AS Level Further Mathematics

Topic:

Vectors in 3D

Teacher:

Muhammad Shafiq ur Rehman

Aitchison College Lahore

Q.1 Oct/Nov/P11+P13/2022

4 The plane Π contains the lines $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \mu(3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$.

(a) Find a Cartesian equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

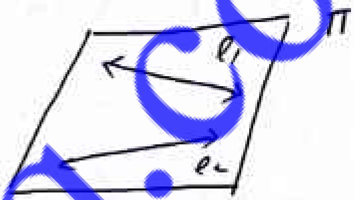
The line l passes through the point P with position vector $2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and is parallel to the vector $\mathbf{j} + \mathbf{k}$.

(b) Find the acute angle between l and Π. [3]

(c) Find the position vector of the foot of the perpendicular from P to Π. [4]

(a) $l_1 \quad r = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

$l_2 \quad r = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$



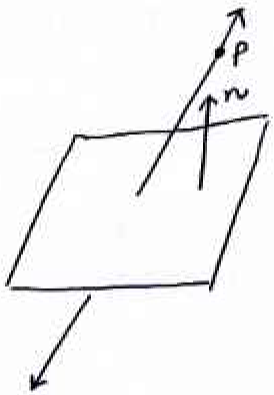
$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 1 \\ 3 & 2 & -1 \end{vmatrix} = -4\hat{i} + 2\hat{j} - 8\hat{k} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$$

Equation of the plane Π

$$2x - y + 4z = 2(3) - (-2) + 4(1)$$

$$2x - y + 4z = 12$$

(b) $l: \quad r = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$



Let θ between l and $\vec{n}$

$$\theta = \cos^{-1}\left(\frac{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}}{\sqrt{2}\,\sqrt{21}} \right)$$

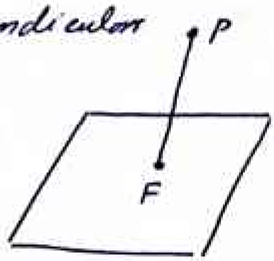
$$\theta = \cos^{-1}\left(\frac{3}{\sqrt{42}}\right) = 62.4^\circ$$

α = be the angle between l and plane Π

$$\alpha = 90^\circ - 62.4^\circ = 27.6^\circ$$

→

(c) Let $F(a,b,c)$ be the foot of perpendicular from P to the plane



Equation of the line through P and F

$$r = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + v \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2+2v \\ 3-v \\ 1+4v \end{pmatrix}$$

For some value of v

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2+2v \\ 3-v \\ 1+4v \end{pmatrix}$$

$a = 2+2v$, $b = 3-v$, $c = 1+4v$

$F(a,b,c)$ lies on the plane

$$2a - b + 4c = 12$$

$$2(2+2v) - (3-v) + 4(1+4v) = 12$$

$$21v + 5 = 12$$

$$v = \frac{1}{3}$$

$$a = 2 + \frac{2}{3} = \frac{8}{3}$$

$$b = 3 - \frac{1}{3} = \frac{8}{3}$$

$$c = 1 + \frac{4}{3} = \frac{7}{3}$$

position vector of F

$$\vec{OF} = \begin{pmatrix} 8/3 \\ 8/3 \\ 7/3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 8 \\ 8 \\ 7 \end{pmatrix}$$

Q.2 Oct/Nov/P12/2022

6 The lines l_1 and l_2 have equations $\mathbf{r} = 2\mathbf{i}+\mathbf{k}+\lambda(\mathbf{i}-\mathbf{j}+2\mathbf{k})$ and $\mathbf{r} = 2\mathbf{j}+6\mathbf{k}+\mu(\mathbf{i}+2\mathbf{j}-2\mathbf{k})$ respectively.

The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2.

(a) Find the length PQ. [5]

The plane Π_1 contains PQ and l_1.

The plane Π_2 contains PQ and l_2.

(b) (i) Write down an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a}+s\mathbf{b}+t\mathbf{c}$. [1]

(ii) Find an equation of Π_2, giving your answer in the form $ax+by+cz=d$. [4]

(c) Find the acute angle between Π_1 and Π_2. [5]

(a) $l_1 \quad r = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

Shortest distance $= \left| \dfrac{(a_2-a_1)\cdot(b_1 \times b_2)}{|b_1 \times b_2|} \right|$

$l_2 \quad r = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

Length of $\overline{PQ}$ = Shortest distance between l_1 and l_2

$$a_2 - a_1 = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 5 \end{pmatrix}$$

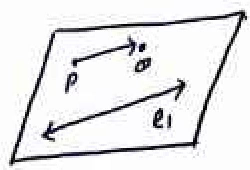
$$b_1 \times b_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & -2 \\ 1 & -1 & 2 \end{vmatrix} = 2i - 4j - 3k = \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix}$$

$$|b_1 \times b_2| = \sqrt{29}$$

$$\text{Length of } \overline{PQ} = \left| \frac{\begin{pmatrix} -2 \\ 2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix}}{\sqrt{29}} \right| = \frac{27}{\sqrt{29}} = 5.01$$

(b)

i) Equation of the plane Π_1



Π_1

$$r = (2i + k) + s(i - j + 2k) + t(2i - 4j - 3k)$$

(ii) $$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & 2 & -2 \\ 2 & -4 & -3 \end{vmatrix} = -14i - j - 8k$$

Π_2

→

Equation of the plane Π_2

$$-14x - y - 8z = -14(0) - (2) - 8(6) = -50$$

$$14x + y + 8z = 50$$

(c)

Let θ be the angle between Π_1 and Π_2

$$\theta = \cos^{-1}\left(\frac{\begin{pmatrix}11\\7\\-2\end{pmatrix}\cdot\begin{pmatrix}14\\1\\8\end{pmatrix}}{\sqrt{174}\sqrt{261}}\right)$$

$$\theta = \cos^{-1}\left(\frac{145}{\sqrt{174}\sqrt{261}}\right)$$

$$\theta = 47.1^\circ$$

Q.3 May/June/P11+P12/2022

2 The points A, B, C have position vectors

$$4\mathbf{i}-4\mathbf{j}+\mathbf{k}, \quad -4\mathbf{i}+3\mathbf{j}-4\mathbf{k}, \quad 4\mathbf{i}-\mathbf{j}-2\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

(b) Find the perpendicular distance from O to the plane ABC. [2]

(c) The point D has position vector $2\mathbf{i}+3\mathbf{j}-3\mathbf{k}$.

Find the coordinates of the point of intersection of the line OD with the plane ABC. [3]

(a) Given $\vec{OA} = \begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix}$ $\quad \vec{OB} = \begin{pmatrix} -4 \\ 3 \\ -4 \end{pmatrix}$ $\quad \vec{OC} = \begin{pmatrix} 4 \\ -1 \\ -2 \end{pmatrix}$

$$\vec{AB} = \begin{pmatrix} -4 \\ 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -8 \\ 7 \\ -5 \end{pmatrix}$$

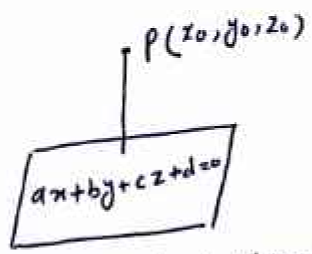
$$\vec{AC} = \begin{pmatrix} 4 \\ -1 \\ -2 \end{pmatrix} - \begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix}$$

$$\vec{n} = \begin{vmatrix} i & j & k \\ -8 & 7 & -5 \\ 0 & 3 & -3 \end{vmatrix} = -6i - 24j - 24k = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$$

Equation of the plane ABC

$$x + 4y + 4z = 1(4) + 4(-4) + 4(1) = -8$$

$$x + 4y + 4z = -8$$



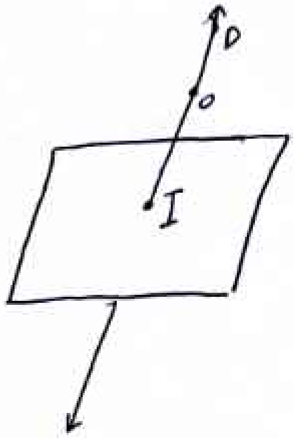
$$\text{Distance} = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

