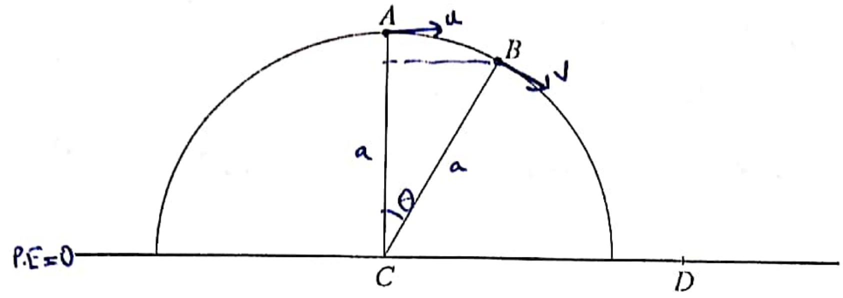
1



A smooth semicircular cylinder of radius a is fixed on a horizontal table with its axis in the plane of the table. C is a point on the axis and CA is a vertical radius (see diagram). A particle is projected horizontally from A, with speed $\sqrt{(0.4ga)}$, in a direction perpendicular to the axis. Show that the particle leaves the surface with speed $\sqrt{(0.8ga)}$ at the point B where $\cos ACB = 0.8$. [7]

Determine the distance from C of the point D at which the particle hits the table, assuming that it moves freely under gravity from B. [7]

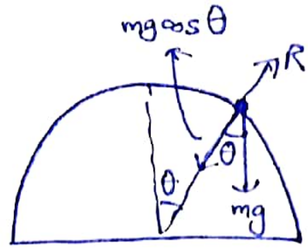
$$(K.E.+P.E.)_A = (K.E.+P.E.)_B$$

$$\frac{mu^2}{2} + mg(a) = \frac{mv^2}{2} + mga\cos\theta$$

$$\frac{u^2}{2} + ga = \frac{v^2}{2} + ga\cos\theta$$

$$u^2 + 2ga = v^2 + 2ga\cos\theta$$

$$v^2 = u^2 + 2ga(1-\cos\theta).$$



R(towards center) at B:

$$mg\cos\theta - R = \frac{mv^2}{a}$$

Since at B, contact is lost $\therefore R = 0$

$$mg\cos\theta = \frac{mv^2}{a}$$

$$ag\cos\theta = v^2$$

$$ag\cos\theta = u^2 + 2ga(1-\cos\theta)$$

$$ag\cos\theta = 0.4ga + 2ga(1-\cos\theta)$$

$$3ga\cos\theta = 0.4ga + 2ga$$

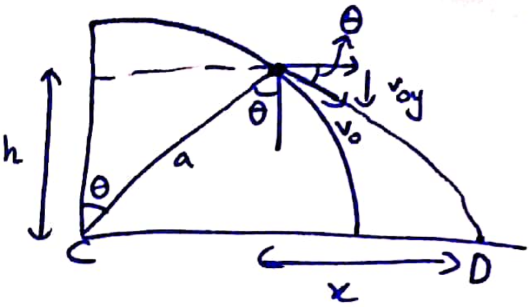
$$\cos\theta = \frac{2.4ga}{3ga} = 0.8 \text{ (Shown)}.$$

$$\therefore v^2 = u^2 + 2ga(1-\cos\theta)$$

$$v^2 = 0.4ga + 2g(1-0.8) = 0.4ga + 0.4ga = 0.8ga$$

$$\Rightarrow v = \sqrt{0.8ga} \text{ (Shown)}$$

$\cos\theta = 0.8 \Rightarrow \sin\theta = 0.6$



Vertically,

$s = h = a\cos\theta = 0.8a.$

$u = v_{oy} = v\sin\theta = \sqrt{0.8ga} \times 0.6 = \sqrt{0.288ga}.$

Using $v^2 = u^2 + 2as$

$v_y^2 = (\sqrt{0.288ga})^2 + 2g(0.8a)$

$v_y^2 = 1.888ga \Rightarrow v_y = \sqrt{1.888ga}.$

$\therefore t = \dfrac{v-u}{a} = \dfrac{\sqrt{1.888ga} - \sqrt{0.288ga}}{g} = \dfrac{\sqrt{ga}\left(\sqrt{1.888} - \sqrt{0.288}\right)}{g}$

Horizontally,

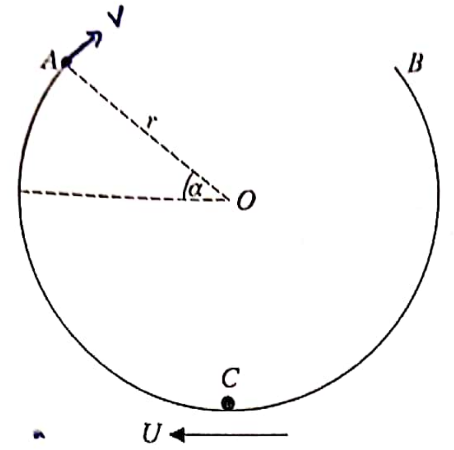
$v_x = v_{xo} = v\cos\theta = \sqrt{0.8ga} \times 0.8 = \sqrt{0.512ga}.$

$\therefore x = v_x \times t = \sqrt{0.512ga} \times \dfrac{\sqrt{ga}\left(\sqrt{1.888} - \sqrt{0.288}\right)}{g}$

$x = \dfrac{\sqrt{0.512}\left(\sqrt{1.888} - \sqrt{0.288}\right)ga}{g} = 0.599a$

$\therefore CD = a\sin\theta + 0.599a = a(0.6) + 0.599a = 1.199a$

$CD \approx 1.20a$



A smooth metal strip with ends A and B is bent into a major arc of a circle with centre O and radius r. The strip is fixed in a vertical plane with AB horizontal and at a height $r\sin\alpha$ above O, where $0 < \alpha < \frac{1}{2}\pi$ (see diagram). A particle is projected from the lowest point C with speed U so that it travels along the inside of the strip, reaching A with speed V. The particle then moves under gravity, and arrives at B. Air resistance may be neglected. Show that

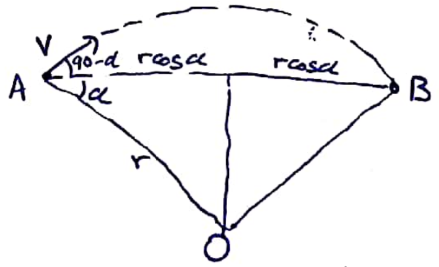
(i) $V^2 = gr\operatorname{cosec}\alpha$, [4]

(ii) $U^2 = gr(\operatorname{cosec}\alpha + 2 + 2\sin\alpha)$. [3]

The mass of the particle is m and the normal reaction between the strip and the particle is R. Show that the greatest value of R is $mg(\operatorname{cosec}\alpha + 3 + 2\sin\alpha)$ and find the least value of R. [7]

i) We require Range $= 2r\cos\alpha$

Note angle of projection $= 90° - \alpha$



$$R = \frac{v^2 \sin 2\theta}{g}$$

$$2r\cos\alpha = \frac{V^2 \sin[2(90° - \alpha)]}{g}$$

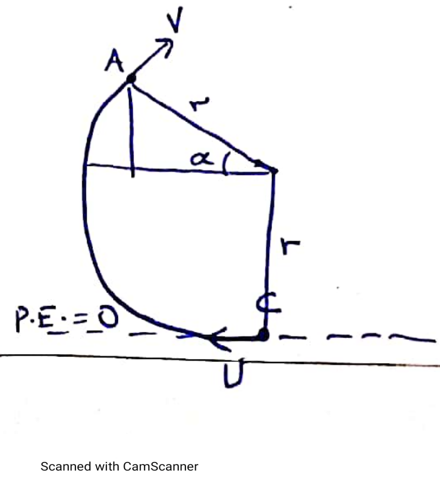
$$2gr\cos\alpha = V^2 \sin(180° - 2\alpha)$$

$$2gr\cos\alpha = V^2 \sin 2\alpha$$

$$2gr\cos\alpha = V^2 [2\sin\alpha\cos\alpha]$$

$$gr = V^2 \sin\alpha$$

$$\Rightarrow V^2 = gr\operatorname{cosec}\alpha.$$



ii) $(K.E. + P.E.)_A = (K.E. + P.E.)_C$

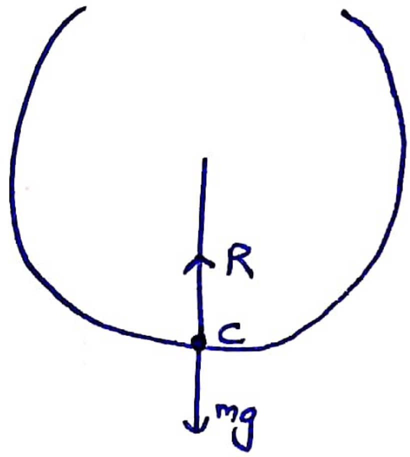
$$\frac{mV^2}{2} + mgr(1+\sin\alpha) = \frac{mU^2}{2} + mg(0)$$

$$V^2 + 2gr(1+\sin\alpha) = U^2$$

$$\therefore\ U^2 = gr\,\text{cosec}\,\alpha + 2gr(1+\sin\alpha)$$

$$U^2 = gr\left[\text{cosec}\,\alpha + 2 + 2\sin\alpha\right].$$

R (towards center) at C :



$$R - mg = \frac{mU^2}{r}$$

$$R = \frac{m}{r}\left\{gr\left[\text{cosec}\,\alpha + 2 + 2\sin\alpha\right]\right\} + mg$$

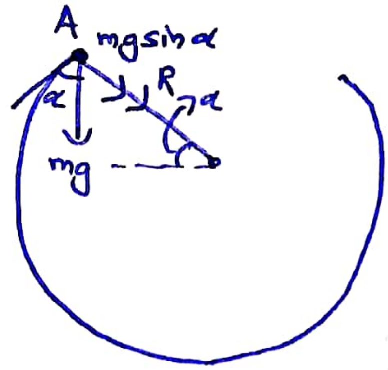
$$R = mg\left[\text{cosec}\,\alpha + 2 + 2\sin\alpha + 1\right]$$

$$R = mg\left[\text{cosec}\,\alpha + 3 + 2\sin\alpha\right].$$

$\therefore$ Maximum value of $R = mg\left[\text{cosec}\,\alpha + 3 + 2\sin\alpha\right].$

For minimum value of R :

R (towards center) at A :

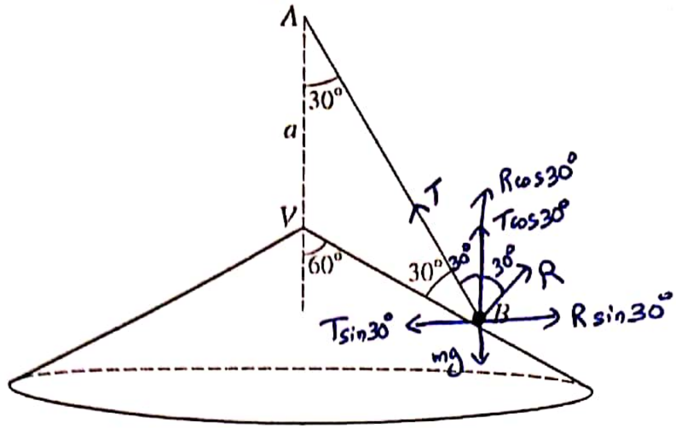


$$R + mg\sin\alpha = \frac{mV^2}{r}$$

$$R = \frac{m}{r}\left[gr\,\text{cosec}\,\alpha\right] - mg\sin\alpha$$

$$R = mg\left[\text{cosec}\,\alpha - \sin\alpha\right]$$

$\therefore$ Minimum value of $R = mg(\text{cosec}\,\alpha - \sin\alpha).$



A particle of mass m is attached to the end B of a light inextensible string. The other end of the string is attached to a fixed point A which is at a distance a above the vertex V of a circular cone of semi-vertical angle 60°. The axis of the cone is vertical. The particle moves with constant speed u in a horizontal circle on the smooth surface of the cone. The string makes a constant angle of 30° with the vertical (see diagram). The tension in the string and the magnitude of the normal force acting on the particle are denoted by T and R respectively. Show that

$$T = \frac{m}{\sqrt{3}}\left(g + \frac{2u^2}{a}\right),$$

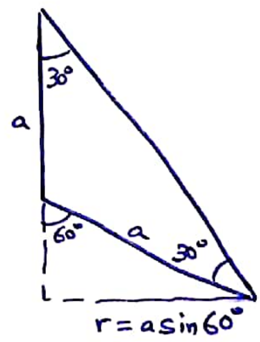
and find a similar expression for R. [6]

Deduce that $u^2 \leqslant \frac{1}{2}ga$. [1]

R(vertically): $T\cos 30° + R\cos 30° = mg$

$$T\left(\frac{\sqrt{3}}{2}\right) + R\left(\frac{\sqrt{3}}{2}\right) = mg$$

$$T + R = \frac{2mg}{\sqrt{3}} \rightarrow (1)$$



R(radially): $T\sin 30° - R\sin 30° = \frac{mu^2}{a\sin 60°}$

$$\frac{T}{2} - \frac{R}{2} = \frac{mu^2}{a\frac{\sqrt{3}}{2}}$$

$$\left(\frac{T-R}{2}\right) \times \frac{a\sqrt{3}}{2} = mu^2$$

$$(a\sqrt{3})(T-R) = 4mu^2 \rightarrow (2)$$

From (1), $R = \frac{2mg}{\sqrt{3}} - T \rightarrow (3)$

Substitute (3) in (2):

$$(a\sqrt{3})\left[T-\left(\frac{2mg}{\sqrt{3}} - T\right)\right] = 4mu^2$$

$$2T - \frac{2mg}{\sqrt{3}} = \frac{4mu^2}{a\sqrt{3}} \Rightarrow 2T = \frac{2mg}{\sqrt{3}} + \frac{4mu^2}{a\sqrt{3}}$$

$$T = \frac{mg}{\sqrt{3}} + \frac{2mu^2}{a\sqrt{3}}$$

$$T = \frac{m}{\sqrt{3}}\left(g + \frac{2u^2}{a}\right).$$

$$\therefore\ R = \frac{2mg}{\sqrt{3}} - \frac{m}{\sqrt{3}}\left(g + \frac{2u^2}{a}\right)$$

$$R = \frac{m}{\sqrt{3}}\left[2g - g - \frac{2u^2}{a}\right] = \frac{m}{\sqrt{3}}\left[g - \frac{2u^2}{a}\right]$$

Since contact is never lost, $R \geq 0$

$$\therefore\ \frac{m}{\sqrt{3}}\left[g - \frac{2u^2}{a}\right] \geq 0$$

$$g \geq \frac{2u^2}{a}$$

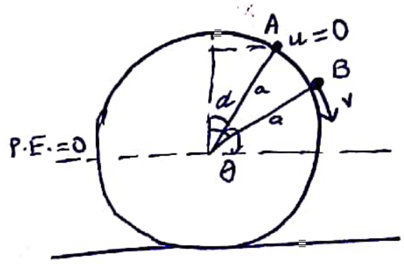
$$\frac{ag}{2} \geq u^2$$

$$\therefore\ u^2 \leq \frac{1}{2}ga.$$

(Shown).

A particle P of mass m is placed at the point Q on the outer surface of a fixed smooth sphere with centre O and radius a. The acute angle between OQ and the upward vertical is α, where $\cos\alpha = \frac{9}{10}$. The particle is released from rest and begins to move in a vertical circle on the surface of the sphere. Show that P loses contact with the sphere when OP makes an angle θ with the upward vertical, where $\cos\theta = \frac{3}{5}$, and find the speed of P at this instant. [6]

Show that, in the subsequent motion, when P is at a distance $\frac{7}{5}a$ from the vertical diameter through O, its distance below the horizontal through O is $\frac{31}{30}a$. [6]



$$(K.E.+P.E.)_A = (K.E.+P.E.)_B$$

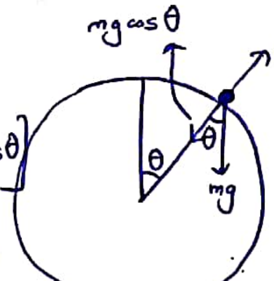
$$\frac{mu^2}{2} + mg(a\cos\alpha) = \frac{mv^2}{2} + mg(a\cos\theta)$$

$$\frac{m(0)^2}{2} + mga \times \frac{9}{10} = \frac{mv^2}{2} + mga\cos\theta$$

$$\frac{9ga}{10} = \frac{v^2}{2} + ga\cos\theta$$

$$\frac{9ga}{5} = v^2 + 2ga\cos\theta$$

$$\Rightarrow v^2 = \frac{9ga}{5} - 2ga\cos\theta$$



R(towards center): $\quad mg\cos\theta - R = \frac{mv^2}{a}$

$$R = mg\cos\theta - \frac{m}{a}\left[\frac{9ga}{5} - 2ga\cos\theta\right]$$

$$R = mg\cos\theta - \frac{9mg}{5} + 2mg\cos\theta$$

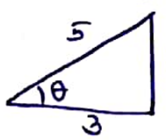
$$R = 3mg\cos\theta - \frac{9mg}{5}$$

To lose contact, $R = 0 \Rightarrow 3mg\cos\theta - \frac{9mg}{5} = 0$

$$\cos\theta = \frac{\frac{9mg}{5}}{3mg} = \frac{3}{5}.$$

$$v^2 = \frac{9ga}{5} - 2ga\left(\frac{3}{5}\right) = \frac{3ga}{5} \Rightarrow v = \sqrt{\frac{3ga}{5}}.$$

$\cos\theta = \frac{3}{5}$

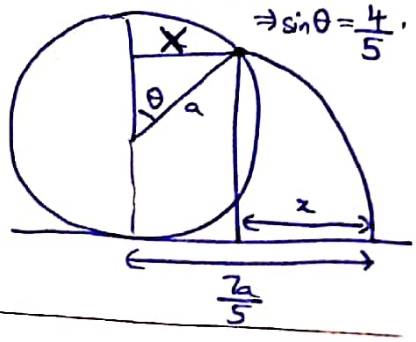


$\Rightarrow \sin\theta = \frac{4}{5}.$

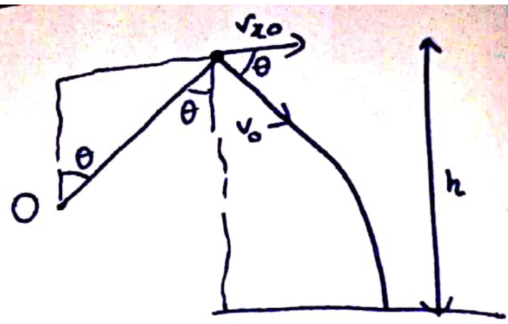
To be a distance $\frac{7a}{5}$ from vertical diameter through O:

Note $X = a\sin\theta = a \times \frac{4}{5} = \frac{4a}{5}$

$$\therefore x = \frac{7a}{5} - \frac{4a}{5} = \frac{3a}{5}.$$



Horizontally, $v_{x0} = v_0 \cos\theta = \sqrt{\frac{3ga}{5}} \times \frac{3}{5}$



$$v_{x0} = \frac{3}{5}\sqrt{\frac{3ga}{5}}.$$

Using $x = v_x t$

$$t = \frac{x}{v_x} = \frac{\frac{3a}{5}}{\frac{3}{5}\sqrt{\frac{3ga}{5}}} = \frac{a}{\sqrt{\frac{3ga}{5}}}.$$

Vertically, $v_{y0} = v_0 \sin\theta = \sqrt{\frac{3ga}{5}} \times \frac{4}{5} = \frac{4}{5}\sqrt{\frac{3ga}{5}}.$

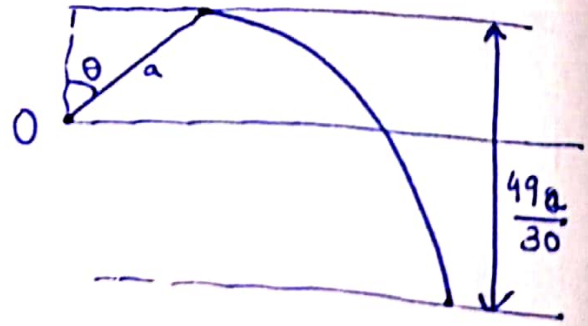
Using $s = ut + \frac{1}{2}at^2$ gives

$$h = \left(\frac{4}{5}\sqrt{\frac{3ga}{5}}\right) \times \frac{a}{\sqrt{\frac{3ga}{5}}} + \frac{1}{2} g \left(\frac{a}{\sqrt{\frac{3ga}{5}}}\right)^2$$

$$h = \frac{4a}{5} + \frac{g a^2}{2 \times \frac{3ga}{5}} = \frac{4a}{5} + \frac{5a}{6} = \frac{49a}{30}.$$

Distance below horizontal through O:

$$= \frac{49a}{30} - a\cos\theta = \frac{49a}{30} - a\left(\frac{3}{5}\right) = \frac{31a}{30}$$

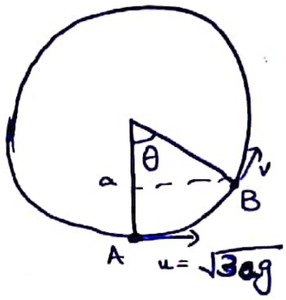


5 A particle P of mass m is suspended from a fixed point O by a light inextensible string of length a. When hanging at rest under gravity, P is given a horizontal velocity of magnitude $\sqrt{(3ag)}$ and subsequently moves freely in a vertical circle. Show that the tension T in the string when OP makes an angle θ with the downward vertical is given by

$$T = mg(1 + 3\cos\theta).$$ [4]

When the string is horizontal, it comes into contact with a small smooth peg Q which is at the same horizontal level as O and at a distance x from O, where $x < a$. Given that P completes a vertical circle about Q, find the least possible value of x. [8]

$(K.E. + P.E.)_A = (K.E. + P.E.)_B$



$\frac{1}{2}mu^2 + mg(0) = \frac{1}{2}mv^2 + mga(1-\cos\theta)$

$\frac{3ag}{2} = \frac{v^2}{2} + ga(1-\cos\theta)$

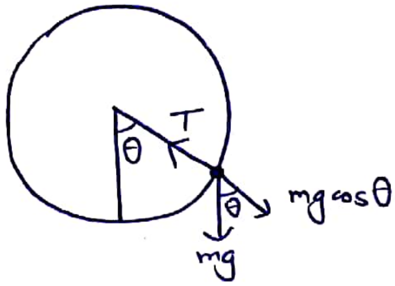
$3ag = v^2 + 2ga(1-\cos\theta)$

$v^2 = 3ag - 2ag(1-\cos\theta)$

$v^2 = ag[3-2+2\cos\theta]$

$v^2 = ag[1+2\cos\theta]$.

R(towards center): $T - mg\cos\theta = \frac{mv^2}{a}$

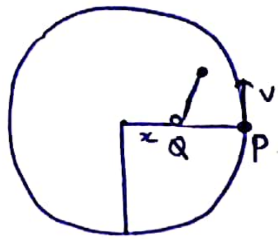


$T = mg\cos\theta + \frac{m}{a}[ag(1+2\cos\theta)]$

$T = mg[\cos\theta + 1 + 2\cos\theta]$

$T = mg[1+3\cos\theta]$

When the particle is horizontal, $\theta = 90°$

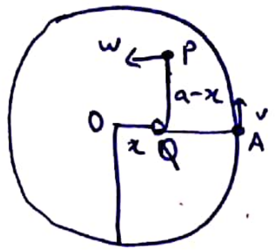


$v^2 = ag[1+2\cos 90°] = ag$.

$v = \sqrt{ag}$.

When P is vertically above peg Q,

$(K.E. + P.E.)_P = (K.E. + P.Q.)_A$



$\frac{mw^2}{2} + mg(a-x) = \frac{mv^2}{2} + mg(0)$

$\frac{w^2}{2} + g(a-x) = \frac{v^2}{2}$

$\frac{w^2}{2} + ag - xg = \frac{ag}{2}$

$\frac{w^2}{2} = xg - \frac{ag}{2} \Rightarrow w^2 = 2xg - ag$

$w^2 = (2x-a)g$.

R (towards center/peg Q):

$$T + mg = \frac{m w^2}{a-x}$$

$$T = \frac{m(2x-a)g}{a-x} - mg$$

To complete a circle, $T \geq 0$ at P:

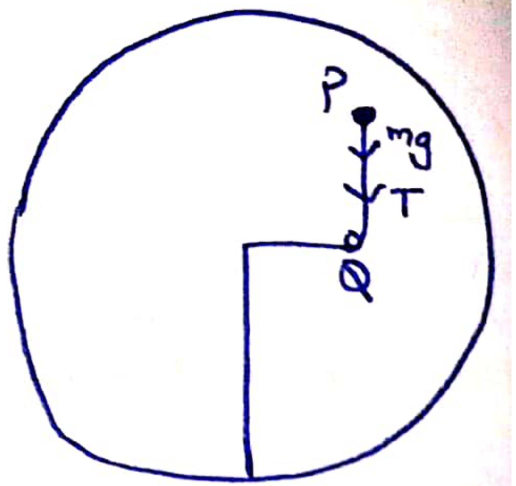
$$\frac{m(2x-a)g}{a-x} - mg \geq 0$$

$$mg\left[\frac{2x-a}{a-x} - 1\right] \geq 0$$

$$2x - a - a + x \geq 0$$

$$3x \geq 2a$$

$$x \geq \frac{2a}{3}$$



$\therefore$ Least possible value of $x = \frac{2a}{3}$.

