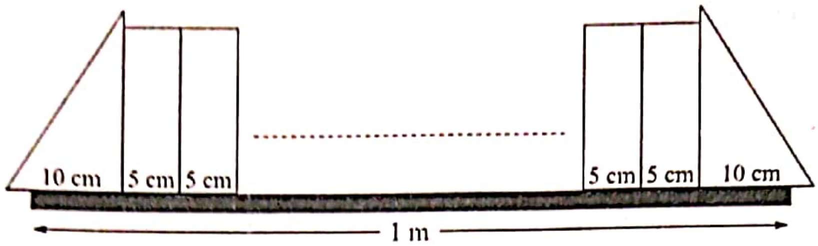
1

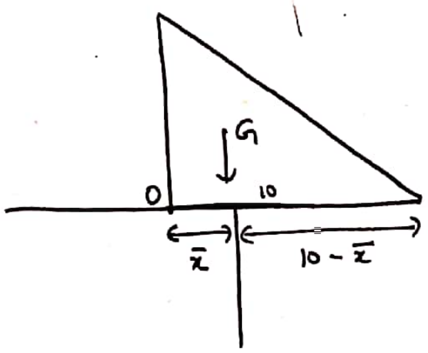


Two identical uniform heavy triangular prisms, each of base width 10 cm, are arranged as shown at the ends of a smooth horizontal shelf of length 1 m. Some books, each of width 5 cm, are placed on the shelf between the prisms.

(i) Find how far the base of a prism can project beyond an end of the shelf without the prism toppling. [2]

(ii) Find the greatest number of books that can be stored on the shelf without either of the prisms toppling. [2]

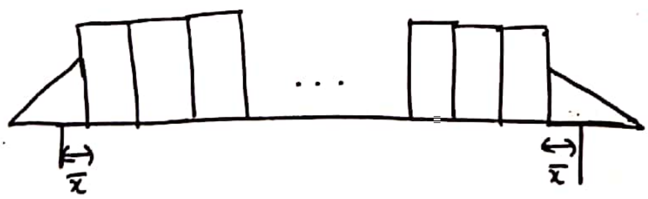
i)



$\bar{x} = \frac{1}{3}(10) = \frac{10}{3}$

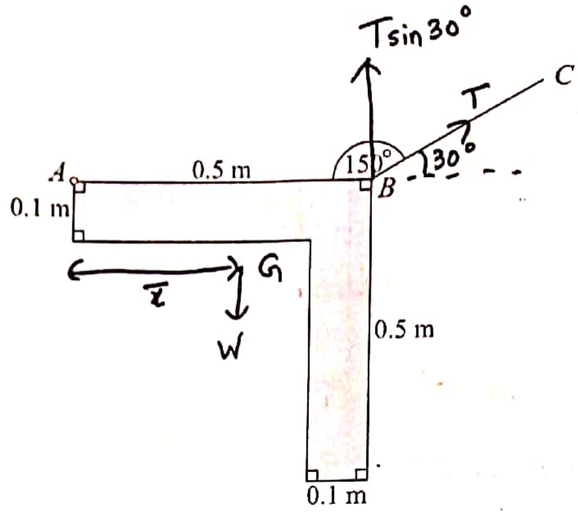
$\Rightarrow$ Prism can project beyond the end of the shelf $10 - \bar{x} = 10 - \frac{10}{3} = \frac{20}{3}$ cm without toppling.

ii)



~~Greatest~~ number of books $< \frac{100 - 2\bar{x}}{5} < \frac{100 - 2\left(\frac{10}{3}\right)}{5} < 18.66$

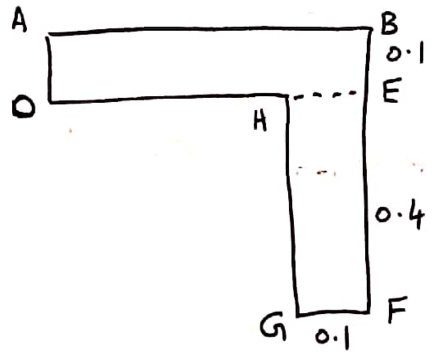
Greatest number of books $= 18$. so neither prism topples.



A uniform lamina of weight 9 N has dimensions as shown in the diagram. The lamina is freely hinged to a fixed point at A. A light inextensible string has one end attached to B, and the other end attached to a fixed point C, which is in the same vertical plane as the lamina. The lamina is in equilibrium with AB horizontal and angle $ABC = 150°$.

(i) Show that the tension in the string is 12.2 N. [5]

(ii) Find the magnitude of the force acting on the lamina at A. [4]



i) Let, ρ = Mass per unit area

~~Are~~ Mass of ABED = $0.5 \times 0.1 \times \rho = 0.05\rho$.

Mass of EFGH = $0.4 \times 0.1 \times \rho = 0.04\rho$.

Total mass of lamina = $0.05\rho + 0.04\rho = 0.09\rho$.

From AD, distance of centre of mass of ABED = 0.25 m

" " " " " " HEFG = $0.4 + \frac{0.1}{2} = 0.45$ m

" " " " " " lamina = $\bar{x}$

Taking moments about AD:

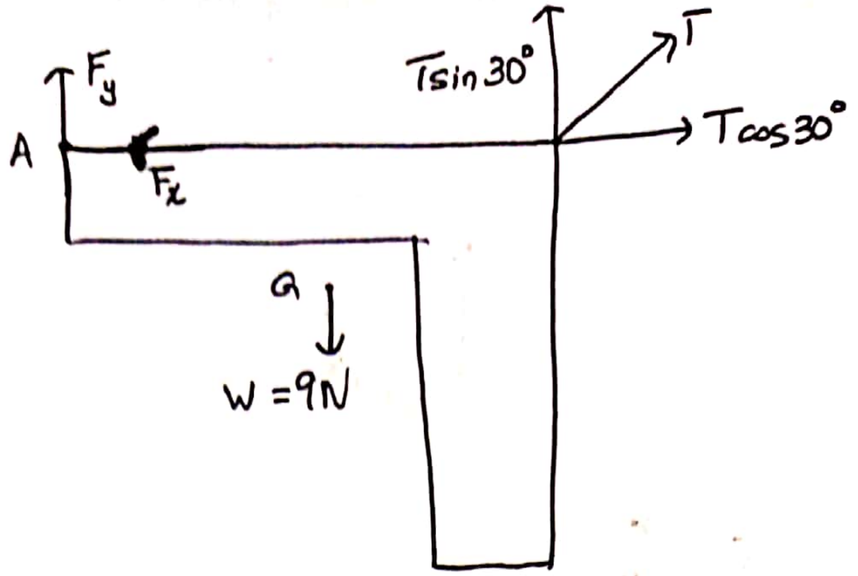
$$0.05\rho \times 0.25 + 0.04\rho \times 0.45 = 0.09\rho \times \bar{x}$$

$$\bar{x} = \frac{61}{180}.$$

Taking moments about A:

$$T \sin 30° \times 0.5 = 9 \times \frac{61}{180} \Rightarrow T = 12.2\text{N}.$$

ii)



R(↑):

$$F_y + T\sin 30° = 9$$
$$F_y + 12.2\sin 30° = 9$$
$$F_y = 2.9\text{ N}$$

R(→):

$$F_x = T\cos 30°$$
$$= 12.2\cos 30°$$
$$F_x = 6.1\sqrt{3}\text{ N.}$$

$$\Rightarrow \quad F = \sqrt{F_x^2 + F_y^2}$$
$$= \sqrt{(6.1\sqrt{3})^2 + 2.9^2} = 10.95\text{ N}$$
$$F \approx 11.0\text{ N}$$

i.e. Magnitude of force on lamina at A = 11.0N.

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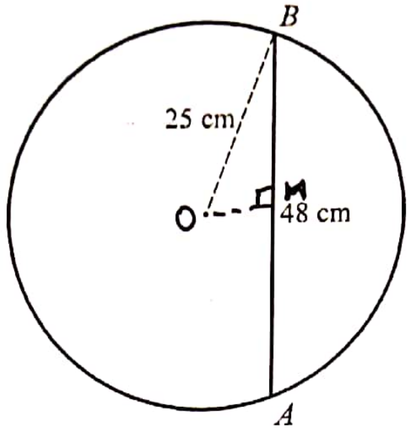

A frame consists of a uniform circular ring of radius 25 cm and mass 1.5 kg, and a uniform rod of length 48 cm and mass 0.6 kg. The ends A and B of the rod are attached to points on the circumference of the ring, as shown in the diagram. Find the distance of the centre of mass of the frame from the centre of the ring. [4]

In ΔOMB, $MB = 24\text{ cm} \Rightarrow OM = \sqrt{25^2 - 24^2} = 7\text{ cm}$.

Object	Mass (kg)	Distance of centre of mass from centre of the ring (cm)
Ring	1.5	0
Rod	0.6	7
Frame	2.1	$\bar{x}$

Taking moments about O:

$$1.5 \times 0 + 0.6 \times 7 = 2.1\,\bar{x}$$

$$2 = \bar{x}$$

i.e. Distance of centre of mass of frame from the centre of the ring $= 2$ cm.

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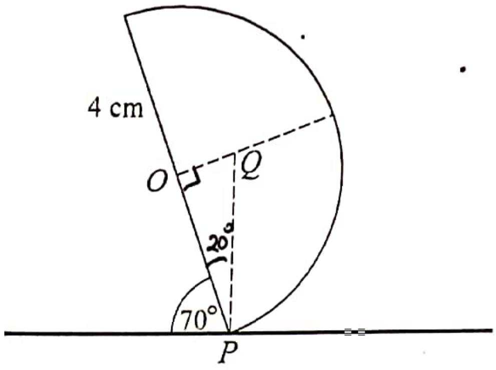


A uniform solid hemisphere, with centre O and radius 4 cm, is held so that a point P of its rim is in contact with a horizontal surface. The plane face of the hemisphere makes an angle of 70° with the horizontal. Q is the point on the axis of symmetry of the hemisphere which is vertically above P. The diagram shows the vertical cross-section of the hemisphere which contains O, P and Q.

(i) Determine whether or not the centre of mass of the hemisphere is between O and Q. [3]

The hemisphere is now released.

(ii) State whether or not the hemisphere falls on to its plane face, giving a reason for your answer. [2]

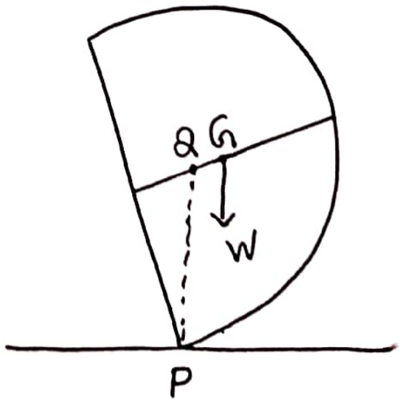
i) In ΔOPQ, $O\hat{P}Q = 20°$, $OP = 4\text{cm}$, $Q\hat{O}P = 90°$

$$\Rightarrow \quad \tan 20° = \frac{OQ}{4} \Rightarrow OQ = 1.455\text{ cm} \approx 1.46\text{cm}.$$

For the hemisphere, centre of mass, G is such that $OG = \frac{3}{8}(4) = 1.5\text{cm}$.

$\Rightarrow$ Centre of mass is not between O and Q.

ii) Hemisphere does not fall on to its plane face as line of action of weight lies to the right of point P and so it produces a clockwise moment.



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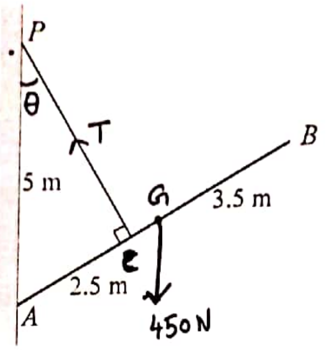


A uniform beam AB has length 6 m and mass 45 kg. One end of a light inextensible rope is attached to the beam at the point 2.5 m from A. The other end of the rope is attached to a fixed point P on a vertical wall. The beam is in equilibrium with A in contact with the wall at a point 5 m below P. The rope is taut and at right angles to AB (see diagram). Find

(i) the tension in the rope, [4]

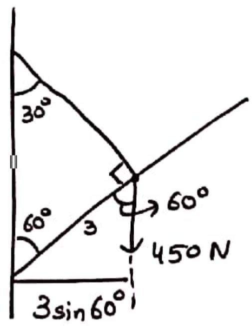
(ii) the horizontal and vertical components of the force exerted by the wall on the beam at A. [3]

i) In ΔAPC, $\sin\theta = \frac{2.5}{5} = 0.5 \Rightarrow \theta = \sin^{-1}(0.5) = 30°$.

Taking moments about A:

$T \times 2.5 = 450 \times 3\sin 60°$

$T = 467.6 \simeq 468\text{ N}$.



ii) $R(\uparrow)$: $F_y + T\cos 30° = 450$

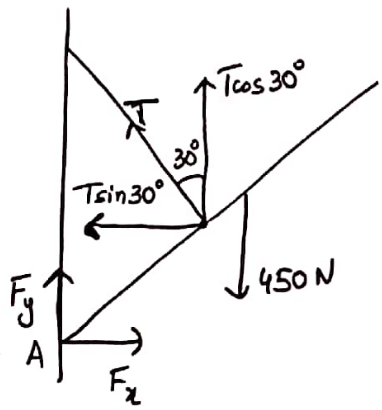
$F_y + 467.6\cos 30° = 450$

$F_y = 45\text{ N}$

$R(\parallel)$: $F_x = T\sin 30°$

$F_x = 467.6\sin 30°$

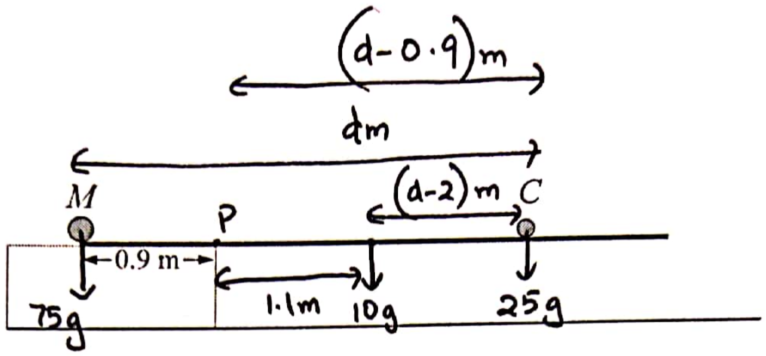
$F_x = 233.8\text{ N} \simeq 234\text{ N}$.



i.e. Horizontal component of force on beam at A = 234 N

Vertical " " " " " " " = 45 N.

6



A uniform rigid plank has mass 10 kg and length 4 m. The plank has 0.9 m of its length in contact with a horizontal platform. A man M of mass 75 kg stands on the end of the plank which is in contact with the platform. A child C of mass 25 kg walks on to the overhanging part of the plank (see diagram). Find the distance between the man and the child when the plank is on the point of tilting. [4]

Taking moments about P:

$$75g \times 0.9 = 10g \times 1.1 + 25g \times (d - 0.9)$$

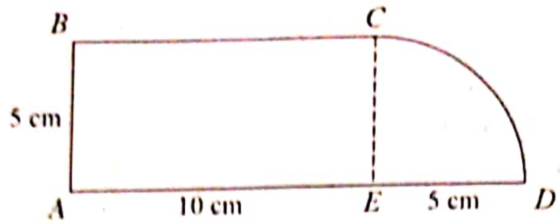
$$67.5 = 11 + 25d - 22.5$$

$$d = \frac{67.5 - 11 + 22.5}{25}$$

$$d = 3.16 \text{ m}$$

$\Rightarrow$ Distance between the man and the child $= 3.16$ m. when plank is on point of tilting

7

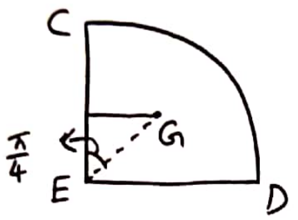


A uniform lamina $ABCDE$ consists of a rectangular part with sides 5 cm and 10 cm, and a part in the form of a quarter of a circle of radius 5 cm, as shown in the diagram.