(b)

Distance from O to the plane ABC

$$= \frac{0+0+0+8}{\sqrt{1^2+4^2+4^2}} = 1.39$$

(c) equation of the line through O and D



$$r = \begin{pmatrix}0\\0\\0\end{pmatrix} + t\begin{pmatrix}2\\3\\-3\end{pmatrix} = t\begin{pmatrix}2\\3\\-3\end{pmatrix}$$

Let $I(a, b, c)$ be the point of intersection

$$a + 4b + 4c = -8$$

For some value of t

$$\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}2t\\3t\\-3t\end{pmatrix}$$

$$a = 2t, \quad b = 3t, \quad c = -3t$$

$$2t + 12t - 12t = -8$$

$$t = -4$$

$$a = -8, \quad b = -12, \quad c = 12$$

$$I(-8, -12, 12)$$

Q.4 May/June/P11+P12/2022

6 Let t be a positive constant.

The line l_1 passes through the point with position vector $t\mathbf{i}+\mathbf{j}$ and is parallel to the vector $-2\mathbf{i}-\mathbf{j}$. The line l_2 passes through the point with position vector $\mathbf{j}+t\mathbf{k}$ and is parallel to the vector $-2\mathbf{j}+\mathbf{k}$.

It is given that the shortest distance between the lines l_1 and l_2 is $\sqrt{21}$.

(a) Find the value of t. [5]

The plane Π_1 contains l_1 and is parallel to l_2.

(b) Write down an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. [1]

The plane Π_2 has Cartesian equation $5x-6y+7z=0$.

(c) Find the acute angle between l_2 and Π_2. [3]

(d) Find the acute angle between Π_1 and Π_2. [3]

(a)

$$l_1 \quad r = \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

$$l_2 \quad r = \begin{pmatrix} 0 \\ 1 \\ t \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

$$a_2 - a_1 = \begin{pmatrix} 0 \\ 1 \\ t \end{pmatrix} - \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix}$$

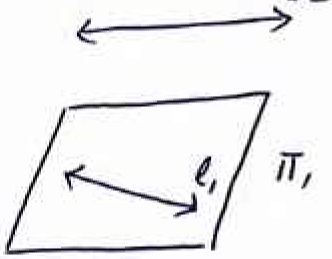
$$b_1 \times b_2 = \begin{vmatrix} i & j & k \\ -2 & -1 & 0 \\ 0 & -2 & 1 \end{vmatrix} = -i + 2j + 4k = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$$

$$\sqrt{21} = \frac{\begin{pmatrix} -t \\ 0 \\ t \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}}{\sqrt{21}} = \frac{5t}{\sqrt{21}}$$

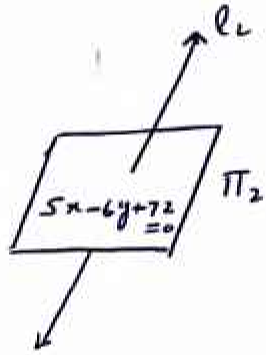
$$5t = 21 \Rightarrow t = 4.2$$

(b) Equation of the plane Π_1

$$r = \left(\tfrac{21}{5} i + j\right) + \lambda(2i + j) + \mu(-2j + k)$$



(c) Let θ be the angle between l_2 and normal vectors



$$\theta = \cos^{-1}\left(\frac{\begin{pmatrix}0\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}5\\-6\\7\end{pmatrix}}{\sqrt{5}\sqrt{110}}\right)$$

$$\theta = \cos^{-1}\left(\frac{19}{\sqrt{5}\sqrt{110}}\right) = 35.8^\circ$$

Let α be the angle between l_2 and plane Π_2

$$\alpha = 90 - 35.8^\circ$$

$$= 54.1^\circ$$

Q.5 May/June/P13/2022

7 The position vectors of the points A, B, C, D are

$$7\mathbf{i}+4\mathbf{j}-\mathbf{k}, \quad 11\mathbf{i}+3\mathbf{j}, \quad 2\mathbf{i}+6\mathbf{j}+3\mathbf{k}, \quad 2\mathbf{i}+7\mathbf{j}+\lambda\mathbf{k}$$

respectively.

(a) Given that the shortest distance between the line AB and the line CD is 3, show that $\lambda^2-5\lambda+4=0$. [7]

Let Π_1 be the plane ABD when $\lambda=1$.

Let Π_2 be the plane ABD when $\lambda=4$.

(b) (i) Write down an equation of Π_1, giving your answer in the form $\mathbf{r}=\mathbf{a}+s\mathbf{b}+t\mathbf{c}$. [2]

(ii) Find an equation of Π_2, giving your answer in the form $ax+by+cz=d$. [4]

Given $\vec{OA}=\begin{pmatrix}7\\4\\-1\end{pmatrix}$, $\vec{OB}=\begin{pmatrix}11\\3\\0\end{pmatrix}$, $\vec{OC}=\begin{pmatrix}2\\6\\3\end{pmatrix}$, $\vec{OD}=\begin{pmatrix}2\\7\\\lambda\end{pmatrix}$

$$\vec{AB}=\begin{pmatrix}11\\3\\0\end{pmatrix}-\begin{pmatrix}7\\4\\-1\end{pmatrix}=\begin{pmatrix}4\\-1\\1\end{pmatrix}$$

$$\vec{CD}=\begin{pmatrix}2\\7\\\lambda\end{pmatrix}-\begin{pmatrix}2\\6\\3\end{pmatrix}=\begin{pmatrix}0\\1\\\lambda-3\end{pmatrix}$$

$$a_2-a_1=\begin{pmatrix}2\\6\\3\end{pmatrix}-\begin{pmatrix}7\\4\\-1\end{pmatrix}=\begin{pmatrix}-5\\2\\4\end{pmatrix}$$

$$b_1\times b_2=\begin{vmatrix}i & j & k\\ 4 & -1 & 1\\ 0 & 1 & \lambda-3\end{vmatrix}=(2-\lambda)i+(12-4\lambda)j+4\hat{k}$$

$$=\begin{pmatrix}2-\lambda\\12-4\lambda\\4\end{pmatrix}$$

$$|b_1\times b_2|=\sqrt{(2-\lambda)^2+(12-4\lambda)^2+(4)^2}$$

$$=\sqrt{17\lambda^2-100\lambda+164}$$

$$3=\left|\frac{\begin{pmatrix}-5\\2\\4\end{pmatrix}\cdot\begin{pmatrix}2-\lambda\\12-4\lambda\\4\end{pmatrix}}{\sqrt{17\lambda^2-100\lambda+164}}\right|=\left|\frac{30-3\lambda}{\sqrt{17\lambda^2-100\lambda+164}}\right|$$

$$9(17\lambda^2-100\lambda+164) = 9(\lambda-10)^2$$

$$17\lambda^2-100\lambda+164 = \lambda^2-20\lambda+100$$

$$16\lambda^2-80\lambda+64=0$$

$$\lambda^2-5\lambda+4=0 \quad \text{proved}$$

For plane ABD when $\lambda=1$

$$OA=\begin{pmatrix}7\\4\\-1\end{pmatrix}, \quad OB=\begin{pmatrix}11\\3\\0\end{pmatrix}, \quad \vec{OD}=\begin{pmatrix}2\\7\\1\end{pmatrix}$$

$$\vec{AB}=\begin{pmatrix}4\\-1\\1\end{pmatrix}$$

$$\vec{AD}=\begin{pmatrix}2\\7\\1\end{pmatrix}-\begin{pmatrix}7\\4\\-1\end{pmatrix}=\begin{pmatrix}-5\\3\\2\end{pmatrix}$$

Equation of the plane Π_1 is

$$r=(7i+4j-k)+s(4i-j+k)+t(-5i+3j+2k)$$

For plane ABD when $\lambda=4$

$$AB=\begin{pmatrix}4\\-1\\1\end{pmatrix}$$

$$\vec{AD}=\begin{pmatrix}2\\7\\4\end{pmatrix}-\begin{pmatrix}7\\4\\-1\end{pmatrix}=\begin{pmatrix}-5\\3\\5\end{pmatrix}$$

$$\vec{n}=\begin{vmatrix}i & j & k\\ 4 & -1 & 1\\ -5 & 3 & 5\end{vmatrix}=-8i-25j+7k$$

Equation of the plane Π_2

$$-8x-25y+7z=-8(7)-25(4)+7(-1)=-163$$

$$8x+25y-7z=163$$

Q.6 May/June/P13/2021

6 The lines l_1 and l_2 have equations $\mathbf{r} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k} + s(2\mathbf{i} - 3\mathbf{j})$ and $\mathbf{r} = 3\mathbf{i} - 2\mathbf{k} + t(3\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ respectively.