6 A particle P of mass m is projected horizontally with speed $\sqrt{(\frac{7}{2}ga)}$ from the lowest point of the inside of a fixed hollow smooth sphere of internal radius a and centre O. The angle between OP and the downward vertical at O is denoted by θ. Show that, as long as P remains in contact with the inner surface of the sphere, the magnitude of the reaction between the sphere and the particle is $\frac{3}{2}mg(1 + 2\cos\theta)$. [4]

Find the speed of P

(i) when it loses contact with the sphere, [3]

(ii) when, in the subsequent motion, it passes through the horizontal plane containing O. (You may assume that this happens before P comes into contact with the sphere again.) [3]

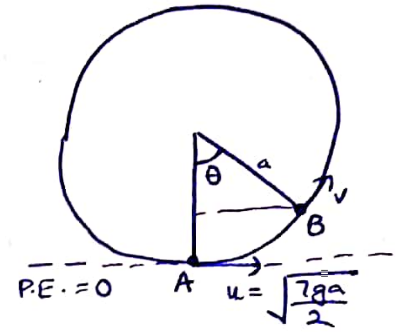
$$(K.E.+P.E.)_A = (K.E.+P.E.)_B$$

$$\frac{1}{2}mu^2 + mg(0) = \frac{1}{2}mv^2 + mga(1-\cos\theta)$$

$$\frac{u^2}{2} = \frac{v^2}{2} + ga(1-\cos\theta)$$

$$u^2 = v^2 + 2ga(1-\cos\theta).$$

$$v^2 = u^2 - 2ga(1-\cos\theta).$$



R (towards center):

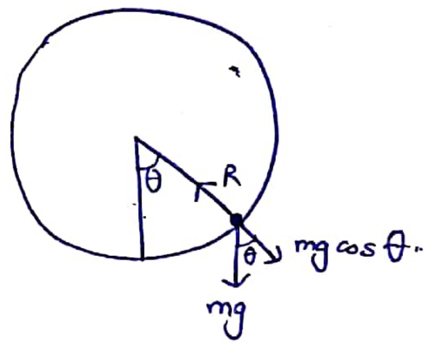
$$R - mg\cos\theta = \frac{mv^2}{a}$$

$$R = mg\cos\theta + \frac{m}{a}\left[u^2 - 2ga(1-\cos\theta)\right]$$

$$R = \frac{mu^2}{a} + mg\cos\theta - 2mg(1-\cos\theta)$$

$$R = \frac{mu^2}{a} + mg\left[\cos\theta - 2 + 2\cos\theta\right]$$

$$R = \frac{m}{a}\left[\frac{7ga}{2}\right] + mg\left[3\cos\theta - 2\right]$$



$$R = \frac{7mg}{2} + mg\left[3\cos\theta - 2\right] = mg\left[\frac{7}{2} + 3\cos\theta - 2\right] = mg\left[\frac{3}{2} + 3\cos\theta\right]$$

$$R = \frac{3}{2}mg\left[1 + 2\cos\theta\right].$$

i) When contact is lost, $R = 0 \Rightarrow \frac{3}{2}mg\left[1+2\cos\theta\right] = 0$

$$\cos\theta = \frac{-1}{2}$$

$$\therefore v^2 = u^2 - 2ga(1-\cos\theta)$$

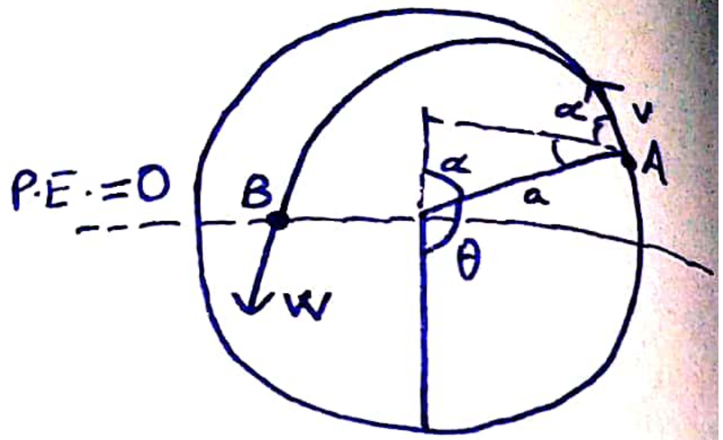
$$v^2 = \frac{7ga}{2} - 2ga\left[1 - \left(\frac{-1}{2}\right)\right] = \frac{7ga}{2} - 2ga\left(\frac{3}{2}\right)$$

$$v^2 = \frac{ga}{2} \Rightarrow v = \sqrt{\frac{ga}{2}}.$$

ii)

$$(K.E.+P.E.)_A = (K.E.+P.E.)_B$$

$$\frac{1}{2}mv^2 + mg(a\cos\alpha) = \frac{1}{2}mw^2 + mg(0)$$



$$\frac{v^2}{2} + ga\left(\frac{1}{2}\right) = \frac{w^2}{2}$$

$$v^2 + ga = w^2$$

$$\alpha = 180° - \theta.$$

$$\frac{ga}{2} + ga = w^2$$

$$\therefore \cos\alpha = -\cos\theta = -\left(\frac{-1}{2}\right) = \frac{1}{2}.$$

$$\frac{3ga}{2} = w^2$$

$$w = \sqrt{\frac{3ga}{2}}.$$

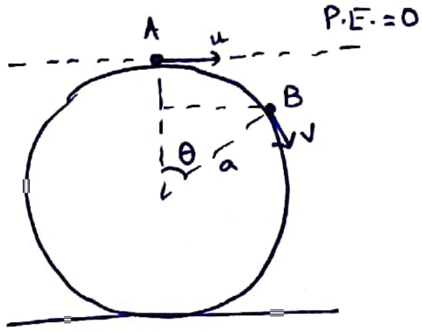
i.e. Speed of particle as it passes through horizontal plane containing O $= \sqrt{\frac{3ga}{2}}$.

7 A smooth sphere, with centre O and radius a, has its lowest point fixed on a horizontal plane. A particle P of mass m is projected horizontally with speed u from the highest point on the outer surface of the sphere. In the subsequent motion, OP makes an angle θ with the upward vertical through O. Show that, while P remains in contact with the sphere, the magnitude of the reaction of the sphere on P is $mg(3\cos\theta - 2) - \frac{mu^2}{a}$. [4]

The particle loses contact with the surface of the sphere when $\theta = \alpha$. Given that $u = \frac{1}{2}\sqrt{(ga)}$, find

(i) $\cos\alpha$, [2]

(ii) the vertical component of the velocity of P as it strikes the horizontal plane. [5]



$$(K.E. + P.E.)_A = (K.E. + P.E.)_B$$

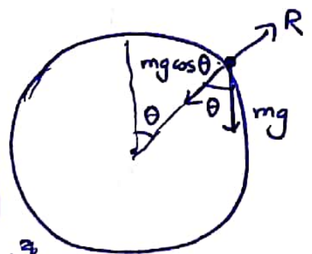
$$\frac{1}{2}mu^2 + mg(0) = \frac{1}{2}mv^2 + mg\left[-a(1-\cos\theta)\right]$$

$$\frac{u^2}{2} = \frac{v^2}{2} - ga(1-\cos\theta)$$

$$u^2 = v^2 - 2ga(1-\cos\theta).$$

$$v^2 = u^2 + 2ga(1-\cos\theta).$$

R(towards center):



$$mg\cos\theta - R = \frac{mv^2}{a}$$

$$R = mg\cos\theta - \frac{mv^2}{a}.$$

$$= mg\cos\theta - \frac{m}{a}\left[u^2 + 2ga(1-\cos\theta)\right]$$

$$= mg\cos\theta - \frac{mu^2}{a} - 2mg(1-\cos\theta)$$

$$= mg\left[\cos\theta - 2 + 2\cos\theta\right] - \frac{mu^2}{a}.$$

$$R = mg\left[3\cos\theta - 2\right] - \frac{mu^2}{a}.$$

(Shown).

When contact is lost, $R = 0, \theta = \alpha$:

i)
$$0 = mg\left[3\cos\alpha - 2\right] - \frac{mu^2}{a}$$

$$0 = mg\left[3\cos\alpha - 2\right] - \frac{m\left(\frac{1}{2}\sqrt{ga}\right)^2}{a}$$

$$\frac{m\left(\frac{1}{4}ga\right)}{a} = mg\left[3\cos\alpha - 2\right]$$

$$\frac{mg}{4} = mg\left[3\cos\alpha - 2\right]$$

$$\frac{1}{4} = 3\cos\alpha - 2 \Rightarrow 1 = 12\cos\alpha - 8$$

$$9 = 12\cos\alpha$$

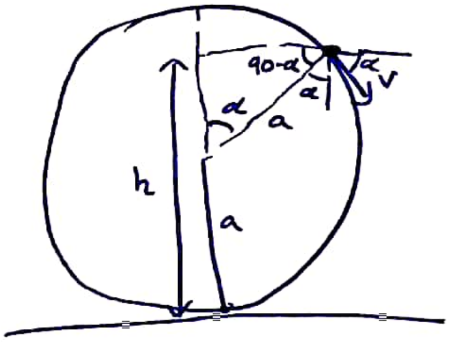
$$\Rightarrow \cos\alpha = \frac{3}{4}.$$

ii) $\cos\alpha = \dfrac{3}{4} \Rightarrow v^2 = u^2 + 2ga(1-\cos\alpha)$

$$v^2 = \left(\frac{1}{2}\sqrt{ga}\right)^2 + 2ga\left(1-\frac{3}{4}\right)$$

$$v^2 = \frac{1}{4}ga + \frac{2ga}{4} = \frac{3ga}{4}$$

$$v^2 = \frac{3ga}{4} \Rightarrow v = \frac{\sqrt{3ga}}{2}.$$



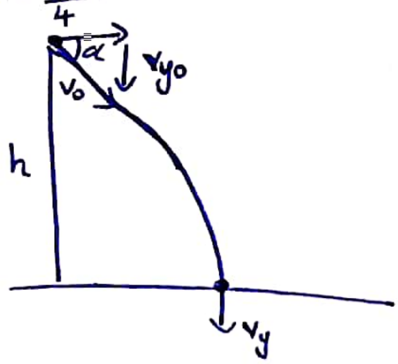
$$h = a + a\cos\alpha = a + a\left(\frac{3}{4}\right) = \frac{7}{4}a.$$

$\cos\alpha = \dfrac{3}{4}$

$\Rightarrow \sin\alpha = \dfrac{\sqrt{7}}{4}.$

$$V_{y_0} = v_0 \sin\alpha = \frac{\sqrt{3ga}}{2} \times \frac{\sqrt{7}}{4}$$

$$V_{y_0} = \frac{\sqrt{21ga}}{8}$$



Vertically, $u = v_{y_0} = \dfrac{\sqrt{21ga}}{8}$, $s = h = \dfrac{7a}{4}$, $a = g$, $v = v_y$

$$v^2 = u^2 + 2as$$

$$v_y^2 = \left(\frac{\sqrt{21ga}}{8}\right)^2 + 2g\left(\frac{7a}{4}\right)$$

$$v_y^2 = \frac{21ga}{64} + \frac{14ga}{4} = \frac{245ga}{64}$$

$$\Rightarrow v_y = \frac{\sqrt{245ga}}{8}.$$

i.e. Vertical component of velocity as it strikes horizontal surface $= \dfrac{\sqrt{245ga}}{8}$.

8 A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. When P is hanging vertically below O, it is given a horizontal speed u. In the subsequent motion, P moves in a complete circle. When OP makes an angle θ with the downward vertical, the tension in the string is T. Show that

$$T = \frac{mu^2}{a} + mg(3\cos\theta - 2).$$ [5]

Given that the ratio of the maximum value of T to the minimum value of T is 3 : 1, find u in terms of a and g. [4]

Assuming this value of u, find the value of $\cos\theta$ when the tension is half of its maximum value. [3]

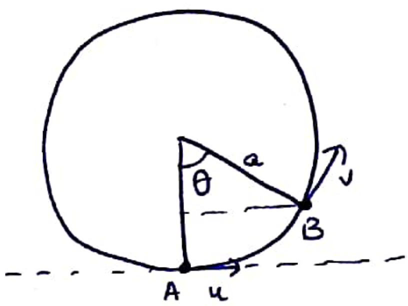
$$(K.E. + P.E.)_A = (K.E. + P.E.)_B$$

$$\frac{1}{2}mu^2 + mg(0) = \frac{1}{2}mv^2 + mga(1-\cos\theta)$$

$$\frac{u^2}{2} = \frac{v^2}{2} + ga(1-\cos\theta)$$

$$u^2 = v^2 + 2ga(1-\cos\theta).$$

$$v^2 = u^2 - 2ga(1-\cos\theta).$$

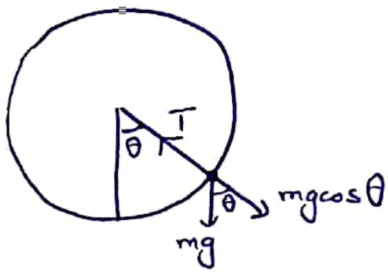


R (towards center): $T - mg\cos\theta = \frac{mv^2}{a}$

$$T = mg\cos\theta + \frac{m}{a}\left[u^2 - 2ga(1-\cos\theta)\right]$$

$$= mg\cos\theta + \frac{mu^2}{a} - 2mg(1-\cos\theta)$$

$$= \frac{mu^2}{a} + mg\left[\cos\theta - 2 + 2\cos\theta\right]$$



$$T = \frac{mu^2}{a} + mg\left[3\cos\theta - 2\right]$$

Maximum tension occurs when $\theta = 0°$,

$$\therefore T_{max} = \frac{mu^2}{a} + mg\left[3\cos 0° - 2\right] = \frac{mu^2}{a} + mg$$

Minimum tension occurs when $\theta = 180°$,

$$\therefore T_{min} = \frac{mu^2}{a} + mg\left[3\cos 180° - 2\right] = \frac{mu^2}{a} + mg\left[-5\right] = \frac{mu^2}{a} - 5mg.$$

$$\frac{T_{max}}{T_{min}} = \frac{3}{1} \Rightarrow \frac{\frac{mu^2}{a} + mg}{\frac{mu^2}{a} - 5mg} = \frac{3}{1}$$

$$\frac{mu^2}{a} + mg = \frac{3mu^2}{a} - 15mg$$

$$16mg = \frac{2mu^2}{a}$$

$$\frac{16mga}{2m} = u^2 \Rightarrow u^2 = 8ga \Rightarrow u = \sqrt{8ga}$$

$$T = \frac{1}{2} T_{max}$$

$$\frac{mu^2}{a} + mg(3\cos\theta - 2) = \frac{1}{2}\left(\frac{mu^2}{a} + mg\right)$$

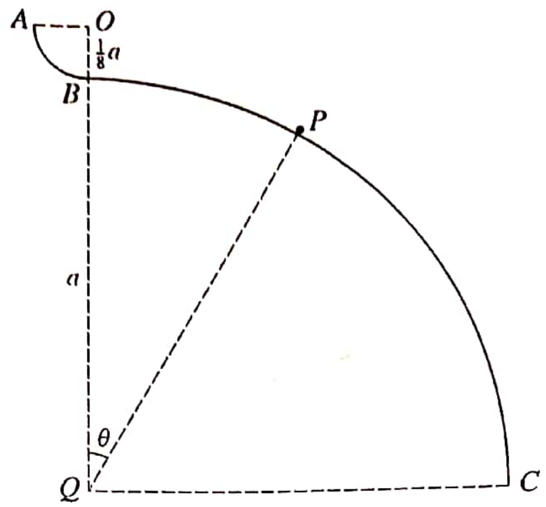
$$\frac{m(8ga)}{a} + mg(3\cos\theta - 2) = \frac{1}{2}\left[\frac{m(8ga)}{a} + mg\right]$$

$$8mg + mg(3\cos\theta - 2) = \frac{1}{2}\left[9mg\right]$$

$$8 + 3\cos\theta - 2 = \frac{9}{2}$$

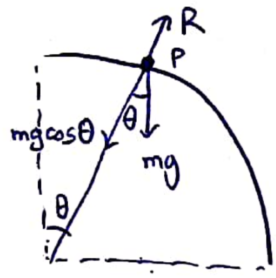
$$\cos\theta = \frac{\frac{9}{2} + 2 - 8}{3}$$

$$\cos\theta = \frac{-1}{2}.$$



A particle P travels on a smooth surface whose vertical cross-section is in the form of two arcs of circles. The first arc AB is a quarter of a circle of radius $\frac{1}{8}a$ and centre O. The second arc BC is a quarter of a circle of radius a and centre Q. The two arcs are smoothly joined at B. The point Q is vertically below O and the two arcs are in the same vertical plane. The particle P is projected vertically downwards from A with speed u. When P is on the arc BC, angle BQP is θ (see diagram). Given that P loses contact with the surface when $\cos\theta = \frac{5}{6}$, find u in terms of a and g. [8]

R (towards center) at P where contact is lost:



$$mg\cos\theta - R = \frac{mv^2}{a}$$

$$R = mg\cos\theta - \frac{mv^2}{a}$$

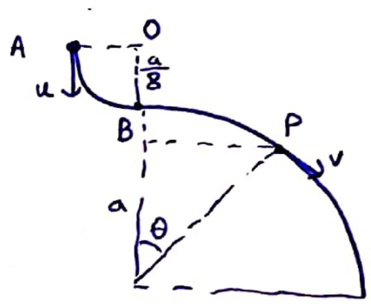
Since contact is lost, $R = 0$ and $\cos\theta = \frac{5}{6}$:

$$0 = mg\left(\frac{5}{6}\right) - \frac{mv^2}{a}$$

$$\frac{mv^2}{a} = \frac{5mg}{6}$$

$$v^2 = \frac{5ag}{6}.$$

$$(K.E. + P.E.)_A = (K.E. + P.E.)_P$$



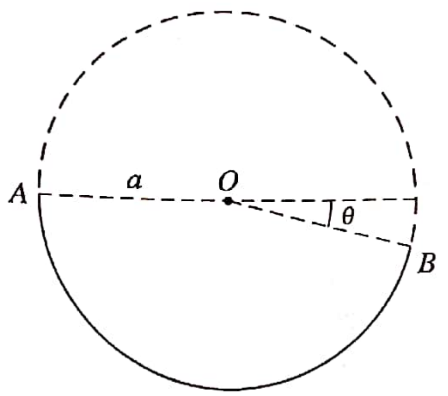
$$\frac{1}{2}mu^2 + mg\left(\frac{9a}{8}\right) = \frac{mv^2}{2} + mga\cos\theta$$

$$\frac{u^2}{2} + \frac{9ag}{8} = \frac{v^2}{2} + ga\left(\frac{5}{6}\right)$$

$$\frac{u^2}{2} + \frac{9ag}{8} = \frac{1}{2}\left(\frac{5ag}{6}\right) + \frac{5ag}{6}$$

$$\frac{u^2}{2} = \frac{ag}{8}$$

$$u^2 = \frac{ag}{4} \Rightarrow u = \frac{\sqrt{ag}}{2}.$$



A smooth wire is in the form of an arc AB of a circle, of radius a, that subtends an obtuse angle $\pi - \theta$ at the centre O of the circle. It is given that $\sin\theta = \frac{1}{4}$. The wire is fixed in a vertical plane, with AO horizontal and B below the level of O (see diagram). A small bead of mass m is threaded on the wire and projected vertically downwards from A with speed $\sqrt{(\frac{3}{10}ga)}$.

(i) Find the reaction between the bead and the wire when the bead is vertically below O. [3]

(ii) Find the speed of the bead as it leaves the wire at B. [3]

(iii) Show that the greatest height reached by the bead is $\frac{1}{8}a$ above the level of O. [4]

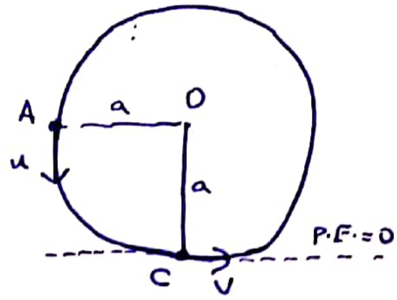
i) $(P.E. + K.E.)_A = (P.E. + K.E.)_C$

$mga + \frac{mu^2}{2} = mg(0) + \frac{mv^2}{2}$

$ga + \frac{u^2}{2} = \frac{v^2}{2}$

$u^2 + 2ga = v^2$

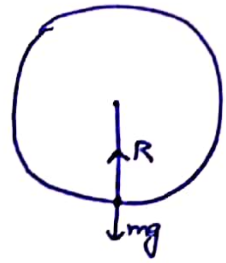
$\Rightarrow v^2 = \frac{3ga}{10} + 2ga = \frac{23ga}{10}$.



R(towards center) at the lowest point:

$R - mg = \frac{mv^2}{a}$

$R = mg + \frac{m}{a}\left[\frac{23ga}{10}\right] = \frac{33mg}{10}$.



ii) $(P.E. + K.E.)_A = (P.E. + K.E.)_B$

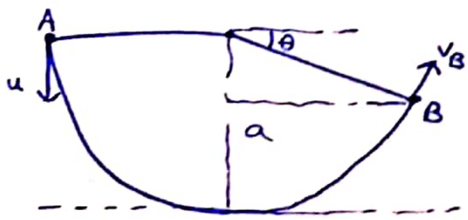
$mga + \frac{mu^2}{2} = mg(a)(1 - \cancel{\sin}\,\theta) + \frac{mv_B^2}{2}$

$mga + \frac{m}{2}\left(\frac{3ga}{10}\right) = mga\left(1 - \frac{1}{4}\right) + \frac{mv_B^2}{2}$

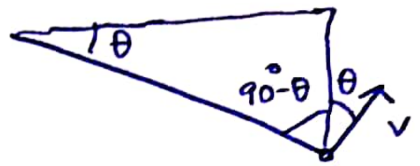
$ga + \frac{3ga}{20} = ga\left(\frac{3}{4}\right) + \frac{v_B^2}{2}$

$\frac{2ga}{5} = \frac{v_B^2}{2}$

$v_B^2 = \frac{4ga}{5} \Rightarrow v_B = \sqrt{\frac{4ga}{5}}$.



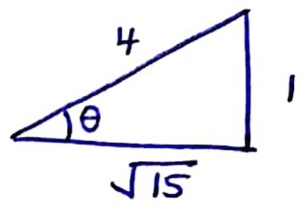
iii) $v_{y0} = v_0 \cos\theta = \sqrt{\frac{4ga}{5}} \times \frac{\sqrt{15}}{4} = \sqrt{\frac{3ga}{4}}$.



For vertical motion,

$u = \sqrt{\frac{3ga}{4}}$, $a = -g$, $v = 0$ since we want maximum height.

$\sin\theta = \frac{1}{4}$



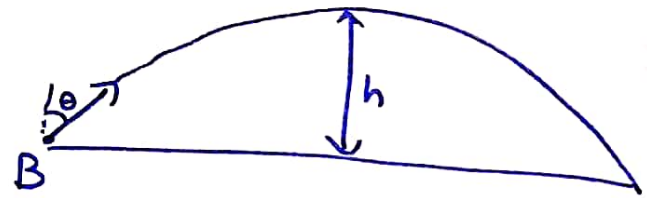
$\therefore \cos\theta = \frac{\sqrt{15}}{4}$.

$$v^2 = u^2 + 2as$$

$$0^2 = \frac{3ga}{4} - 2gh$$

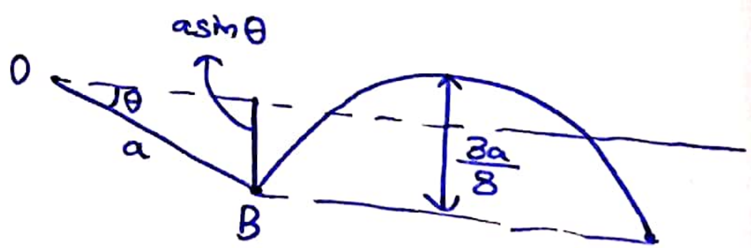
$$h = \frac{\frac{3ga}{4}}{2g} = \frac{3a}{8}.$$

i.e. Height reached above B $= \frac{3a}{8}$



Height reached above O

$= \frac{3a}{8} - a\sin\theta = \frac{3a}{8} - a\left(\frac{1}{4}\right) = \frac{a}{8}$.



11 A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. When the particle is hanging in equilibrium, with the string vertical, it is given a horizontal impulse of magnitude J. When the string makes an angle θ with the downward vertical at O, the tension is T.

(i) Show that $T = \dfrac{J^2}{ma} - mg(2 - 3\cos\theta)$. [5]

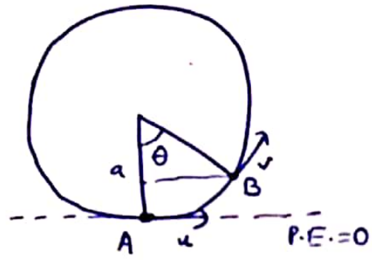
(ii) It is given that $J = m\sqrt{(kga)}$, where k is a positive constant. Justifying your answers, describe the motion of P in each of the following cases:

(a) $k = 1$,

(b) $k = 6$.