(i) Show that the distance of the centre of mass of the part CDE of the lamina is $\frac{20}{3\pi}$ cm from CE. [2]

(ii) Find the distance of the centre of mass of the lamina $ABCDE$ from the edge AB. [4]

i)



Let, G be centre of mass of quarter circle CED.

$$EG = \frac{2r \sin\alpha}{3\alpha} = \frac{2(5)\sin\left(\frac{\pi}{4}\right)}{3\left(\frac{\pi}{4}\right)} = \frac{10\left(\frac{\sqrt{2}}{2}\right)}{\frac{3\pi}{4}} = \frac{5\sqrt{2}\times 4}{3\pi} = \frac{20\sqrt{2}}{3\pi}.$$

$\Rightarrow$ Distance of centre of mass G from CE $= EG\cos\frac{\pi}{4} = \frac{20\sqrt{2}}{3\pi} \times \frac{1}{\sqrt{2}} = \frac{20}{3\pi}$.

(Shown).

ii) Let, ρ = Mass per unit area.

Mass of rectangle ABCE $= 5(10)\rho = 50\rho$.

Mass of sector CED $= \frac{\pi(5)^2}{4}\rho = \frac{25\pi\rho}{4}$.

Total mass of composite figure $= 50\rho + \frac{25\pi\rho}{4} = \left(50 + \frac{25\pi}{4}\right)\rho$.

From AB, distance of centre of mass of rectangle = 5 cm

" " " " " " sector $= 10 + \frac{20}{3\pi}$

" " " " " " composite figure $= \bar{x}$.

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Taking moments about AB:

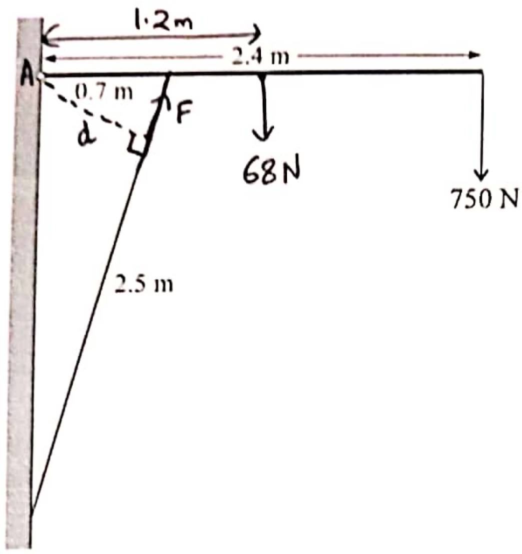
$$50\rho \times 5 + \frac{25\pi\rho}{4} \times \left(10 + \frac{20}{3\pi}\right) = \left(50 + \frac{25\pi}{4}\right)\rho \times \bar{x}$$

$$250 + \frac{250\pi}{4} + \frac{125}{3} = \left(\frac{4(50) + 25\pi}{4}\right)\bar{x}$$

$$\bar{x} = 7.008$$

$$\bar{x} \approx 7.01 \text{ cm.}$$

i.e. Distance of centre of mass of lamina from AB = 7.01 cm.



A uniform beam has length 2.4 m and weight 68 N. The beam is hinged at a fixed point of a vertical wall, and held in a horizontal position by a light rod of length 2.5 m. One end of the rod is attached to the beam at a point 0.7 m from the wall, and the other end of the rod is attached to the wall at a point vertically below the hinge. The beam carries a load of 750 N at its end (see diagram).

(i) Find the force in the rod. [4]

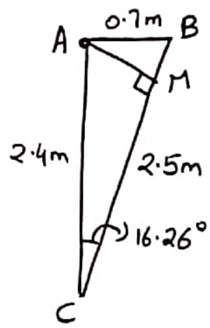
The components of the force exerted by the hinge on the beam are X N horizontally towards the wall and Y N vertically downwards.

(ii) Find the values of X and Y. [3]

i) In ΔABC, $AC = \sqrt{2.5^2 - 0.7^2} = 2.4\,m$.

$\tan \hat{C} = \frac{0.7}{2.4} \Rightarrow \hat{C} = \tan^{-1}\left(\frac{7}{24}\right) = 16.26°$

In ΔAMC, $\sin 16.26° = \frac{AM}{2.4} \Rightarrow AM = 0.672\,m.$



Taking moments about the hinge at A:

$F \times 0.672 = 68 \times 1.2 + 750 \times 2.4$

$F = 2800$ N.

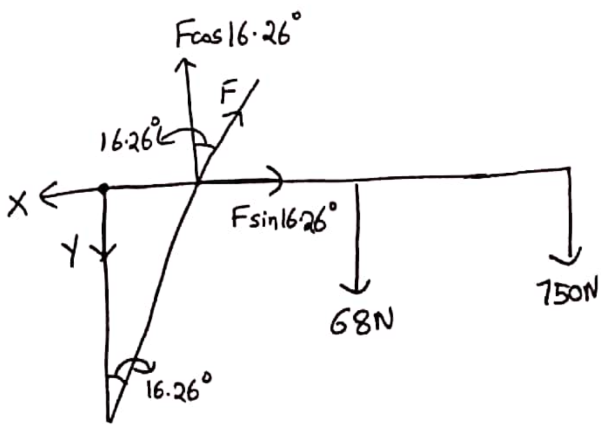
ii) $R(\uparrow)$: $F\cos 16.26° = Y + 68 + 750$

$2800 \cos 16.26° = Y + 818$

$Y = 1870$ N

$R(\rightarrow)$: $X = F \sin 16.26°$

$X = 2800 \sin 16.26° = 784$ N



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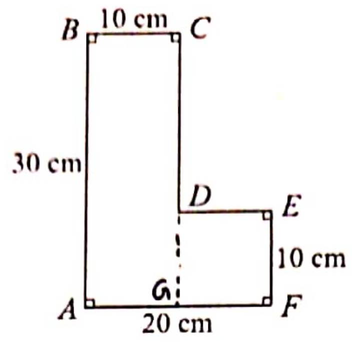


Fig. 1

ABCDEF is the L-shaped cross-section of a uniform solid. This cross-section passes through the centre of mass of the solid and has dimensions as shown in Fig. 1.

(i) Find the distance of the centre of mass of the solid from the edge *AB* of the cross-section. [3]

Let, ρ = Mass per unit area

Mass of rectangle ABCG = $30(10)\rho = 300\rho$.

Mass of square DEFG = $10(10)\rho = 100\rho$.

Total mass of cross-section = 400ρ.

From AB, distance of centre of mass of rectangle ABCG = 5cm

" " " " " " square DEFG = 10+5 = 15cm

" " " " " " cross-section = $\bar{x}$.

Taking moments about AB:

$$400\rho \times \bar{x} = 300\rho \times 5 + 100\rho \times 15$$

$$\bar{x} = 7.5\text{cm}$$

$\Rightarrow$ Distance of centre of mass of solid from edge AB = 7.5cm.

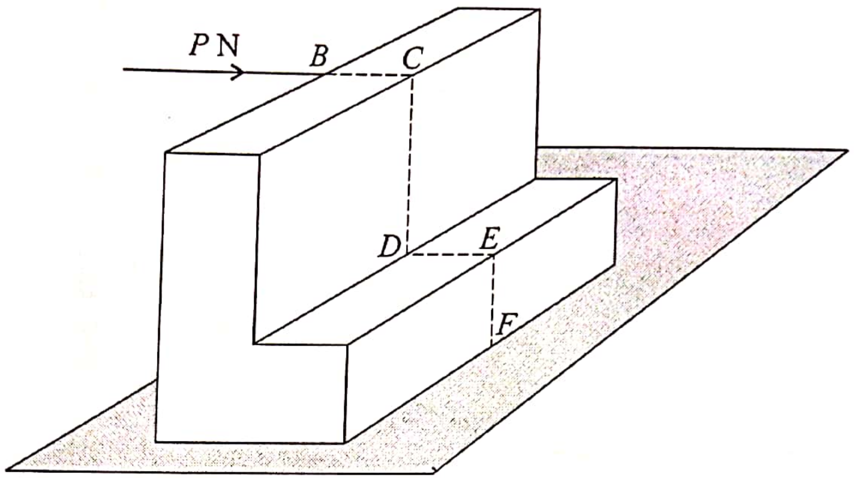


Fig. 2

The solid rests in equilibrium with the face containing the edge AF of the cross-section in contact with a horizontal table. The weight of the solid is W N. A horizontal force of magnitude P N is applied to the solid at the point B, in the direction of BC (see Fig. 2). The table is sufficiently rough to prevent sliding.

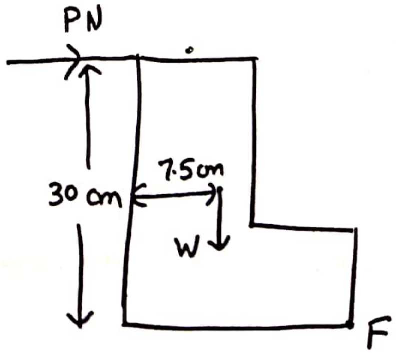
(ii) Find P in terms of W, given that the equilibrium of the solid is about to be broken. [3]

Taking moments about F,

$$P \times 30 = W \times (20 - 7.5)$$

$$P = \frac{W \times 12.5}{30}$$

$$P = \frac{5W}{12}.$$



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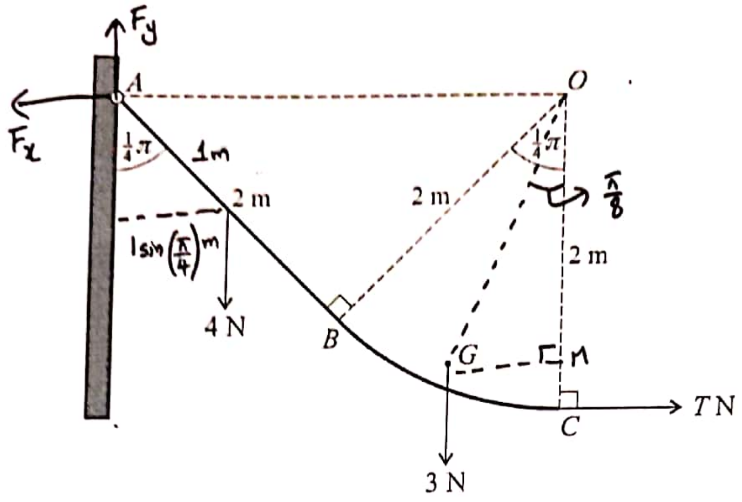


A rigid rod consists of two parts. The part BC is in the form of an arc of a circle of radius 2 m and centre O, with angle $BOC = \frac{1}{4}\pi$ radians. BC is uniform and has weight 3 N. The part AB is straight and of length 2 m; it is uniform and has weight 4 N. The part AB of the rod is a tangent to the arc BC at B. The end A of the rod is freely hinged to a fixed point of a vertical wall. The rod is held in equilibrium, with the straight part AB making an angle of $\frac{1}{4}\pi$ radians with the wall, by means of a horizontal string attached to C. The string is in the same vertical plane as the rod, and the tension in the string is T N (see diagram).

(i) Show that the centre of mass G of the part BC of the rod is at a distance of 2.083 m from the wall, correct to 4 significant figures. [4]

(ii) Find the value of T. [3]

(iii) State the magnitude of the horizontal component and the magnitude of the vertical component of the force exerted on the rod by the hinge. [1]

i) $OG = \dfrac{r\sin\alpha}{\alpha} = \dfrac{2\sin\left(\frac{\pi}{8}\right)}{\frac{\pi}{8}} = 1.9489\,m$

In $\triangle OGM$, $\sin\left(\frac{\pi}{8}\right) = \dfrac{GM}{OG} \Rightarrow \sin\left(\frac{\pi}{8}\right) = \dfrac{GM}{1.9489} \Rightarrow GM = 0.7458\,m$

$OA = \sqrt{2^2 + 2^2} = \sqrt{8}$

$\Rightarrow$ Distance of G from wall $= OA - GM = \sqrt{8} - 0.7458 = 2.0825 \approx 2.083\,m$ (Shown).

ii) Taking moments about A:

$4 \times 1\sin\left(\frac{\pi}{4}\right) + 3 \times 2.083 = T \times 2$

$4.538\text{ N} = T$

$\Rightarrow T \approx 4.54\text{ N}.$

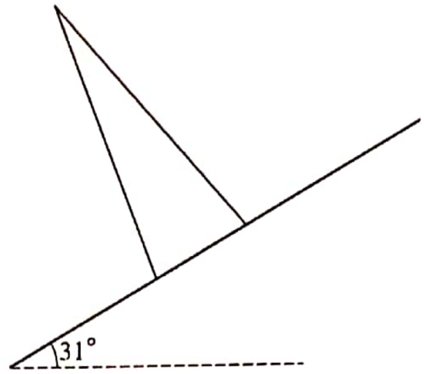
iii) $R(\uparrow)$: $F_y = 4+3 \Rightarrow F_y = 7N$ and $R(\rightarrow)$: $F_x = T = 4.54N$.

$\Rightarrow$ Horizontal component $= 4.54\,N$

Vertical component $= 7\,N$.

11 A uniform solid cone has height 38 cm.

(i) Write down the distance of the centre of mass of the cone from its base. [1]



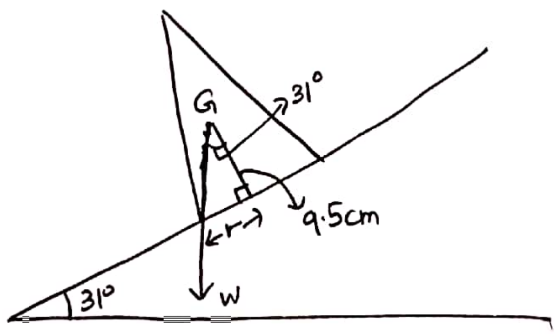
The cone is placed with its axis vertical on a rough horizontal plane. The plane is slowly tilted, and the cone remains in equilibrium until the angle of inclination of the plane reaches 31° (see diagram), when the cone topples.

(ii) Find the radius of the cone. [2]

(iii) Show that $\mu \geqslant 0.601$, correct to 3 significant figures, where μ is the coefficient of friction between the cone and the plane. [2]

i) Distance of centre of mass of cone $= \frac{1}{4}(38\,\text{cm}) = 9.5\,\text{cm}$.

ii)



$\tan 31° = \dfrac{r}{9.5} \Rightarrow r = 9.5 \tan 31° = 5.708\text{ cm} \approx 5.71\text{ cm}.$

iii) $R(\perp)$: $R = mg\cos 31°$

$R(\parallel)$: $f = mg\sin 31°$

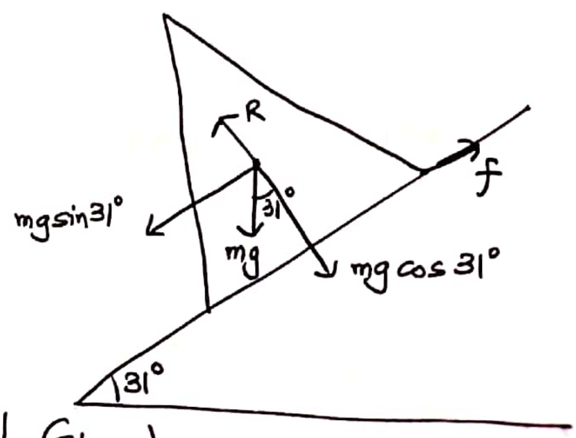
Since cone does not slip,

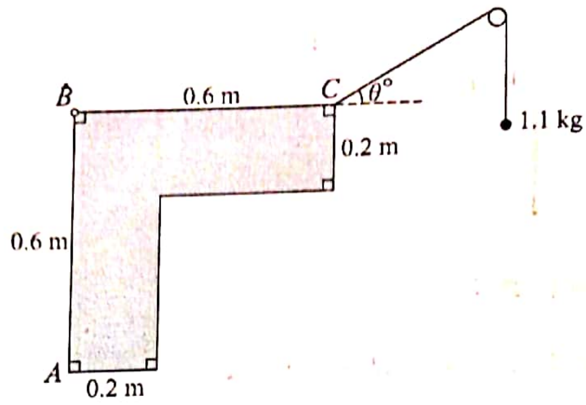
$f \leq \mu R$

$mg\sin 31° \leq \mu(mg\cos 31°)$

$\tan 31° \leq \mu \Rightarrow \mu \geq \tan 31°$

$\mu \geq 0.6008 \approx 0.601$ (Shown).