The plane Π_1 contains l_1 and the point P with position vector $-2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$.

(a) Find an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. [2]

The plane Π_2 contains l_2 and is parallel to l_1.

(b) Find an equation of Π_2, giving your answer in the form $ax + by + cz = d$. [4]

(c) Find the acute angle between Π_1 and Π_2. [5]

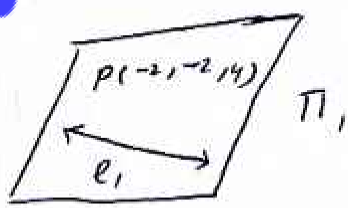
(d) The point Q is such that $\overrightarrow{OQ} = -5\overrightarrow{OP}$.

Find the position vector of the foot of the perpendicular from the point Q to Π_2. [4]

(a) $l_1 \quad r = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} + s\begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$

$l_2 \quad r = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$

$a = \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$

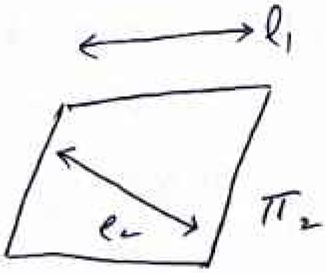


$c = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$

Equation of the plane Π_1

$r = (-2i - 2j + 4k) + \lambda(2i - 3j) + \mu(i - 3k)$

(b) $\vec{n} = \begin{vmatrix} i & j & k \\ 2 & -3 & 0 \\ 3 & -1 & 3 \end{vmatrix} = -9i + 6j + 7k = \begin{pmatrix} -9 \\ -6 \\ 7 \end{pmatrix}$



Equation of the plane Π_2

$-9x + 6y + 7z = -9(3) - 6(0) + 7(-2) = -41$

$9x + 6y - 7z = 41$

(c) Let θ be the angle between Π_1 and Π_2

$$n_1 = \begin{vmatrix} i & j & k \\ 2 & -3 & 0 \\ 1 & 0 & -3 \end{vmatrix} = 9i + 6j + 3k = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$n_2 = \begin{pmatrix} 9 \\ 6 \\ -7 \end{pmatrix}$$

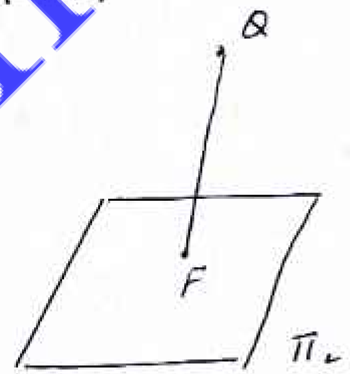
$$\theta = \cos^{-1}\left(\frac{\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 6 \\ -7 \end{pmatrix}}{\sqrt{14}\,\sqrt{166}} \right) = \cos^{-1} \frac{32}{\sqrt{14}\sqrt{166}}$$

$$\theta = 48.4^\circ$$

(d) Given $OQ = -5\vec{OP} = -5\begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \\ -20 \end{pmatrix}$

Let $F(a,b,c)$

Equation of line through Q and F



$9x + 6y - 7z = 41$

$$r = \begin{pmatrix} 10 \\ 10 \\ -20 \end{pmatrix} + t\begin{pmatrix} 9 \\ 6 \\ -7 \end{pmatrix} = \begin{pmatrix} 10+9t \\ 10+6t \\ -20-7t \end{pmatrix}$$

For some value of t

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 10+9t \\ 10+6t \\ -20-7t \end{pmatrix}$$

$a = 10+9t$, $b = 10+6t$, $c = -20-7t$

$F(a,b,c)$ also lies on Π_2

$$9a + 6b - 7c = 41$$

$$9(10+9t) + 6(10+6t) - 7(-20-7t) = 41$$

$$290 + 166t = 41$$

$$t = -\frac{3}{2}$$

$a = 10 - \frac{27}{2} = -\frac{7}{2}$, $b = 10 - \frac{18}{2} = 1$, $c = -20 + \frac{21}{2} = -\frac{19}{2}$

$$\vec{OF} = \begin{pmatrix} -\frac{7}{2} \\ 1 \\ -\frac{19}{2} \end{pmatrix}$$

Q.7 Oct/Nov/P11+P13/2021

5 The plane Π has equation $\mathbf{r} = -2\mathbf{i}+3\mathbf{j}+3\mathbf{k}+\lambda(\mathbf{i}+\mathbf{k})+\mu(2\mathbf{i}+3\mathbf{j})$.

(a) Find a Cartesian equation of Π, giving your answer in the form $ax+by+cz=d$. [4]

The line l passes through the point P with position vector $2\mathbf{i}-3\mathbf{j}+5\mathbf{k}$ and is parallel to the vector $\mathbf{k}$.

(b) Find the position vector of the point where l meets Π. [3]

(c) Find the acute angle between l and Π. [3]

(d) Find the perpendicular distance from P to Π. [3]

(a) $$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 2 & 3 & 0 \end{vmatrix} = -3i+2j+3k = \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix}$$

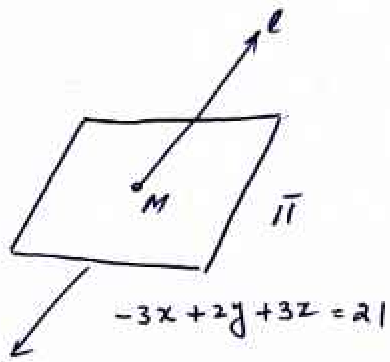
Equation of the plane Π

$$-3x+2y+3z = -3(-2)+2(3)+3(3)$$

$$-3x+2y+3z = 21$$

(b) Equation of the line l

$$r = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 5+t \end{pmatrix}$$



Let $M(a,b,c)$

M lies on l and the plane Π

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 5+t \end{pmatrix} \quad \text{For some value of } t$$

$$a=2, \quad b=-3, \quad c=5+t$$

$$-3a+2b+3c=21$$

$$-3(2)+2(-3)+3(5+t)=21 \Rightarrow t=6$$

$$a=2, \quad b=-3, \quad c=11$$

$$\vec{OM} = \begin{pmatrix} 2 \\ -3 \\ 11 \end{pmatrix}$$

$\rightarrow$

(c) Let θ be the angle between l and normal vector of plane

$$\theta = \cos^{-1}\left(\frac{\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}-3\\2\\3\end{pmatrix}}{\sqrt{1}\sqrt{22}}\right) = \cos^{-1}\left(\frac{3}{\sqrt{22}}\right) = 50.2^\circ$$

α be the angle b/w l and the plane

$$\alpha = 90 - 50.2$$
$$= 39.8^\circ$$

(d) $P(2, -3, 5)$

plane π $\quad -3x + 2y + 3z = 21$

Distance from P to plane π $= \dfrac{|-3(2) + 2(-3) + 3(5) - 21|}{\sqrt{9+4+9}}$

$$= \frac{18}{\sqrt{22}} = 3.84$$

7 The points A, B, C have position vectors

$$2\mathbf{i}+2\mathbf{j}, \quad -\mathbf{j}+\mathbf{k} \quad \text{and} \quad 2\mathbf{i}+\mathbf{j}-7\mathbf{k}$$

respectively, relative to the origin O.

(a) Find an equation of the plane OAB, giving your answer in the form $\mathbf{r.n} = p$. [3]

The plane Π has equation $x-3y-2z=1$.