[4]

i) $(K.E. + P.E.)_A = (K.E. + P.E.)_B$



$$\frac{1}{2}mu^2 + mg(0) = \frac{1}{2}mv^2 + mga(1-\cos\theta)$$

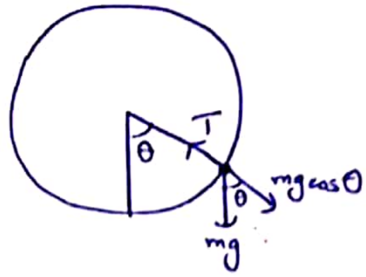
$$\frac{u^2}{2} = \frac{v^2}{2} + ga - ga\cos\theta$$

$$u^2 = v^2 + 2ga - 2ga\cos\theta$$

$$v^2 = u^2 - 2ga + 2ga\cos\theta.$$

R(towards center) at B:

$$T - mg\cos\theta = \frac{mv^2}{a}$$



$$T = mg\cos\theta + \frac{m}{a}\left[u^2 - 2ga + 2ga\cos\theta\right]$$

$$T = mg\cos\theta + \frac{mu^2}{a} - 2mg + 2mg\cos\theta$$

$$T = 3mg\cos\theta - 2mg + \frac{mu^2}{a}.$$

Impulse, $J = mu \Rightarrow u = \dfrac{J}{m}$

$$\therefore T = mg(3\cos\theta - 2) + \frac{m}{a}\left(\frac{J}{m}\right)^2$$

$$T = mg(3\cos\theta - 2) + \frac{mJ^2}{am^2} = mg(3\cos\theta - 2) + \frac{J^2}{ma}$$

$$T = \frac{J^2}{ma} - mg(2 - 3\cos\theta).$$

---

ii) a) $J = m\sqrt{ga} \Rightarrow u = \dfrac{J}{m} = \sqrt{ga}$.

$$v^2 = u^2 - 2ga + 2ga\cos\theta = (\sqrt{ga})^2 - 2ga + 2ga\cos\theta$$

$$v^2 = 2ga\cos\theta - ga = ga(2\cos\theta - 1)$$

Setting $v = 0 \Rightarrow ga(2\cos\theta - 1) = 0$

$$\cos\theta = \frac{1}{2} \Rightarrow \theta = 60°.$$

Then particle P oscillates.

We also observe $T = 3mg\cos 60° - 2mg + \dfrac{mga}{a} = \dfrac{mg}{2} > 0 \therefore$ remains in circular path.

b) $J = m\sqrt{6ga} \Rightarrow u = \frac{J}{m} = \sqrt{6ga}$.

Then $v^2 = u^2 - 2ga + 2ga\cos\theta$

$= 6ga - 2ga + 2ga\cos\theta$

$v^2 = 4ga + 2ga\cos\theta$

Setting $v = 0$ gives $4ga + 2ga\cos\theta = 0$

$\cos\theta = \frac{-4ga}{2ga} = -2$

Then $v \neq 0$.

We observe $T = 3mg\cos\theta - 2mg + \frac{m(6ga)}{a}$

$= 3mg\cos\theta - 2mg + 6mg$

$T = 3mg\cos\theta + 4mg$

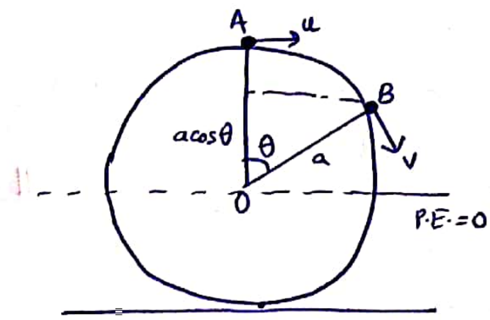
We observe that $T > 0$ since $T = 0$ gives $\cos\theta = \frac{-4}{3}$ No solutions.

or we could observe that when $\theta = 180°$, $T = 3mg(-1) + 4mg = mg > 0$

Then $T, v > 0$ $\therefore$ particle P does full circles.

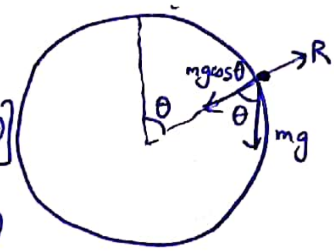
12 A particle P, of mass m, is placed at the highest point of a fixed solid smooth sphere with centre O and radius a. The particle P is given a horizontal speed u and it moves in part of a vertical circle, with centre O, on the surface of the sphere. When OP makes an angle θ with the upward vertical, and P is still in contact with the surface of the sphere, the speed of P is v and the reaction of the sphere on P has magnitude R. Show that $R = mg(3\cos\theta - 2) - \dfrac{mu^2}{a}$. [5]

The particle loses contact with the sphere at the instant when $v = 2u$. Find u in terms of a and g. [4]



$$(K.E. + P.E.)_A = (K.E. + P.E.)_B$$
$$\frac{1}{2}mu^2 + mga = \frac{mv^2}{2} + mga\cos\theta$$
$$\frac{u^2}{2} + ga = \frac{v^2}{2} + ga\cos\theta$$
$$u^2 + 2ga = v^2 + 2ga\cos\theta$$
$$v^2 = u^2 + 2ga - 2ga\cos\theta.$$

R (towards center) at B:



$$mg\cos\theta - R = \frac{mv^2}{a}$$
$$R = mg\cos\theta - \frac{m}{a}\left[u^2 + 2ga - 2ga\cos\theta\right]$$
$$R = mg\cos\theta - \frac{mu^2}{a} - 2mg + 2mg\cos\theta$$
$$R = 3mg\cos\theta - 2mg - \frac{mu^2}{a}.$$
$$R = mg(3\cos\theta - 2) - \frac{mu^2}{a}.$$

When $v = 2u$,

$$v^2 = u^2 + 2ga - 2ga\cos\theta$$
$$(2u)^2 = u^2 + 2ga - 2ga\cos\theta$$
$$4u^2 - u^2 = 2ga(1 - \cos\theta)$$
$$\frac{3u^2}{2ga} = 1 - \cos\theta$$
$$\cos\theta = 1 - \frac{3u^2}{2ga} \rightarrow (1)$$

Since when $v = 2u$, contact is lost then $R = 0$

$$\Rightarrow mg(3\cos\theta - 2) - \frac{mu^2}{a} = 0$$
$$g(3\cos\theta - 2) = \frac{u^2}{a}$$
$$u^2 = ag(3\cos\theta - 2) \rightarrow (2).$$

Substituting (1) in (2) gives:

$$u^2 = ag\left[3 \times \left(1 - \frac{3u^2}{2ga}\right) - 2\right]$$

$$u^2 = ag\left[3 - \frac{9u^2}{2ga} - 2\right]$$

$$u^2 = ag\left[1 - \frac{9u^2}{2ga}\right]$$

$$u^2 = ag - \frac{9u^2}{2}$$

$$u^2 + \frac{9u^2}{2} = ag$$

$$\frac{11u^2}{2} = ag$$

$$u^2 = \frac{2ag}{11}$$

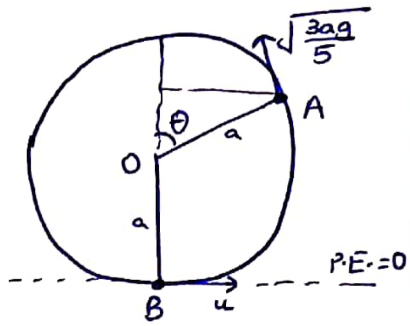
$$u = \sqrt{\frac{2ag}{11}}.$$

13 A particle P is at rest at the lowest point on the smooth inner surface of a hollow sphere with centre O and radius a. The particle is projected horizontally with speed u and begins to move in a vertical circle on the inner surface of the sphere. The particle loses contact with the sphere at the point A, where OA makes an angle θ with the upward vertical through O. Given that the speed of P at A is $\sqrt{(\frac{3}{5}ag)}$, find u in terms of a and g. [5]

Find, in terms of a, the greatest height above the level of O achieved by P in its subsequent motion. (You may assume that P achieves its greatest height before it makes any further contact with the sphere.) [5]

$$(K.E.+P.E.)_B = (K.E.+P.E.)_A$$

$$\frac{mu^2}{2} + mg(0) = \frac{m}{2}\left(\sqrt{\frac{3ag}{5}}\right)^2 + mga(1+\cos\theta)$$



$$\frac{mu^2}{2} = \frac{m}{2}\times\frac{3ag}{5} + mga(1+\cos\theta)$$

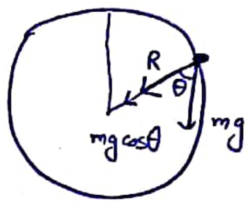
$$\frac{u^2}{2} = \frac{3ag}{10} + ga + ga\cos\theta$$

$$u^2 = \frac{3ag}{5} + 2ga + 2ga\cos\theta$$

$$u^2 = \frac{13ag}{5} + 2ag\cos\theta$$

R (towards center) at A:

$$R + mg\cos\theta = \frac{m}{a}\left(\frac{3ag}{5}\right)$$



$$R = \frac{3mg}{5} - mg\cos\theta.$$

Since contact is lost at A, $R=0$

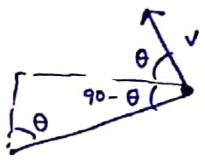
$$\Rightarrow \frac{3mg}{5} = mg\cos\theta$$

$$\Rightarrow \cos\theta = \frac{3}{5}.$$

Then $u^2 = \frac{13ag}{5} + 2ag\left(\frac{3}{5}\right) = \frac{13ag}{5} + \frac{6ag}{5} = \frac{19ag}{5}.$

$$\therefore u = \sqrt{\frac{19ag}{5}}.$$

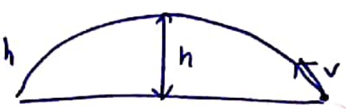
$$V_y = V\sin\theta = \sqrt{\frac{3ag}{5}}\times\frac{4}{5} = \frac{4}{5}\sqrt{\frac{3ag}{5}}.$$



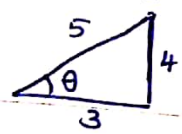
For maximum height, $v_y = 0$

Use $v^2 = u^2 + 2as$

$$0^2 = \left(\frac{4}{5}\sqrt{\frac{3ag}{5}}\right)^2 + 2(-g)h$$



$\cos\theta = \frac{3}{5}$

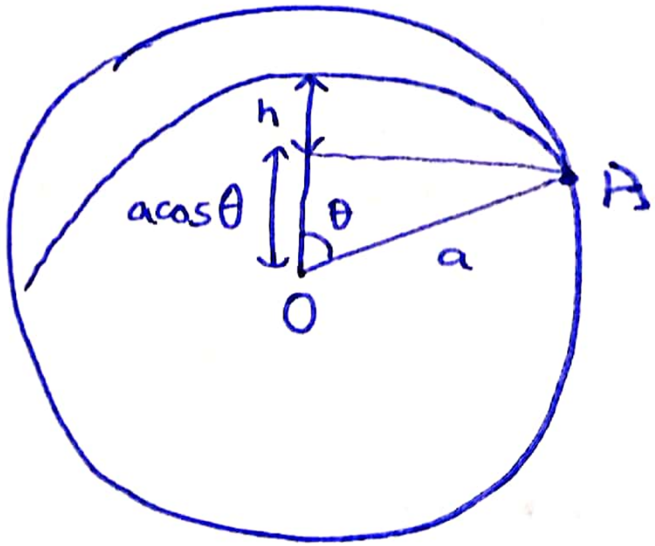


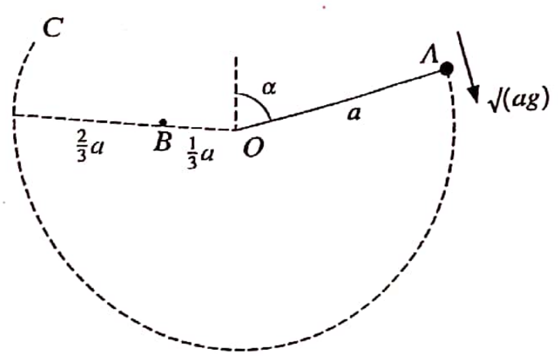
$\Rightarrow \sin\theta = \frac{4}{5}.$

$$h = \frac{\frac{16}{25}\left(\frac{3ag}{5}\right)}{2g} = \frac{24a}{125}.$$

~~Grete~~ Greatest height reached above O

$$= a\cos\theta + h = a\left(\frac{3}{5}\right) + \frac{24a}{125} = \frac{99a}{125}.$$





A particle of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The point A is such that $OA = a$ and OA makes an angle α with the upward vertical through O. The particle is held at A and then projected downwards with speed $\sqrt{(ag)}$ so that it begins to move in a vertical circle with centre O. There is a small smooth peg at the point B which is at the same horizontal level as O and at a distance $\frac{1}{3}a$ from O on the opposite side of O to A (see diagram).

(i) Show that, when the string first makes contact with the peg, the speed of the particle is $\sqrt{(ag(1 + 2\cos\alpha))}$. [2]

$(K.E. + P.E.)_A = (K.E. + P.E.)_D$

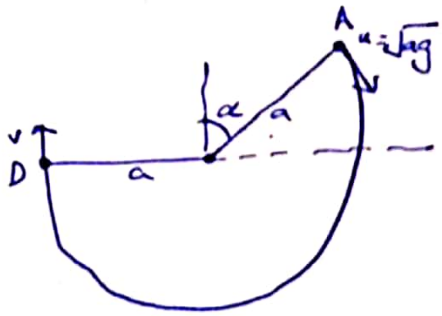
$\frac{m(ag)}{2} + mga\cos\alpha = \frac{mv^2}{2} + mg(0)$

$\frac{mga}{2} + mga\cos\alpha = \frac{mv^2}{2}$

$ga + 2ga\cos\alpha = v^2$

$ag(1 + 2\cos\alpha) = v^2$

$\Rightarrow v = \sqrt{ag(1 + 2\cos\alpha)}.$



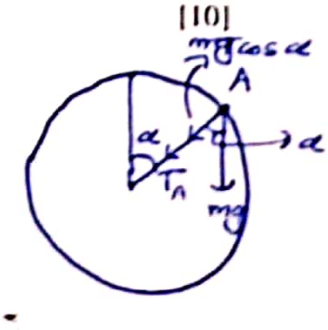
The particle now begins to move in a vertical circle with centre B. When the particle is at the point C where angle $CBO = 150°$, the tension in the string is the same as it was when the particle was at the point A.

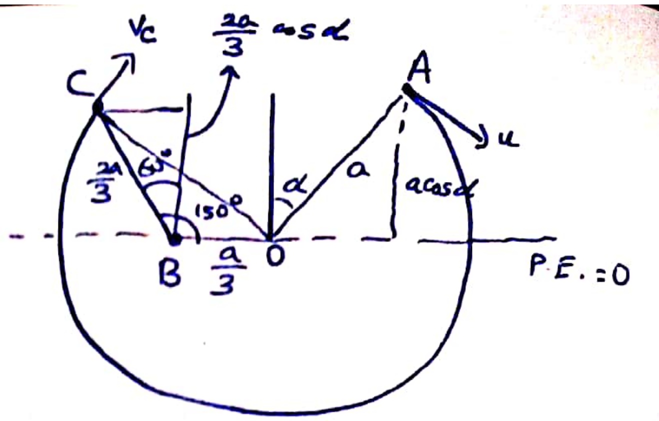
(ii) Find the value of $\cos\alpha$. [10]

R(towards center) at A:

$T_A + mg\cos\alpha = \frac{m(ag)}{a}$

$T_A = mg - mg\cos\alpha = mg(1 - \cos\alpha).$





$$(K.E + P.E.)_A = (K.E. + P.E.)_C$$

$$\frac{mu^2}{2} + mg(a\cos\alpha) = \frac{mv_c^2}{2} + mg\left(\frac{2a}{3}\cos 60°\right)$$

$$\frac{u^2}{2} + ga\cos\alpha = \frac{v_c^2}{2} + \frac{2ga}{3}\left(\frac{1}{2}\right)$$

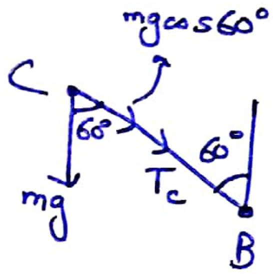
$$u^2 + 2ga\cos\alpha = v_c^2 + \frac{2ga}{3}$$

Substituting $u = \sqrt{ag}$ gives:

$$ag + 2ga\cos\alpha = v_c^2 + \frac{2ga}{3}$$

$$v_c^2 = 2ga\cos\alpha + \frac{ga}{3}.$$

R(towards center) at B gives:



$$T_c + mg\cos 60° = \frac{mv_c^2}{r}$$

$$T_c + mg\left(\frac{1}{2}\right) = \frac{m\left[2ga\cos\alpha + \frac{ga}{3}\right]}{\frac{2a}{3}}$$

$$T_c = \frac{3mg\left(2\cos\alpha + \frac{1}{3}\right)}{2} - \frac{mg}{2}$$

Since $T_c = T_A$

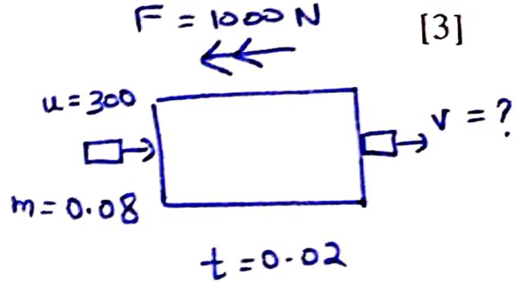
$$\therefore \frac{3mg\left(2\cos\alpha + \frac{1}{3}\right)}{2} - \frac{mg}{2} = mg(1-\cos\alpha) \Rightarrow \frac{mg}{2}\left[6\cos\alpha + 1 - 1\right] = mg(1-\cos\alpha)$$

$$6\cos\alpha \cancel{-1} = 2 - 2\cos\alpha$$

$$8\cos\alpha = \cancel{3}\ 2$$

$$\cos\alpha = \frac{2}{8} = \frac{1}{4}.$$

15 A bullet of mass 0.08 kg is fired horizontally into a fixed vertical barrier. It enters the barrier horizontally with speed $300\ \mathrm{m\,s^{-1}}$ and emerges horizontally after 0.02 s. There is a constant horizontal resisting force of magnitude 1000 N. Find the speed with which the bullet emerges from the barrier. [3]



$$F = \frac{mv - mu}{t}$$

$$-1000 = \frac{0.08(v) - 0.08(300)}{0.02}$$

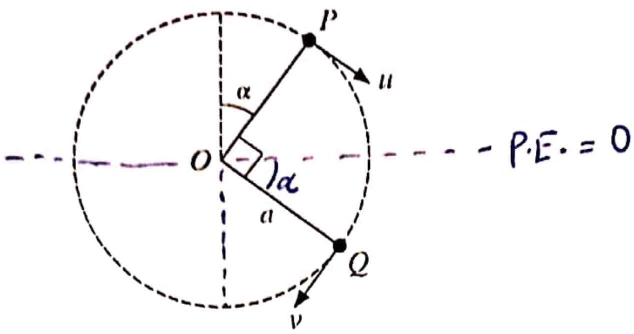
$$-1000(0.02) = 0.08v - 24$$

$$-20 = 0.08v - 24$$

$$4 = 0.08v$$

$$\frac{4}{0.08} = v$$

$$v = 50\ \mathrm{ms^{-1}}.$$



A particle of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle is moving in complete vertical circles with the string taut. When the particle is at the point P, where OP makes an angle α with the upward vertical through O, its speed is u. When the particle is at the point Q, where angle $QOP = 90°$, its speed is v (see diagram). It is given that $\cos\alpha = \frac{4}{5}$.

(i) Show that $v^2 = u^2 + \frac{14}{5}ag$. [2]

$$(P.E. + K.E.)_P = (P.E. + K.E.)_Q$$

$$mga\cos\alpha + \frac{mu^2}{2} = mg(-a\sin\alpha) + \frac{mv^2}{2}$$

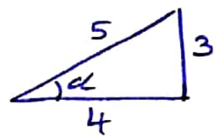
$$ag\cos\alpha + \frac{u^2}{2} = -ga\sin\alpha + \frac{v^2}{2}$$

$$2ag\cos\alpha + u^2 = -2ga\sin\alpha + v^2$$

$$v^2 = u^2 + 2ag(\cos\alpha + \sin\alpha)$$

$$v^2 = u^2 + 2ag\left(\frac{4}{5} + \frac{3}{5}\right) = u^2 + 2ag\left(\frac{7}{5}\right) = u^2 + \frac{14ag}{5}.$$

$\cos\alpha = \frac{4}{5}$



$\therefore \sin\alpha = \frac{3}{5}.$

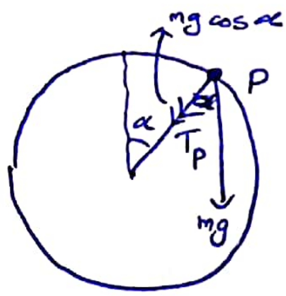
The tension in the string when the particle is at Q is twice the tension in the string when the particle is at P.

(ii) Obtain another equation relating u^2, v^2, a and g, and hence find u in terms of a and g. [5]

R (towards center) at P:

$$T_P + mg\cos\alpha = \frac{mu^2}{a}$$

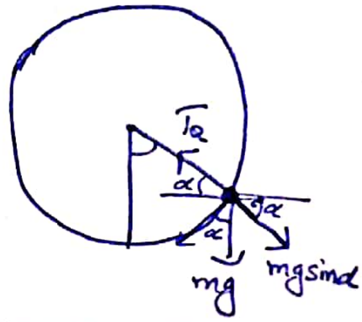
$$T_P = \frac{mu^2}{a} - mg\left(\frac{4}{5}\right) = \frac{mu^2}{a} - \frac{4mg}{5}.$$



R (towards center) at Q:

$$T_Q - mg\sin\alpha = \frac{mv^2}{a}$$

$$T_Q = \frac{mv^2}{a} + mg\left(\frac{3}{5}\right) = \frac{mv^2}{a} + \frac{3mg}{5}.$$



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$$T_Q = 2T_p$$

$$\frac{mv^2}{a} + \frac{3mg}{5} = 2\left(\frac{mu^2}{a} - \frac{4mg}{5}\right) \Rightarrow \frac{mv^2}{a} = \frac{2mu^2}{a} - \frac{8mg}{5} - \frac{3mg}{5}$$

$$\frac{v^2}{a} = \frac{2u^2}{a} - \frac{11g}{5} \Rightarrow v^2 = 2u^2 - \frac{11ag}{5}.$$

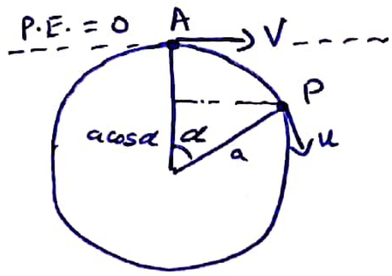
$$\therefore \quad v^2 = u^2 + \frac{14ag}{5} = 2u^2 - \frac{11ag}{5}$$

$$\therefore \quad \frac{25ag}{5} = u^2$$

$$\Rightarrow \quad u^2 = 5ag.$$

$$u = \sqrt{5ag}.$$

(iii) Find the least tension in the string during the motion. [3]



$$(P.E. + K.E.)_A = (P.E. + K.E.)_P$$

$$mg(0) + \frac{mV^2}{2} = mg\left[-a(1-\cos\alpha)\right] + \frac{mu^2}{2}$$

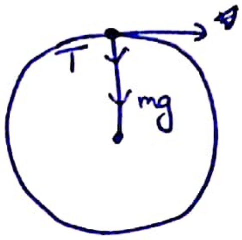
$$\frac{V^2}{2} = ga(\cos\alpha - 1) + \frac{u^2}{2}$$

$$V^2 = 2ga(\cos\alpha - 1) + u^2$$

$$V^2 = 2ga\cos\alpha - 2ga + 5ga$$

$$V^2 = 2ga\cos\alpha + 3ga = 2ga\left(\frac{4}{5}\right) + 3ga = \frac{23ga}{5}.$$

R (towards center) at highest point:



$$T_{min} + mg = \frac{mV^2}{a}$$

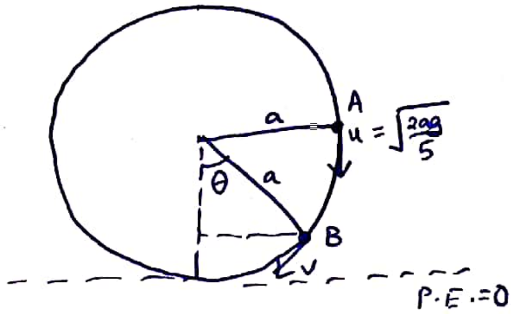
$$T_{min} + mg = \frac{m \times \frac{23ga}{5}}{a}$$

$$T_{min} = \frac{18mg}{5}.$$

17 A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle is held so that the string is taut, with OP horizontal. The particle is projected downwards with speed $\sqrt{\left(\frac{2}{5}ag\right)}$ and begins to move in a vertical circle. The string breaks when its tension is equal to $\frac{11}{5}mg$.