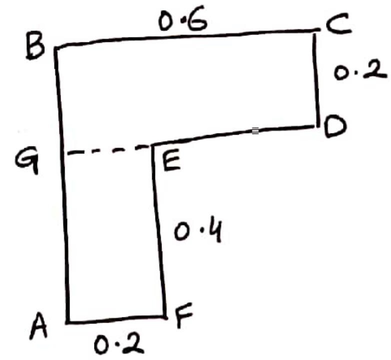


A uniform lamina of weight 15 N has dimensions as shown in the diagram.

(i) Show that the distance of the centre of mass of the lamina from AB is 0.22 m. [4]

The lamina is freely hinged at B to a fixed point. One end of a light inextensible string is attached to the lamina at C. The string passes over a fixed smooth pulley and a particle of mass 1.1 kg is attached to the other end of the string. The lamina is in equilibrium with BC horizontal. The string is taut and makes an angle of $\theta°$ with the horizontal at C, and the particle hangs freely below the pulley (see diagram).

(ii) Find the value of θ. [3]



i) Let, ρ = Mass per unit area

Mass of rectangle BCDG $= 0.6 \times 0.2 \times \rho = 0.12\rho$

Mass of rectangle GEAF $= 0.4 \times 0.2 \times \rho = 0.08\rho$.

Total mass of lamina $= 0.20\rho$.

From AB, distance of centre of mass of:

rectangle BCDG $= \frac{0.6}{2} = 0.3\text{ m}$

rectangle GEAF $= \frac{0.2}{2} = 0.1\text{ m}$

composite lamina $= \bar{x}$

Taking moments about AB:

$$0.20\rho \times \bar{x} = 0.12\rho \times 0.3 + 0.08\rho \times 0.1$$

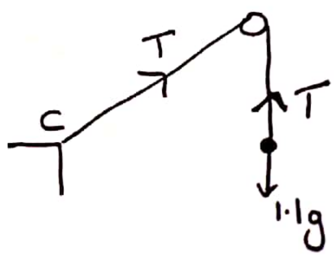
$$0.2\bar{x} = 0.036 + 0.008$$

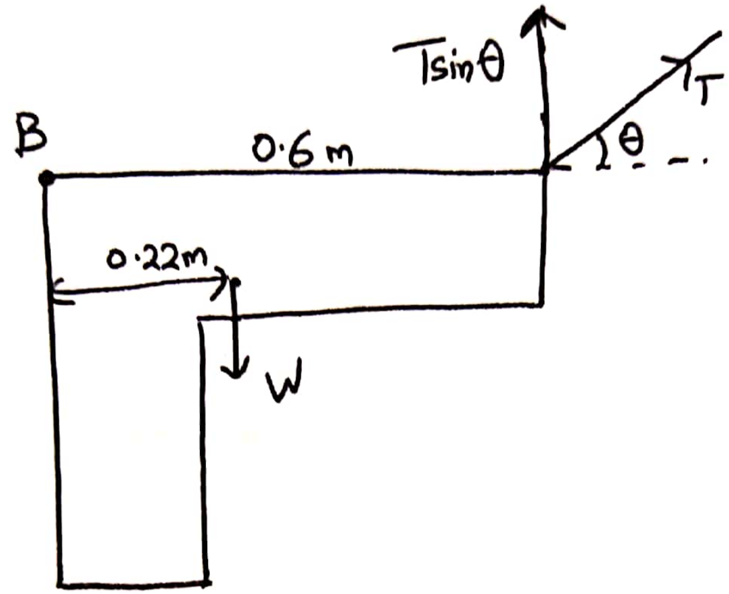
$$\bar{x} = 0.22$$

$\Rightarrow$ Distance of centre of mass of lamina from AB $= 0.22\text{ m}$ (Shown).

ii) $R(\uparrow)$: $T = 1.1g$

$\Rightarrow T = 11\text{ N}$.





Taking moments about B :

$$T\sin\theta \times 0\cdot6 = W \times 0\cdot22$$

$$\sin\theta = \frac{15 \times 0\cdot22}{0\cdot6 \times 11}$$

$$\sin\theta = \frac{1}{2}$$

$$\theta = 30^\circ.$$

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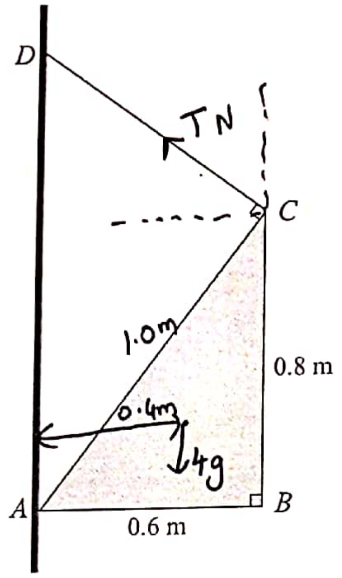


A uniform triangular lamina ABC is right-angled at B and has sides $AB = 0.6$ m and $BC = 0.8$ m. The mass of the lamina is 4 kg. One end of a light inextensible rope is attached to the lamina at C. The other end of the rope is attached to a fixed point D on a vertical wall. The lamina is in equilibrium with A in contact with the wall at a point vertically below D. The lamina is in a vertical plane perpendicular to the wall, and AB is horizontal. The rope is taut and at right angles to AC (see diagram). Find

(i) the tension in the rope, [4]

(ii) the horizontal and vertical components of the force exerted at A on the lamina by the wall. [3]

i) In ΔABC, $AC = \sqrt{0.6^2+0.8^2} = 1$ m.

For ΔABC, coordinates of centre of mass of $\Delta ABC = \left(\frac{0+0.6+0.6}{3}, \frac{0+0+0.8}{3}\right)$

$= \left(0.4, \frac{0.8}{3}\right)$

Taking moments about A :

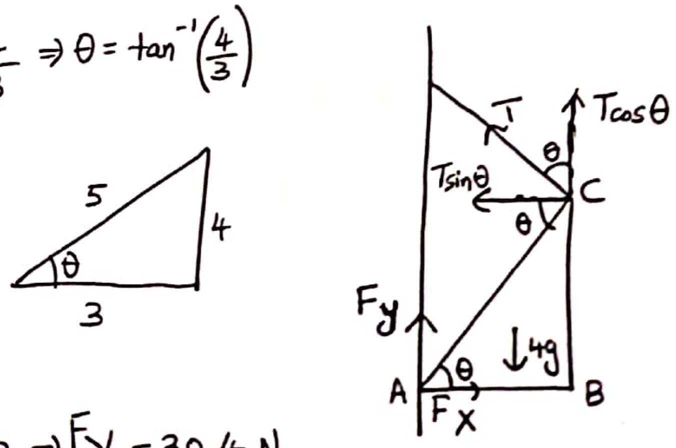
$4g \times 0.4 = T \times 1$

$T = 16$ N

ii) In ΔABC, $\tan\theta = \frac{0.8}{0.6} = \frac{4}{3} \Rightarrow \theta = \tan^{-1}\left(\frac{4}{3}\right)$

$\sin\theta = \frac{4}{5}$

$\cos\theta = \frac{3}{5}$



$R(\uparrow)$: $F_y + T\cos\theta = 4g$

$F_y + 16\left(\frac{3}{5}\right) = 4 \times 10 \Rightarrow F_y = 30.4$ N

$R(\rightarrow)$:

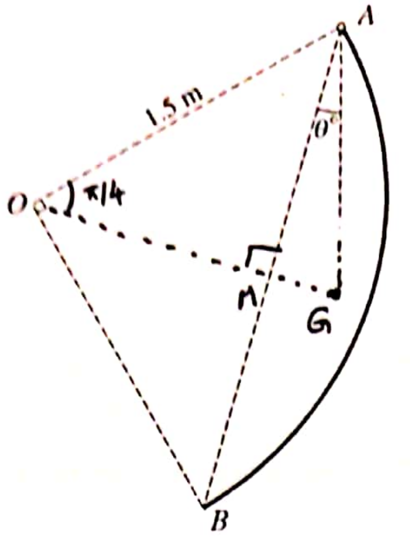
$$F_X = T\sin\theta$$

$$F_X = 16\left(\frac{4}{5}\right)$$

$$F_X = 12.8\ N$$

i.e. ~~Vertical~~ Horizontal component of force on lamina by wall = 12.8 N

Vertical " " " " " " " = 30.4 N.



A uniform rigid wire AB is in the form of a circular arc of radius 1.5 m with centre O. The angle AOB is a right angle. The wire is in equilibrium, freely suspended from the end A. The chord AB makes an angle of $\theta°$ with the vertical (see diagram).

(i) Show that the distance of the centre of mass of the arc from O is 1.35 m, correct to 3 significant figures. [2]

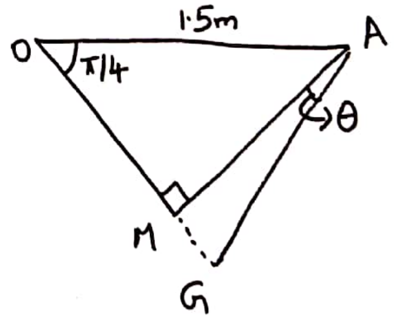
(ii) Find the value of θ. [3]

i) $OG = \dfrac{r \sin\alpha}{\alpha} = \dfrac{1.5 \sin\left(\frac{\pi}{4}\right)}{\frac{\pi}{4}} = \dfrac{4 \times 1.5 \sin\left(\frac{\pi}{4}\right)}{\pi} = \dfrac{6 \sin\left(\frac{\pi}{4}\right)}{\pi} = 1.350\text{ m}$

$OG \approx 1.35\text{m}$ (3 s.f.)

ii) In ΔOAM, $OM = 1.5\cos\left(\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{4}$ m

$AM = 1.5 \sin\left(\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{4}$ m.



$\Rightarrow MG = OG - OM$

$= 1.350 - \frac{3\sqrt{2}}{4} = 0.2898$

In ΔAMG,

$\tan\theta = \dfrac{MG}{AM} = \dfrac{0.2898}{\frac{3\sqrt{2}}{4}} \Rightarrow \tan\theta = 0.2732$

$\theta = 15.28° \approx 15.3°$

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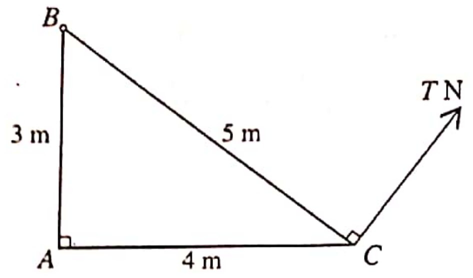


Uniform rods AB, AC and BC have lengths 3 m, 4 m and 5 m respectively, and weights 15 N, 20 N and 25 N respectively. The rods are rigidly joined to form a right-angled triangular frame ABC. The frame is hinged at B to a fixed point and is held in equilibrium, with AC horizontal, by means of an inextensible string attached at C. The string is at right angles to BC and the tension in the string is T N (see diagram).

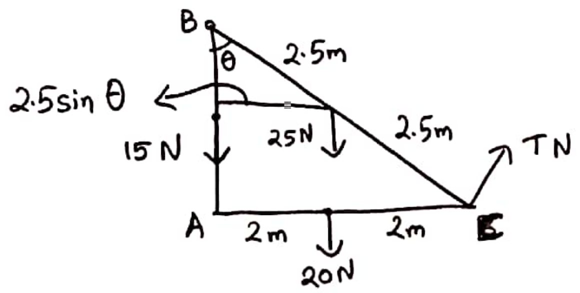
(i) Find the value of T. [2]

A uniform triangular lamina PQR, of weight 60 N, has the same size and shape as the frame ABC. The lamina is now attached to the frame with P, Q and R at A, B and C respectively. The composite body is held in equilibrium with A, B and C in the same positions as before. Find

(ii) the new value of T, [2]

(iii) the magnitude of the vertical component of the force acting on the composite body at B. [2]

i)



In ΔABC, $\cos\theta = \frac{3}{5}$ and $\sin\theta = \frac{4}{5}$

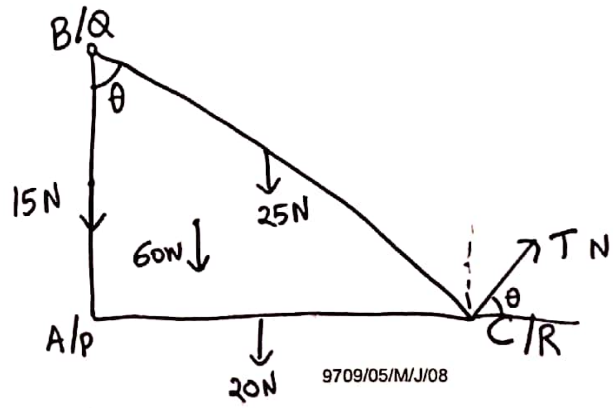
Taking moments about B:

$$25 \times 2.5 \sin\theta + 20 \times 2 = T \times 5$$

$$25 \times 2.5 \times \frac{4}{5} + 40 = 5T$$

$$\Rightarrow T = 18\text{ N}.$$

ii)



Taking moments about B:

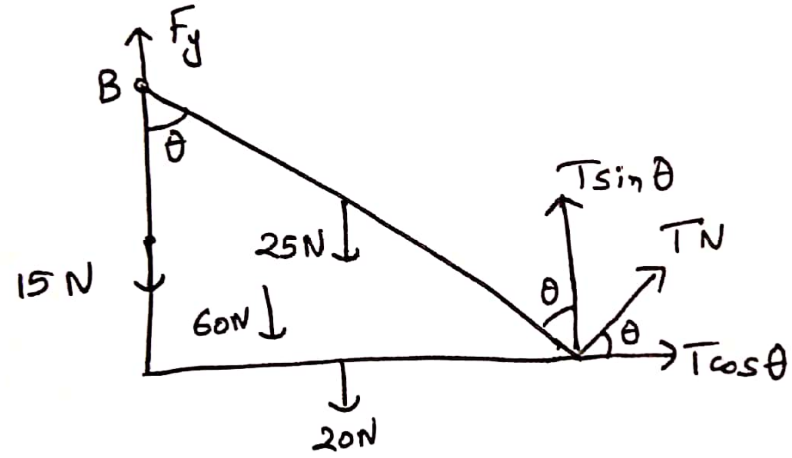
$$15 \times 0 + 25 \times 2.5 \times \sin\theta + 20 \times 2 + 60 \times \left(\frac{0+0+4}{3}\right) = T \times 5$$

$$50 + 40 + 80 = 5T$$

$$T = 34\,N.$$

x-coordinate of centre of mass of triangular lamina

iii)



R(↑):

$$F_y + T\sin\theta = 15 + 25 + 20 + 60$$

$$F_y + 34\left(\frac{4}{5}\right) = 120$$

$$F_y = 92.8N$$

i.e. Vertical component of force on composite body at B = 92.8N.