(b) Find the perpendicular distance of Π from the origin. [1]

(c) Find the acute angle between the planes OAB and Π. [3]

(d) Find an equation for the common perpendicular to the lines OC and AB. [10]

(a) Given $\vec{OA} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$, $\vec{OB} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, $\vec{OC} = \begin{pmatrix} 2 \\ 1 \\ -7 \end{pmatrix}$ $\quad r \cdot n = p$

p: distance from origin to the plane

$$\vec{n} = \begin{vmatrix} i & j & k \\ 2 & 2 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 2i - 2j - 2k = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

Equation of the plane OAB:

$$r \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 0$$

(b)

$$\text{Distance of } \Pi \text{ from the origin} = \frac{|(0)-3(0)-2(0)-1|}{\sqrt{(1)^2+(-3)^2+(-2)^2}} = \frac{1}{\sqrt{14}}$$

(c) Let θ be the angle b/w the planes OAB and Π

$$\theta = \cos^{-1}\left[\frac{\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}}{\sqrt{3}\sqrt{14}}\right] = \cos^{-1}\left(\frac{6}{\sqrt{3}\sqrt{14}}\right) = 22.2^\circ$$

(d)

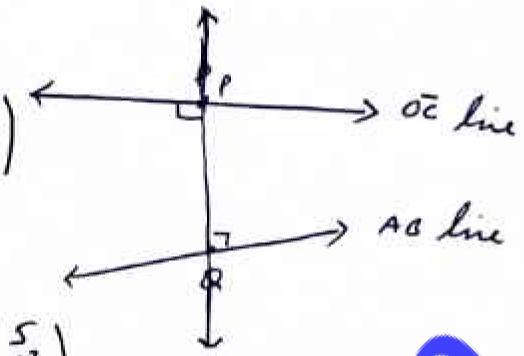
$$\vec{OC} = \begin{pmatrix} 2 \\ 1 \\ -7 \end{pmatrix}$$

$$\vec{AB} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$$

→

Equation of the line OC : $\gamma = \lambda\begin{pmatrix}2\\1\\-7\end{pmatrix}$

Equation of the line AB $\gamma = \begin{pmatrix}2\\2\\0\end{pmatrix} + \mu\begin{pmatrix}-2\\-3\\1\end{pmatrix}$



Normal vector on the both lines OC and AB

$$= \begin{vmatrix} i & j & k \\ 2 & 1 & -7 \\ -2 & -3 & 1 \end{vmatrix} = -20i + 12j - 4k = \begin{pmatrix}5\\-3\\1\end{pmatrix}$$

Let $\vec{OP} = \lambda\begin{pmatrix}2\\1\\-7\end{pmatrix}$, $\vec{OQ} = \begin{pmatrix}2-2\mu\\2-3\mu\\\mu\end{pmatrix}$ For some value of λ and μ

$$\vec{PQ} = \begin{pmatrix}2-2\mu\\2-3\mu\\\mu\end{pmatrix} - \lambda\begin{pmatrix}2\\1\\-7\end{pmatrix}$$

$$= \begin{pmatrix}2-2\mu-2\lambda\\2-3\mu-\lambda\\\mu+7\lambda\end{pmatrix}$$

Since $\vec{PQ}$ is perpendicular to OC and AB, so

$\vec{PQ}\cdot\begin{pmatrix}2\\1\\-7\end{pmatrix} = 0$ and $\vec{PQ}\cdot\begin{pmatrix}-2\\-3\\1\end{pmatrix} = 0$

$$\begin{pmatrix}2-2\mu-2\lambda\\2-3\mu-\lambda\\\mu+7\lambda\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-7\end{pmatrix} = 0$$

$$14\mu + 54\lambda = 6 \quad \text{—} ①$$

$$\begin{pmatrix}2-2\mu-2\lambda\\2-3\mu-\lambda\\\mu+7\lambda\end{pmatrix}\cdot\begin{pmatrix}-2\\-3\\1\end{pmatrix} = 0$$

$$14\mu + 14\lambda = 10 \quad \text{—} ②$$

By solving ① and ② $\mu = \frac{57}{70}$ $\lambda = -\frac{1}{10}$

$$\vec{OP} = -\frac{1}{10}\begin{pmatrix}2\\1\\-7\end{pmatrix} = \begin{pmatrix}-0.2\\-0.1\\0.7\end{pmatrix}$$

Equation of common perpendicular to the lines OC and AB is

$$\gamma = \begin{pmatrix}-0.2\\-0.1\\0.7\end{pmatrix} + t\begin{pmatrix}5\\-3\\1\end{pmatrix}$$

5 The lines l_1 and l_2 have equations $\mathbf{r} = 3\mathbf{i} + 3\mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} + 4\mathbf{k})$ and $\mathbf{r} = 3\mathbf{i} - 5\mathbf{j} - 6\mathbf{k} + \mu(5\mathbf{j} + 6\mathbf{k})$ respectively.

(a) Find the shortest distance between l_1 and l_2. [5]

The plane Π contains l_1 and is parallel to the vector $\mathbf{i} + \mathbf{k}$.

(b) Find the equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

(c) Find the acute angle between l_2 and Π. [3]

(a)
$$l_1 = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$$

$$l_2 = \begin{pmatrix} 3 \\ -5 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$$

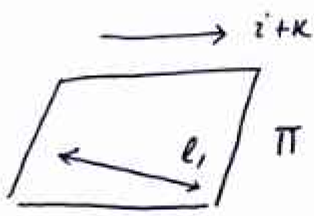
$$a_2 - a_1 = \begin{pmatrix} 3 \\ -5 \\ -6 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ -9 \end{pmatrix}$$

$$b_1 \times b_2 = \begin{vmatrix} i & j & k \\ 1 & 4 & 4 \\ 0 & 5 & 6 \end{vmatrix} = 4i - 6j + 5k = \begin{pmatrix} 4 \\ -6 \\ 5 \end{pmatrix}$$

Shortest distance b/w l_1 and l_2 $= \dfrac{|(a_2 - a_1) \cdot (b_1 \times b_2)|}{|b_1 \times b_2|}$

$$= \frac{\left| \begin{pmatrix} 0 \\ -5 \\ -9 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -6 \\ 5 \end{pmatrix} \right|}{\sqrt{77}} = \frac{15}{\sqrt{77}} = 1.71$$

(b)
$$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & 4 & 4 \\ 1 & 0 & 1 \end{vmatrix} = 4i + 3j - 4k = \begin{pmatrix} 4 \\ 3 \\ -4 \end{pmatrix}$$



Equation of the plane Π

$$4x + 3y - 4z = 4(3) + 3(0) - 4(3)$$

$$4x + 3y - 4z = 0$$

$\rightarrow$

(c) Let θ be the angle b/w $\overset{dv}{}$ of l_2 and normal vector of plane π

$$\theta = \cos^{-1}\left[\frac{\left|\begin{pmatrix}0\\5\\6\end{pmatrix}\cdot\begin{pmatrix}4\\3\\-4\end{pmatrix}\right|}{\sqrt{61}\,\sqrt{41}}\right] = \cos^{-1}\frac{+9}{\sqrt{61}\,\sqrt{41}} = 79.6^\circ$$

α = be the angle b/w l_2 and plane

$\alpha = 90 - 79.6 = 10.4^\circ$

7 The lines l_1 and l_2 have equations $\mathbf{r} = -5\mathbf{j} + \lambda(5\mathbf{i} + 2\mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} + \mu(\mathbf{j} + \mathbf{k})$ respectively. The plane Π contains l_1 and is parallel to l_2.

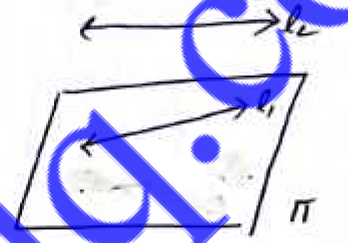
(a) Find the equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

(b) Find the distance between l_2 and Π. [3]

The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2.