(i) Show that the string breaks when OP makes an angle θ with the downward vertical through O, where $\cos\theta = \frac{3}{5}$. Find the speed of P at this instant. [6]

$$(K.E.+P.E.)_A = (K.E.+P.E.)_B$$



$$\frac{mu^2}{2} + mg(a) = \frac{mv^2}{2} + mg(a)(1-\cos\theta)$$

$$\frac{u^2}{2} + ga = \frac{v^2}{2} + ga - ga\cos\theta$$

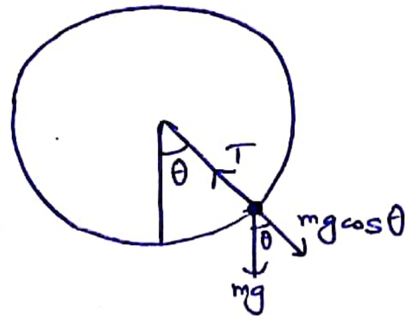
$$u^2 + 2ga = v^2 + 2ga - 2ga\cos\theta$$

$$u^2 = v^2 - 2ga\cos\theta$$

$$\Rightarrow v^2 = u^2 + 2ga\cos\theta.$$

R (towards center) at B:

$$T - mg\cos\theta = \frac{mv^2}{a}$$



$$T - mg\cos\theta = \frac{m}{a}\left[u^2 + 2ga\cos\theta\right]$$

$$T = \frac{mu^2}{a} + 2mg\cos\theta + mg\cos\theta$$

$$T = \frac{mu^2}{a} + 3mg\cos\theta.$$

For spring to break, $T = \frac{11mg}{5}$ and $u^2 = \frac{2ag}{5}$.

$$\frac{11mg}{5} = \frac{m}{a}\left(\frac{2ag}{5}\right) + 3mg\cos\theta$$

$$\frac{9mg}{5} = 3mg\cos\theta$$

$$\cos\theta = \frac{9mg/5}{3mg}$$

$$\cos\theta = \frac{3}{5}.$$

$$v^2 = u^2 + 2ga\cos\theta = \frac{2ga}{5} + 2ga\left(\frac{3}{5}\right)$$

$$v^2 = \frac{8ga}{5} \Rightarrow v = \sqrt{\frac{8ga}{5}}$$

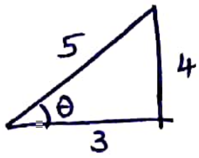
i.e. Speed of P when string breaks $= \sqrt{\frac{8ga}{5}}$.

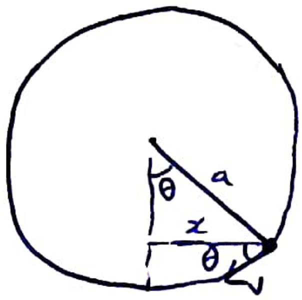
(ii) For the subsequent motion after the string breaks, find the distance OP when the particle P is vertically below O. [6]

$V_x = v\cos\theta = v \times \frac{3}{5} = \sqrt{\frac{8ga}{5}} \times \frac{3}{5} = \frac{3}{5}\sqrt{\frac{8ga}{5}}$.

$\cos\theta = \frac{3}{5}$.

$V_y = v\sin\theta = v \times \frac{4}{5} = \sqrt{\frac{8ga}{5}} \times \frac{4}{5} = \frac{4}{5}\sqrt{\frac{8ga}{5}}$.



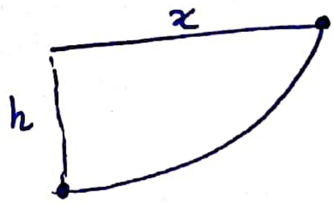


For particle to be vertically below O:

Horizontal distance covered: $x = a\sin\theta$

$\therefore \sin\theta = \frac{4}{5}$.

$= a\left(\frac{4}{5}\right) = \frac{4a}{5}$.



Using $x = V_x \times t$

$$t = \frac{x}{V_x} = \frac{\frac{4a}{5}}{\frac{3}{5}\sqrt{\frac{8ga}{5}}} = \frac{4a}{3\sqrt{\frac{8ga}{5}}}$$

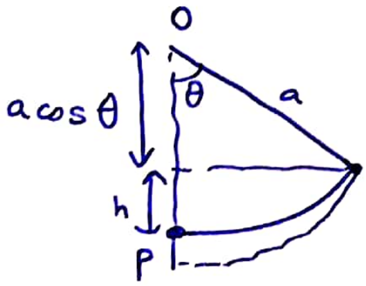
Vertically, using $s = ut + \frac{1}{2}at^2$ gives:

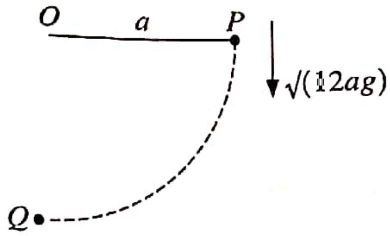
$$h = \left(\frac{4}{5}\sqrt{\frac{8ga}{5}}\right)\left[\frac{4a}{3\sqrt{\frac{8ga}{5}}}\right] + \frac{1}{2}g\left[\frac{4a}{3\sqrt{\frac{8ga}{5}}}\right]^2$$

$$h = \frac{16a}{15} + \frac{5a}{9} = \frac{73a}{45}$$

$$\therefore OP = a\cos\theta + h = a\left(\frac{3}{5}\right) + \frac{73a}{45} = \frac{100a}{45}$$

$$= \frac{20a}{9}.$$



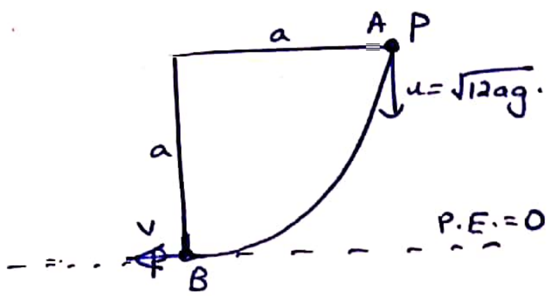


A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle is held with the string taut and horizontal. It is projected downwards with speed $\sqrt{(12ag)}$. At the lowest point of its motion, P collides directly with a particle Q of mass km which is at rest (see diagram). In the collision, P and Q coalesce. The tension in the string immediately after the collision is half of its value immediately before the collision. Find the possible values of k. [11]

$$(K.E. + P.E.)_A = (K.E. + P.E.)_B$$

$$\frac{m}{2} \times 12ag + mga = \frac{mv^2}{2} + mg(0)$$

$$7ag = \frac{v^2}{2} \Rightarrow v^2 = 14ag.$$

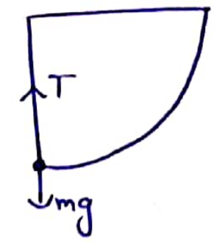


R (towards center) at B :

$$T - mg = \frac{mv^2}{a}$$

$$T = \frac{m(14ag)}{a} + mg = 15mg.$$

Then tension is 15mg before collision.

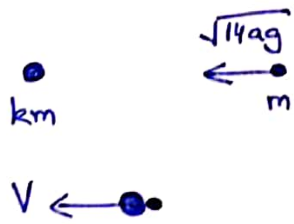


Using principle of conservation of momentum,

$$m(\sqrt{14ag}) + km(0) = (m+km)V$$

$$\frac{\sqrt{14ag}\,m}{m+km} = V$$

$$V = \frac{\sqrt{14ag}}{(1+k)}.$$

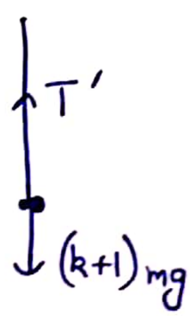


R (towards center) immediately after collision:

$$T' - (k+1)mg = \frac{(k+1)m \times V^2}{a}$$

$$T' = (k+1)mg + \frac{(k+1)m}{a}\left(\frac{\sqrt{14ag}}{1+k}\right)^2$$

$$T' = (k+1)mg + \frac{m \times 14ag}{a(1+k)} = (k+1)mg + \frac{14mg}{(k+1)}$$



Since $T' = \frac{1}{2} T$

$$(k+1)mg + \frac{14mg}{k+1} = \frac{1}{2}\left[15mg\right]$$

$$mg\left[k+1+\frac{14}{k+1}\right] = \frac{15}{2}mg$$

$$\frac{(k+1)^2+14}{k+1} = \frac{15}{2}$$

$$2(k^2+2k+15) = 15k+15$$

$$2k^2+4k+30 = 15k+15$$

$$2k^2-11k+15 = 0$$

$$2k^2-6k-5k+15 = 0$$

$$(2k-5)(k-3) = 0$$

$$k = \frac{5}{2}, 3.$$

19 A particle P, of mass m, is able to move in a vertical circle on the smooth inner surface of a sphere with centre O and radius a. Points A and B are on the inner surface of the sphere and AOB is a horizontal diameter. Initially, P is projected vertically downwards with speed $\sqrt{\left(\frac{21}{2}ag\right)}$ from A and begins to move in a vertical circle. At the lowest point of its path, vertically below O, the particle P collides with a stationary particle Q, of mass $4m$, and rebounds. The speed acquired by Q, as a result of the collision, is just sufficient for it to reach the point B.

(i) Find the speed of P and the speed of Q immediately after their collision. [7]

Using principal of conservation of energy,

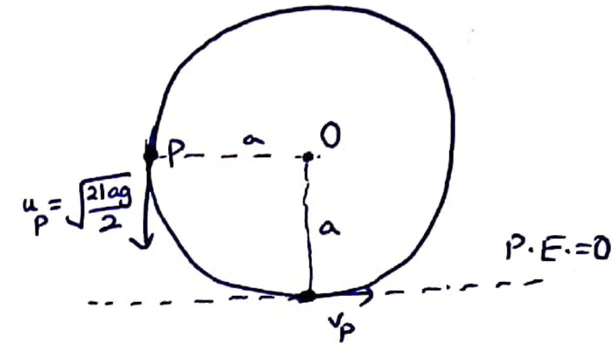
$$mg(a) + \frac{m}{2}(u_P)^2 = mg(0) + \frac{m}{2}v_P^2$$

$$mga + \frac{m}{2}\left(\frac{21ag}{2}\right) = \frac{m}{2}v_P^2$$

$$ga + \frac{21ga}{4} = \frac{v_P^2}{2}$$

$$\frac{25ga}{2} = v_P^2$$

$$\Rightarrow v_P = \sqrt{\frac{25ga}{2}}.$$



i.e. P reaches lowest point with speed $\sqrt{\frac{25ga}{2}}$.

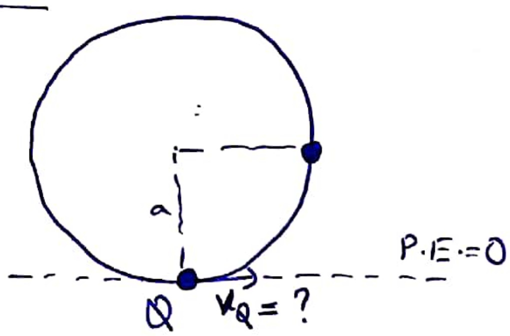
Using principle of conservation of energy,

$$4mg(0) + 4m \times \frac{v_Q^2}{2} = 4mga + \frac{4m \times 0^2}{2}$$

$$2m v_Q^2 = 4mga$$

$$v_Q^2 = 2ga$$

$$v_Q = \sqrt{2ga}.$$



$\therefore$ After collision, speed of $Q = \sqrt{2ga}$.

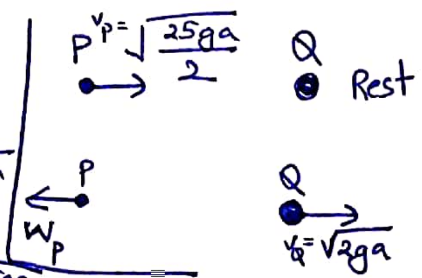
Using the principle of conservation of momentum,

$$m\left(\sqrt{\frac{25ga}{2}}\right) + 4m(0) = m(-w_P) + 4m\sqrt{2ga}$$

$$m w_P = 4m\sqrt{2ga} - m\sqrt{\frac{25ga}{2}}$$

$$w_P = 4\sqrt{2ga} - \sqrt{\frac{25ga}{2}} = \left(4\sqrt{2} - \frac{5}{\sqrt{2}}\right)\sqrt{ga} = \frac{3}{\sqrt{2}}\sqrt{ga}$$

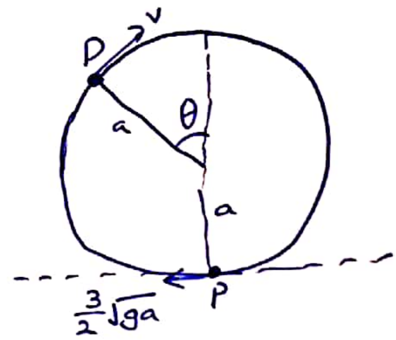
Collision



i.e. Speed of P after collision $= \frac{3}{\sqrt{2}}\sqrt{ga}$.

In its subsequent motion, P loses contact with the inner surface of the sphere at the point D, where the angle between OD and the upward vertical through O is θ.

(ii) Find $\cos\theta$. [5]



$$(K.E.+P.E.)_P = (K.E.+P.E.)_D$$

$$\frac{m}{2}\left(\frac{3}{\sqrt{2}}\sqrt{ga}\right)^2 + mg(0) = \frac{mv^2}{2} + mga(1+\cos\theta)$$

$$\frac{1}{2}\times\frac{9}{\cancel{2}}ga + 0 = \frac{v^2}{2} + ga(1+\cos\theta)$$

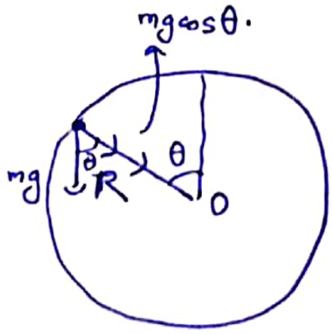
$$2\left(\frac{9ga}{\cancel{4}}\right) = v^2 + 2ga(1+\cos\theta)$$

$$v^2 = 2\left(\frac{9ga}{\cancel{4}}\right) - 2ga(1+\cos\theta)$$

$$v^2 = \frac{18ga}{\cancel{4}} - 2ga - 2ga\cos\theta$$

$$v^2 = \cancel{\frac{7ga}{8}}\ \frac{5ga}{\cancel{2}} - 2ga\cos\theta.$$

R(towards center) at O:



$$R + mg\cos\theta = \frac{mv^2}{a}$$

$$R = \frac{m}{a}\left[\frac{5ga}{\cancel{2}} - 2ga\cos\theta\right] - mg\cos\theta$$

$$R = \frac{5mg}{\cancel{2}} - 2mg\cos\theta - mg\cos\theta$$

$$R = \frac{5mg}{\cancel{2}} - 3mg\cos\theta.$$

Since at D, contact with surface is lost, $R = 0$

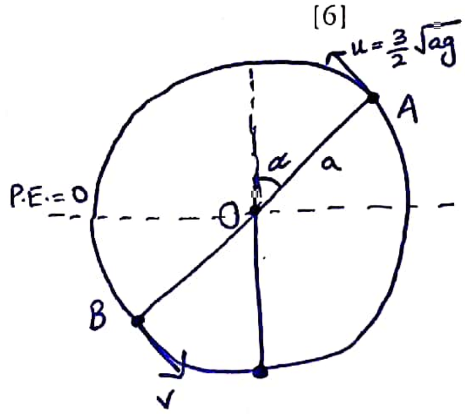
$$\Rightarrow \frac{5mg}{\cancel{2}} - 3mg\cos\theta = 0$$

$$\cos\theta = \frac{\frac{5mg}{\cancel{2}}}{3mg} = \frac{5}{\cancel{6}}.$$

20 A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle P is moving in a complete vertical circle about O. The points A and B are on the circle, at opposite ends of a diameter, and such that OA makes an acute angle α with the upward vertical through O. The speed of P as it passes through A is $\frac{3}{2}\sqrt{(ag)}$. The tension in the string when P is at B is four times the tension in the string when P is at A.

(i) Show that $\cos\alpha = \frac{3}{4}$. [6]

$$(P.E. + K.E.)_A = (P.E. + K.E.)_B$$



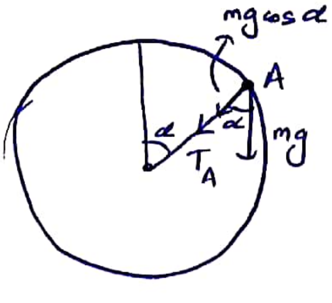
$$mg\,a\cos\alpha + \frac{1}{2}m\left(\frac{3}{2}\sqrt{ag}\right)^2 = mg(-a\cos\alpha) + \frac{mv^2}{2}$$

$$ga\cos\alpha + \frac{1}{2}\times\frac{9}{4}ag = -ga\cos\alpha + \frac{v^2}{2}$$

$$2ga\cos\alpha + \frac{9ga}{8} = \frac{v^2}{2}$$

$$v^2 = 4ga\cos\alpha + \frac{9ga}{4} = ga\left[4\cos\alpha + \frac{9}{4}\right].$$

R (towards center) at A:



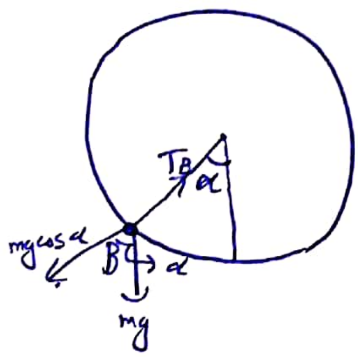
$$T_A + mg\cos\alpha = \frac{mu^2}{a}$$

$$T_A = \frac{m}{a}\left(\frac{3}{2}\sqrt{ag}\right)^2 - mg\cos\alpha$$

$$T_A = \frac{m}{a}\times\frac{9}{4}ag - mg\cos\alpha$$

$$T_A = \frac{9mg}{4} - mg\cos\alpha = mg\left(\frac{9}{4} - \cos\alpha\right).$$

R (towards center) at B:



$$T_B - mg\cos\alpha = \frac{mv^2}{a}$$

$$T_B = mg\cos\alpha + \frac{m}{a}\left[ga\left(4\cos\alpha + \frac{9}{4}\right)\right]$$

$$= mg\cos\alpha + mg\left(4\cos\alpha + \frac{9}{4}\right)$$

$$T_B = 5mg\cos\alpha + \frac{9mg}{4}$$

Since $T_B = 4T_A$

$$5mg\cos\alpha + \frac{9mg}{4} = 4\left(\frac{9mg}{4} - mg\cos\alpha\right)$$

$$mg\left[5\cos\alpha+\frac{9}{4}\right]=4mg\left[\frac{9}{4}-\cos\alpha\right]$$

$$5\cos\alpha+\frac{9}{4}=4\left(\frac{9}{4}-\cos\alpha\right)$$

$$5\cos\alpha+\frac{9}{4}=9-4\cos\alpha$$

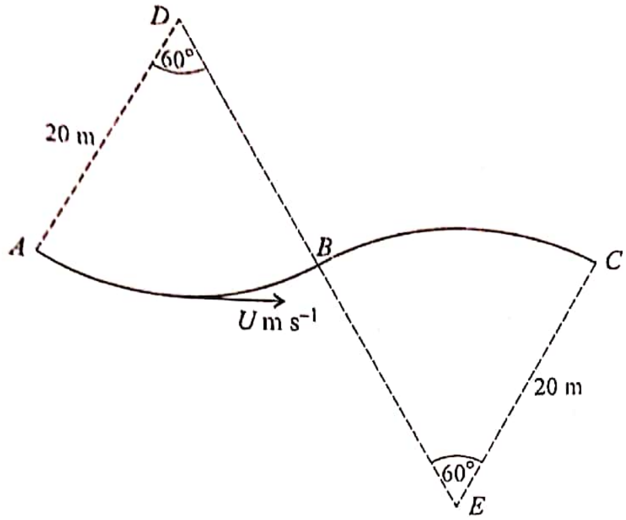
$$9\cos\alpha=\frac{27}{4}$$

$$\cos\alpha=\frac{3}{4}.$$

(ii) Find the tension in the string when P is at B. [2]

$$T_B=5mg\cos\alpha+\frac{9mg}{4}$$

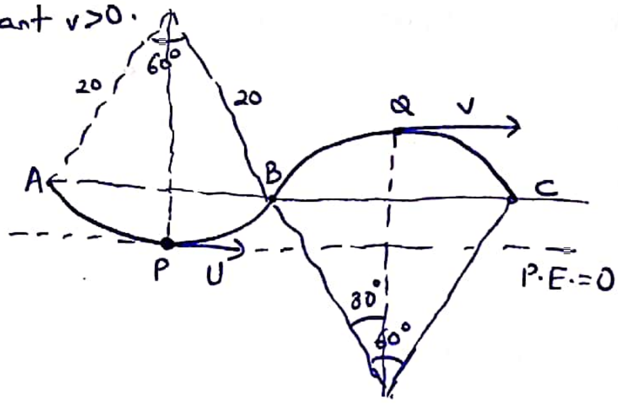
$$T_B=5mg\left(\frac{3}{4}\right)+\frac{9mg}{4}=6mg$$



A toboggan travels along the path ABC shown in the diagram. The path lies in a vertical plane, and consists of two circular arcs AB and BC. The arcs are of radius 20 m, and subtend angles of 60° at their centres, D and E respectively. The line ABC is horizontal, and there is no friction between the toboggan and the snow. Air resistance is negligible, and the toboggan may be treated as a particle. The speed of the toboggan at its lowest point is $U \text{ m s}^{-1}$. Find the range of values of U for which the toboggan will reach C without losing contact with the snow. [10]

Find also the value of U for which the toboggan will leave the snow at B, and travel as a projectile until it lands again at C. [4]

At the highest point on circular arc BC, we want $v>0$.



$$(P.E.+K.E.)_P = (P.E.+K.E.)_Q$$

$$mg(0) + \frac{m}{2}(U^2) = mg[2\times 20(1-\cos 30°)] + \frac{mv^2}{2}$$

$$\frac{U^2}{2} = g(40)(1-\cos 30°) + \frac{v^2}{2}$$

$$v^2 = U^2 - 80g(1-\cos 30°)$$

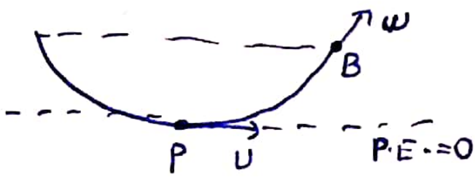
Since $v>0$, $v^2>0$

$$\Rightarrow U^2 - 80g(1-\cos 30°) > 0$$

$$U > \sqrt{80g(1-\cos 30°)} = 10.35$$

$$U > 10.4 \text{ ms}^{-1}.$$

Secondly, we want the particle to not escape at B i.e. we want $R>0$ at B.



$$(P.E.+K.E.)_P = (P.E.+K.E.)_B$$

$$mg(0) + \frac{mU^2}{2} = mg[20(1-\cos 30°)] + \frac{m\omega^2}{2}$$

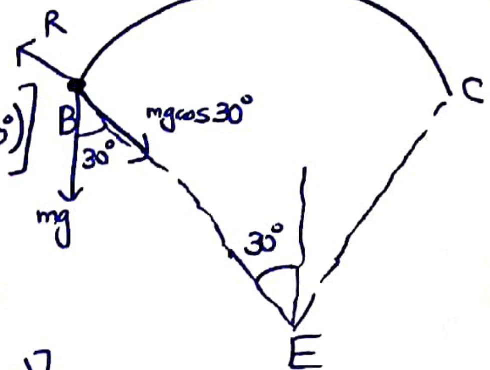
$$\frac{U^2}{2} = 20g(1-\cos 30°) + \frac{\omega^2}{2}$$

$$\Rightarrow \omega^2 = U^2 - 40g(1-\cos 30°)$$

R(towards center) with center at E:

$$mg\cos 30° - R = \frac{m\omega^2}{20}$$

$$R = mg\cos 30° - \frac{m}{20}\left[U^2 - 40g(1-\cos 30°)\right]$$



Since at B, we want $R > 0$

$$\therefore \quad mg\cos 30° - \frac{m}{20}\left[U^2 - 40g(1-\cos 30°)\right] > 0$$

$$\cancel{m}g\cos 30° > \frac{\cancel{m}}{20}\left[U^2 - 40g(1-\cos 30°)\right]$$

$$20g\cos 30° > U^2 - 40g(1-\cos 30°)$$

$$\sqrt{20g\cos 30° + 40g(1-\cos 30°)} > U$$

$$\therefore U < 15.05$$

$$U < 15.1$$

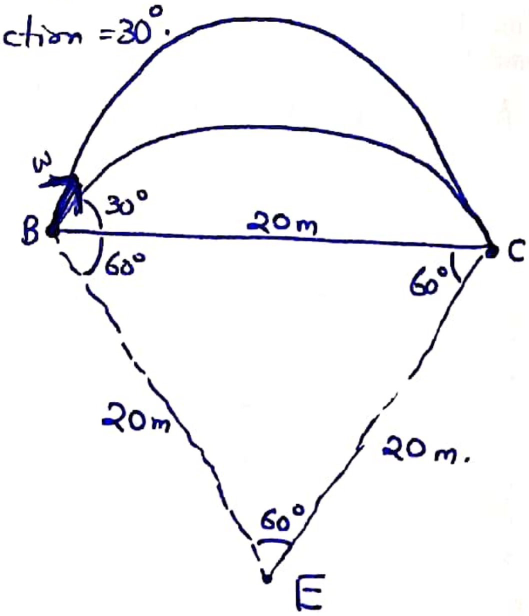
$\therefore$ toboggan reaches C without losing contakt if $10.4 < U < 15.1$

Note ΔBEC is equilateral $\therefore BC = 20\,m$.

We want toboggan to lose contact at B and land at C.

Then we want a range $= 20\,m$. Note angle of projection $= 30°$.