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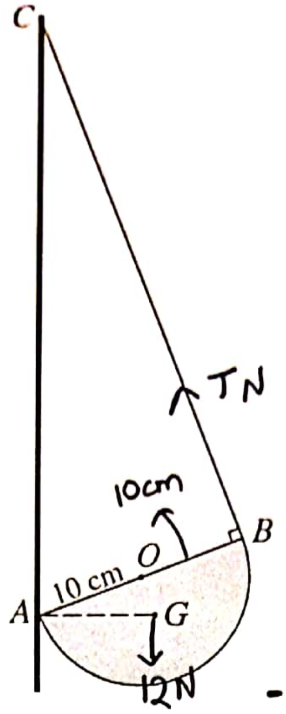

AB is a diameter of a uniform solid hemisphere with centre O, radius 10 cm and weight 12 N. One end of a light inextensible string is attached to the hemisphere at B and the other end is attached to a fixed point C of a vertical wall. The hemisphere is in equilibrium with A in contact with the wall at a point vertically below C. The centre of mass G of the hemisphere is at the same horizontal level as A, and angle ABC is a right angle (see diagram). Calculate the tension in the string. [4]

For the hemisphere, $OG = \frac{3}{8}(10) = 3.75\text{cm}$.

In ΔAOG, $AG = \sqrt{10^2 + 3.75^2} = 10.68$ cm.

Taking moments about A,

$$12 \times 10.68 = T \times 20$$

$$T = \frac{12 \times 10.68}{20} = 6.408\text{N}$$

$$T \approx 6.41\text{ N}$$

i.e. Tension in the string $= 6.41$N.

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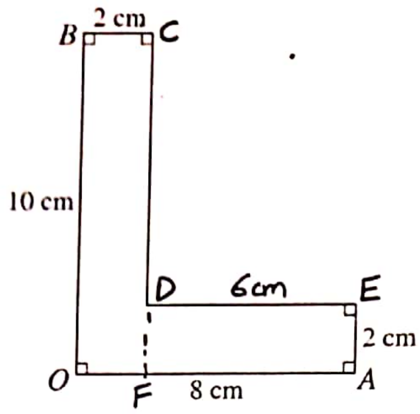


Fig. 1

A uniform solid body has a cross-section as shown in Fig. 1.

(i) Show that the centre of mass of the body is 2.5 cm from the plane face containing OB and 3.5 cm from the plane face containing OA. [4]

Let, ρ = Mass per unit area.

Mass of OBCF $= 10 \times 2 \times \rho = 20\rho$; Mass of AFDE $= 6(2)\rho = 12\rho$

$\Rightarrow$ Mass of composite figure $= 20\rho + 12\rho = 32\rho$.

Coordinates of centre of mass of OBCF $= (1, 5)$

" " " " " " AFDE $= (5, 1)$.

" " " " " " composite figure $= (\bar{x}, \bar{y})$

Taking moments about OB:

$20\rho \times 1 + 12\rho \times 5 = 32\rho \times \bar{x} \Rightarrow \bar{x} = 2.5$ cm

Taking moments about OA:

$20\rho \times 5 + 12\rho \times 1 = 32\rho \times \bar{y} \Rightarrow \bar{y} = 3.5$ cm

ie. distance of centre of mass from OB = 2.5 cm (Shown)

" " " " " " OA = 3.5 cm (Shown).

(ii) The solid is placed on a rough plane which is initially horizontal. The coefficient of friction between the solid and the plane is μ.

(a)

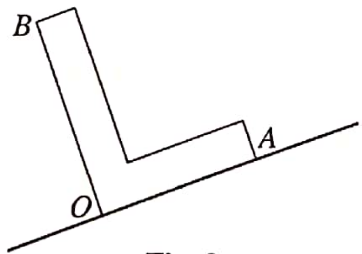
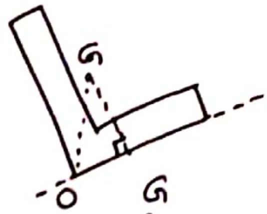


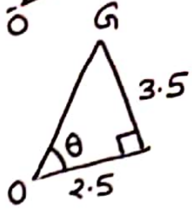
Fig. 2

The solid is placed with OA in contact with the plane, and then the plane is tilted so that OA lies along a line of greatest slope with A higher than O (see Fig. 2). When the angle of inclination is sufficiently great the solid starts to topple (without sliding). Show that $\mu > \frac{5}{7}$. [5]

When solid starts to topple, the centre of mass is vertically above O.

$\tan\theta = \frac{2.5}{3.5} \Rightarrow \theta = \tan^{-1}\left(\frac{5}{7}\right)$.





Since object does not slide, $f \leq \mu R$.

$R(//)$: $f = W\sin\theta$.

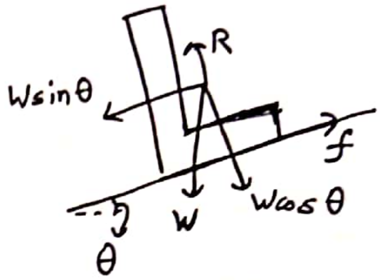
$R(\perp)$: $R = W\cos\theta$.

Since $f \leq \mu R$

$W\sin\theta \leq \mu(W\cos\theta)$

$\frac{\sin\theta}{\cos\theta} \leq \mu$

$\Rightarrow \mu \geq \tan\theta$



But since $\theta = \tan^{-1}\left(\frac{5}{7}\right) \Rightarrow \tan\theta = \frac{5}{7}$

$\Rightarrow \mu \geq \frac{5}{7}$. (shown).

(b)

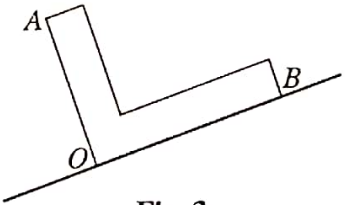
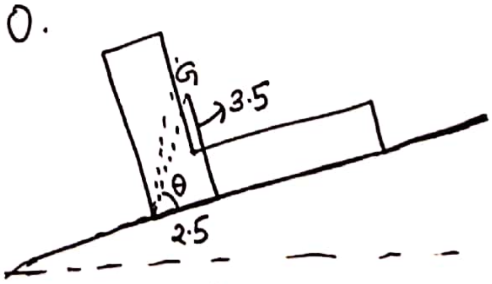


Fig. 3

Instead, the solid is placed with OB in contact with the plane, and then the plane is tilted so that OB lies along a line of greatest slope with B higher than O (see Fig. 3). When the angle of inclination is sufficiently great the solid starts to slide (without toppling). Find another inequality for μ. [2]

Toppling occurs when G lies vertically above O.

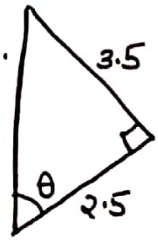
~~Since~~ Then $\tan\theta = \frac{3.5}{2.5} = \frac{7}{5}$



But since toppling does not occur,

$\hat{\theta}$, Angle of inclination $< \theta$

$\Rightarrow$ ~~tan~~ (angle of inclination) $< \tan\theta = \frac{7}{5} \Rightarrow \tan\hat{\theta} < \frac{7}{5}$.



At point of sliding, $f = \mu R$

$\Rightarrow W\sin\hat{\theta} = \mu(W\cos\hat{\theta})$

$\mu = \tan\hat{\theta}$ where $\hat{\theta}$ is angle of inclination.

Then $\mu = \tan\hat{\theta} < \frac{7}{5}$ i.e. $\mu < \frac{7}{5}$.

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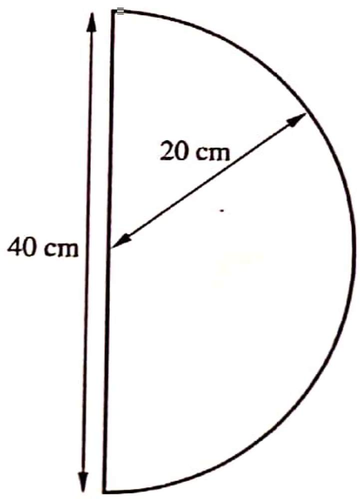

A frame consists of a uniform semicircular wire of radius 20 cm and mass 2 kg, and a uniform straight wire of length 40 cm and mass 0.9 kg. The ends of the semicircular wire are attached to the ends of the straight wire (see diagram). Find the distance of the centre of mass of the frame from the straight wire. [4]

Object	Mass	Distance of centre of mass from straight wire
Semicircular wire	2 kg	$\dfrac{20\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \dfrac{40}{\pi}$
Straight wire	0.9 kg	0
Composite wire	2.9 kg	$\bar{x}$

Taking moments about straight wire :

$$2.9\bar{x} = 2 \times \frac{40}{\pi} + 0.9 \times 0$$

$$\bar{x} = 8.780 \approx 8.78 \text{ cm}.$$

$\Rightarrow$ Distance of centre of mass of frame from straight wire $= 8.78$ cm.

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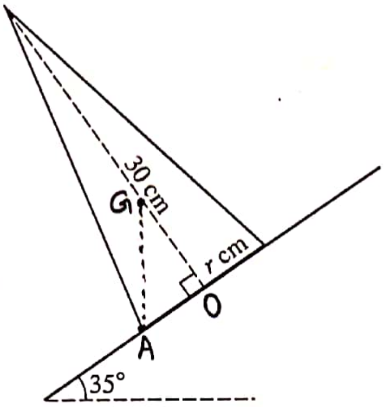


A uniform solid cone has height 30 cm and base radius r cm. The cone is placed with its axis vertical on a rough horizontal plane. The plane is slowly tilted and the cone remains in equilibrium until the angle of inclination of the plane reaches 35°, when the cone topples. The diagram shows a cross-section of the cone.

(i) Find the value of r. [3]

(ii) Show that the coefficient of friction between the cone and the plane is greater than 0.7. [2]

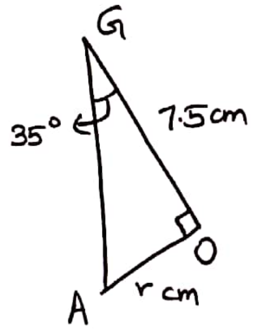
i) When the cone topples, G is vertically above A.

$OG = \frac{1}{4}(30) = 7.5$ cm.

In ΔOGA, $\tan 35^\circ = \frac{r}{7.5}$

$r = 7.5 \tan 35^\circ$

$r = 5.251 \approx 5.25$ cm.



ii) $R(\parallel)$: $f = W \sin 35^\circ$

$R(\perp)$: $R = W \cos 35^\circ$

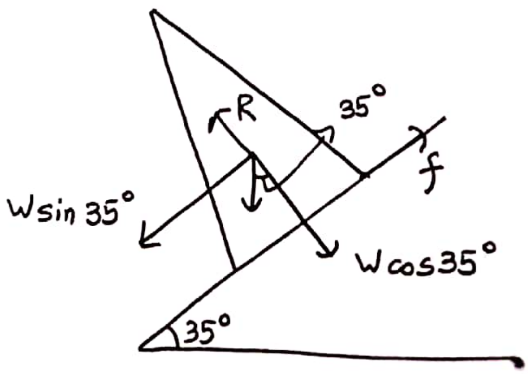
Since cone does not slide,

$f \leq \mu R$

$W \sin 35^\circ \leq \mu (W \cos 35^\circ)$

$\tan 35^\circ \leq \mu$

$\Rightarrow \mu \geq 0.7002 \Rightarrow \mu > 0.700$ (shown).



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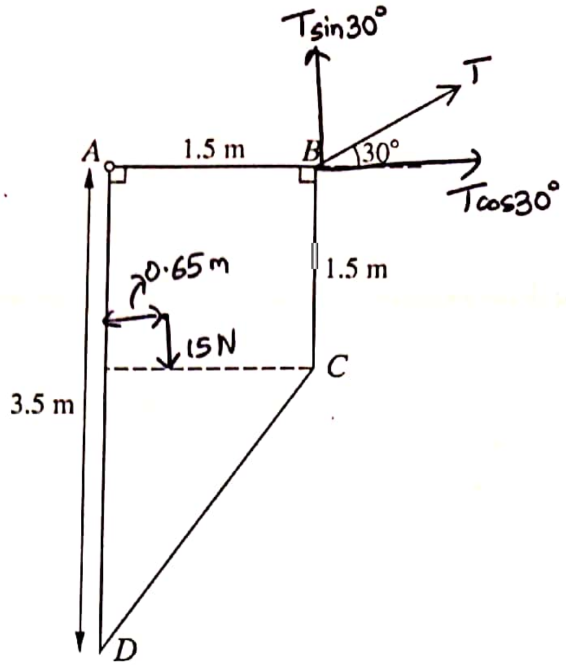

A uniform lamina of weight 15 N is in the form of a trapezium $ABCD$ with dimensions as shown in the diagram. The lamina is freely hinged at A to a fixed point. One end of a light inextensible string is attached to the lamina at B. The lamina is in equilibrium with AB horizontal; the string is taut and in the same vertical plane as the lamina, and makes an angle of 30° upwards from the horizontal (see diagram). Find the tension in the string. Let, ρ = Mass per unit area. [5]

Object	Mass of object	Distance of centre of mass from AD
Square	$1.5^2 \times \rho = 2.25\rho$	$\frac{1.5}{2} = 0.75$
Triangle	$\frac{1}{2} \times 2 \times 1.5\rho = 1.5\rho$	$\frac{(0+0+1.5)}{3} = 0.5$
Composite lamina	3.75ρ	$\bar{x}$

Taking moments about AD:

$$2.25\rho \times 0.75 + 1.5\rho \times 0.5 = 3.75\rho \times \bar{x}$$

$$2.4375 = 3.75\,\bar{x}$$

$$\bar{x} = 0.65 \text{ m}$$

Taking moments about A:

$$T\sin 30° \times 1.5 = 15 \times 0.65$$

$$T = 13\text{N}.$$

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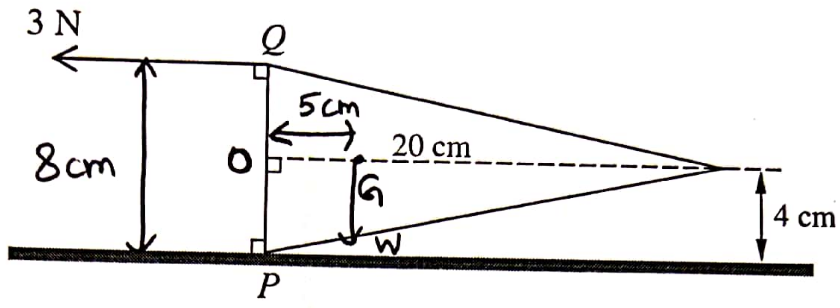


A uniform solid cone has height 20 cm and base radius 4 cm. PQ is a diameter of the base of the cone. The cone is held in equilibrium, with P in contact with a horizontal surface and PQ vertical, by a force applied at Q. This force has magnitude 3 N and acts parallel to the axis of the cone (see diagram). Calculate the mass of the cone. [4]

For the cone, $OG = \frac{1}{4}(20) = 5$ cm.