(c) Show that P has position vector $\frac{55}{27}\mathbf{i} - 5\mathbf{j} + \frac{22}{27}\mathbf{k}$ and state a vector equation for PQ. [8]

(a) $l_1 \quad r = \begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$



$l_2 \quad r = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$$\vec{n} = \begin{vmatrix} i & j & k \\ 5 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = -2i - 5j + 5k = \begin{pmatrix} -2 \\ -5 \\ 5 \end{pmatrix}$$

Equation of the plane Π is

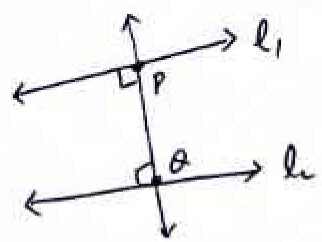
$-2x - 5y + 5z = -2(0) - 5(-5) + 5(0)$

$-2x - 5y + 5z = 25$

(b) Distance b/w l_2 and Π $= \dfrac{|-2(4) - 5(2) + 5(-2) - 25|}{\sqrt{(-2)^2 + (-5)^2 + (5)^2}}$

$= \dfrac{53}{\sqrt{54}} = 7 \cdot 21$

(c) $\vec{OP} = \begin{pmatrix} 5\lambda \\ -5 \\ 2\lambda \end{pmatrix}$ For some value of λ



$\vec{OQ} = \begin{pmatrix} 4 \\ 2+\mu \\ -2+\mu \end{pmatrix}$ For some value of μ

$\vec{PQ} = \begin{pmatrix} 4 - 5\lambda \\ 7 + \mu \\ -2 + \mu - 2\lambda \end{pmatrix}$

$\rightarrow$

$$\vec{PQ}\cdot\begin{pmatrix}5\\0\\2\end{pmatrix}=0 \qquad \text{and} \qquad \vec{PQ}\cdot\begin{pmatrix}0\\1\\1\end{pmatrix}=0$$

$$\begin{pmatrix}4-5\lambda\\7+\mu\\-2+\mu-2\lambda\end{pmatrix}\cdot\begin{pmatrix}5\\0\\2\end{pmatrix}=0 \qquad \begin{pmatrix}4-5\lambda\\7+\mu\\-2+\mu-2\lambda\end{pmatrix}\cdot\begin{pmatrix}0\\1\\1\end{pmatrix}=0$$

$$-29\lambda+2\mu=-16 \quad - (1) \qquad\qquad -2\lambda+2\mu=-5 \quad - (2)$$

By solving (1) and (2)

$$\lambda=\frac{11}{27}, \quad \mu=-\frac{113}{54}$$

$$\vec{OP}=\begin{pmatrix}5\lambda\\-5\\2\lambda\end{pmatrix}=\begin{pmatrix}\frac{55}{27}\\-5\\\frac{22}{27}\end{pmatrix}=\frac{55}{27}i-5j+\frac{22}{27}k$$

Equation of PQ is

$$r=\begin{pmatrix}\frac{55}{27}\\-5\\\frac{22}{27}\end{pmatrix}+t\begin{pmatrix}-2\\-5\\5\end{pmatrix}$$

4 The points A, B, C have position vectors

$$-\mathbf{i}+\mathbf{j}+2\mathbf{k}, \quad -2\mathbf{i}-\mathbf{j}, \quad 2\mathbf{i}+2\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

(b) Find the perpendicular distance from O to the plane ABC. [2]

(c) Find the acute angle between the planes OAB and ABC. [4]

(a) Given $\vec{OA} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, $\vec{OB} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$, $\vec{OC} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$

$$\vec{AB} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\vec{AC} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$$

$$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & 2 & 2 \\ 3 & -1 & 0 \end{vmatrix} = -2i - 6j + 7k = \begin{pmatrix} -2 \\ -6 \\ 7 \end{pmatrix}$$

Equation of the plane ABC

$$-2x - 6y + 7z = -2(-1) - 6(1) + 7(2) = 10$$

$$-2x - 6y + 7z = 10$$

(b)

perpendicular distance from O to the plane ABC $= \dfrac{|0 - 0 + 0 - 10|}{\sqrt{4+36+49}} = \dfrac{10}{\sqrt{89}}$

(c) normal vector of the plane OAB

$$\vec{n} = \begin{vmatrix} i & j & k \\ -1 & 1 & 2 \\ -2 & -1 & 0 \end{vmatrix} = 2\hat{i} - 4\hat{j} + 3\hat{k} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$$

let θ = Angle b/w planes OAB and ABC $= \cos^{-1} \dfrac{\begin{pmatrix} -2 \\ -6 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}}{\sqrt{89}\,\sqrt{29}} = 36.2^\circ$

7 The line l_1 passes through the points $A(-3, 1, 4)$ and $B(-1, 5, 9)$. The line l_2 passes through the points $C(-2, 6, 5)$ and $D(-1, 7, 5)$.

(i) Find the shortest distance between the lines l_1 and l_2. [5]

(ii) Find the acute angle between the line l_2 and the plane containing A, B and D. [5]

(a) $\vec{AB} = \begin{pmatrix} -1+3 \\ 5-1 \\ 9-4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$, $\vec{CD} = \begin{pmatrix} -1+2 \\ 7-6 \\ 5-5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

l_1 $r = \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$

l_2 $r = \begin{pmatrix} -2 \\ 6 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$a_2 - a_1 = \begin{pmatrix} -2 \\ 6 \\ 5 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$

$b_1 \times b_2 = \begin{vmatrix} i & j & k \\ 2 & 4 & 5 \\ 1 & 1 & 0 \end{vmatrix} = -5i + 5j - 2k = \begin{pmatrix} -5 \\ 5 \\ -2 \end{pmatrix}$

Shortest distance b/w l_1 and l_2 $= \dfrac{\begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ 5 \\ -2 \end{pmatrix}}{\sqrt{25+25+4}} = \dfrac{18}{\sqrt{54}} = 2.45$

(b) $\vec{AD} = \begin{pmatrix} -1+3 \\ 7-1 \\ 5-4 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix}$

$\vec{n} = \vec{AB} \times \vec{AD} = \begin{vmatrix} i & j & k \\ 2 & 4 & 5 \\ 2 & 6 & 1 \end{vmatrix} = -26i + 8j + 4k = \begin{pmatrix} -13 \\ 4 \\ 2 \end{pmatrix}$

θ = be the angle b/w direction vector of l_2 and $\vec{n}$

$\theta = \cos^{-1} \left| \dfrac{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -13 \\ 4 \\ 2 \end{pmatrix}}{\sqrt{2}\sqrt{189}} \right| = \cos^{-1} \dfrac{9}{\sqrt{378}} = 62.4$

α = be the angle b/w l_2 and plane $= 90 - 62.4 = 27.6^\circ$

Q.12 May/June/P11+P12/2019

3 The lines l_1 and l_2 have equations $\mathbf{r} = 6\mathbf{i} + 2\mathbf{j} + 7\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j})$ and $\mathbf{r} = 4\mathbf{i} + 4\mathbf{j} + \mu(-6\mathbf{j} + \mathbf{k})$ respectively. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Find the position vectors of P and Q. [8]

$l_1: \quad r = \begin{pmatrix} 6 \\ 2 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6+\lambda \\ 2+\lambda \\ 7 \end{pmatrix}$

$l_2 \quad r = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -6 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4-6\mu \\ \mu \end{pmatrix}$

$\vec{OP} = \begin{pmatrix} 6+\lambda \\ 2+\lambda \\ 7 \end{pmatrix}, \quad \vec{OQ} = \begin{pmatrix} 4 \\ 4-6\mu \\ \mu \end{pmatrix}$

$\Rightarrow \vec{PQ} = \begin{pmatrix} -2-\lambda \\ 2-\lambda-6\mu \\ -7+\mu \end{pmatrix}$

$\vec{PQ} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$ and $\vec{PQ} \cdot \begin{pmatrix} 0 \\ -6 \\ 1 \end{pmatrix} = 0$

$-2-\lambda+2-\lambda-6\mu=0$, $-12+6\lambda+36\mu-7+\mu=0$

$-2\lambda-6\mu=0$ — (1) , $6\lambda+37\mu=19$ — (2)

By solving (1) and (2)

$\lambda=-3, \mu=1$

$\vec{OP} = \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix}$

$\vec{OQ} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$

Q.13 Oct/Nov/P11+P12+P13/2019

6 With O as the origin, the points A, B, C have position vectors

$$\mathbf{i}-\mathbf{j}, \quad 2\mathbf{i}+\mathbf{j}+7\mathbf{k}, \quad \mathbf{i}-\mathbf{j}+\mathbf{k}$$

respectively.