Using $\quad R = \dfrac{v^2 \sin 2\theta}{g}$



$$20 = \frac{\omega^2 \sin(2\times 30°)}{g}$$

$$\omega^2 = \frac{20g}{\sin 60°}$$

We know $\omega^2 = U^2 - 40g(1-\cos 30°)$, Then:

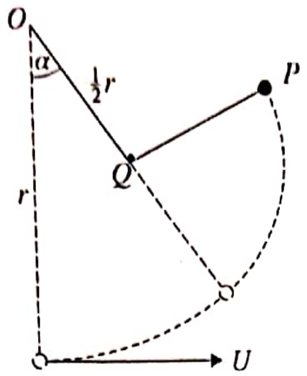
$$U^2 - 40g(1-\cos 30°) = \frac{20g}{\sin 60°}$$

$$U^2 = 40g(1-\cos 30°) + \frac{20g}{\sin 60°}$$

$$U = \sqrt{40g(1-\cos 30°) + \frac{20g}{\sin 60°}}.$$

$$U = 16.86\,ms^{-1} \approx 16.9\,ms^{-1}.$$

22



A particle P of mass m is connected to a fixed point O by a light inextensible string OP of length r, and is moving in a vertical circle, centre O. At its lowest point, P has speed U. When the string makes an angle α with the downward vertical it encounters a small fixed peg Q, where $OQ = \frac{1}{2}r$. The string proceeds to wrap itself around the peg, so that P begins to move in a vertical circle with centre Q (see diagram). Given that the particle describes a complete circle about Q, show that

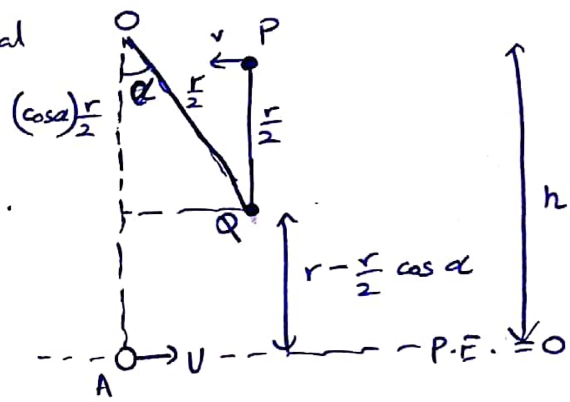
$$U^2 \geqslant gr(\tfrac{7}{2} - \cos\alpha).$$ [7]

Consider P at its highest point in the vertical circle about Q.

At this position,

$$h = r - \frac{r}{2}\cos\alpha + \frac{r}{2} = \frac{3r}{2} - \frac{r}{2}\cos\alpha.$$

$$h = \frac{r}{2}(3 - \cos\alpha).$$



$$(K.E. + P.E.)_A = (K.E. + P.E.)_P$$

$$\frac{1}{2}mU^2 + mg(0) = \frac{1}{2}mv^2 + mg\left[\frac{r}{2}(3 - \cos\alpha)\right]$$

$$\frac{U^2}{2} = \frac{v^2}{2} + \frac{gr}{2}(3 - \cos\alpha)$$

$$U^2 = v^2 + gr(3 - \cos\alpha)$$

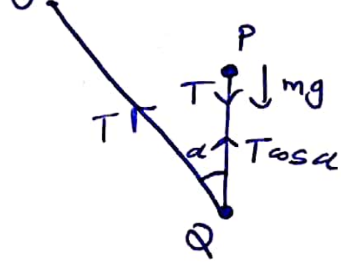
$$v^2 = U^2 - gr(3 - \cos\alpha).$$

R(towards center):

$$T + mg - T\cos\alpha = \frac{mv^2}{\frac{r}{2}}$$

$$T(1 - \cos\alpha) + mg = \frac{2mv^2}{r}$$

$$T = \frac{\frac{2mv^2}{r} - mg}{1 - \cos\alpha}$$



For complete circles, we require $T \geq 0$ at the highest point

$$\Rightarrow \frac{\frac{2mv^2}{r} - mg}{1-\cos\alpha} \geq 0 \Rightarrow \frac{2mv^2}{r} - mg \geq 0$$

$$\frac{2mv^2}{r} \geq mg$$

$$v^2 \geq \frac{gr}{2}$$

Substituting $v^2 = U^2 - gr(3-\cos\alpha)$

$$\therefore \quad U^2 - gr(3-\cos\alpha) \geq \frac{gr}{2}$$

$$U^2 \geq gr(3-\cos\alpha) + \frac{gr}{2}$$

$$U^2 \geq gr\left[3 - \cos\alpha + \frac{1}{2}\right]$$

$$U^2 \geq gr\left[\frac{7}{2} - \cos\alpha\right].$$

23 H is a point on horizontal ground and C is the point vertically above H such that $CH = r$. One end of a light inextensible string of length r is attached to C and a particle of mass m is attached to the other end. When the particle is hanging in equilibrium at H, it is given a horizontal speed U.

(i) Show that when the string makes an angle θ with the downward vertical the tension is

$$m\left[\frac{U^2}{r} - g(2 - 3\cos\theta)\right].$$ [4]

(ii) Deduce that, for the particle to describe a complete circle,

$$U^2 \geqslant 5gr.$$ [2]

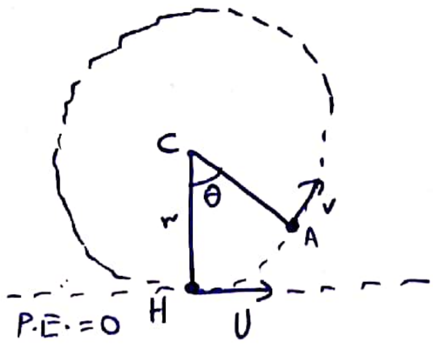
It is given that $U^2 = 6gr$ and that when the particle reaches its highest point it becomes detached from the string. The particle then travels freely until it reaches the ground at the point K. Show that $HK = (2\sqrt{2})r$. [5]

$$(K.E. + P.E.)_H = (K.E. + P.E.)_A$$

$$\frac{1}{2}mU^2 + mg(0) = \frac{1}{2}mv^2 + mgr(1-\cos\theta)$$

$$\frac{U^2}{2} = \frac{v^2}{2} + gr(1-\cos\theta)$$

$$v^2 = U^2 - 2gr(1-\cos\theta)$$



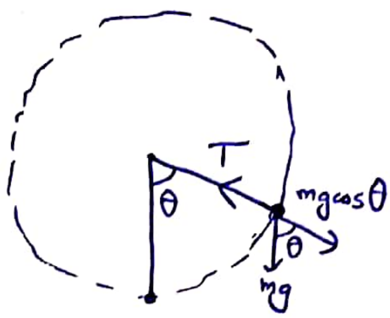
R(towards center) at A:

$$T - mg\cos\theta = \frac{mv^2}{r}$$

$$T = mg\cos\theta + \frac{m}{r}\left[U^2 - 2gr(1-\cos\theta)\right]$$

$$= \frac{mU^2}{r} + mg\cos\theta - 2mg(1-\cos\theta)$$

$$= \frac{mU^2}{r} + mg\left[\cos\theta - 2(1-\cos\theta)\right]$$



$$T = \frac{mU^2}{r} + mg\left[3\cos\theta - 2\right] = m\left[\frac{U^2}{r} - g(2-3\cos\theta)\right].$$

ii) To describe a complete circle, $T \geqslant 0$ when $\theta = 180°$

$$\Rightarrow m\left[\frac{U^2}{r} - g(2 - 3\cos 180°)\right] \geq 0$$

$$\frac{U^2}{r} - g(5) \geq 0$$

$$U^2 \geqslant 5gr$$

At the highest point, $\theta = 180^\circ$

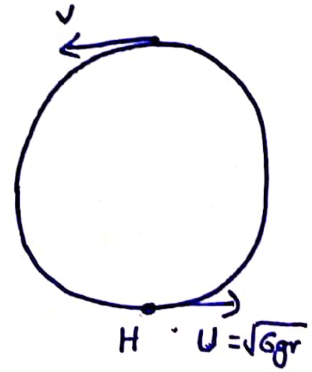
$$\therefore v^2 = u^2 - 2gr(1-\cos\theta)$$

$$= 6gr - 2gr(1-\cos 180^\circ)$$

$$v^2 = 6gr - 2gr(2) = 2gr$$

$$v = \sqrt{2gr}$$

i.e. Speed at highest point $= \sqrt{2gr}$.



Vertically, $v_{oy} = 0$, $s = 2r$, $a = g$.

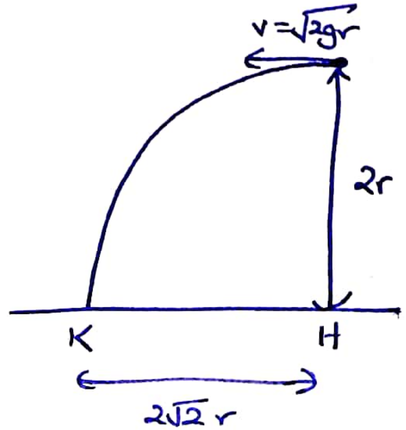
Using $s = ut + \frac{1}{2}at^2$

$$2r = 0(t) + \frac{1}{2}gt^2$$

$$2r = \frac{gt^2}{2}$$

$$\sqrt{\frac{4r}{g}} = t.$$

i.e. Particle takes time $= \sqrt{\frac{4r}{g}}$ s to reach horizontal surface.

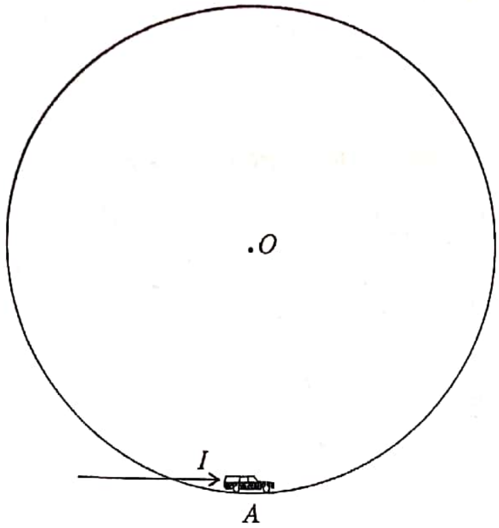


Horizontally, $v_x = x_{ox} = \sqrt{2gr}$, $t = \sqrt{\frac{4r}{g}}$

$$x = v_x \times t = \sqrt{2gr} \times \sqrt{\frac{4r}{g}} = \sqrt{\frac{8gr^2}{g}} = \sqrt{8}\,r = 2\sqrt{2}\,r.$$

i.e. $HK = 2\sqrt{2}\,r$

Scanned with CamScanner



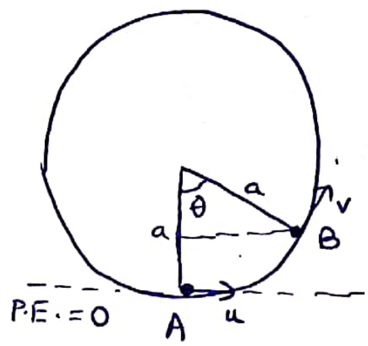
The diagram shows a toy car of mass m which can move freely on a circular track of radius a and centre O. The track is fixed with its plane vertical. The car is given a horizontal impulse of magnitude I when it is at rest at the lowest point A of the track. The car may be modelled as a particle moving without resistance. Show that the force that the track exerts on the car when the radius to the car makes an angle θ with OA has magnitude

$$mg(3\cos\theta - 2) + \frac{I^2}{ma}. \quad [6]$$

Find the set of values of I for which the car does not lose contact with the track. [4]

$I = mu$

$\therefore \quad u = \frac{I}{m}$.



$(K.E. + P.E.)_A = (K.E. + P.E.)_B$

$\frac{1}{2}mu^2 + mg(0) = \frac{1}{2}mv^2 + mg\left[a(1-\cos\theta)\right]$

$\frac{u^2}{2} = \frac{v^2}{2} + ga(1-\cos\theta)$

$u^2 = v^2 + 2ga(1-\cos\theta)$.

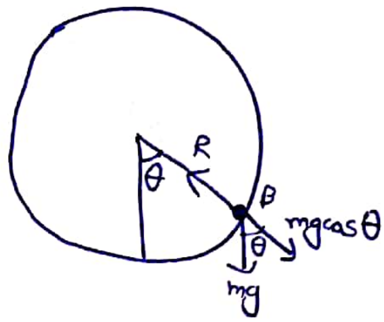
$v^2 = u^2 - 2ga(1-\cos\theta)$.

R (towards center) at B :

$R - mg\cos\theta = \frac{mv^2}{a}$

$R = mg\cos\theta + \frac{m}{a}\left[u^2 - 2ga(1-\cos\theta)\right]$

$R = mg\cos\theta + \frac{mu^2}{a} - 2mg(1-\cos\theta)$



$$R = mg\left[\cos\theta - 2(1-\cos\theta)\right] + \frac{mu^2}{a}$$

$$= mg\left[3\cos\theta - 2\right] + \frac{m}{a}\left[\frac{I}{m}\right]^2$$

$$R = mg(3\cos\theta - 2) + \frac{I^2}{ma}.$$

We require that $R > 0$ when $\theta = 180°$. This ensures that contact is not lost at the highest point:

$$mg(3\cos 180° - 2) + \frac{I^2}{ma} > 0$$

$$-5mg > \frac{-I^2}{ma}$$

$$5mg \times ma < I^2$$

$$5m^2ag < I^2$$

$$\sqrt{5m^2ag} < I$$

$$m\sqrt{5ag} < I$$

Secondly, we could also require that the toy car ~~doesn't~~ starts oscillating. This requires $v < 0$ when $\theta = 90°$ as this ensures that the car doesn't cross the horizontal.

$v < 0 \Rightarrow v^2 < 0 \Rightarrow \quad u^2 - 2ga(1-\cos 90°) < 0$

$$u^2 - 2ga < 0$$

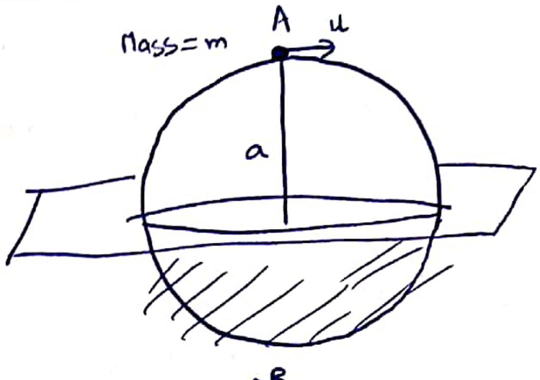
$$u < \sqrt{2ga}$$

Using $u = \frac{I}{m}$ gives $\frac{I}{m} < \sqrt{2ga}$

$$I < m\sqrt{2ga}.$$

25 A smooth hemisphere of radius a has its circular face fixed to the surface of a horizontal table. The centre of this face is O. A particle P of mass m is placed at the highest point of the hemisphere and is given a horizontal velocity u. The particle does not immediately lose contact with the surface of the hemisphere. Assuming all resistances to motion are neglected, find the set of possible values of u. [4]

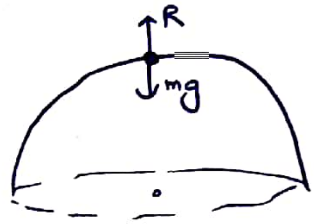
Show that P loses contact with the surface before OP becomes horizontal. [5]



R(towards center) at A:

$$mg - R = \frac{mu^2}{a}$$

$$R = mg - \frac{mu^2}{a}.$$



Since particle does not lose contact at highest point A,

$$R > 0$$

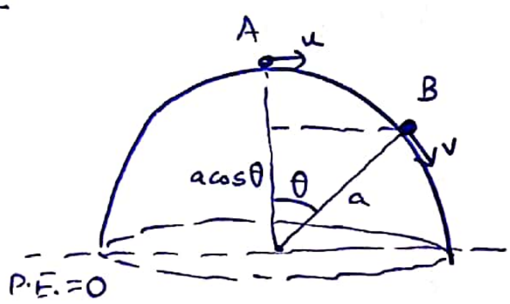
$$\therefore mg - \frac{mu^2}{a} > 0$$

$$mg > \frac{mu^2}{a}$$

$$ag > u^2$$

$$\sqrt{ag} > u$$

i.e. $u < \sqrt{ag}$.



$$(P.E. + K.E.)_A = (P.E. + K.E.)_B$$

$$mga + \frac{1}{2}mu^2 = mga\cos\theta + \frac{1}{2}mv^2$$

$$ga + \frac{u^2}{2} = ga\cos\theta + \frac{v^2}{2}$$

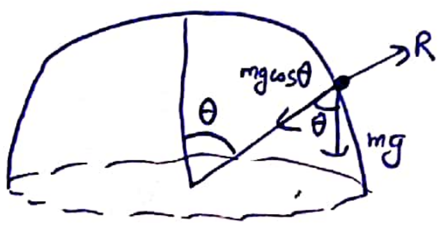
$$2ga + u^2 = 2ga\cos\theta + v^2$$

$$v^2 = u^2 + 2ga - 2ga\cos\theta.$$

R(towards center) at B:

$$mg\cos\theta - R = \frac{mv^2}{a}$$

$$R = mg\cos\theta - \frac{mv^2}{a}$$



For particle to lose contact, $R=0$

$$\Rightarrow mg\cos\theta - \frac{mv^2}{a} = 0$$

$$mg\cos\theta = \frac{mv^2}{a}$$

$$ga\cos\theta = v^2$$

$$ga\cos\theta = u^2+2ga-2ga\cos\theta$$

$$3ga\cos\theta = u^2+2ga$$

$$\cos\theta = \frac{u^2}{3ga} + \frac{2ga}{3ga}$$

$$\cos\theta = \frac{u^2}{3ga} + \frac{2}{3}$$

$$0 < u < \sqrt{ga} \Rightarrow 0 < u^2 < ga \Rightarrow 0 < \frac{u^2}{3ga} < \frac{ga}{3ga} \Rightarrow 0 < \frac{u^2}{3ga} < \frac{1}{3}.$$

Then $$0+\frac{2}{3} < \frac{u^2}{3ga} + \frac{2}{3} < \frac{1}{3} + \frac{2}{3}$$

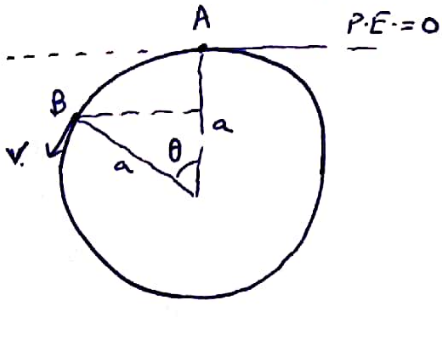
$$\frac{2}{3} < \frac{u^2}{3ga} + \frac{2}{3} < 1$$

$$\frac{2}{3} < \cos\theta < 1$$

$$0 < \theta < 48.18^\circ$$

Then particle loses contact before OP becomes horizontal.

26 A small bead B of mass m is threaded on a smooth wire fixed in a vertical plane. The wire forms a circle of radius a and centre O. The highest point of the circle is A. The bead is slightly displaced from rest at A. When angle $AOB = \theta$, where $\theta < \cos^{-1}(\frac{2}{3})$, the force exerted on the bead by the wire has magnitude R_1. When angle $AOB = \pi + \theta$, the force exerted on the bead by the wire has magnitude R_2. Show that $R_2 - R_1 = 4mg$. [8]



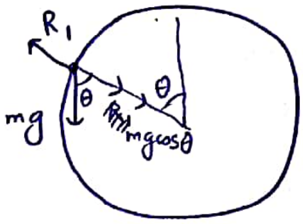
$$(P.E. + K.E.)_A = (P.E. + K.E.)_B$$

$$mg(0) + \frac{m}{2}(0)^2 = mg(-a(1-\cos\theta)) + \frac{mv^2}{2}$$

$$0 = mga(\cos\theta - 1) + \frac{mv^2}{2}$$

$$0 = ga(\cos\theta - 1) + \frac{v^2}{2}$$

$$v^2 = 2ga(1-\cos\theta).$$



R (towards center at B):

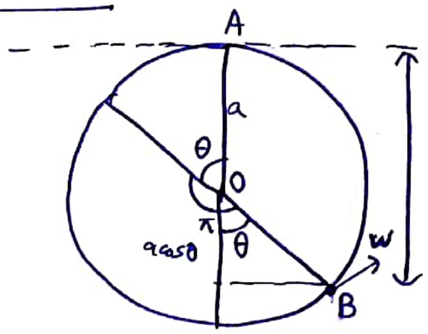
$$mg\cos\theta - R_1 \cancel{mg\cos\theta} = \frac{mv^2}{a}$$

$$R_1 = -\frac{mv^2}{a} + mg\cos\theta$$

$$= -\frac{m}{a}\left[2ga(1-\cos\theta)\right] + mg\cos\theta$$

$$= -2mg + 2mg\cos\theta + mg\cos\theta$$

$$R_1 = -2mg + 3mg\cos\theta = 3mg\cos\theta - 2mg.$$



$$(P.E. + K.E.)_A = (P.E. + K.E.)_B$$

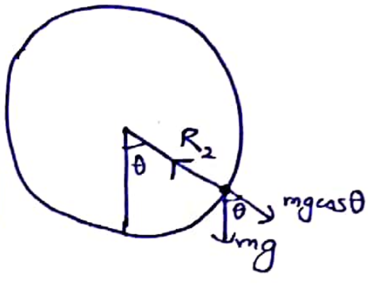
$$mg(0) + \frac{1}{2}m(0)^2 = mg[-a(1+\cos\theta)] + \frac{mw^2}{2}$$

$$0 = -mga(1+\cos\theta) + \frac{mw^2}{2}$$

$$\cancel{m}ga(1+\cos\theta) = \frac{\cancel{m}w^2}{2}$$

$$2ga(1+\cos\theta) = w^2$$

R (towards center at B):



$$R_2 - mg\cos\theta = \frac{mw^2}{a}$$

$$R_2 = mg\cos\theta + \frac{m}{a}\left[2ga(1+\cos\theta)\right]$$

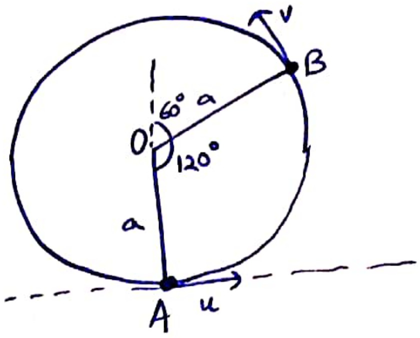
$$R_2 = mg\cos\theta + 2mg + 2mg\cos\theta$$

$$\Rightarrow R_2 = 3mg\cos\theta + 2mg.$$

$$\therefore R_2 - R_1 = (3mg\cos\theta + 2mg) - (3mg\cos\theta - 2mg)$$

$$= 4mg.$$

27 A particle of mass m is attached to one end A of a light inextensible string of length a. The other end of the string is attached to a fixed point O and the particle hangs in equilibrium under gravity. The particle is projected horizontally so that it starts to move in a vertical circle. The string slackens after turning through an angle of 120°. Show that the speed of the particle is then $\sqrt{(\frac{1}{2}ga)}$ and find the initial speed of projection. [5]



$$(K.E + P.E.)_A = (K.E + P.E.)_B$$

$$\frac{1}{2}mu^2 + mg(0) = \frac{1}{2}mv^2 + mg\left[a + a\cos 60°\right]$$

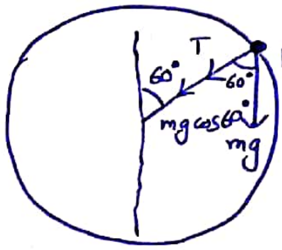
$$\frac{u^2}{2} = \frac{v^2}{2} + g\left[\frac{3a}{2}\right]$$

$$u^2 = v^2 + 3ag$$

$$v^2 = u^2 - 3ag.$$

R (towards center): $T + mg\cos 60° = \frac{mv^2}{a}$

$$T + mg\left(\frac{1}{2}\right) = \frac{m}{a}\left[\cancel{u^2 - 3ag}\right] \quad v^2$$



Since at B, string slackens, $T = 0$

$$\Rightarrow \quad 0 + \frac{mg}{2} = \frac{mv^2}{a}$$

$$\frac{ag}{2} = v^2$$

$$\Rightarrow \therefore v = \sqrt{\frac{ag}{2}}.$$

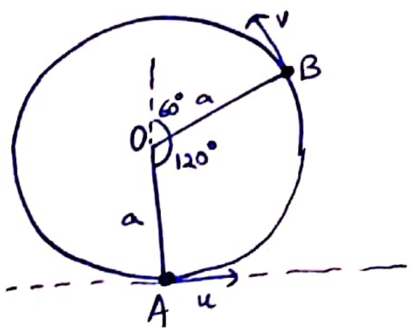
And $\quad u^2 = v^2 + 3ag$

$$u^2 = \frac{ag}{2} + 3ag$$

$$u^2 = \frac{7ag}{2}$$

$$u = \sqrt{\frac{7ag}{2}}.$$

27 A particle of mass m is attached to one end A of a light inextensible string of length a. The other end of the string is attached to a fixed point O and the particle hangs in equilibrium under gravity. The particle is projected horizontally so that it starts to move in a vertical circle. The string slackens after turning through an angle of $120°$. Show that the speed of the particle is then $\sqrt{(\frac{1}{2}ga)}$ and find the initial speed of projection. [5]



$$(K.E + P.E.)_A = (K.E. + P.E.)_B$$

$$\frac{1}{2}mu^2 + mg(0) = \frac{1}{2}mv^2 + mg\left[a + a\cos 60°\right]$$

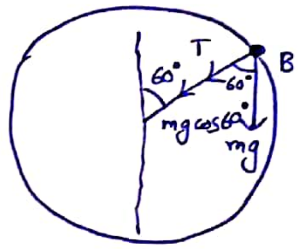
$$\frac{u^2}{2} = \frac{v^2}{2} + g\left[\frac{3a}{2}\right]$$

$$u^2 = v^2 + 3ag$$

$$v^2 = u^2 - 3ag.$$

R (towards center): $T + mg\cos 60° = \frac{mv^2}{a}$

$$T + mg\left(\frac{1}{2}\right) = \frac{m}{a}\left[\overset{v^2}{\cancel{u^2 - 3ag}}\right]$$