Taking moments about P:

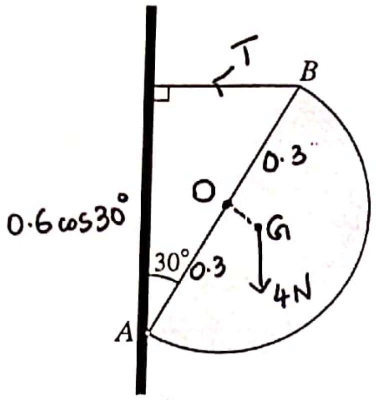
$$W \times 5 = 3 \times 8$$

$$W = 4.8 \text{ N}$$

$$\Rightarrow mg = 4.8$$

$$m = \frac{4.8}{10} = 0.48 \text{ kg}.$$

i.e. Mass of the cone = 0.48 kg

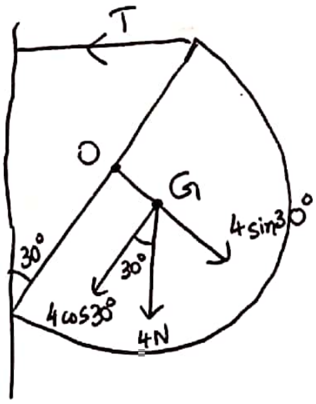


AB is the diameter of a uniform semicircular lamina which has radius 0.3 m and mass 0.4 kg. The lamina is hinged to a vertical wall at A with AB inclined at 30° to the vertical. One end of a light inextensible string is attached to the lamina at B and the other end of the string is attached to the wall vertically above A. The lamina is in equilibrium in a vertical plane perpendicular to the wall with the string horizontal (see diagram).

(i) Show that the tension in the string is 2.00 N correct to 3 significant figures. [4]

(ii) Find the magnitude and direction of the force exerted on the lamina by the hinge. [3]

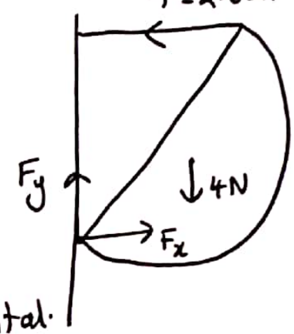
i) $OG = \dfrac{2r\sin\alpha}{3\alpha} = \dfrac{2(0.3)\sin\left(\frac{\pi}{2}\right)}{3\left(\frac{\pi}{2}\right)} = \dfrac{0.6}{3\left(\frac{\pi}{2}\right)} = \dfrac{1.2}{3\pi} = \dfrac{0.4}{\pi} = \dfrac{2}{5\pi}$.



Taking moments about A:

$$T \times 0.6\cos 30^\circ = 4\cos 30^\circ \times \frac{2}{5\pi} + 4\sin 30^\circ \times 0.3$$

$$T = 2.003 \approx 2.00\text{ N}.$$

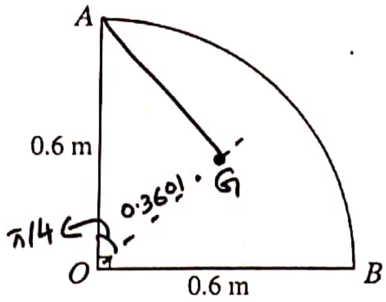


ii) $R(\uparrow)$: $F_y = 4$ N

$R(\rightarrow)$: $F_x = 2.00$ N

$\Rightarrow F = \sqrt{F_x^2 + F_y^2} = \sqrt{4^2+2^2} = 4.472\text{ N} \approx 4.47\text{N}$

$\tan\alpha = \dfrac{F_y}{F_x} = \dfrac{4}{2.00} \Rightarrow \alpha = \tan^{-1}(1.997) = 63.4^\circ$ with horizontal.



AOB is a uniform lamina in the shape of a quadrant of a circle with centre *O* and radius 0.6 m (see diagram).

(i) Calculate the distance of the centre of mass of the lamina from *A*. [3]

The lamina is freely suspended at *A* and hangs in equilibrium.

(ii) Find the angle between the vertical and the side *AO* of the lamina. [3]

i) $OG = \dfrac{2r\sin\alpha}{3\alpha} = \dfrac{2(0.6)\sin\left(\frac{\pi}{4}\right)}{3\left(\frac{\pi}{4}\right)} = \dfrac{1.2\left(\frac{\sqrt{2}}{2}\right)}{\frac{3\pi}{4}} = 0.3601$

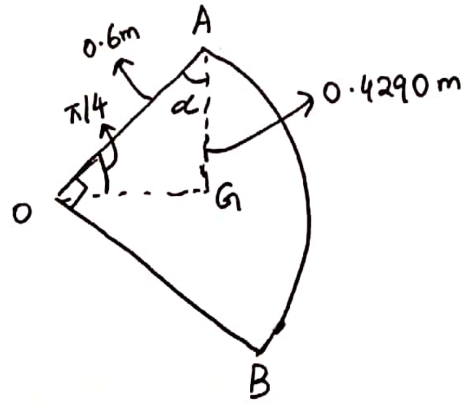
In ΔAOG,

$AG^2 = AO^2 + OG^2 - 2(AO)(OG)\cos A\hat{O}G$

$AG^2 = 0.6^2 + 0.3601^2 - 2(0.6)(0.3601)\cos\left(\frac{\pi}{4}\right)$

$AG = 0.4290 \approx 0.429\text{ m}.$

ii)



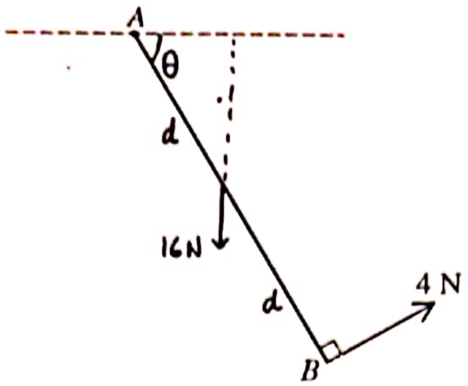
When lamina hangs in equilibrium, G lies vertically below A.

In ΔAOG, $\dfrac{\sin\hat{A}}{OG} = \dfrac{\sin\hat{O}}{AG}$

$\dfrac{\sin\alpha}{0.3601} = \dfrac{\sin\left(\frac{\pi}{4}\right)}{0.4290}$

$\sin\alpha = 0.5934$

$\alpha = 36.40° \approx 36.4°.$



A uniform rod AB of weight 16 N is freely hinged at A to a fixed point. A force of magnitude 4 N acting perpendicular to the rod is applied at B (see diagram). Given that the rod is in equilibrium.

(i) calculate the angle the rod makes with the horizontal, [2]

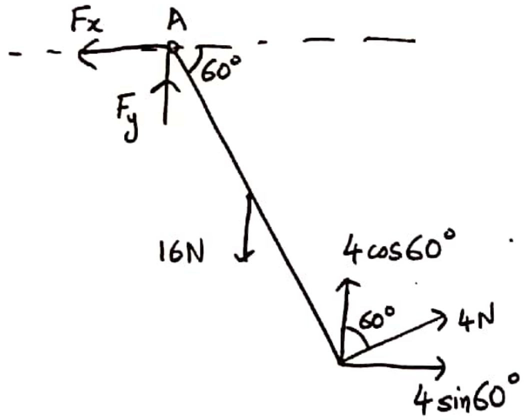
(ii) find the magnitude and direction of the force exerted on the rod at A. [4]

i) Taking moments about A:

$$16 \times d\cos\theta = 4 \times 2d$$

$$\cos\theta = \frac{8d}{16d} = \frac{1}{2} \Rightarrow \theta = 60^\circ.$$

ii)



R(→): $F_x = 4\sin 60^\circ = 2\sqrt{3}$ N.

R(↑): $F_y + 4\cos 60^\circ = 16$

$$F_y = 16 - 4\cos 60^\circ = 14\text{ N}$$

$$\Rightarrow F = \sqrt{F_x^2 + F_y^2} = \sqrt{(2\sqrt{3})^2 + 14^2} = 14.42 \approx 14.4\text{N}.$$

$$\tan\alpha = \frac{F_y}{F_x} = \frac{14}{2\sqrt{3}} \Rightarrow \alpha = \tan^{-1}\left(\frac{7}{\sqrt{3}}\right) = 76.1^\circ$$

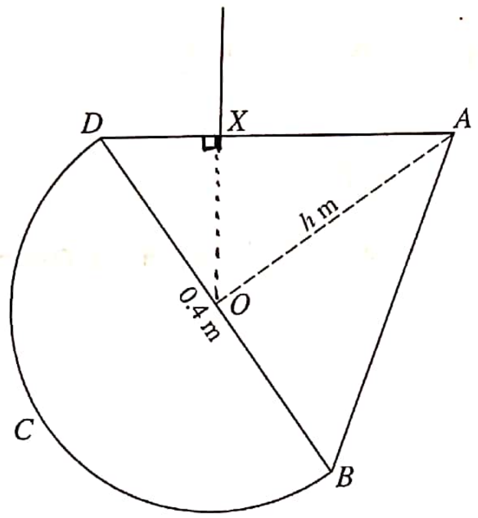
Magnitude of force exerted on rod at A = 14.4 N

Direction of force exerted on rod at A = 76.1° with horizontal.

25 A uniform lamina $ABCD$ consists of a semicircle BCD with centre O and diameter 0.4 m, and an isosceles triangle ABD with base $BD = 0.4$ m and perpendicular height h m. The centre of mass of the lamina is at O.

(i) Find the value of h. [4]

(ii)



The lamina is suspended from a vertical string attached to a point X on the side AD of the triangle (see diagram). Given the lamina is in equilibrium with AD horizontal, calculate XD. [3]

i) For semicircle, $OG = \frac{2r\sin\alpha}{3\alpha} = \frac{2(0.2)\sin\left(\frac{\pi}{2}\right)}{3\left(\frac{\pi}{2}\right)} = \frac{0.4(1)}{\frac{3\pi}{2}} = \frac{0.8}{3\pi} = \frac{4}{15\pi}$.

For isosceles Δ, $OG' = \frac{1}{3}h$

Let, ρ = Mass per unit area.

Mass of semicircle $= \frac{\pi(0.2)^2\rho}{2} = \frac{\pi\rho}{50}$.

Mass of $\Delta = \frac{1}{2} \times 0.4 \times h\rho = \frac{h\rho}{5}$

Total mass of composite figure $= \frac{\pi\rho}{50} + \frac{h\rho}{5}$.

Taking moments about BD:

$$\left(\frac{\pi\rho}{50} + \frac{h\rho}{5}\right) \times 0 = \frac{\pi\rho}{50} \times \frac{-4}{15\pi} + \frac{1}{3}h \times \frac{h\rho}{5}$$

$$0 = \frac{-2}{375} + \frac{h^2}{15} \Rightarrow h^2 = \frac{2}{375} \times 15$$

$$h = 0.2828 \approx 0.283\text{ m}.$$

ii) Since lamina is in equilibrium, centre of mass O is vertically below X.

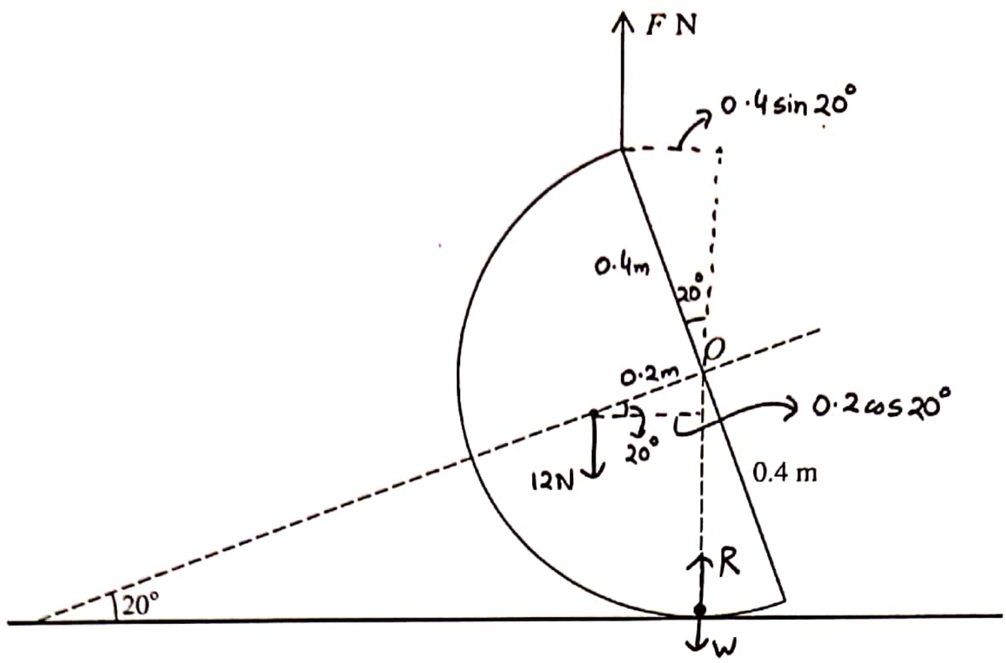
In ΔAOD, $\tan\hat{D} = \frac{0.2828}{0.2} \Rightarrow \hat{D} = 54.7°$

In ΔDXO,

$$\cos \hat{D} = \frac{DX}{DO} \Rightarrow \cos 54.7^\circ = \frac{XD}{0.2}$$

$$XD = 0.2 \cos 54.7^\circ$$

$$XD = 0.1154 \approx 0.115\ m$$



A smooth hemispherical shell, with centre O, weight 12 N and radius 0.4 m, rests on a horizontal plane. A particle of weight W N lies at rest on the inner surface of the hemisphere vertically below O. A force of magnitude F N acting vertically upwards is applied to the highest point of the hemisphere, which is in equilibrium with its axis of symmetry inclined at 20° to the horizontal (see diagram).

(i) Show, by taking moments about O, that $F = 16.48$ correct to 4 significant figures. [3]

(ii) Find the normal contact force exerted by the plane on the hemisphere in terms of W. Hence find the least possible value of W. [3]

i) For hemisphere, $OG = \frac{r}{2} = \frac{0.4}{2} = 0.2\text{m}$

Taking moments about O:

$$F \times 0.4 \sin 20^\circ = 12 \times 0.2 \cos 20^\circ$$

$$F = 16.484 \approx 16.48 \text{ N (Shown)}.$$

ii) R(↑):

$$F + R = 12 + W$$

$$16.48 + R = 12 + W$$

$$R = 12 + W - 16.48$$

$$R = W - 4.48$$

For least value of W, set $R = 0 \Rightarrow W - 4.48 = 0 \Rightarrow W = 4.48$ N.

i.e. Least possible value of $W = 4.48$ N.

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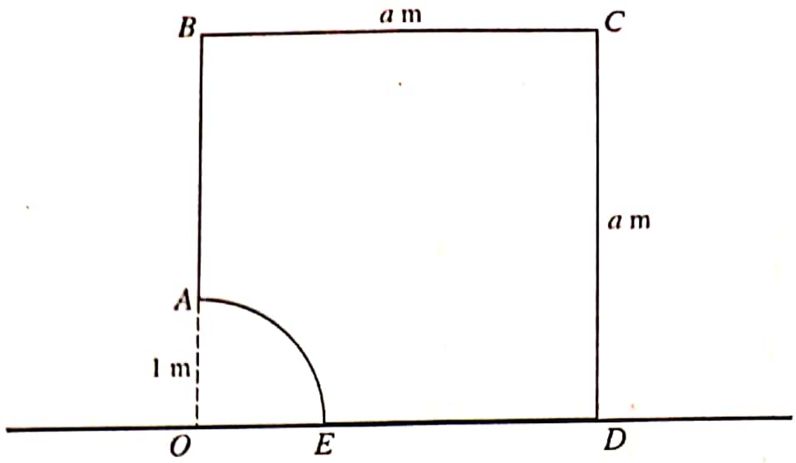


$ABCDE$ is the cross-section through the centre of mass of a uniform prism resting in equilibrium with DE on a horizontal surface. The cross-section has the shape of a square $OBCD$ with sides of length a m, from which a quadrant OAE of a circle of radius 1 m has been removed (see diagram).

(i) Find the distance of the centre of mass of the prism from O, giving the answer in terms of a, π and $\sqrt{2}$. [5]

(ii) Hence show that

$$3a^2(2-a) < \tfrac{3}{2}\pi - 2,$$

and verify that this inequality is satisfied by $a = 1.68$ but not by $a = 1.67$. [4]

i) Let, ρ = Mass per unit area.