(i) Find the shortest distance between the lines OC and AB. [5]

(ii) Find the cartesian equation of the plane containing the line OC and the common perpendicular of the lines OC and AB. [4]

(a) Given $\vec{OA} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$, $\vec{OB} = \begin{pmatrix}2\\1\\7\end{pmatrix}$, $\vec{OC} = \begin{pmatrix}1\\-1\\1\end{pmatrix}$

$$\vec{AB} = \begin{pmatrix}2\\1\\7\end{pmatrix} - \begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}1\\2\\7\end{pmatrix}$$

l_{OC} $\quad r = \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}$

l_{AB} $\quad r = \begin{pmatrix}1\\-1\\0\end{pmatrix} + \mu\begin{pmatrix}1\\2\\7\end{pmatrix}$

$$a_2 - a_1 = \begin{pmatrix}1\\-1\\0\end{pmatrix} - \begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$b_1 \times b_2 = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 1 & 2 & 7 \end{vmatrix} = -9i - 6j + 3k = \begin{pmatrix}3\\2\\-1\end{pmatrix}$$

$$|b_1 \times b_2| = \sqrt{9+4+1} = \sqrt{14}$$

Shortest distance b/w lines OC and AB $= \dfrac{\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}3\\2\\-1\end{pmatrix}}{\sqrt{14}} = \dfrac{1}{\sqrt{14}} = 0.267$

(b) $\vec{n} = \vec{OC} \times (\vec{OC} \times \vec{AB}) = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 3 & 2 & -1 \end{vmatrix} = -i + 4j + 5k$

Equation of the plane

$$-x + 4y + 5z = -1(0) + 4(0) + 5(0)$$

$$-x + 4y + 5z = 0$$

Q.14 May/June/P11+P12/2018

10 The line l_1 is parallel to the vector $a\mathbf{i} - \mathbf{j} + \mathbf{k}$, where a is a constant, and passes through the point whose position vector is $9\mathbf{j} + 2\mathbf{k}$. The line l_2 is parallel to the vector $-a\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and passes through the point whose position vector is $-6\mathbf{i} - 5\mathbf{j} + 10\mathbf{k}$.

(i) It is given that l_1 and l_2 intersect.

(a) Show that $a = -\frac{6}{13}$. [3]

(b) Find a cartesian equation of the plane containing l_1 and l_2. [4]

(ii) Given instead that the perpendicular distance between l_1 and l_2 is $3\sqrt{(30)}$, find the value of a. [5]

Given l_1: $r = \begin{pmatrix} 0 \\ 9 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} a \\ -1 \\ 1 \end{pmatrix}$

l_2: $r = \begin{pmatrix} -6 \\ -5 \\ 10 \end{pmatrix} + \mu \begin{pmatrix} -a \\ 2 \\ 4 \end{pmatrix}$

given l_1 and l_2 intersect, so for some value of λ and μ

$$\begin{pmatrix} \lambda a \\ 9 - \lambda \\ 2 + \lambda \end{pmatrix} = \begin{pmatrix} -6 - a\mu \\ -5 + 2\mu \\ 10 + 4\mu \end{pmatrix}$$

$\lambda a = -6 - a\mu$ — (1)

$9 - \lambda = -5 + 2\mu$ — (2)

$2 + \lambda = 10 + 4\mu$ — (3)

From (2) $\lambda + 2\mu = 14$

From (3) $\lambda - 4\mu = 8$

$\lambda = 12$, $\mu = 1$

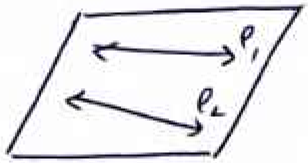
putting these values in (1)

$12a = -6 - a$

$13a = -6$

$a = \frac{-6}{13}$

(b) $\vec{n} = \begin{vmatrix} i & j & k \\ \frac{-6}{13} & -1 & 1 \\ \frac{6}{13} & 2 & 4 \end{vmatrix}$



$= -6i + \frac{30}{13}j - \frac{6}{13}k = \begin{pmatrix} -13 \\ 5 \\ -1 \end{pmatrix}$

Equation of the plane

$-13x + 5y - z = -13(0) + 5(9) - 1(2) = 43$

$-13x + 5y - z = 43$

(c) $a_2 - a_1 = \begin{pmatrix} -6 \\ -5 \\ 10 \end{pmatrix} - \begin{pmatrix} 0 \\ 9 \\ 2 \end{pmatrix} = \begin{pmatrix} -6 \\ -14 \\ 8 \end{pmatrix}$

$b_1 \times b_2 = \begin{vmatrix} i & j & k \\ a & -1 & 1 \\ -a & 2 & 4 \end{vmatrix} = -6i - 5aj + ak = \begin{pmatrix} -6 \\ -5a \\ a \end{pmatrix}$

$|b_1 \times b_2| = \sqrt{36 + 25a^2 + a^2} = \sqrt{36 + 26a^2}$

$(a_2 - a_1)\cdot(b_1 \times b_2) = \begin{pmatrix} -6 \\ -14 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} -6 \\ -5a \\ a \end{pmatrix} = 36 + 78a$

$3\sqrt{30} = \frac{|36 + 78a|}{\sqrt{36 + 26a^2}}$

$540(18 + 13a^2) = 36(6 + 13a)^2$

$15(18 + 13a^2) = (6 + 13a)^2$

$270 + 195a^2 = 36 + 156a + 169a^2$

$26a^2 - 156a + 234 = 0$

$a^2 - 6a + 9 = 0$

$(a-3)^2 = 0 \Rightarrow a = 3$

Q.15 May/June/P13/2018

7 The lines l_1 and l_2 have vector equations

$$\mathbf{r} = a\mathbf{i} + 9\mathbf{j} + 13\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \quad \text{and} \quad \mathbf{r} = -3\mathbf{i} + 7\mathbf{j} - 2\mathbf{k} + \mu(-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$$

respectively. It is given that l_1 and l_2 intersect.

(i) Find the value of the constant a. [3]

The point P has position vector $3\mathbf{i} + \mathbf{j} + 6\mathbf{k}$.

(ii) Find the perpendicular distance from P to the plane containing l_1 and l_2. [4]

(iii) Find the perpendicular distance from P to l_2. [4]

(a) (i) l_1: $r = \begin{pmatrix} a \\ 9 \\ 13 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

l_2: $r = \begin{pmatrix} -3 \\ 7 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$

It is given that l_1 and l_2 intersect, so

$$\begin{pmatrix} a+\lambda \\ 9+2\lambda \\ 13+3\lambda \end{pmatrix} = \begin{pmatrix} -3-\mu \\ 7+2\mu \\ -2-3\mu \end{pmatrix}$$

$a + \lambda = -3 - \mu$ — ①

$2\lambda - 2\mu = -2$ — ②

$3\lambda + 3\mu = -15$ — ③

From ② and ③

$\lambda = -3, \quad \mu = -2$

put these values in ①

$a - 3 = -3 + 2 \Rightarrow a = 2$

(b) (ii) $n = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ -1 & 2 & -3 \end{vmatrix} = -12i + 4k = \begin{pmatrix} -12 \\ 0 \\ 4 \end{pmatrix}$

perpendicular distance from $(3,1,6)$ = ?

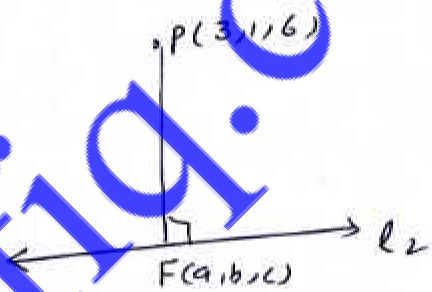
Equation of the plane containing l_1 and l_2

$$-12x + 4z = -12(2) + 4(13)$$

$$-12x + 4z - 28 = 0$$

$$\text{Distance} = \frac{|-12(3) + 4(6) - 28|}{\sqrt{144+16}} = \frac{40}{\sqrt{160}} = 3.16$$

(iii) Let $F(a,b,c)$ be the foot of perpendicular from P to l_2



$$\vec{PF} = \begin{pmatrix} a-3 \\ b-1 \\ c-6 \end{pmatrix}$$

$$\vec{PF} \cdot \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} = 0$$

$$-a+3+2b-2-3c+18=0$$

$$-a+2b-3c=-19 \quad \text{— (1)}$$

For some value of μ

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -3-\mu \\ 7+2\mu \\ -2-3\mu \end{pmatrix}$$

$$a = -3-\mu, \quad b = 7+2\mu, \quad c = -2-3\mu$$

put these in (1)

$$3+\mu+14+4\mu+6+9\mu=-19$$

$$14\mu+23=-19$$

$$\mu = -\frac{43}{14}$$

$$a = -3+\frac{23}{14} = \frac{-19}{7}, \quad b = 7-\frac{4}{7} = \frac{45}{7}, \quad c = -2+\frac{6}{7} = -\frac{8}{7}$$

$$\vec{PF} = \begin{pmatrix} -\frac{40}{7} \\ \frac{38}{7} \\ -\frac{50}{7} \end{pmatrix} \qquad |\vec{PF}|$$

Q.16 Oct/Nov/P11+P13/2018

8 The plane Π_1 has equation

$$\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

(i) Find a cartesian equation of Π_1. [3]

The plane Π_2 has equation $3x + y - z = 3$.