Since at B, string slackens, $T = 0$

$$\Rightarrow \quad 0 + \frac{mg}{2} = \frac{mv^2}{a}$$

$$\frac{ag}{2} = v^2$$

$$\Rightarrow \therefore v = \sqrt{\frac{ag}{2}}.$$

And $u^2 = v^2 + 3ag$

$$u^2 = \frac{ag}{2} + 3ag$$

$$u^2 = \frac{7ag}{2}$$

$$u = \sqrt{\frac{7ag}{2}}.$$

28 A particle P of mass m is projected horizontally with speed u from the lowest point on the inside of a fixed hollow sphere with centre O. The sphere has a smooth internal surface of radius a. Assuming that the particle does not lose contact with the sphere, show that when the speed of the particle has been reduced to $\frac{1}{2}u$ the angle θ between OP and the downward vertical satisfies the equation

$$8ga(1 - \cos\theta) = 3u^2. \qquad [2]$$

Find, in terms of m, u, a and g, an expression for the magnitude of the contact force acting on the particle in this position. [4]

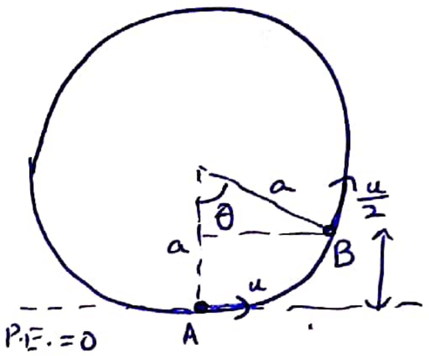
$$(K.E. + P.E.)_A = (K.E. + P.E.)_B$$

$$\frac{1}{2}mu^2 + mg(0) = \frac{1}{2}m\left(\frac{u}{2}\right)^2 + mga(1-\cos\theta)$$

$$\frac{u^2}{2} = \frac{u^2}{8} + ga - ga\cos\theta$$

$$4u^2 = u^2 + 8ga(1-\cos\theta)$$

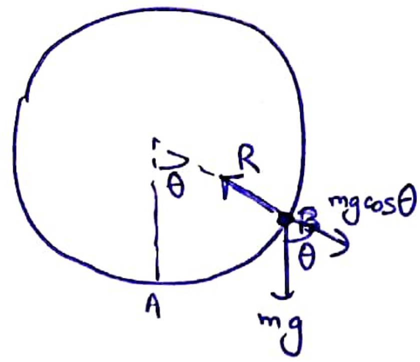
$$3u^2 = 8ga(1-\cos\theta).$$



R(towards center):

$$R - mg\cos\theta = \frac{m\left(\frac{u}{2}\right)^2}{a}$$

$$R = mg\cos\theta + \frac{mu^2}{4a}$$



From $3u^2 = 8ga(1 - \cos\theta)$

$$\Rightarrow \frac{3u^2}{8ga} = 1 - \cos\theta \Rightarrow \cos\theta = 1 - \frac{3u^2}{8ga}.$$

$$\therefore R = mg\left[1 - \frac{3u^2}{8ga}\right] + \frac{mu^2}{4a}$$

$$= mg - \frac{3mgu^2}{8ga} + \frac{mu^2}{4a}$$

$$= mg - \frac{3mu^2}{8a} + \frac{mu^2}{4a}$$

$$= mg + \frac{-3mu^2 + 2mu^2}{8a}$$

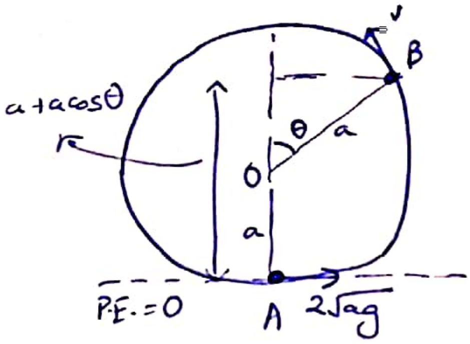
$$R = mg - \frac{mu^2}{8a}$$

29 A fixed hollow sphere with centre O has a smooth inner surface of radius a. A particle P of mass m is projected horizontally with speed $2\sqrt{(ag)}$ from the lowest point of the inner surface of the sphere. The particle loses contact with the inner surface of the sphere when OP makes an angle θ with the upward vertical.

(i) Show that $\cos\theta = \frac{2}{3}$. [5]

(ii) Find the greatest height that P reaches above the level of O. [4]

i) $(P.E. + K.E.)_A = (P.E. + K.E.)_B$



$$mg(0) + \frac{m}{2}\left(2\sqrt{ag}\right)^2 = mg\left[a(1+\cos\theta)\right] + \frac{mv^2}{2}$$

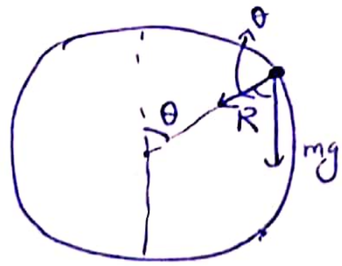
$$\frac{m}{2} \times 4ag = mga + mga\cos\theta + \frac{mv^2}{2}$$

$$2ag = ag + ag\cos\theta + \frac{v^2}{2}$$

$$ag - ag\cos\theta = \frac{v^2}{2}$$

$$2ag - 2ag\cos\theta = v^2$$

R (towards center): $R + mg\cos\theta = \frac{mv^2}{a}$



$$R = \frac{mv^2}{a} - mg\cos\theta.$$

Since particle loses contact when angle with upward vertical is θ, $R = 0$

$$\Rightarrow \frac{mv^2}{a} - mg\cos\theta = 0$$

$$\frac{m}{a}\left[2ag - 2ag\cos\theta\right] = mg\cos\theta$$

$$2mg - 2mg\cos\theta = mg\cos\theta$$

$$2mg = 3mg\cos\theta$$

$$\Rightarrow \cos\theta = \frac{2mg}{3mg} = \frac{2}{3}.$$

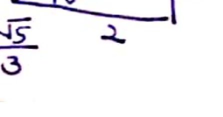
ii) When $\cos\theta = \frac{2}{3} \Rightarrow v^2 = 2ag - 2ag\cos\theta = 2ag - 2ag\left(\frac{2}{3}\right) = \frac{2ag}{3}.$

$$h = a(1+\cos\theta) = a\left(1+\frac{2}{3}\right) = \frac{5a}{3}.$$

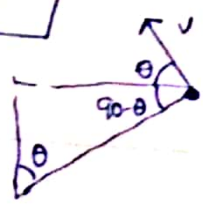
Vertically,

$$V_y = v\sin\theta = \sqrt{\frac{2ag}{3}} \times \frac{\sqrt{5}}{3} = \sqrt{\frac{10ag}{27}}.$$

$\cos\theta = \frac{2}{3}$



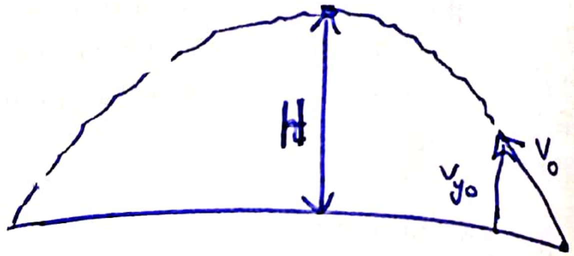
$\Rightarrow \sin\theta = \frac{\sqrt{5}}{3}$



For highest point, $v_y = 0$

Using $\quad v^2 = u^2 + 2as$

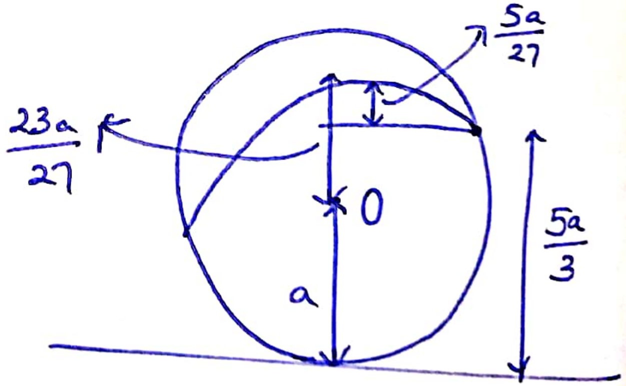
$$0^2 = \left(\sqrt{\frac{10ag}{27}}\right)^2 + 2(-g)(H)$$



$$2gH = \frac{10ag}{27}$$

$$H = \frac{\frac{10ag}{27}}{2g} = \frac{5a}{27}.$$

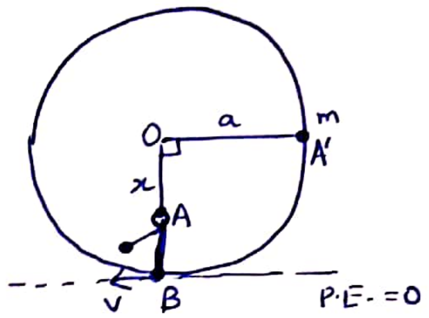
$\therefore$ Greatest height reached above $O = \frac{5a}{3} + \frac{5a}{27} - a = \frac{23a}{27}.$



30 A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle is held with the string taut and horizontal and is then released. When the string is vertical, it comes into contact with a small smooth peg A which is vertically below O and at a distance x ($< a$) from O. In the subsequent motion, when AP makes an angle θ with the downward vertical, the tension in the string is T. Show that

$$T = mg\left(3\cos\theta + \frac{2x}{a-x}\right). \qquad [7]$$

Given that P completes a vertical circle about A, find the least possible value of $\frac{x}{a}$. [2]



$$(P.E. + K.E)_{A'} = (P.E. + K.E.)_B$$

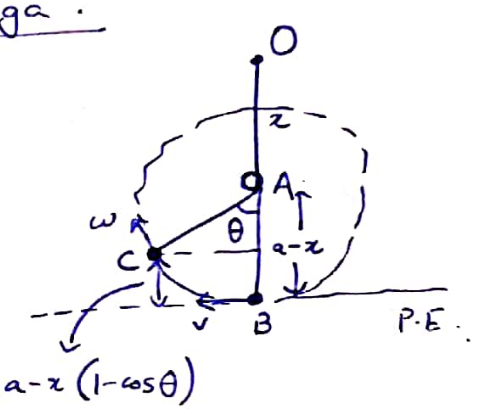
$$mga + \frac{1}{2}m(0)^2 = mg(0) + \frac{1}{2}mv^2$$

$$mga = \frac{mv^2}{2}$$

$$v^2 = 2ga.$$

$$v = \sqrt{2ga}$$

i.e. Speed when particle is vertically below $O = \sqrt{2ga}$.



$$(P.E. + K.E.)_B = (P.E. + K.E.)_C$$

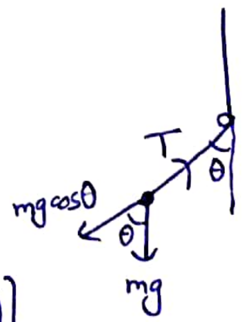
$$mg(0) + \frac{m}{2}\left(\sqrt{2ga}\right)^2 = mg\left[(a-x)(1-\cos\theta)\right] + \frac{m\omega^2}{2}$$

$$\frac{m}{2} \times 2ga = mg(a-x)(1-\cos\theta) + \frac{m\omega^2}{2}$$

$$ga = g(a-x)(1-\cos\theta) + \frac{\omega^2}{2}$$

$$\omega^2 = 2ga - 2g(a-x)(1-\cos\theta).$$

R(towards center / A / peg):



$$T - mg\cos\theta = \frac{m\omega^2}{(a-x)}$$

$$T = mg\cos\theta + \frac{m}{a-x}\left[\omega^2\right]$$

$$\Rightarrow T = mg\cos\theta + \frac{m}{a-x}\left[2ga - 2g(a-x)(1-\cos\theta)\right]$$

$$= mg\cos\theta + \frac{2mga}{a-x} - 2mg(1-\cos\theta) = mg\cos\theta + \frac{2mga}{a-x} - 2mg + 2mg\cos\theta$$

$$= 3mg\cos\theta + \frac{2mga}{a-x} - 2mg = mg\left[3\cos\theta + \frac{2a}{a-x} - 2\right]$$

$$= mg\left[3\cos\theta + \frac{2a-2(a-x)}{a-x}\right] = mg\left[3\cos\theta + \frac{2x}{a-x}\right].$$

For particle to complete circles, $T \geq 0$ when $\theta = 180°$

$$\Rightarrow \quad mg\left[3\cos 180° + \frac{2x}{a-x}\right] \geq 0$$

$$mg\left[-3 + \frac{2x}{a-x}\right] \geq 0$$

$$\frac{2x}{a-x} \geq 3$$

$$2x \geq 3a - 3x$$

$$5x \geq 3a$$

$$\frac{x}{a} \geq \frac{3}{5}.$$

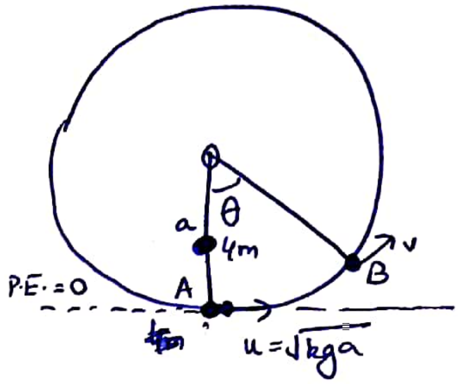
$\therefore$ Minimum value of $\frac{x}{a} = \frac{3}{5}$.

31 A small bead of mass m is threaded on a thin smooth wire which forms a circle of radius a. The wire is fixed in a vertical plane. A light inextensible string is attached to the bead and passes through a small smooth ring fixed at the centre of the circle. The other end of the string is attached to a particle of mass $4m$ which hangs freely under gravity. The bead is projected from the lowest point of the wire with speed $\sqrt{(kga)}$. Show that, when the angle between the two parts of the string is θ, the normal force exerted on the bead by the wire is $mg(3\cos\theta + k - 6)$, towards the centre. [5]

Given that the bead reaches the highest point of the wire, find an inequality which must be satisfied by k. [2]

For small bead,

$$(P.E. + K.E.)_A = (P.E. + K.E.)_B$$



$$mg(0) + \frac{1}{2}m(kga) = mga(1-\cos\theta) + \frac{1}{2}mv^2$$

$$\frac{kga}{2} = ga - ga\cos\theta + \frac{v^2}{2}$$

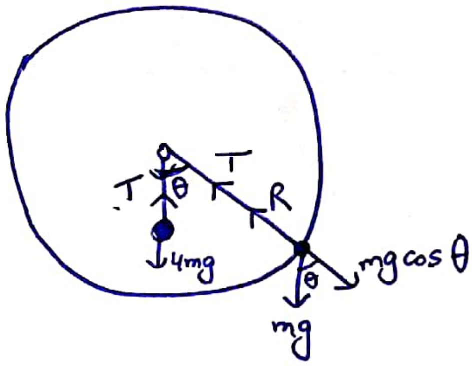
$$kga = 2ga - 2ga\cos\theta + v^2$$

$$v^2 = kga - 2ga + 2ga\cos\theta$$

$$v^2 = (k-2)ga + 2ga\cos\theta$$

For the particle of mass 4m,

R(towards center/vertically): $T = 4mg$.



Generally, (towards center):

$$F_{net} = \frac{mv^2}{a}$$

$$T + R - mg\cos\theta = \frac{mv^2}{a}$$

$$4mg + R - mg\cos\theta = \frac{mv^2}{a}.$$

$$R = mg\cos\theta - 4mg + \frac{mv^2}{a}.$$

$$= mg\cos\theta - 4mg + \frac{m}{a}\left[(k-2)ga + 2ga\cos\theta\right]$$

$$= mg\cos\theta - 4mg + (k-2)mg + 2mg\cos\theta$$

$$= 3mg\cos\theta - 4mg + (k-2)mg$$

$$= (3\cos\theta + k - 2 - 4)mg$$

$$R = (3\cos\theta + k - 6)mg$$

For bead to reach highest point on the wire,

$$v \geq 0 \text{ when } \theta = 180^\circ$$

$$(k-2)ga + 2ga\cos\theta \geq 0$$

$$(k-2)ga + 2ga\cos 180^\circ \geq 0$$

$$(k-2)ga - 2ga \geq 0$$

$$k-2 \geq \frac{2ga}{ga}$$

$$k-2 \geq 2$$

$$\boxed{k \geq 4}$$

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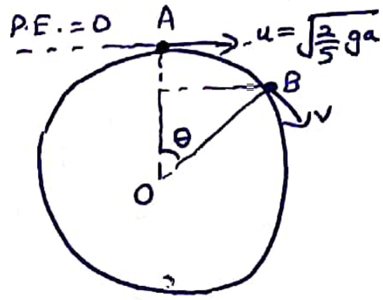
32 A smooth sphere, with centre O and radius a, is fixed on a smooth horizontal plane Π. A particle P of mass m is projected horizontally from the highest point of the sphere with speed $\sqrt{(\frac{2}{5}ga)}$. While P remains in contact with the sphere, the angle between OP and the upward vertical is denoted by θ. Show that P loses contact with the sphere when $\cos\theta = \frac{4}{5}$. [6]

Subsequently the particle collides with the plane Π. The coefficient of restitution between P and Π is $\frac{5}{9}$. Find the vertical height of P above Π when the vertical component of the velocity of P first becomes zero. [6]



$$(P.E. + K.E.)_A = (P.E. + K.E.)_B$$

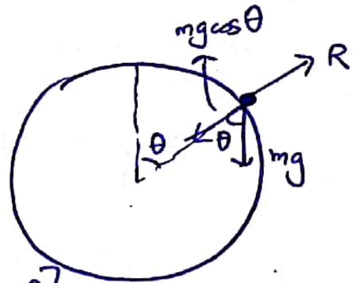
$$mg(0) + \frac{m}{2} \times \frac{2ga}{5} = mg\left[-a(1-\cos\theta)\right] + \frac{mv^2}{2}$$

$$\frac{2mga}{10} = mga(\cos\theta - 1) + \frac{mv^2}{2}$$

$$\frac{ga}{5} = ga\cos\theta - ga + \frac{v^2}{2}$$

$$\frac{2ga}{5} = 2ga\cos\theta - 2ga + v^2$$

$$v^2 = \frac{12ga}{5} - 2ga\cos\theta.$$



R(↙, towards center):

$$mg\cos\theta - R = \frac{mv^2}{r}$$

$$R = mg\cos\theta - \frac{mv^2}{r}$$

$$= mg\cos\theta - \frac{m}{a}\left[\frac{12ga}{5} - 2ga\cos\theta\right]$$

$$= mg\cos\theta - \frac{12mg}{5} + 2mg\cos\theta$$

$$R = 3mg\cos\theta - \frac{12mg}{5}$$

When particle loses contact,

$$R = 0$$

$$\Rightarrow 3mg\cos\theta - \frac{12mg}{5} = 0$$

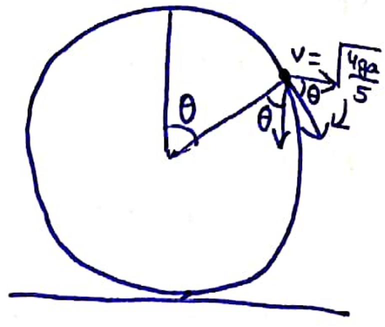
$$\cos\theta = \frac{\frac{12mg}{5}}{3mg} = \frac{4}{5}.$$

(Shown).

When $\cos\theta = \frac{4}{5}$,

$$v^2 = \frac{12ga}{5} - 2ga\left[\frac{4}{5}\right] = \frac{12ga}{5} - \frac{8ga}{5} = \frac{4ga}{5}. \Rightarrow v = \sqrt{\frac{4ga}{5}}.$$

$$h = a + a\cos\theta = a + a\left(\frac{4}{5}\right) = \frac{9a}{5}.$$



For vertical motion, $V_y = v\sin\theta = \sqrt{\frac{4ga}{5}} \times \frac{3}{5} = \frac{3}{5}\sqrt{\frac{4ga}{5}}$

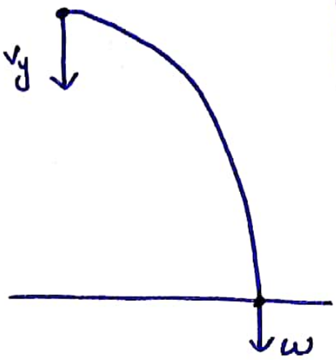
Using $v^2 = u^2 + 2as$

$$\omega^2 = \left[\frac{3}{5}\left(\sqrt{\frac{4ga}{5}}\right)\right]^2 + 2g\left(\frac{9a}{5}\right)$$

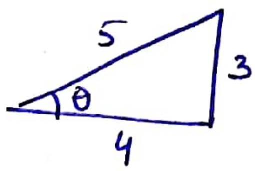
$$= \frac{9}{25} \times \frac{4ga}{5} + \frac{18ga}{5}$$

$$\omega^2 = \frac{486ga}{125}.$$

$$\omega = \sqrt{\frac{486ga}{125}}.$$



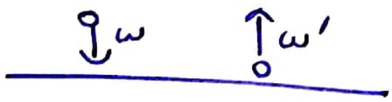
$\cos\theta = \frac{4}{5}$



$\Rightarrow \sin\theta = \frac{3}{5}$.