Mass of square $= a^2\rho$.

Mass of quarter circle $= \dfrac{\pi(1)^2}{4}\rho = \dfrac{\pi\rho}{4}$.

Total mass of lamina $= a^2\rho - \dfrac{\pi\rho}{4} = \left(a^2 - \dfrac{\pi}{4}\right)\rho$.

From O, distance of centre of mass of square $= \sqrt{\left(\dfrac{a}{2}\right)^2 + \left(\dfrac{a}{2}\right)^2} = \sqrt{\dfrac{a^2}{4} + \dfrac{a^2}{4}} = \dfrac{\sqrt{2}}{2}a$.

distance of centre of mass of quarter circle $= \dfrac{2r\sin\alpha}{3\alpha} = \dfrac{2(1)\sin\left(\frac{\pi}{4}\right)}{3\left(\frac{\pi}{4}\right)} = \dfrac{4\sqrt{2}}{3\pi}$.

distance of centre of mass of lamina $= x$

Taking moments about O:

$$\left(a^2 - \frac{\pi}{4}\right)\rho \times x = a^2\rho \times \frac{\sqrt{2}}{2}a - \frac{\pi\rho}{4} \times \frac{4\sqrt{2}}{3\pi}$$

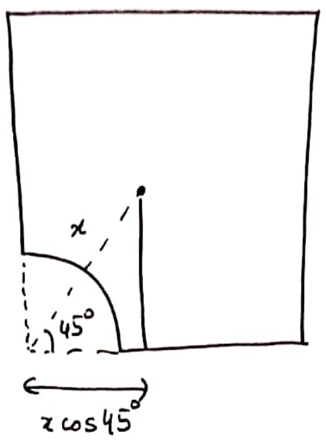
$$\left(a^2 - \frac{\pi}{4}\right)x = \frac{\sqrt{2}a^3}{2} - \frac{\sqrt{2}}{3}$$

$$\left(\frac{4a^2 - \pi}{4}\right)x = \frac{3\sqrt{2}a^3 - 2\sqrt{2}}{6}$$

$$x = \frac{(3\sqrt{2}\,a^3 - 2\sqrt{2})(4)}{6(4a^2 - \pi)} = \frac{2(3\sqrt{2}a^3 - 2\sqrt{2})}{3(4a^2 - \pi)}$$

$$x = \frac{2\sqrt{2}\,(3a^3 - 2)}{3(4a^2 - \pi)}$$

ii)



Since prism is in equilibrium, the line of action of weight should lie to the right of the quarter circle. Then we require $x\cos 45° > 1$

$$\Rightarrow \quad x > \frac{1}{\cos 45°}$$

$$x > \sqrt{2}$$

$$\Rightarrow \frac{2\sqrt{2}(3a^3 - 2)}{3(4a^2 - \pi)} > \sqrt{2}$$

$$2(3a^3 - 2) > 3(4a^2 - \pi)$$

$$6a^3 - 4 > 12a^2 - 3\pi$$

$$6a^3 - 12a^2 > 4 - 3\pi$$

$$6a^2(a - 2) > 4 - 3\pi$$

$$-2\left[3a^2(2 - a)\right] > 4 - 3\pi$$

$$3a^2(2 - a) < -2 + \frac{3\pi}{2}$$

$$3a^2(2 - a) < \frac{3\pi}{2} - 2. \quad \text{(Shown)}.$$

$a = 1.68 \Rightarrow \quad 3 \times 1.68^2(2 - 1.68) < \frac{3\pi}{2} - 2$

$2.7095 < 2.712$ (True)

$a = 1.67 \Rightarrow 3 \times 1.67^2(2 - 1.67) < \frac{3\pi}{2} - 2$

$2.7610 < 2.712$ (False) (Shown).

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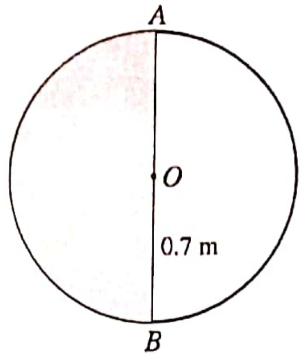


The diagram shows a circular object formed from a uniform semicircular lamina of weight 11 N and a uniform semicircular arc of weight 9 N. The lamina and the arc both have centre O and radius 0.7 m and are joined at the ends of their common diameter AB.

(i) Show that the distance of the centre of mass of the object from O is 0.0371 m, correct to 3 significant figures. [3]

The object hangs in equilibrium, freely suspended at A.

(ii) Find the angle between AB and the vertical and state whether the lowest point of the object is on the lamina or on the arc. [3]

i) For semicircular lamina, mass = 1.1 kg

For semicircular arc, mass = 0.9 kg.

Total mass of object = 1.1 + 0.9 = 2 kg.

From O, distance of centre of mass of

semicircular lamina $= \dfrac{2 \times 0.7 \times \sin\left(\frac{\pi}{2}\right)}{3\left(\frac{\pi}{2}\right)} = \dfrac{14}{15\pi}$.

semicircular arc $= \dfrac{0.7 \times \sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \dfrac{7}{5\pi}$.

object $= \bar{x}$

Taking moments about AB:

$$2\bar{x} = 1.1 \times \frac{14}{15\pi} + 0.9 \times \frac{-7}{5\pi}$$

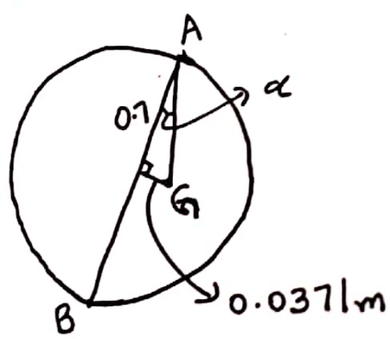
$$\bar{x} = -0.03713 \approx -0.0371 \text{ m}. \text{ (shown)}.$$

Centre of mass lies 0.0371 m to the right of O.

ii) $\tan\alpha = \dfrac{0.0371}{0.7} \Rightarrow \alpha = 3.036° \approx 3.04°$.

i.e. Angle between AB and vertical = 3.04°.

The object rotates clockwise so the lowest point lies on the arc.



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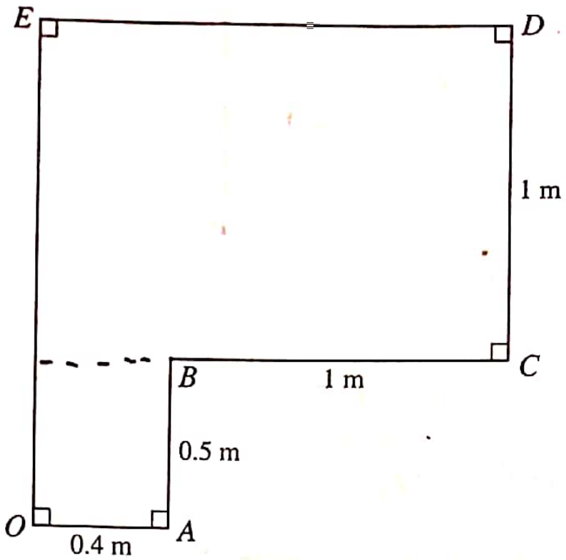

The diagram shows the cross-section $OABCDE$ through the centre of mass of a uniform prism. The interior angles of the cross-section at O, A, C, D and E are all right angles. $OA = 0.4$ m, $AB = 0.5$ m and $BC = CD = 1$ m.

(i) Calculate the distance of the centre of mass of the prism from OE. [3]

The weight of the prism is 120 N. A force of magnitude F N acting along DE holds the prism in equilibrium when OA rests on a rough horizontal surface.

(ii) Find the set of possible values of F. [6]

i) Let, ρ = Mass per unit area.

Mass of larger rectangle = $1.4 \times 1 \times \rho = 1.4\rho$.

Mass of smaller rectangle = $0.4 \times 0.5\rho = 0.2\rho$.

Total mass of cross-section = 1.6ρ.

From OE, distance of centre of mass of

larger rectangle = $\frac{1.4}{2} = 0.7$

smaller rectangle = $\frac{0.4}{2} = 0.2$

cross-section = $\bar{x}$

Taking moments about OE:

$$1.6\rho \times \bar{x} = 1.4\rho \times 0.7 + 0.2\rho \times 0.2$$

$$\bar{x} = 0.6375 \text{ m}$$

$\Rightarrow$ Distance of centre of mass of prism from OE = 0.6375 m.

ii) There are two cases: prism is about to topple about point O or about point A.

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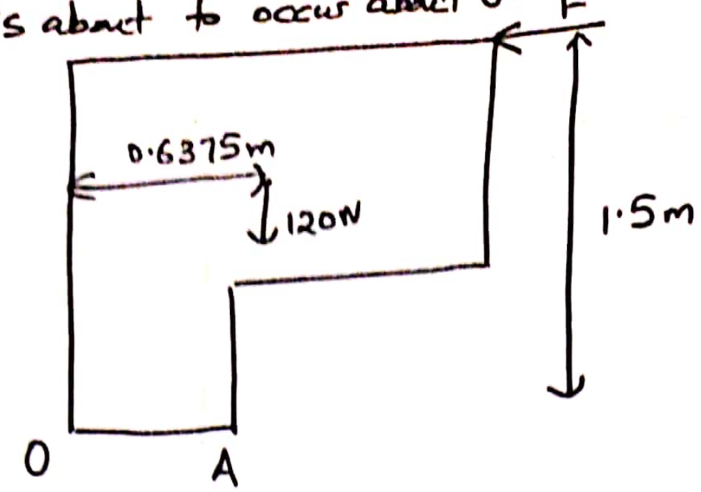
Case 1 : Toppling is about to occur about O:

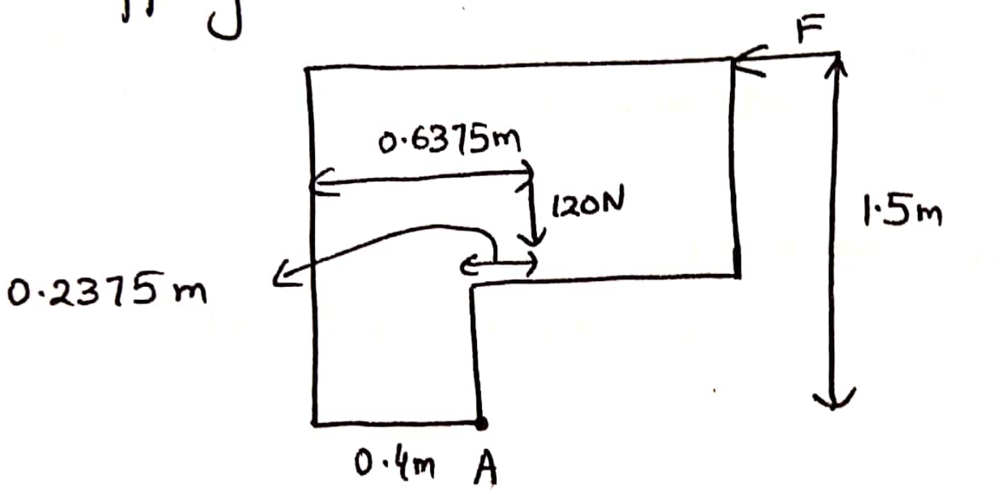


Taking moments about O :

$$120 \times 0.6375 = F \times 1.5$$

$$51N = F$$

Case 2 : Toppling is about to occur about A:



Taking moments about A:

$$120 \times 0.2375 = F \times 1.5$$

$$F = 19\,N.$$

$\Rightarrow$ Set of possible values of F: $19 < F < 51$

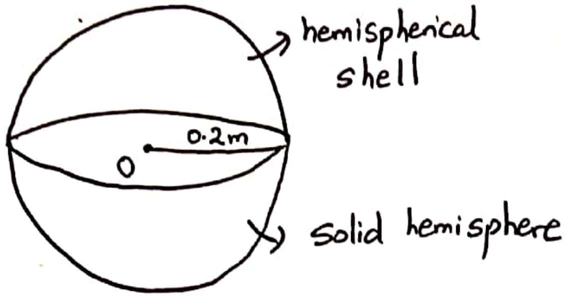
30 A uniform hemispherical shell of weight 8 N and a uniform solid hemisphere of weight 12 N are joined along their circumferences to form a non-uniform sphere of radius 0.2 m.

(i) Show that the distance between the centre of mass of the sphere and the centre of the sphere is 0.005 m. [3]

This sphere is placed on a horizontal surface with its axis of symmetry horizontal. The equilibrium of the sphere is maintained by a force of magnitude F N acting parallel to the axis of symmetry applied to the highest point of the sphere.

(ii) Calculate F. [3]

i)



Object	Mass	Distance of centre of mass from O
Hemispherical shell	0.8	$\frac{0.2}{2} = 0.1$
Solid hemisphere	1.2	$\frac{3(0.2)}{8} = 0.075$
Sphere	2.0	$\bar{y}$

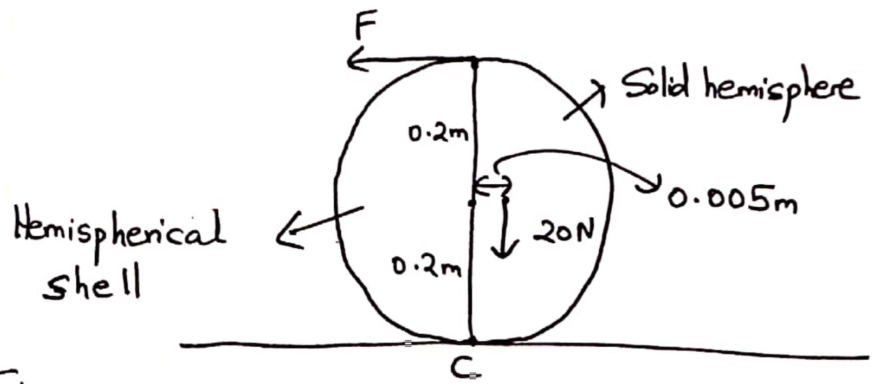
Taking moments about O:

$$2.0 \times \bar{y} = 0.8 \times 0.1 + 1.2(-0.075)$$

$$\bar{y} = -0.005 \text{ m}$$

$\Rightarrow$ Distance of centre of mass from centre of sphere = 0.005m.

ii)



Taking moments about point C, (point of contact):

$$F \times 0.4\text{m} = 20 \times 0.005 \text{ m}$$

$$F = 0.25 \text{ N}.$$

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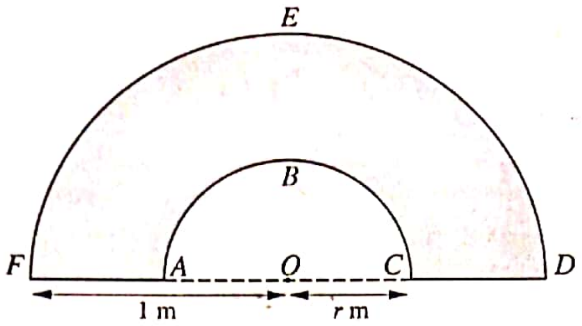


The diagram shows a uniform lamina $ABCDEF$, formed from a semicircle with centre O and radius 1 m by removing a semicircular part with centre O and radius r m.

(i) Show that the distance in metres of the centre of mass of the lamina from O is

$$\frac{4(1+r+r^2)}{3\pi(1+r)}.$$ [4]

The centre of mass of the lamina lies on the arc ABC.

(ii) Show that $r = 0.494$, correct to 3 significant figures. [3]

The lamina is freely suspended at F and hangs in equilibrium.

(iii) Find the angle between the diameter of the lamina and the vertical. [2]

i) Let, ρ = Mass per unit area.