(ii) Find the acute angle between Π_1 and Π_2, giving your answer in degrees. [2]

(iii) Find an equation of the line of intersection of Π_1 and Π_2, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. [5]

(i) n_1 = Normal vector of the plane Π_1

$$= \begin{vmatrix} i & j & k \\ -4 & 1 & 3 \\ 0 & 1 & 2 \end{vmatrix} = -i + 8j - 4k = \begin{pmatrix} -1 \\ 8 \\ -4 \end{pmatrix}$$

Cartesian equation of Π_1

$$-x + 8y - 4z = -5 + 8 + 0$$

$$-x + 8y - 4z = 3$$

(ii) n_2 = Normal vector of Π_2

$$n_2 = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

Let θ be the angle b/w Π_1 and Π_2

$$\theta = \cos^{-1}\left[\frac{\begin{pmatrix} -1 \\ 8 \\ -4 \end{pmatrix}\cdot\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}}{\sqrt{81}\sqrt{11}}\right] = \cos^{-1}\left(\frac{9}{9\sqrt{11}}\right) = 72.5^\circ$$

(iii) Π_1 $\quad -x + 8y - 4z = 3$

Π_2 $\quad 3x + y - z = 3$

point $(1,1,1)$ lies on both planes

Direction vector of intersecting line of planes

$$= \begin{vmatrix} i & j & k \\ 3 & 1 & -1 \\ -1 & 8 & -4 \end{vmatrix} = 4i + 13j + 25k$$

so the required line is

$$r = (i + j + k) + \lambda(4i + 13j + 25k)$$

Q.17 Oct/Nov/P12/2018

10 The position vectors of the points A, B, C, D are

$$\mathbf{i}+\mathbf{j}+3\mathbf{k}, \quad 3\mathbf{i}+4\mathbf{j}+5\mathbf{k}, \quad -\mathbf{i}+3\mathbf{k}, \quad m\mathbf{j}+4\mathbf{k},$$

respectively, where m is a constant.

(i) Show that the lines AB and CD are parallel when $m = \frac{3}{2}$. [1]

(ii) Given that $m \neq \frac{3}{2}$, find the shortest distance between the lines AB and CD. [5]

(iii) When $m = 2$, find the acute angle between the planes ABC and ABD, giving your answer in degrees. [6]

(i) $\vec{OA} = \begin{pmatrix}1\\1\\3\end{pmatrix}$, $\vec{OB} = \begin{pmatrix}3\\4\\5\end{pmatrix}$, $\vec{OC} = \begin{pmatrix}-1\\0\\3\end{pmatrix}$, $\vec{OD} = \begin{pmatrix}0\\m\\4\end{pmatrix}$

$\vec{AB} = \begin{pmatrix}3\\4\\5\end{pmatrix} - \begin{pmatrix}1\\1\\3\end{pmatrix} = \begin{pmatrix}2\\3\\2\end{pmatrix}$

$\vec{CD} = \begin{pmatrix}0\\\frac{3}{2}\\4\end{pmatrix} - \begin{pmatrix}-1\\0\\3\end{pmatrix} = \begin{pmatrix}1\\1.5\\1\end{pmatrix}$

$\vec{CD} = \frac{1}{2}\vec{AB}$

This shows that $\vec{AB}$ and $\vec{CD}$ are parallel

(ii) $a_2 - a_1 = \begin{pmatrix}-1\\0\\3\end{pmatrix} - \begin{pmatrix}1\\1\\3\end{pmatrix} = \begin{pmatrix}-2\\-1\\0\end{pmatrix}$

$b_1 \times b_2 = \begin{vmatrix}i & j & k\\2 & 3 & 2\\1 & m & 1\end{vmatrix} = (3-2m)i + (2m-3)k$

$(a_2 - a_1)\cdot(b_1 \times b_2) = \begin{pmatrix}-2\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}3-2m\\0\\2m-3\end{pmatrix} = -6+4m$

$|b_1 \times b_2| = \sqrt{(3-2m)^2 + (2m-3)^2} = \sqrt{2}\,(2m-3)$

Shortest distance $= \dfrac{|2(2m-3)|}{\sqrt{2}\,(2m-3)} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$

$$\vec{AB} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \quad \vec{AC} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}, \quad \vec{AD} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$n_1 = \text{Normal vector of } ABC = \begin{vmatrix} i & j & k \\ 2 & 3 & 2 \\ -2 & -1 & 0 \end{vmatrix} = 2i - 4j + 4k = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

$$n_2 = \text{Normal vector of } ABD = \begin{vmatrix} i & j & k \\ 2 & 3 & 2 \\ -1 & 1 & 1 \end{vmatrix} = i - 4j + 5k = \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}$$

$$n_1 \cdot n_2 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix} = 1 + 8 + 10 = 19$$

Let θ be the angles b/w planes

$$\theta = \cos^{-1}\left(\frac{19}{\sqrt{9}\sqrt{42}}\right) = 12.2^\circ$$

Q.18 May/June/P11+P12/2017

The position vectors of the points A, B, C, D are

$$\mathbf{i}+\mathbf{j}+3\mathbf{k},\quad 3\mathbf{i}-\mathbf{j}+5\mathbf{k},\quad 3\mathbf{i}-\mathbf{j}+\mathbf{k},\quad 5\mathbf{i}-5\mathbf{j}+\alpha\mathbf{k},$$

respectively, where α is a positive integer. It is given that the shortest distance between the line AB and the line CD is equal to $2\sqrt{2}$.

(i) Show that the possible values of α are 3 and 5. [7]

(ii) Using $\alpha = 3$, find the shortest distance of the point D from the line AC, giving your answer correct to 3 significant figures. [3]

(iii) Using $\alpha = 3$, find the acute angle between the planes ABC and ABD, giving your answer in degrees. [4]

$$\vec{OA} = \begin{pmatrix}1\\1\\3\end{pmatrix},\quad \vec{OB} = \begin{pmatrix}3\\-1\\5\end{pmatrix},\quad \vec{OC} = \begin{pmatrix}3\\-1\\1\end{pmatrix},\quad \vec{OD} = \begin{pmatrix}5\\-5\\\alpha\end{pmatrix}$$

$$\vec{AB} = \begin{pmatrix}3\\-1\\5\end{pmatrix} - \begin{pmatrix}1\\1\\3\end{pmatrix} = \begin{pmatrix}2\\-2\\2\end{pmatrix} = \begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$\vec{CD} = \begin{pmatrix}5\\-5\\\alpha\end{pmatrix} - \begin{pmatrix}3\\-1\\1\end{pmatrix} = \begin{pmatrix}2\\-4\\\alpha-1\end{pmatrix}$$

$$l_{AB} \qquad r = \begin{pmatrix}1\\1\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$l_{CD} \qquad r = \begin{pmatrix}3\\-1\\1\end{pmatrix} + \mu\begin{pmatrix}2\\-4\\\alpha-1\end{pmatrix}$$

$$a_2 - a_1 = \begin{pmatrix}3\\-1\\1\end{pmatrix} - \begin{pmatrix}1\\1\\3\end{pmatrix} = \begin{pmatrix}2\\-2\\-2\end{pmatrix}$$