Rebound speed, $\omega' = \frac{5}{9}\omega$

$$\omega' = \frac{5}{9}\sqrt{\frac{486ga}{125}}$$



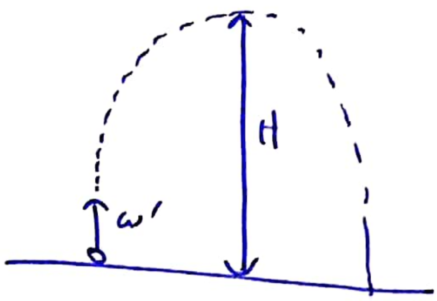
For H, vertical height reached, we want final vertical component to be zero. Use

$$v^2 = u^2 + 2as$$

$$0^2 = \left(\frac{5}{9}\sqrt{\frac{486ga}{125}}\right)^2 + 2(-g)H$$

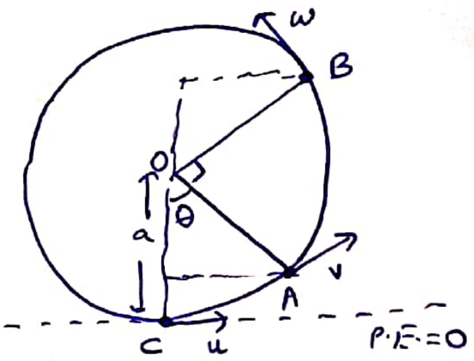
$$2gH = \frac{25}{81} \times \frac{486ga}{125}$$

$$H = \frac{3a}{5}.$$



33 A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The path of the particle is a complete vertical circle with centre O. When P is at its lowest point, its speed is u. When P is at the point A, the tension in the string is T and the string makes an angle θ with the downward vertical, where $\cos\theta = \frac{3}{5}$. When P is at the point B, above the level of O, the tension in the string is $\frac{1}{8}T$ and angle $BOA = 90°$. Find u in terms of a and g. [9]

$$((K.E) + P.E.)_C = (K.E. + P.E.)_A$$



$$\frac{1}{2}mu^2 + mg(0) = \frac{1}{2}mv^2 + mga(1-\cos\theta)$$

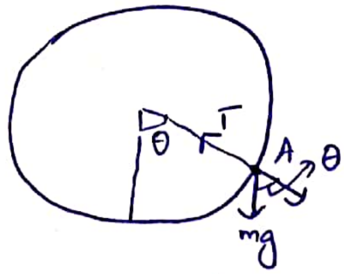
$$\frac{u^2}{2} = \frac{v^2}{2} + ga(1-\cos\theta)$$

$$v^2 = u^2 - 2ga(1-\cos\theta)$$

$$v^2 = u^2 - 2ga\left(1-\frac{3}{5}\right)$$

$$v^2 = u^2 - \frac{4ga}{5}$$

R(towards center) at A:

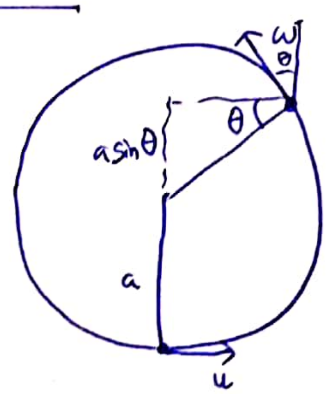


$$T - mg\cos\theta = \frac{mv^2}{a}$$

$$T = \frac{m}{a}\left[u^2 - \frac{4ga}{5}\right] + mg\cos\theta$$

$$T = \frac{mu^2}{a} - \frac{4mg}{5} + mg\left[\frac{3}{5}\right] = \frac{mu^2}{a} - \frac{mg}{5}$$

$$(K.E. + P.E)_C = (K.E + P.E.)_B$$



$$\frac{1}{2}mu^2 + mg(0) = \frac{1}{2}m\omega^2 + mg[a(1+\sin\theta)]$$

$$\frac{u^2}{2} = \frac{\omega^2}{2} + ga(1+\sin\theta)$$

$$u^2 = \omega^2 + 2ga(1+\sin\theta)$$

$$\omega^2 = u^2 - 2ga(1+\sin\theta)$$

$$\omega^2 = u^2 - 2ga\left(1+\frac{4}{5}\right)$$

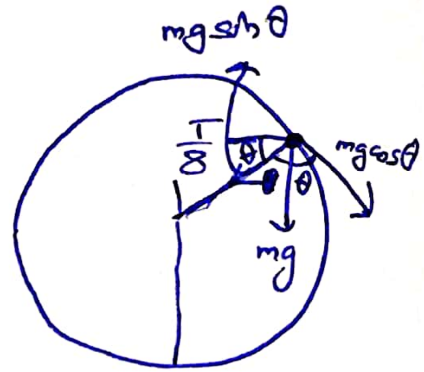
$$\omega^2 = u^2 - 2ga\left(\frac{9}{5}\right) = u^2 - \frac{18ga}{5}.$$

$$\cos\theta = \frac{3}{5}$$

$$\Rightarrow \sin\theta = \frac{4}{5}.$$

R (towards center) at B:

$$mg\sin\theta + \frac{T}{8} = \frac{m\omega^2}{a}$$



$$\frac{T}{8} = \frac{m}{a}\left[u^2 - \frac{18ga}{5}\right] - mg\sin\theta$$

$$T = 8\left(\frac{mu^2}{a} - \frac{18mg}{5} - mg\sin\theta\right)$$

$$T = \frac{8mu^2}{a} - \frac{144mg}{5} - 8mg\sin\theta$$

$$T = \frac{8mu^2}{a} - \frac{144mg}{5} - 8mg\left(\frac{4}{5}\right) = \frac{8mu^2}{a} - \frac{176mg}{5}$$

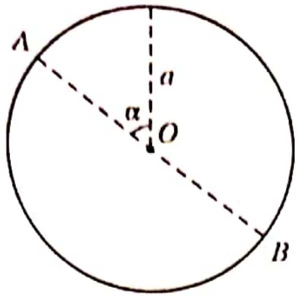
Then $$\frac{mu^2}{a} - \frac{mg}{5} = \frac{8mu^2}{a} - \frac{176mg}{5}$$

$$35mg = \frac{7mu^2}{a}$$

$$\frac{35mga}{7m} = u^2$$

$$u^2 = 5ga.$$

$$\Rightarrow u = \sqrt{5ga}.$$



A smooth cylinder of radius a is fixed with its axis horizontal. The point O is the centre of a circular cross-section of the cylinder. The line AOB is a diameter of this circular cross-section and the radius OA makes an angle α with the upward vertical (see diagram). It is given that $\cos\alpha = \frac{3}{5}$. A particle P of mass m moves on the inner surface of the cylinder in the plane of the cross-section. The particle passes through A with speed u along the surface in the downwards direction. The magnitude of the reaction between P and the inner surface of the sphere is R_A when P is at A, and is R_B when P is at B. It is given that $R_B = 10R_A$. Show that $u^2 = ag$. [6]

The particle loses contact with the surface of the cylinder when OP makes an angle θ with the upward vertical. Find the value of $\cos\theta$. [4]

$(K.E + P.E.)_A = (K.E. + P.E.)_B$

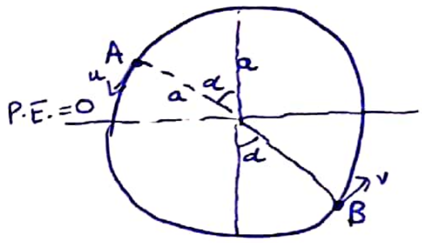
$\frac{1}{2}mu^2 + mg\,a\cos\alpha = \frac{1}{2}mv^2 + mg(-a\cos\alpha)$

$\frac{u^2}{2} + ga\cos\alpha = \frac{v^2}{2} \cancel{g} - ga\cos\alpha$

$u^2 + 4ga\cos\alpha = v^2$

$u^2 + 4ga\left(\frac{3}{5}\right) = v^2$

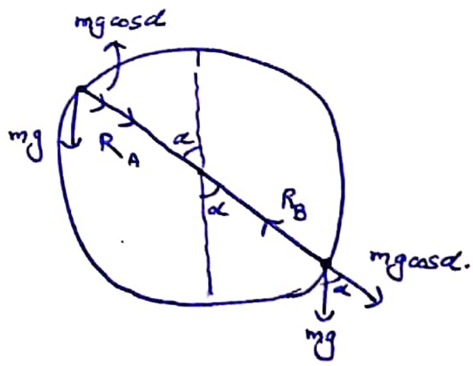
$\Rightarrow v^2 = u^2 + \frac{12ga}{5}$.



R(towards center) at A gives:

$R_A + mg\cos\alpha = \frac{mu^2}{r}$

$R_A = \frac{mu^2}{a} - mg\left(\frac{3}{5}\right) = \frac{mu^2}{a} - \frac{3mg}{5}$.



R(towards center) at B gives:

$R_B - mg\cos\alpha = \frac{mv^2}{a}$

$R_B = \frac{m}{a}\left[u^2 + \frac{12ga}{5}\right] + mg\left[\frac{3}{5}\right]$

$R_B = \frac{mu^2}{a} + \frac{12mg}{5} + \frac{3mg}{5} = \frac{mu^2}{a} + 3mg$.

$R_B = 10R_A$

$\Rightarrow \frac{mu^2}{a} + 3mg = 10\left(\frac{mu^2}{a} - \frac{3mg}{5}\right) \Rightarrow \frac{mu^2}{a} + 3mg = \frac{10mu^2}{a} - \frac{30mg}{5}$

$9\cancel{m}g = \frac{9\cancel{m}u^2}{a} \Rightarrow u^2 = ag$.

35 A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. When P is hanging at rest vertically below O, it is projected horizontally. In the subsequent motion P completes a vertical circle. The speed of P when it is at its highest point is u. Show that the least possible value of u is $\sqrt{(ag)}$. [2]

It is now given that $u = \sqrt{(ag)}$. When P passes through the lowest point of its path, it collides with, and coalesces with, a stationary particle of mass $\frac{1}{4}m$. Find the speed of the combined particle immediately after the collision. [4]

In the subsequent motion, when OP makes an angle θ with the upward vertical the tension in the string is T. Find an expression for T in terms of m, g and θ. [5]

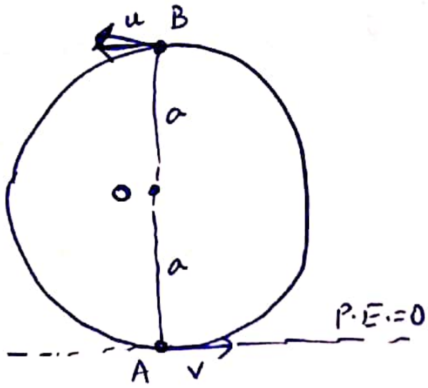
Find the value of $\cos\theta$ when the string becomes slack. [2]

$(P.E.+K.E.)_A = (P.E.+K.E.)_B$

$mg(0) + \frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mg(2a)$

$\frac{v^2}{2} = \frac{u^2}{2} + 2ga$

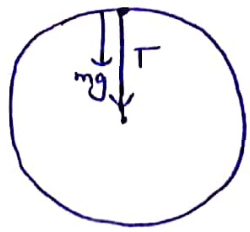
$v^2 = u^2 + 4ga. \Rightarrow u^2 = v^2 - 4ga.$



R (towards center) at top:

$T + mg = \frac{mv^2}{r}$

$T = \frac{mu^2}{a} - mg$



To complete a circle, $T \geq 0$

$\Rightarrow \frac{mu^2}{a} - mg \geq 0$

$\frac{\cancel{m}u^2}{a} \geq \cancel{m}g$

$u^2 \geq ag.$

Then $u \geq \sqrt{ag}$

i.e. Least possible value of $u = \sqrt{ag}$.

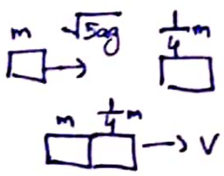
$u = \sqrt{ag} \Rightarrow u^2 = ag \therefore v^2 = u^2 + 4ag = ag + 4ag = 5ag$. Then $v = \sqrt{5ag}$.

Using principle of conservation of momentum,

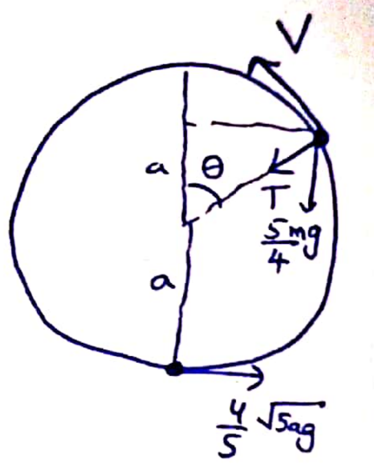
$m(\sqrt{5ag}) + \frac{1}{4}m(0) = (m + \frac{1}{4}m)V$

$\frac{m\sqrt{5ag}}{\frac{5}{4}m} = V$

$V = \frac{4\sqrt{5ag}}{5}.$



$$\frac{1}{2}\left(\frac{5}{4}m\right)V^2 + \left(\frac{5}{4}m\right)g\left[a(1+\cos\theta)\right] = \frac{1}{2}\left(\frac{5}{4}m\right)\left(\frac{4}{5}\sqrt{5ag}\right)^2$$



$$\frac{5mV^2}{8} + \frac{5mga}{4} + \frac{5mga\cos\theta}{4} = \cancel{\frac{5m}{8}} \times \cancel{\frac{16}{25}}^{2} \times 5ag$$

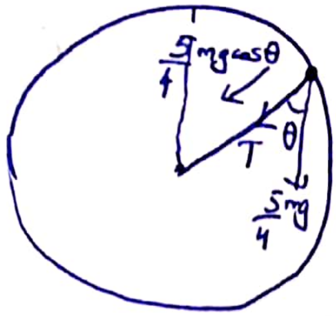
$$\frac{5mV^2}{8} + \frac{5mga}{4} + \frac{5mga\cos\theta}{4} = 2mag$$

$$5V^2 + 10ga + 10ga\cos\theta = 16ga$$

$$5V^2 = 6ga - 10ga\cos\theta$$

$$V^2 = ga\left(\frac{6}{5} - 2\cos\theta\right).$$

R(towards center): $T + \frac{5}{4}mg\cos\theta = \frac{\left(\frac{5}{4}m\right)V^2}{a}$



$$T = \frac{5}{4a}mV^2 - \frac{5}{4}mg\cos\theta$$

$$= \frac{5m}{4a}\left[ga\left(\frac{6}{5} - 2\cos\theta\right)\right] - \frac{5}{4}mg\cos\theta$$

$$= \frac{3}{2}mg - \frac{5}{2}mg\cos\theta - \frac{5}{4}mg\cos\theta$$

$$T = \frac{3mg}{2} - \frac{15mg\cos\theta}{4}$$

When string becomes slack,

$$T = 0$$

$$\frac{3mg}{2} - \frac{15mg\cos\theta}{4} = 0$$

$$\cos\theta = \frac{\frac{3mg}{2}}{\frac{15mg}{4}} = \frac{12}{30} = \frac{2}{5} = 0.4$$

i.e. $\cos\theta = 0.4$.

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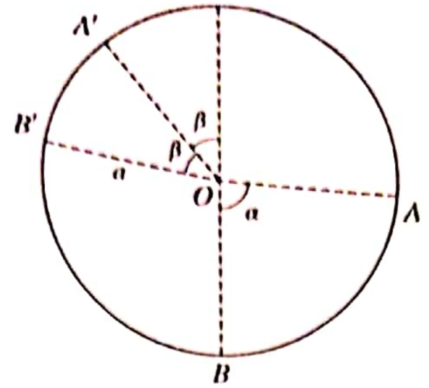
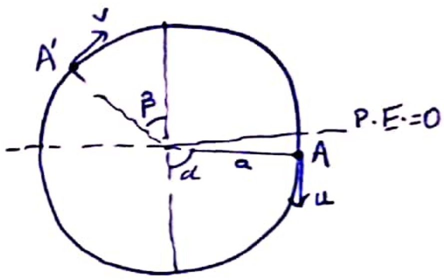


A particle P of mass m is free to move on the smooth inner surface of a fixed hollow sphere of radius a. The centre of the sphere is O. The points A and A' are on the inner surface of the sphere, on opposite sides of the vertical through O; the radius OA makes an angle α with the downward vertical and the radius OA' makes an angle β with the upward vertical. The point B is on the inner surface of the sphere, vertically below O. The point B' is on the inner surface of the sphere and such that OB' makes an angle 2β with the upward vertical through O (see diagram). It is given that $\cos\alpha = \frac{1}{16}$.

(i) P is projected from A with speed u along the surface of the sphere downwards towards B. Subsequently it loses contact with the sphere at A'. Show that $u^2 = \frac{1}{8}ag(1 + 24\cos\beta)$. [5]



$$(P.E.+K.E.)_A = (P.E.+K.E.)_{A'}$$

$$mg(-a\cos\alpha) + \frac{1}{2}mu^2 = mg(a\cos\beta) + \frac{1}{2}mv^2$$

$$-ga\cos\alpha + \frac{u^2}{2} = ga\cos\beta + \frac{v^2}{2}.$$

$$-2ga\cos\alpha + u^2 = 2ga\cos\beta + v^2$$

$$v^2 = u^2 - 2ga\cos\alpha - 2ga\cos\beta$$

$$v^2 = u^2 - 2ga(\cos\alpha + \cos\beta)$$

R (towards center) at A':

$$R + mg\cos\beta = \frac{mv^2}{r}$$

Since contact is lost at A, $R = 0$

$$\Rightarrow \cancel{m}g\cos\beta = \frac{\cancel{m}v^2}{a}$$

$$ag\cos\beta = u^2 - 2ga(\cos\alpha + \cos\beta)$$

$$u^2 = 2ga\left[\frac{1}{16} + \cos\beta\right] + ag\cos\beta$$

$$= \frac{2ga}{16} + 2ga\cos\beta + ga\cos\beta$$

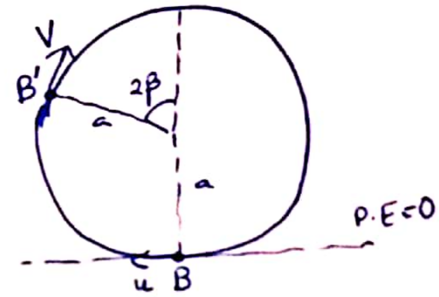
$$u^2 = \frac{ga}{8} + 3ga\cos\beta$$

$$u^2 = \frac{1}{8}ga\left[1 + 24\cos\beta\right].$$

(ii) P is now projected from B with speed u along the surface of the sphere towards B'. Subsequently it loses contact with the sphere at B'. Find $\cos\beta$. **[6]**

$(P.E. + K.E.)_B = (P.E. + K.E.)_{B'}$

$mg(0) + \frac{1}{2} m u^2 = mg(a + a\cos 2\beta) + \frac{1}{2} m V^2$



$\frac{mu^2}{2} = mga(1+\cos 2\beta) + \frac{mV^2}{2}$

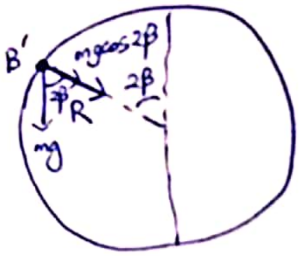
$\frac{u^2}{2} = ga(1+\cos 2\beta) + \frac{V^2}{2}$

$V^2 = u^2 - 2ga(1+\cos 2\beta)$.

At B'

R (towards center): $R + mg\cos 2\beta = \frac{mV^2}{a}$

At B', $R = 0 \Rightarrow$ $mg\cos 2\beta = \frac{mV^2}{a}$



$ag\cos 2\beta = V^2$

$ag\cos 2\beta = u^2 - 2ag(1+\cos 2\beta)$

$u^2 = 2ag(1+\cos 2\beta) + ag\cos 2\beta$

$u^2 = 2ag + 3ag\cos 2\beta = ag(2+3\cos 2\beta)$

Then $u^2 = \frac{1}{8} ag(1+24\cos\beta) = ag(2+3\cos 2\beta)$

$1 + 24\cos\beta = 16 + 24\cos 2\beta$

$1 + 24\cos\beta = 16 + 24[2\cos^2\beta - 1]$

$1 + 24\cos\beta = 16 + 48\cos^2\beta - 24$

$-48\cos^2\beta + 24\cos\beta + 9 = 0$

Let, $\cos\beta = x$

$\Rightarrow -48x^2 + 24x + 9 = 0$

$16x^2 - 8x - 3 = 0$

$16x^2 - 12x + 4x - 3 = 0$

$(4x+1)(4x-3) = 0$

$x = \frac{-1}{4}$ or $x = \frac{3}{4}$.

$\cos\beta = \frac{-1}{4}$ or $\boxed{\cos\beta = \frac{3}{4}}$

(Ignore since $\beta < 90°$)

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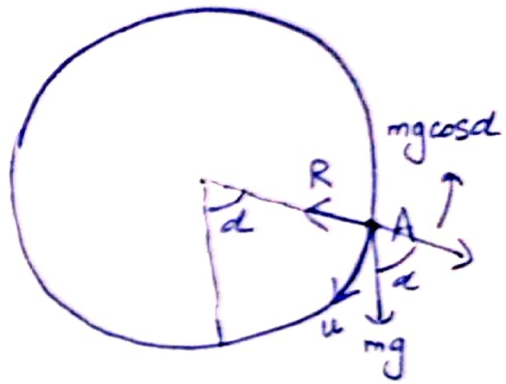
(iii) In part **(i)**, the reaction of the sphere on P when it is initially projected at A is R. Find R in terms of m and g. [3]

Since $\cos\beta = \frac{+3}{4}$,



$$u^2 = \frac{1}{8}ga\left[1 + 24\cos\beta\right] = \frac{1}{8}ga\left[1 + 24 \times \frac{3}{4}\right]$$

$$u^2 = \frac{19ga}{8}$$

R(towards center):

$$R - mg\cos\alpha = \frac{mu^2}{a}$$

$$R = mg\cos\alpha + \frac{mu^2}{a}$$

$$= mg\left[\frac{1}{16}\right] + \frac{m}{a}\left[\frac{19ga}{8}\right]$$

$$R = \frac{mg}{16} + \frac{19mg}{8} = \frac{39mg}{16}.$$

37 A particle P of mass m is free to move on the smooth inner surface of a fixed hollow sphere of radius a. The centre of the sphere is O and the point C is on the inner surface of the sphere, vertically below O. The points A and B on the inner surface of the sphere are the ends of a diameter of the sphere. The diameter AOB makes an acute angle α with the vertical, where $\cos\alpha = \frac{4}{5}$, with A below the horizontal level of B. The particle is projected from A with speed u, and moves along the inner surface of the sphere towards C. The normal reaction forces on the particle at A and C are in the ratio 8 : 9.

(i) Show that $u^2 = 4ag$. [6]

$$(K.E. + P.E.)_A = (K.E. + P.E.)_C$$

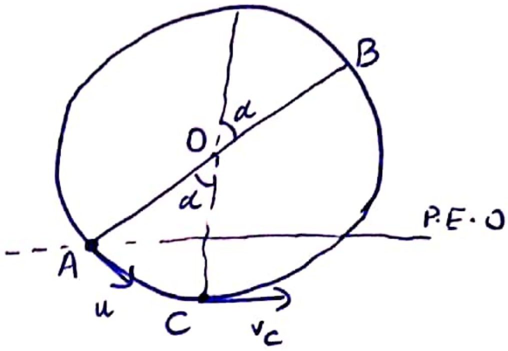
$$\frac{1}{2}mu^2 + mg(0) = \frac{1}{2}mv_c^2 + mg(-a(1-\cos\alpha))$$

$$\frac{u^2}{2} = \frac{v_c^2}{2} - ga(1-\cos\alpha)$$

$$v_c^2 = u^2 + 2ga(1-\cos\alpha).$$

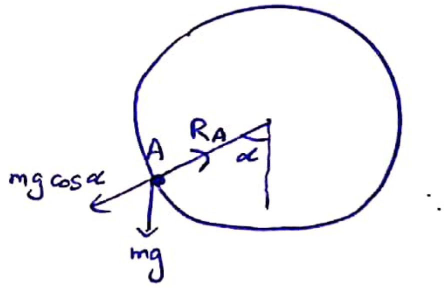
$$= u^2 + 2ga\left(1 - \frac{4}{5}\right)$$

$$v_c^2 = u^2 + \frac{2ga}{5}$$



At A, $R(\nearrow)$: $R_A - mg\cos\alpha = \frac{mu^2}{a}$

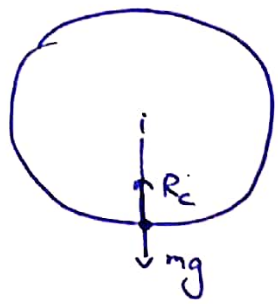
$$R_A = mg\cos\alpha + \frac{mu^2}{a}.$$



At C, $R(\uparrow)$: $R_c - mg = \frac{mv_c^2}{a}$

$$R_c = \frac{m}{a}\left[u^2 + \frac{2ga}{5}\right] + mg$$

$$R_c = \frac{mu^2}{a} + \frac{7mg}{5}.$$



$$\frac{R_A}{R_C} = \frac{8}{9}$$

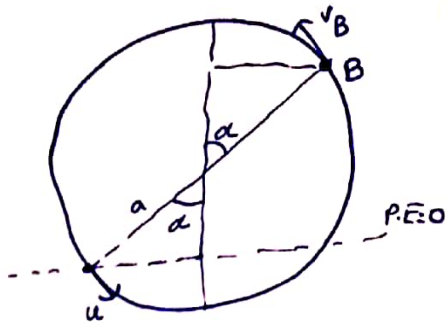
$$9\left(mg\cos\alpha + \frac{mu^2}{a}\right) = 8\left(\frac{mu^2}{a} + \frac{7mg}{5}\right)$$

$$9mg\cos\alpha + \frac{9mu^2}{a} = \frac{8mu^2}{a} + \frac{56mg}{5}$$

$$\frac{mu^2}{a} = \frac{56mg}{5} - 9mg\left(\frac{4}{5}\right) \Rightarrow \frac{mu^2}{a} = 4mg$$

$$u^2 = \frac{4mga}{m} = 4ag.$$

(ii) Determine whether P reaches B without losing contact with the inner surface of the sphere. [6]



$$(K.E. + P.E.)_A = (K.E + P.E.)_C$$

$$\frac{1}{2}mu^2 + mg(0) = \frac{1}{2}mv_B^2 + mg(2a\cos\alpha)$$

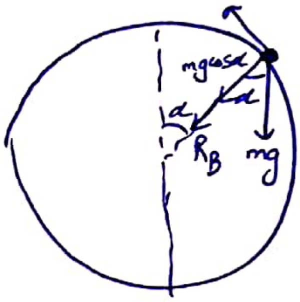
$$\frac{u^2}{2} = \frac{v_B^2}{2} + 2ga\cos\alpha$$

$$v_B^2 = u^2 - 4ga\cos\alpha.$$

$$= 4ag - 4ag\left(\frac{4}{5}\right)$$

$$v_B^2 = 4ag - \frac{16ag}{5} = \frac{4ag}{5}$$

Since $v_B^2 \geq 0$ $\therefore$ P reaches the height of B.



R (towards center): $\quad R_B + mg\cos\alpha = \frac{mv_B^2}{r}$

$$R_B = \frac{mv_B^2}{a} - mg\cos\alpha$$

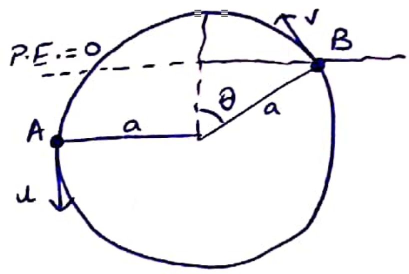
$$= \frac{m\left(\frac{4ag}{5}\right)}{a} - mg\left(\frac{4}{5}\right)$$

$$R_B = \frac{4mg}{5} - \frac{4mg}{5} = 0$$

Then at B, $R_B \geq 0$ $\therefore$ P is still just in contact with inner surface of sphere.

38 A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O and P is held with the string taut and horizontal. The particle P is projected vertically downwards with speed $\sqrt{(2ag)}$ so that it begins to move along a circular path. The string becomes slack when OP makes an angle θ with the upward vertical through O.