Mass of semicircle DEF $= \frac{\pi(1)^2}{2} = \frac{\pi}{2}\rho$.

Mass of semicircle ABC $= \frac{\pi r^2 \rho}{2}$

Mass of lamina $= \left(\frac{\pi}{2} - \frac{\pi r^2}{2}\right)\rho = \frac{\pi\rho}{2}(1-r^2)$.

From O, distance of centre of mass of

Semicircle DEF $= \frac{2(1)\sin\left(\frac{\pi}{2}\right)}{3\left(\frac{\pi}{2}\right)} = \frac{4}{3\pi}$.

Semicircle ABC $= \frac{2r\sin\left(\frac{\pi}{2}\right)}{3\left(\frac{\pi}{2}\right)} = \frac{4r}{3\pi}$.

lamina $= \bar{x}$.

Taking moments about DF:

$$\frac{\pi\rho}{2}(1-r^2)\,\bar{x} = \frac{\pi\rho}{2}\times\frac{4}{3\pi} - \frac{\pi r^2\rho}{2}\left[\frac{4r}{3\pi}\right]$$

$$\frac{\pi}{2}(1-r^2)\,\bar{x} = \frac{2-2r^3}{3} \Rightarrow \bar{x} = \frac{2(1-r^3)\times 2}{3\pi(1-r^2)}$$

$$\Rightarrow \bar{x} = \frac{4(1-r)(1+r+r^2)}{3\pi(1-r)(1+r)} = \frac{4(1+r+r^2)}{3\pi(1+r)}.$$

II) Since centre of mass lies on arc ABC, $\bar{x} = r$

$$\Rightarrow \frac{4(1+r+r^2)}{3\pi(1+r)} = r$$

$$4 + 4r + 4r^2 = 3\pi(r) + 3\pi r^2$$

$$0 = (3\pi - 4)r^2 + (3\pi - 4)r - 4$$

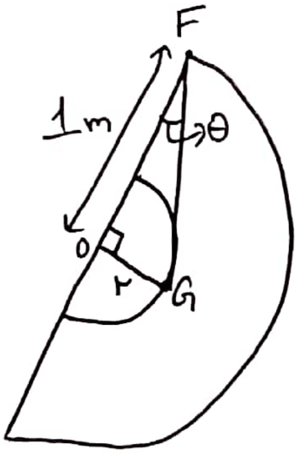
$$r = \frac{-(3\pi-4) \pm \sqrt{(3\pi-4)^2 - 4(3\pi-4)(-4)}}{2(3\pi-4)}$$

$r = 0.4936$ or $r = -1.4936$

$r = 0.494$ or $r = -1.49$ (Ignore)

$$\Rightarrow r = 0.494$$

III) When suspended freely, the centre of ~~grav~~ mass of the object lies vertically below F.



In ΔFOG,

$$\tan\theta = \frac{r}{1} = \frac{0.4936}{1}.$$

$$\theta = \tan^{-1}(0.4936) = 26.27° \approx 26.3°$$

ie. Angle between diameter and vertical $= 26.3°$.

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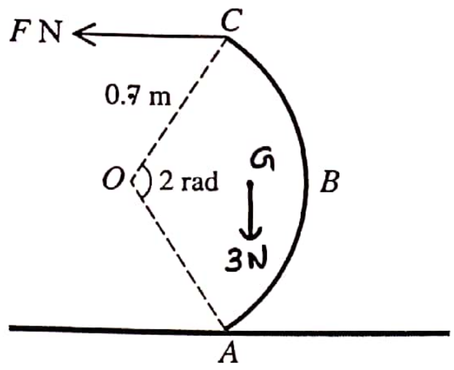


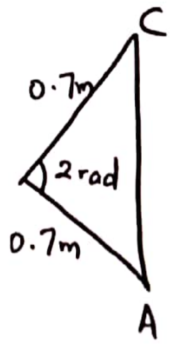
The diagram shows a uniform object ABC of weight 3 N in the form of an arc of a circle with centre O and radius 0.7 m. The angle AOC is 2 radians. The object rests in equilibrium with A on a horizontal surface and C vertically above A. Equilibrium is maintained by a horizontal force of magnitude F N applied at C in the plane of the object. Calculate F. [4]

Let, G be centre of mass of arc of circle.

$$OG = \frac{r \sin \alpha}{\alpha} = \frac{0.7 \times \sin(1)}{1} = 0.5890$$

$$AC = \sqrt{0.7^2 + 0.7^2 - 2(0.7)(0.7)\cos 2}$$

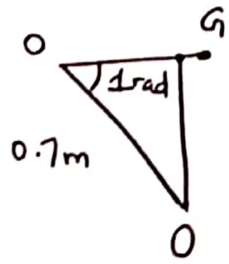
$$AC = 1.178$$



Horizontal distance between A and G

$$= OG - 0.7\cos(1) = 0.5890 - 0.7\cos(1)$$

$$= 0.2108 \text{ m.}$$



Taking moments about A:

$$F \times 1.178 = 3 \times 0.2108$$

$$F = 0.5368 \text{ N} \approx 0.537 \text{ N.}$$

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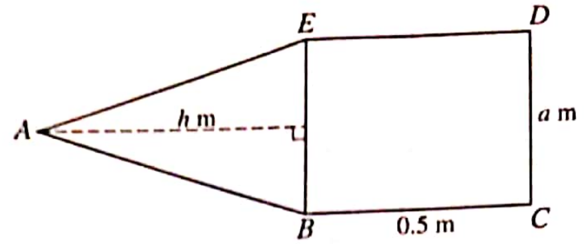


A uniform lamina $ABCDE$ consists of a rectangle $BCDE$ and an isosceles triangle ABE joined along their common edge BE. For the triangle, $AB = AE$, $BE = a$ m and the perpendicular height is h m. For the rectangle, $BC = DE = 0.5$ m and $CD = BE = a$ m (see diagram).

(i) Show that the distance in metres of the centre of mass of the lamina from BE towards CD is

$$\frac{3-4h^2}{12+12h}.$$ [4]

The lamina is freely suspended at E and hangs in equilibrium.

(ii) Given that DE is horizontal, calculate h. [2]

(iii) Given instead that $h = 0.5$ and AE is horizontal, calculate a. [3]

i) Let, ρ = Mass per unit area

Mass of $\Delta = \frac{1}{2} \times a \times h\,\rho = \frac{ah\rho}{2}$.

Mass of rectangle $= 0.5a \times \rho = 0.5a\rho$

Total mass of lamina $= \frac{ah\rho}{2} + 0.5a\rho = 0.5a\rho(h+1)$.

From EB, distance of centre of mass of

triangle ABE $= \frac{h}{3}$

rectangle BCDE $= 0.25$

lamina $= \bar{x}$

Taking moments about BE:

$$0.5a\rho(h+1) \times \bar{x} = \frac{ah\rho}{2} \times \frac{-h}{3} + \frac{a\rho}{2}(0.25)$$

$$\frac{(h+1)}{2}[\bar{x}] = \frac{-h^2}{6} + \frac{1}{8}$$

$$\bar{x} = \left(\frac{-h^2}{3} + \frac{1}{4}\right) \times \frac{1}{h+1} = \left(\frac{-4h^2+3}{12}\right) \times \frac{1}{(h+1)}$$

$$\bar{x} = \frac{3-4h^2}{12(h+1)}. \text{ (Shown)}.$$

ii) When freely suspended, if DE is horizontal, the centre of mass lies vertically below E and so it lies on BE. Then $\bar{x} = 0$.

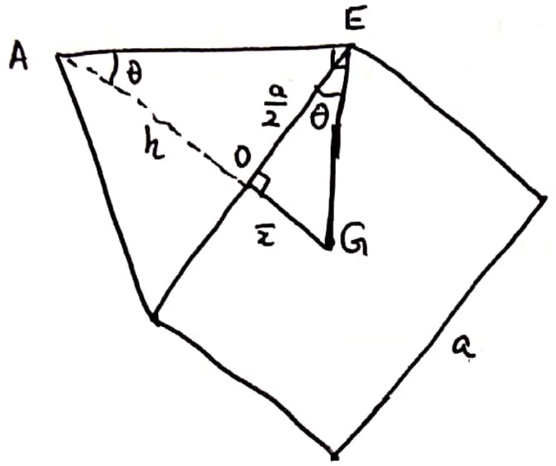
$$\Rightarrow \frac{3-4h^2}{12+12h} = 0$$

$$3 = 4h^2$$

$$h^2 = \frac{3}{4} \Rightarrow h = \frac{\sqrt{3}}{2} = 0.8660\ m$$

$$h \approx 0.866\ m.$$

III) If AE is horizontal and $h = 0.5$.



$$\text{In } \Delta OGE,\ \tan\theta = \frac{\bar{x}}{\frac{a}{2}} = \frac{2\bar{x}}{a} \rightarrow (1)$$

$$\text{In } \Delta OEA,\ \tan\theta = \frac{\frac{a}{2}}{h} = \frac{a}{2h} \rightarrow (2)$$

Substituting (1) in (2) : $\frac{2\bar{x}}{a} = \frac{a}{2h}$

$$a^2 = 4\bar{x}h$$

Substituting $\bar{x} = \frac{3-4h^2}{12(h+1)}$ and $h = 0.5$

$$a^2 = 4\left[\frac{3-4(0.5)^2}{12(0.5+1)}\right] \times 0.5$$

$$a^2 = \frac{2}{9}$$

$$a = 0.4714 \approx 0.471\ m.$$

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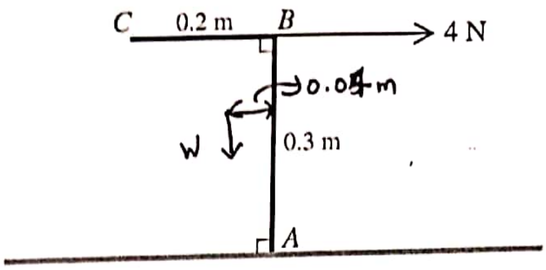


A uniform object ABC is formed from two rods AB and BC joined rigidly at right angles at B. The rod AB has length 0.3 m and the rod BC has length 0.2 m. The object rests with the end A on a rough horizontal surface and the rod AB vertical. The object is held in equilibrium by a horizontal force of magnitude 4 N applied at B and acting in the direction CB (see diagram).

(i) Find the distance of the centre of mass of the object from AB. [3]

(ii) Calculate the weight of the object. [2]

(iii) Find the least possible value of the coefficient of friction between the surface and the object. [2]

i) Let, ρ = Mass per unit length.

Mass of rod AB = 0.3ρ

Mass of rod BC = 0.2ρ

Total mass = $0.3\rho + 0.2\rho = 0.5\rho$.

From AB, distance of centre of mass of

rod AB = 0

rod BC = 0.1

object ABC = $\bar{x}$

Taking moments about AB:

$$0.5\rho \times \bar{x} = 0.3\rho \times 0 + 0.2\rho \times 0.1$$

$$0.5\bar{x} = 0.02$$

$$\bar{x} = 0.04 \text{ m}$$

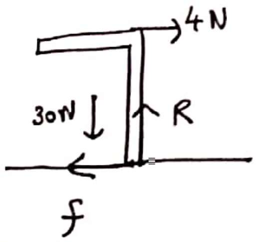
i.e. Distance of centre of mass of object from AB = 0.04 m.

ii) Taking moments about A:

$$W \times 0.04 = 4 \times 0.3$$

$$W = 30 \text{ N}$$

iii) Since object rests in equilibrium, $f \leq \mu R$.



$R(\uparrow)$: $R = 30$ N

$R(\rightarrow)$: $f = 4$ N

Then $f \leq \mu R \Rightarrow 4 \leq \mu(30) \Rightarrow \mu \geq \frac{2}{15}$

i.e. Least value of $\mu = \frac{2}{15} = 0.133$.

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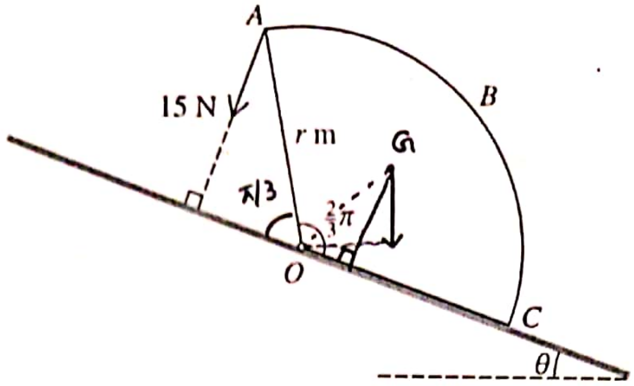


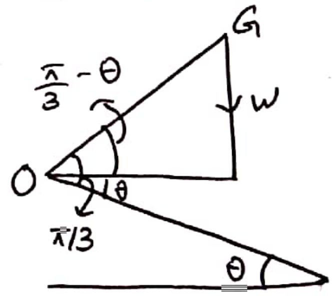
OABC is the cross-section through the centre of mass of a uniform prism of weight 20 N. The cross-section is in the shape of a sector of a circle with centre *O*, radius $OA = r$ m and angle $AOC = \frac{2}{3}\pi$ radians. The prism lies on a plane inclined at an angle θ radians to the horizontal, where $\theta < \frac{1}{3}\pi$. *OC* lies along a line of greatest slope with *O* higher than *C*. The prism is freely hinged to the plane at *O*. A force of magnitude 15 N acts at *A*, in a direction towards to the plane and at right angles to it (see diagram). Given that the prism remains in equilibrium, find the set of possible values of θ. [9]

For sector of circle,

$$OG = \frac{2r\sin\alpha}{3\alpha} = \frac{2r\left(\sin\left(\frac{\pi}{3}\right)\right)}{3\left(\frac{\pi}{3}\right)} = \frac{2r\left(\frac{\sqrt{3}}{2}\right)}{\pi} = \frac{\sqrt{3}r}{\pi}.$$

Horizontal distance from O to line of action of weight of prism

$$= OG \times \cos\left(\frac{\pi}{3} - \theta\right) = \frac{\sqrt{3}r}{\pi} \times \cos\left(\frac{\pi}{3} - \theta\right).$$



$\Rightarrow$ Moment due to weight about point O $= 20 \times \frac{\sqrt{3}r}{\pi}\cos\left(\frac{\pi}{3} - \theta\right).$

Moment due to 15N force about point O $= 15 \times r\cos\left(\frac{\pi}{3}\right)$

$$= 15r \times \frac{1}{2} = \frac{15r}{2}.$$

Since prism remains in equilibrium,

Moment due to weight $\geq$ Moment due to 15 N force

$$20\frac{\sqrt{3}r}{\pi}\cos\left(\frac{\pi}{3} - \theta\right) \geq \frac{15r}{2}$$

$$\cos\left(\frac{\pi}{3} - \theta\right) \geq 0.6801$$

$$\frac{\pi}{3} - \theta \leq \cos^{-1}(0.6801)$$

$\Rightarrow$ Inequality switches direction as cosine is a decreasing function.

$$\frac{\pi}{3} - \theta \leq 0.8227$$

$$0.2244 \leq \theta \Rightarrow \boxed{\theta \geq 0.224}$$

36 A uniform semicircular lamina of radius 0.25 m has diameter AB. It is freely suspended at A from a fixed point and hangs in equilibrium.

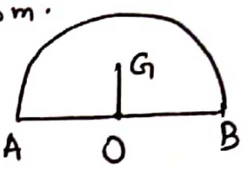
(i) Find the distance of the centre of mass of the lamina from the diameter AB. [1]

(ii) Calculate the angle which the diameter AB makes with the vertical. [2]

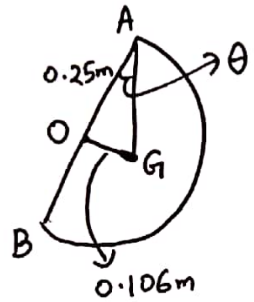
The lamina is now held in equilibrium with the diameter AB vertical by means of a force applied at B. This force has magnitude 6 N and acts at 45° to the upward vertical in the plane of the lamina.