$$b_1 \times b_2 = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 2 & -4 & \alpha-1 \end{vmatrix} = \begin{pmatrix}\alpha-5\\\alpha-3\\2\end{pmatrix}$$

$$2\sqrt{2} = \frac{\left|\begin{pmatrix}2\\-2\\-2\end{pmatrix}\cdot\begin{pmatrix}\alpha-5\\\alpha-3\\2\end{pmatrix}\right|}{\sqrt{(\alpha-5)^2+(\alpha-3)^2+(2)^2}} = \frac{|2\alpha-10-2\alpha+6-4|}{\sqrt{2\alpha^2-16\alpha+38}}$$

$$2\sqrt{2}\sqrt{2\alpha^2-16\alpha+38} = 8$$

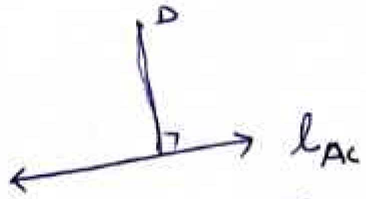
$$\alpha^2 - 8\alpha + 15 = 0$$

$$(\alpha-3)(\alpha+5) = 0$$

$$\alpha = 3 \quad \text{or} \quad \alpha = 5$$

(ii) Given $\alpha = 3$, so

$D(5, -5, 3)$



$$\text{Shortest distance} = \frac{|\vec{AD} \times \vec{AC}|}{|\vec{AC}|}$$

$$\vec{AD} = \begin{pmatrix} 5 \\ -5 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \\ 0 \end{pmatrix}$$

$$\vec{AC} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$\vec{AD} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 4 & -6 & 0 \\ 1 & -1 & 1 \end{vmatrix} = -6i - 4j + 2k$$

$$|\vec{AD} \times \vec{AC}| = \sqrt{36 + 16 + 4} = \sqrt{56}$$

$$\text{Shortest distance} = \frac{\sqrt{56}}{\sqrt{1+1+1}} = \frac{\sqrt{56}}{\sqrt{3}} = 4.32$$

(iii) $\vec{AB} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

$$\vec{AC} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$n_1 = \text{Normal vector of the plane } ABC = \begin{vmatrix} i & j & k \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$$

$$\vec{AD} = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$$

$$n_2 = \text{Normal vector of the plane } ABD = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 2 & -3 & 0 \end{vmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

θ = angle between planes ABC and ABD

$$= \cos^{-1}\left(\frac{\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}}{\sqrt{2}\,\sqrt{14}} \right) = 19.1^\circ$$

9 The plane Π_1 passes through the points (1, 2, 1) and (5, −2, 9) and is parallel to the vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

(i) Find the cartesian equation of Π_1. [4]

The plane Π_2 contains the lines

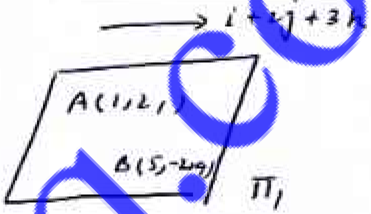
$$\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \quad \text{and} \quad \mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} + \mu(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}).$$

(ii) Find the cartesian equation of Π_2. [4]

(iii) Find the acute angle between Π_1 and Π_2. [3]

(i)

$$\vec{AB} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$



$$\vec{n}_1 = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -7i - j + 3k = \begin{pmatrix} -7 \\ -1 \\ 3 \end{pmatrix}$$

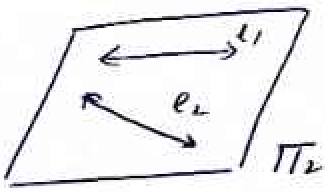
Equation of the plane Π_1

$$-7x - y + 3z = -7(1) - (2) + 3(1) = -6$$

$$7x + y - 3z = 6$$

(ii)

$$n_2 = \begin{vmatrix} i & j & k \\ 1 & -2 & -1 \\ 2 & 3 & -1 \end{vmatrix}$$



$$= 5i - j + 7k = \begin{pmatrix} 5 \\ -1 \\ 7 \end{pmatrix}$$

Equation of the plane Π_2

$$5x - y + 7z = 5(2) - (-3) + 7(1)$$

$$5x - y + 7z = 20$$

(iii) θ = be the angle b/w Π_1 and Π_2

$$\theta = \cos^{-1}\left|\frac{\begin{pmatrix} -7 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -1 \\ 7 \end{pmatrix}}{\sqrt{59}\sqrt{75}}\right| = \cos^{-1}\left(\frac{13}{\sqrt{59}\sqrt{75}}\right) = 78.7^\circ$$

6 The points A, B and C have position vectors $2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and $-\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ respectively.

(i) Find the area of the triangle ABC. [4]

(ii) Find the perpendicular distance of the point A from the line BC. [3]

(iii) Find the cartesian equation of the plane through A, B and C. [2]

(i) $\vec{OA} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, $\vec{OB} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$, $\vec{OC} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$

$$\vec{AB} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix}$$

$$\vec{BC} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ 5 \end{pmatrix}$$

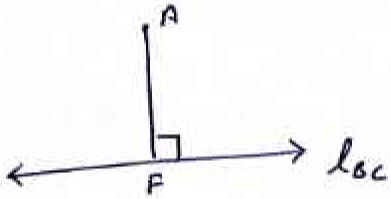
Area of the triangle $ABC = \frac{1}{2}\left|\vec{AB} \times \vec{BC}\right|$

$$\vec{AB} \times \vec{BC} = \begin{vmatrix} i & j & k \\ 1 & 5 & -2 \\ -4 & -2 & 5 \end{vmatrix} = 21i + 3j + 18k = \begin{pmatrix} 21 \\ 3 \\ 18 \end{pmatrix}$$

$$\left|\vec{AB} \times \vec{BC}\right| = \sqrt{(21)^2 + (3)^2 + (18)^2} = 3\sqrt{86}$$

$$\text{Area} = \frac{1}{2}(3\sqrt{86}) = 13.9 \text{ unit}^2$$

(ii) l_{BC} $\quad r = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} + t\begin{pmatrix} -4 \\ -2 \\ 5 \end{pmatrix}$



Let $F(a,b,c)$ be the foot of perpendicular from A to the line BC

$$\vec{AF} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} a-2 \\ b+1 \\ c-1 \end{pmatrix}$$

For some value of t

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3-4t \\ 4-2t \\ -1+5t \end{pmatrix}$$

$$a = 3-4t, \quad b = 4-2t, \quad c = -1+5t$$

Also $\vec{AF} \cdot \begin{pmatrix} -4 \\ -2 \\ 5 \end{pmatrix} = 0$

$$\begin{pmatrix} a-2 \\ b+1 \\ c-1 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ -2 \\ 5 \end{pmatrix} = 0$$

$$-4a + 8 - 2b - 2 + 5c - 5 = 0$$

$$-4a - 2b + 5c = -1$$

putting the values of a, b and c

$$-4(3-4t) - 2(4-2t) + 5(-1+5t) = -1$$

$$-12 + 16t - 8 + 4t - 5 + 25t = -1$$

$$45t = 24$$

$$t = \frac{8}{15}$$

$\vec{AF}$: $a = 3 - \frac{32}{15} = \frac{13}{15}$, $b = 4 - \frac{16}{15} = \frac{44}{15}$, $c = -1 + \frac{40}{15} = \frac{5}{3}$

$$\vec{AF} = \begin{pmatrix} \frac{13}{15} - 2 \\ \frac{44}{15} + 1 \\ \frac{5}{3} - 1 \end{pmatrix} = \begin{pmatrix} -\frac{17}{15} \\ \frac{59}{15} \\ \frac{2}{3} \end{pmatrix}$$

perpendicular distance from A to line BC $= \sqrt{\left(-\frac{17}{15}\right)^2 + \left(\frac{59}{15}\right)^2 + \left(\frac{2}{3}\right)^2} = 4.15$

(iii) $n = \vec{AB} \times \vec{BC} = \begin{pmatrix} 21 \\ 3 \\ 18 \end{pmatrix}$

$$21x + 3y + 18z = 21(2) + 3(-1) + 18(1)$$

$$21x + 3y + 18z = 57$$

$$7x + y + 6z = 19$$