(i) Show that $\cos\theta = \frac{2}{3}$. [5]

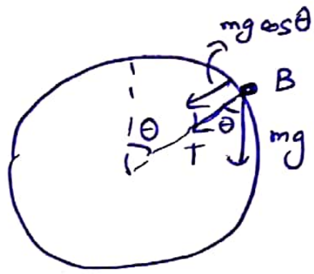


$$(K.E. + P.E.)_B = (K.E. + P.E.)_A$$

$$\frac{1}{2}mv^2 + mg(0) = \frac{1}{2}mu^2 + mg(-a\cos\theta)$$

$$\frac{v^2}{2} = \frac{u^2}{2} - ga\cos\theta$$

$$v^2 = u^2 - 2ga\cos\theta \longrightarrow (1)$$



$$R(\swarrow): \quad mg\cos\theta + T = \frac{mv^2}{a}$$

$$T = \frac{mv^2}{a} - mg\cos\theta$$

At B, string becomes slack $\therefore T = 0$

$$\Rightarrow \frac{\cancel{m}v^2}{a} = \cancel{m}g\cos\theta$$

$$v^2 = ga\cos\theta \rightarrow (2)$$

Substituting (2) in (1) gives:

$$ga\cos\theta = (\sqrt{2ag})^2 - 2ga\cos\theta$$

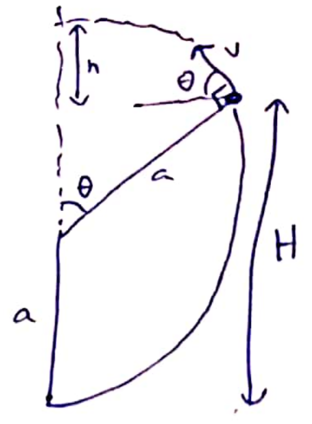
$$3ga\cos\theta = 2ga$$

$$\cos\theta = \frac{2ga}{3ga}$$

$$\cos\theta = \frac{2}{3}.$$

(ii) Find the greatest height, above the horizontal through O, reached by P in its subsequent motion. [4]

$\cos\theta = \frac{2}{3} \Rightarrow \sin\theta = \frac{\sqrt{5}}{3}. \Rightarrow H = a + a\cos\theta = a + a\left(\frac{2}{3}\right) = \frac{5a}{3}.$



$v^2 = ga\cos\theta = ga \times \frac{2}{3} = \frac{2ga}{3}.$

$v = \sqrt{\frac{2ga}{3}}.$

Then as string becomes slack,

$v_y = v\sin\theta = \sqrt{\frac{2ga}{3}} \times \frac{\sqrt{5}}{3} = \sqrt{\frac{10ga}{27}}.$

Vertically, ~~for~~ maximum height,

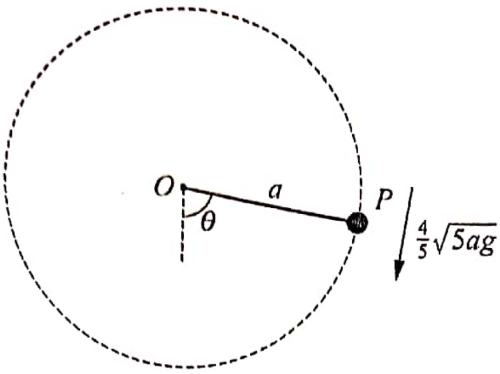
$v^2 = u^2 + 2as$

$0^2 = \left(\sqrt{\frac{10ga}{27}}\right)^2 + 2(-g)h$

$h = \dfrac{\frac{10ga}{27}}{2g} = \frac{5ga}{27g} = \frac{5a}{27}$

$\therefore$ Total height reached $= \frac{5a}{3} + \frac{5a}{27} = \frac{50a}{27}.$

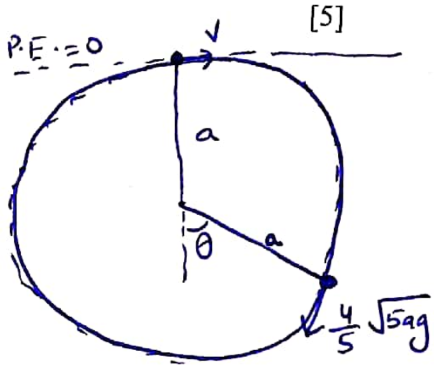
Height reached above $O = \frac{50a}{27} - a = \frac{23a}{27} = 0.852a.$



A particle P is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle P is held with the string taut and making an angle θ with the downward vertical. The particle P is then projected with speed $\frac{4}{5}\sqrt{5ag}$ perpendicular to the string and just completes a vertical circle (see diagram).

Find the value of $\cos\theta$. [5]

Let, top of the vertical circle be level of zero potential.

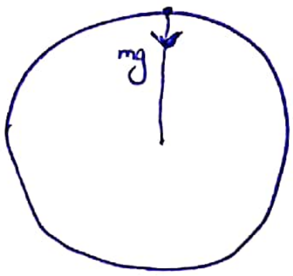


$$\frac{1}{2}mv^2 + mg(0) = \frac{1}{2}mu^2 + mg\left(-a\left(1+\cos\theta\right)\right)$$

$$\frac{v^2}{2} = \frac{u^2}{2} - \cancel{m}ga\left(1+\cos\theta\right).$$

$$v^2 = u^2 - 2ga\left(1+\cos\theta\right).$$

At top, tension is zero since particle just complete a vertical circle:



$$R(\downarrow): \quad mg = \frac{mv^2}{r}$$

$$g = \frac{v^2}{a}$$

$$ga = v^2.$$

Substituting gives:

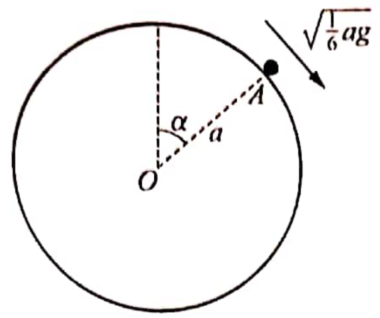
$$ga = \left(\frac{4}{5}\sqrt{5ag}\right)^2 - 2ag\left(1+\cos\theta\right)$$

$$ga = \frac{16}{\cancel{25}\,5} \times \cancel{5}ag - 2ag\left(1+\cos\theta\right)$$

$$ga = \frac{16ga}{5} - 2ag - 2ag\cos\theta$$

$$\frac{-ga}{5} = -2ag\cos\theta$$

$$\Rightarrow \cos\theta = \frac{\frac{-ga}{5}}{-2ga} = \frac{1}{10} \quad \text{i.e. } \cos\theta = \frac{1}{10}.$$



A fixed smooth solid sphere has centre O and radius a. A particle of mass m is projected downwards with speed $\sqrt{\frac{1}{6}ag}$ from the point A on the surface of the sphere, where OA makes an angle α with the upward vertical through O (see diagram). The particle moves in part of a vertical circle on the surface of the sphere. It loses contact with the sphere at the point B, where OB makes an angle β with the upward vertical through O.

Given that $\cos\alpha = \frac{2}{3}$, find the value of $\cos\beta$. [5]

$(K.E. + P.E.)_A = (K.E. + P.E.)_B$

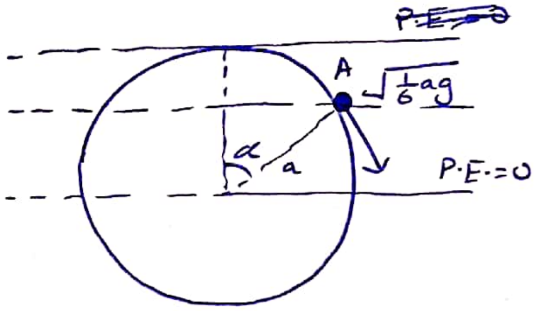
$\frac{1}{2}m\left(\sqrt{\frac{ag}{6}}\right)^2 + mga(1-\cos\alpha) = \frac{1}{2}mv^2 + mg(a)(1-\cos\beta)$

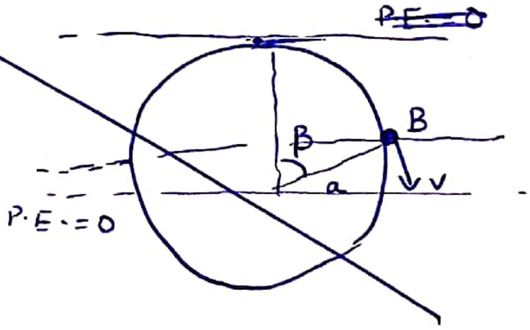
$\frac{m}{2}\left(\frac{ag}{6}\right) + mga\left[1-\frac{2}{3}\right] = \frac{mv^2}{2} + mga(1-\cos\beta)$

$\frac{ag}{12} + \frac{ga}{3} = \frac{v^2}{2} + ga(1-\cos\beta)$

$\frac{5ga}{12} - ga(1-\cos\beta) = \frac{v^2}{2}$

$\frac{5ga}{6} - 2ga(1-\cos\beta) = v^2$





At B,

R(↙): $\cancel{k}\ mg\cos\beta - R = \frac{mv^2}{r}$

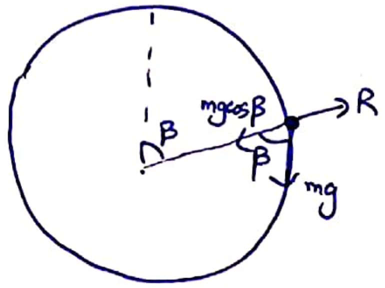
$R = mg\cos\beta - \frac{mv^2}{a}$.

Since at B, R=0 (since particle loses contact):

$0 = mg\cos\beta - \frac{mv^2}{a}$

$g\cos\beta = \frac{v^2}{a}$

$ga\cos\beta = v^2$

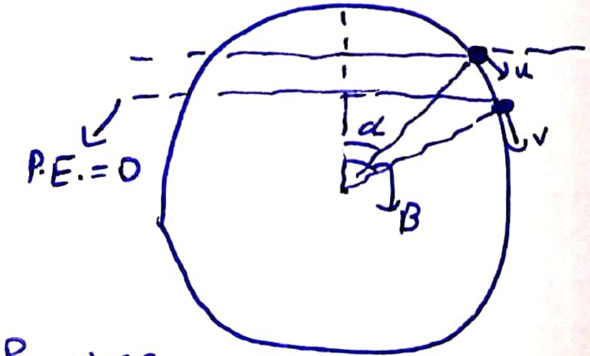


$$\cancel{\frac{5ga}{6} - 2ga(1-\cos\beta) = ga\cos\beta}$$

$$\cancel{\frac{5ga}{6} - 2ga + 2ga\cos\beta = ga\cos\beta}$$

$$\cancel{= -ga\cos\beta}$$

At B, At A,



$$\frac{1}{2}mv^2 + mg(0) = \frac{1}{2}mu^2 + mg(a\cos\alpha - a\cos\beta)$$

$$\frac{v^2}{2} = \frac{u^2}{2} + ga(\cos\alpha - \cos\beta)$$

$$v^2 = u^2 + 2ga(\cos\alpha - \cos\beta).$$

Substituting $u = \sqrt{\frac{ga}{6}}$, $\cos\alpha = \frac{2}{3}$ and $v^2 = ga\cos\beta$ gives:

$$ga\cos\beta = \left(\sqrt{\frac{ga}{6}}\right)^2 + 2ga\left(\frac{2}{3} - \cos\beta\right).$$

$$ga\cos\beta = \frac{ga}{6} + \frac{4ga}{3} - 2ga\cos\beta$$

$$3ga\cos\beta = \frac{3}{2}ga.$$

$$\cos\beta = \frac{3/2\ ga}{3ga} = \frac{1}{2}.$$

41 A particle P of mass m is moving in a horizontal circle with angular speed ω on the smooth inner surface of a hemispherical shell of radius r. The angle between the vertical and the normal reaction of the surface on P is θ.

(a) Show that $\cos\theta = \frac{g}{\omega^2 r}$. [3]

$R(\uparrow)$: $R\cos\theta = mg \Rightarrow R = \frac{mg}{\cos\theta}$.

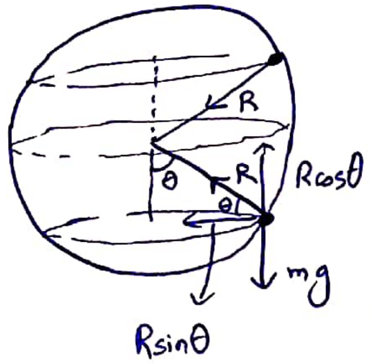
$R(\leftarrow)$: $R\sin\theta = m\cancel{r}\omega^2$

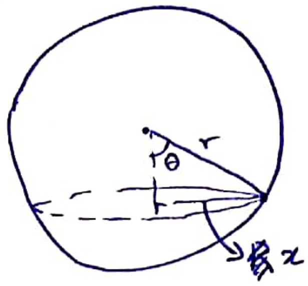
$R\sin\theta = m(r\sin\theta)\omega^2$

$\frac{mg}{\cos\theta} = mr\omega^2$

$\cos\theta = \frac{mg}{mr\omega^2}$

$\cos\theta = \frac{g}{r\omega^2}$.



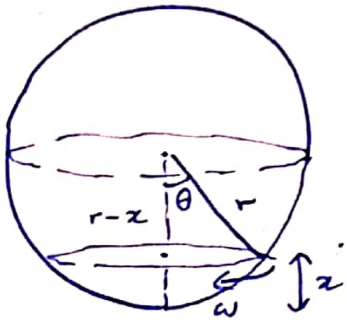


$x = r\sin\theta$

The plane of the circular motion is at a height x above the lowest point of the shell. When the angular speed is doubled, the plane of the motion is at a height $4x$ above the lowest point of the shell.

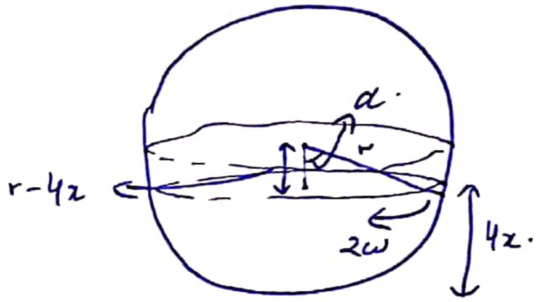
(b) Find x in terms of r. [4]

$$\cos\theta = \frac{r-x}{\cancel{r}} = \frac{g}{\omega^2 r} \rightarrow (1)$$



In the new situation,

$$\cos\alpha = \frac{r-4x}{r}$$



$$\text{and } \cos\alpha = \frac{g}{(2\omega)^2 \times r} = \frac{g}{4\omega^2 r}.$$

$$\text{Then } \frac{r-4x}{r} = \frac{g}{4\omega^2 r}$$

$$\frac{4(r-4x)}{r} = \frac{g}{\omega^2 r} \rightarrow (2)$$

Substituting gives:

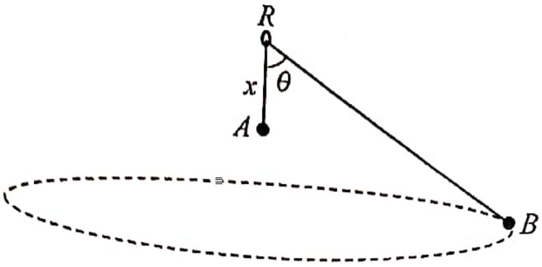
$$\frac{r-x}{\cancel{r}} = \frac{4(r-4x)}{r}$$

$$r - x = 4r - 16x$$

$$15x = 3r$$

$$x = \frac{3r}{15}$$

$$x = \frac{r}{5}$$



A light inextensible string of length a is threaded through a fixed smooth ring R. One end of the string is attached to a particle A of mass $3m$. The other end of the string is attached to a particle B of mass m. The particle A hangs in equilibrium at a distance x vertically below the ring. The angle between AR and BR is θ (see diagram). The particle B moves in a horizontal circle with constant angular speed $2\sqrt{\frac{g}{a}}$.

Show that $\cos\theta = \frac{1}{3}$ and find x in terms of a. [5]

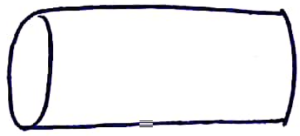
Same as 44.

43 A hollow cylinder of radius a is fixed with its axis horizontal. A particle P, of mass m, moves in part of a vertical circle of radius a and centre O on the smooth inner surface of the cylinder. The speed of P when it is at the lowest point A of its motion is $\sqrt{\frac{7}{2}ga}$.

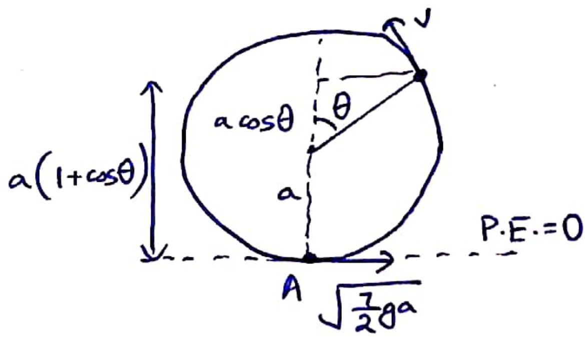
The particle P loses contact with the surface of the cylinder when OP makes an angle θ with the upward vertical through O.

(a) Show that $\theta = 60°$. [5]

Using principle of conservation of energy,



$$\frac{1}{2} m \times \left(\sqrt{\frac{7}{2}ga}\right)^2 + 0 = mg\left[a(1+\cos\theta)\right] + \frac{1}{2}mv^2$$

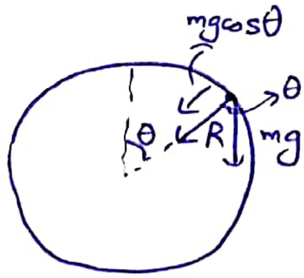


$$\frac{\cancel{m}}{2}\left(\frac{7}{2}ga\right) = \cancel{m}ga(1+\cos\theta) + \frac{\cancel{m}v^2}{2}$$

$$\frac{7ga}{4} = ga + ga\cos\theta + \frac{v^2}{2}$$

$$v^2 = \frac{7ga}{2} - 2ga - 2ga\cos\theta$$

R (towards center):



$$R + mg\cos\theta = \frac{mv^2}{r}$$

$$R + mg\cos\theta = \frac{m}{a}\left[\frac{7ga}{2} - 2ga - 2ga\cos\theta\right]$$

$$R + mg\cos\theta = \frac{7mg}{2} - 2mg - 2mg\cos\theta$$

$$R = \frac{3mg}{2} - 2mg\cos\theta - mg\cos\theta$$

$$R = \frac{3mg}{2} - 3mg\cos\theta.$$

When P loses contact, $R = 0$

$$\Rightarrow 0 = \frac{3mg}{2} - 3mg\cos\theta$$

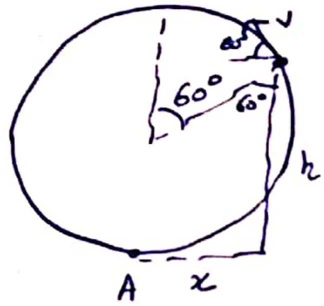
$$\cos\theta = \frac{\frac{3mg}{2}}{3mg} = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = 60°.$$

(b) Show that in its subsequent motion P strikes the cylinder at the point A. [5]

$$v^2 = \frac{7ga}{2} - 2ga - 2ga\cos\theta = \frac{3ga}{2} - 2ga\cos\theta$$

When $\theta = 60°$, $v^2 = \frac{3ga}{2} - 2ga\cos 60° = \frac{ga}{2}$.



$$v = \sqrt{\frac{ga}{2}}.$$

$$x = a\sin 60° = \frac{a\sqrt{3}}{2}.$$

i.e. The particle must move $\frac{a\sqrt{3}}{2}$m horizontally to reach A.

$$V_x = V\cos 60° = \sqrt{\frac{ga}{2}} \times \frac{1}{2} = \frac{1}{2}\sqrt{\frac{ga}{2}}.$$

$$T = \frac{x}{V_x} = \frac{\frac{a\sqrt{3}}{2}}{\frac{1}{2}\sqrt{\frac{ga}{2}}} = \frac{a\sqrt{3} \times \cancel{2}\sqrt{2}}{\cancel{2}\sqrt{ga}} = \sqrt{\frac{6a}{g}}.$$

Vertically:

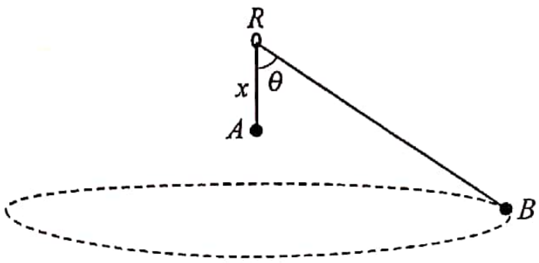
$$V_y = V\sin 60° = \sqrt{\frac{ga}{2}} \times \frac{\sqrt{3}}{2} = \sqrt{\frac{3ga}{8}}.$$

Using ~~$h=$~~ $s = ut + \frac{1}{2}at^2$

$$h = \left(\sqrt{\frac{3ga}{8}}\right) \times \sqrt{\frac{6a}{g}} - \frac{1}{2}g\left(\sqrt{\frac{6a}{g}}\right)^2 = \sqrt{\frac{18ga^2}{8\cancel{g}}} - \frac{g}{2} \times \frac{6a}{g}$$

$$= \frac{3}{2}a - 3a = \frac{-3a}{2}.$$

This corresponds to point A since height of P when it leaves cylinder is $h = a + a\cos 60° = a + a \times \frac{1}{2} = \frac{3a}{2}$.



A light inextensible string of length a is threaded through a fixed smooth ring R. One end of the string is attached to a particle A of mass $3m$. The other end of the string is attached to a particle B of mass m. The particle A hangs in equilibrium at a distance x vertically below the ring. The angle between AR and BR is θ (see diagram). The particle B moves in a horizontal circle with constant angular speed $2\sqrt{\frac{g}{a}}$.

Show that $\cos\theta = \frac{1}{3}$ and find x in terms of a. [5]

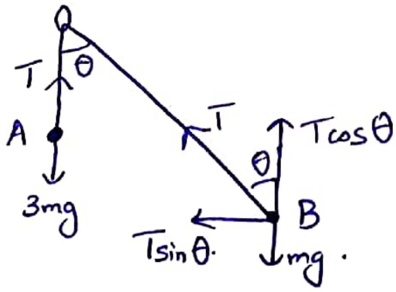
For A, $R(\uparrow)$: $T = 3mg$

For B, $R(\uparrow)$: $T\cos\theta = mg$

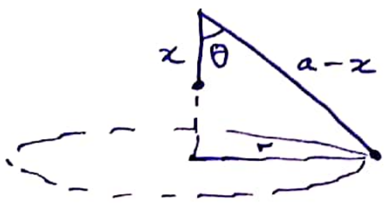
$\Rightarrow 3mg\cos\theta = mg$

$\Rightarrow \cos\theta = \frac{mg}{3mg}$

$\cos\theta = \frac{1}{3}$.



$r = (a-x)\sin\theta$.



For B, $R(\leftarrow)$: $T\sin\theta = mr\omega^2$

$T\sin\theta = m \times (a-x)\sin\theta \times \left(2\sqrt{\frac{g}{a}}\right)^2$

$3\cancel{m}\cancel{g}\,\cancel{\sin\theta} = \cancel{m} \times (a-x)\cancel{\sin\theta} \times \frac{4\cancel{g}}{a}$

$3a = 4(a-x)$

$3a = 4a - 4x$

$4x = a$

$x = \frac{a}{4}$.

45 A particle Q of mass m is attached to a fixed point O by a light inextensible string of length a. The particle moves in complete vertical circles about O. The points A and B are on the path of Q with AB a diameter of the circle. OA makes an angle of $60°$ with the downward vertical through O and OB makes an angle of $60°$ with the upward vertical through O. The speed of Q when it is at A is $2\sqrt{ag}$.

Given that T_A and T_B are the tensions in the string at A and B respectively, find the ratio $T_A : T_B$. [6]

$$(K.E. + P.E)_A = (K.E. + P.E.)_B$$

$$\frac{1}{2} m \times (2\sqrt{ag})^2 + mg(0) = \frac{1}{2} m v^2 + mg(2a\cos 60°)$$

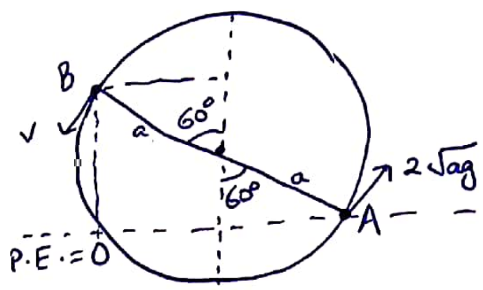
$$\frac{\cancel{m}}{2} \times 4ag = \frac{\cancel{m}v^2}{2} + 2\cancel{m}ga\left(\frac{1}{2}\right)$$

$$2ag = \frac{v^2}{2} + ga$$

$$\frac{v^2}{2} = ag$$

$$v^2 = 2ag$$

$$v = \sqrt{2ag}.$$



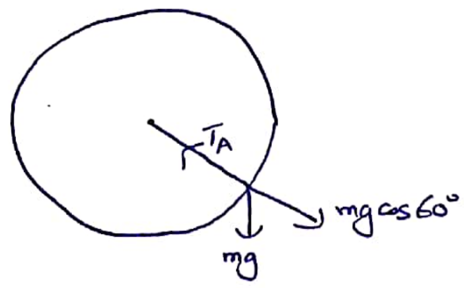
At A,

$$R(\nwarrow): T_A - mg\cos 60° = \frac{m(2\sqrt{ag})^2}{a}$$

$$T_A - \frac{mg}{2} = \frac{m \times 4\cancel{a}g}{\cancel{a}}$$

$$T_A = 4mg + \frac{mg}{2}$$

$$T_A = \frac{9mg}{2}.$$

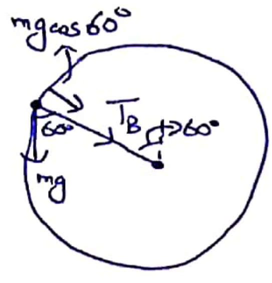


At B,

$$R(\searrow): T_B + mg\cos 60° = \frac{m(\sqrt{2ag})^2}{a}$$

$$T_B + \frac{mg}{2} = 2mg$$

$$T_B = \frac{3mg}{2}.$$



$$T_A : T_B = \frac{9mg}{2} : \frac{3mg}{2} = \boxed{3:1}$$