For $n=1$

$$(a+x)^1=\binom{1}{0}a^1+\binom{1}{1}x^1 \quad (\text{True})$$

suppose result is true for $n=k$

$$(a+x)^k=\binom{k}{0}a^k+\binom{k}{1}a^{k-1}x+\binom{k}{2}a^{k-2}x^2+\ldots+\binom{k}{k}x^k$$

Now for $n=k+1$

$$(a+x)^{k+1}=(a+x)(a+x)^k$$

$$=(a+x)\left[\binom{k}{0}a^k+\binom{k}{1}a^{k-1}x+\binom{k}{2}a^{k-2}x^2+\ldots+\binom{k}{k}x^k\right]$$

$$=\binom{k}{0}a^{k+1}+\binom{k}{1}a^kx+\binom{k}{2}a^{k-1}x^2+\ldots+\binom{k}{k}ax^k$$
$$+\binom{k}{0}a^kx+\binom{k}{1}a^{k-1}x^2+\ldots+\binom{k}{k}x^{k+1}$$

$\rightarrow$

$$= \binom{k}{0}a^{k+1} + \left[\binom{k}{1}+\binom{k}{0}\right]a^{k}x + \left[\binom{k}{2}+\binom{k}{1}\right]a^{k-1}x^{2} + \cdots + \binom{k}{k}x^{k+1}$$

$$= \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^{k}x + \binom{k+1}{2}a^{k-1}x^{2} + \cdots + \binom{k+1}{k+1}x^{k+1}$$

Hence induction is complete.