(iii) Calculate the weight of the lamina. [3]

i) $OG = \dfrac{2r\sin\alpha}{3\alpha} = \dfrac{2(0.25)\sin\left(\frac{\pi}{2}\right)}{3\left(\frac{\pi}{2}\right)} = \dfrac{0.5\times 2}{3\pi} = 0.1061 \approx 0.106\,m.$



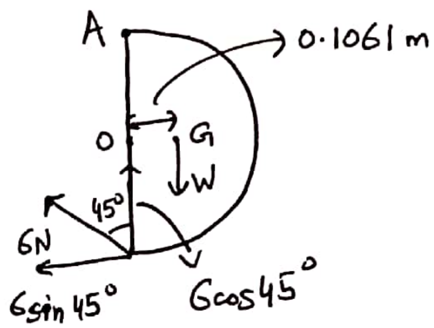
ii)



When suspended freely, the centre of mass G lies vertically below A.

In $\triangle AOG$, $\tan\theta = \dfrac{0.106}{0.25} \Rightarrow \theta = 22.99° \approx 23.0°.$

iii)



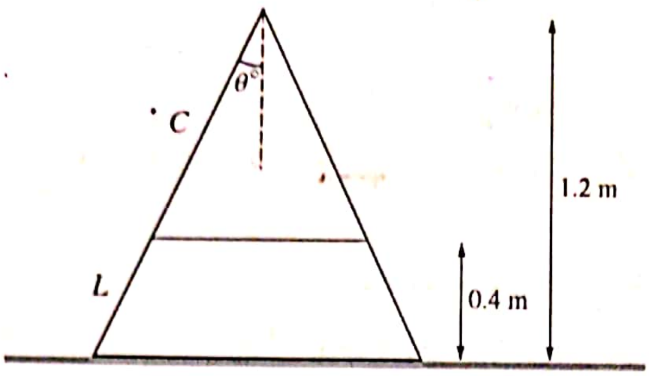
Taking moments about A:

$6\sin 45° \times 2 \times 0.25 = W \times 0.1061$

$\dfrac{6\sin 45° \times 2 \times 0.25}{0.1061} = W$

$W = 19.99\,N \approx 20.0\,N$

37



A uniform solid cone of height 1.2 m and semi-vertical angle $\theta°$ is divided into two parts by a cut parallel to and 0.4 m from the circular base. The upper conical part, C, has weight 16 N, and the lower part, L, has weight 38 N. The two parts of the solid rest in equilibrium with the larger plane face of L on a horizontal surface and the smaller plane face of L covered by the base of C (see diagram).

(i) Calculate the distance of the centre of mass of L from its larger plane face. [3]

An increasing horizontal force is applied to the vertex of C. Equilibrium is broken when the magnitude of this force first exceeds 4 N, and C begins to slide on L.

(ii) By considering the forces on C,

(a) find the coefficient of friction between C and L, [1]

(b) show that $\theta > 14.0$, correct to 3 significant figures. [2]

C is removed and L is placed with its curved surface on the horizontal surface.

(iii) Given that L is on the point of toppling, calculate θ. [3]

i)

Object	Mass	Distance of centre of mass from plane face
C	1.6	$0.4 + \frac{1}{4}(0.8) = 0.6$
L	3.8	~~$\frac{1}{4}(1.2) = 0.3$~~ $\bar{x}$
C+L	5.4	$\frac{1}{4} \times 1.2 = 0.3$

Taking moments about plane face:

$$1.6 \times 0.6 + 3.8 \times \bar{x} = 5.4 \times 0.3$$

$$\bar{x} = 0.1736 \approx 0.174 \text{ m}.$$

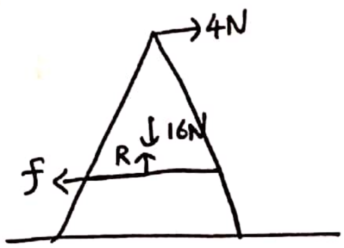
ii) a) For C, $R(\uparrow)$: $R = 16$ N

$R(\rightarrow)$: $f = 4$ N

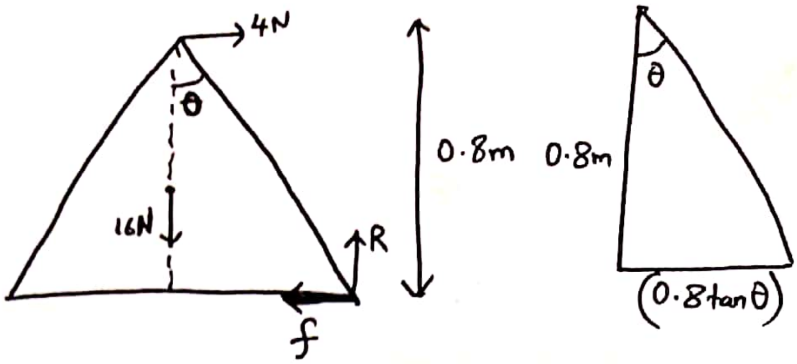
Since C begins to slide,

$$f = \mu R$$

$$4 = \mu(16) \Rightarrow \mu = 0.25$$



b)



Since C does not topple, we require

moment due to weight > moment due to 4 N force

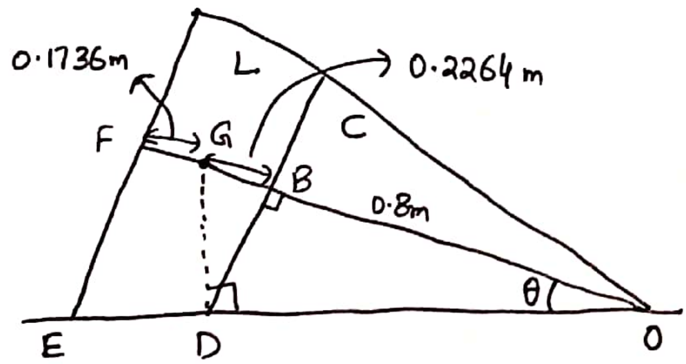
$$16 \times 0.8 \tan\theta > 4 \times 0.8$$

$$\tan\theta > \frac{1}{4} \Rightarrow \theta > \tan^{-1}\left(\frac{1}{4}\right)$$

$$\Rightarrow \theta > 14.036^\circ$$

$$\Rightarrow \theta > 14.0^\circ \text{ (3 s.f.)}.$$

iii)



In ΔOBD, $\cos\theta = \frac{0.8}{OD} \Rightarrow OD = \frac{0.8}{\cos\theta} \rightarrow (1)$

In ΔODG, $\cos\theta = \frac{OD}{OG} \Rightarrow \cos\theta = \frac{OD}{1.0264} \Rightarrow OD = 1.0264 \cos\theta \rightarrow (2)$

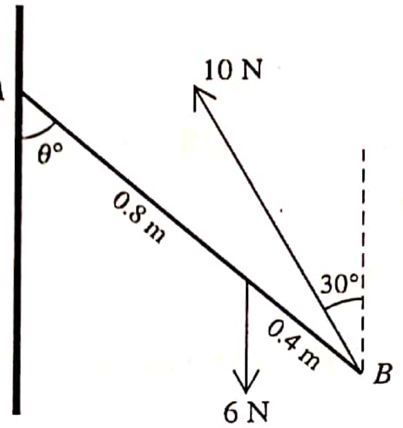
Substituting (1) in (2):

$$\frac{0.8}{\cos\theta} = 1.0264 \cos\theta$$

$$0.7794 = \cos^2\theta$$

$$\Rightarrow \cos\theta = 0.8828$$

$$\theta = 28.00^\circ \approx 28.0^\circ$$

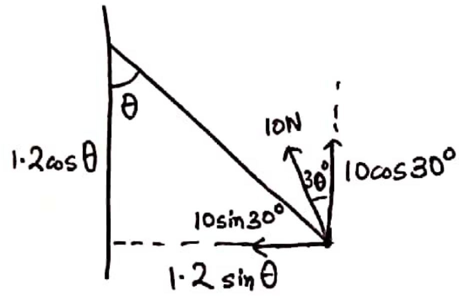


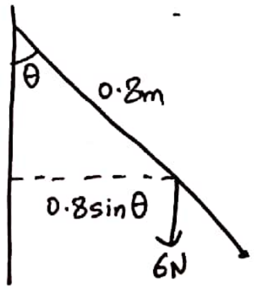
A non-uniform rod AB of weight 6 N rests in limiting equilibrium with the end A in contact with a rough vertical wall. $AB = 1.2$ m, the centre of mass of the rod is 0.8 m from A, and the angle between AB and the downward vertical is $\theta°$. A force of magnitude 10 N acting at an angle of 30° to the upwards vertical is applied to the rod at B (see diagram). The rod and the line of action of the 10 N force lie in a vertical plane perpendicular to the wall. Calculate

(i) the value of θ, [4]

(ii) the coefficient of friction between the rod and the wall. [2]

i)





Taking moments about A:

$$10\cos 30° \times 1.2 \sin\theta = 6\times 0.8\sin\theta + 10\sin 30° \times 1.2\cos\theta$$

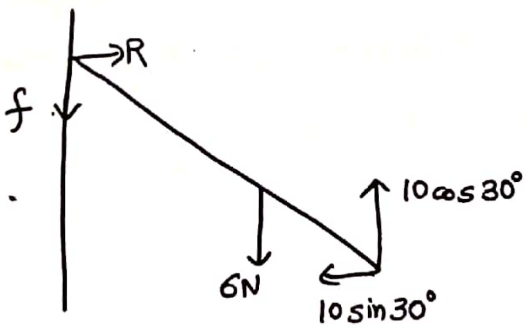
$$6\sqrt{3}\sin\theta = 4.8\sin\theta + 6\cos\theta$$

$$(6\sqrt{3}-4.8)\sin\theta = 6\cos\theta$$

$$\tan\theta = \frac{6}{6\sqrt{3}-4.8}$$

$$\theta = 47.01° \approx 47.0°.$$

ii)



$R(\uparrow)$: $f + 6 = 10\cos 30^\circ$

$f = 2.660\ N$

$R(\rightarrow)$: $R = 10\sin 30^\circ = 5N.$

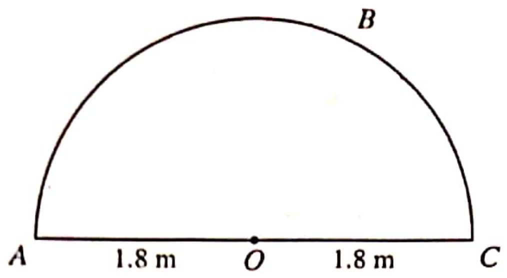
Since rod is in limiting equilibrium,

$f = \mu R$

$2.660 = \mu(5)$

$\mu = 0.5320 \approx 0.532.$

39



A uniform metal frame $OABC$ is made from a semicircular arc ABC of radius 1.8 m, and a straight rod AOC with $AO = OC = 1.8$ m (see diagram).

(i) Calculate the distance of the centre of mass of the frame from O. [3]

A uniform semicircular lamina of radius 1.8 m has weight 27.5 N. A non-uniform object is formed by attaching the frame $OABC$ around the perimeter of the lamina. The object is freely suspended from a fixed point at A and hangs in equilibrium. The diameter AOC of the object makes an angle of 22° with the vertical.

(ii) Calculate the weight of the frame. [5]

i) Let, ρ = Mass per unit length

Length of arc $ABC = \pi(1.8)\rho = 1.8\pi\rho$.

Length of rod $AC = 3.6\rho$.

Total length of frame $= 1.8\pi\rho + 3.6\rho = 1.8\rho(\pi+2)$.

From AC, distance of centre of mass of

rod $AC = 0$

arc $ABC = \dfrac{r\sin\alpha}{\alpha} = \dfrac{1.8\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \dfrac{3.6}{\pi}$.

frame $= \bar{x}$

Taking moments about AC:

$$1.8\rho(\pi+2)\,\bar{x} = 1.8\pi\rho \times \cancel{0}\frac{3.6}{\pi} + 3.6\rho \times \cancel{\frac{3.6}{\pi}}\ 0$$

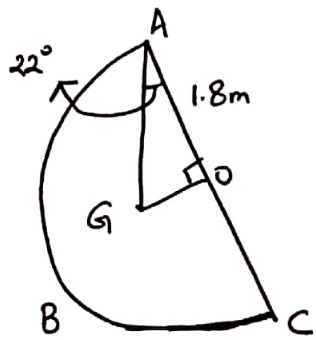
$$\bar{x} = \frac{6.48}{1.8(\pi+2)} = 0.7001\text{ m}$$

$\Rightarrow$ Distance of centre of mass from $O = 0.700$ m.

ii) ~~I~~ When suspended freely, the centre of mass of composite object lies vertically below A.

$\tan 22° = \dfrac{OG}{1.8} \Rightarrow OG = 1.8\tan 22°$

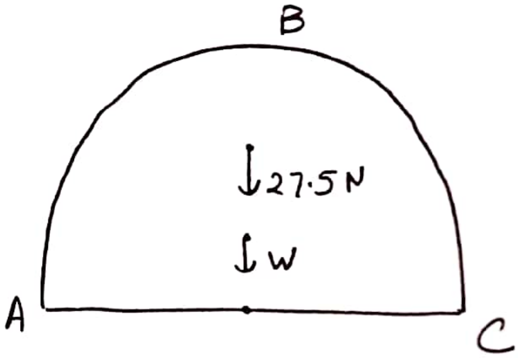
$OG = 0.7272$ m



Let, weight of ~~lamina~~ frame $= W \Rightarrow$ Mass of ~~lamina~~ frame $= \frac{W}{10}$.

weight of lamina $= 27.5 \Rightarrow$ Mass of lamina $= 2.75$

Weight of composite object $= W + 27.5 \Rightarrow$ Mass of composite object $= \frac{W}{10} + 2.75$.



From AC, distance of centre of mass of frame $= 0.700$ m

" " " " " " lamina $= \dfrac{2(1.8)\sin\left(\frac{\pi}{2}\right)}{3\left(\frac{\pi}{2}\right)}$

$= 0.7639$ m

" " " " " " composite object $= 0.7272$ m

Taking moments about AC:

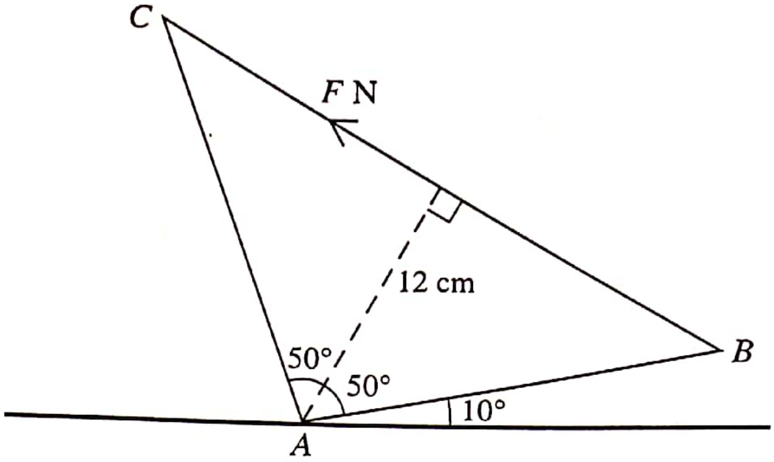
$$\left(\frac{W}{10} + 2.75\right) \times 0.7272 = \frac{W}{10} \times 0.700 + 2.75 \times 0.7639$$

$$0.07272W + 1.9998 = 0.07001W + 2.1008$$

$$0.00271W = 0.101$$

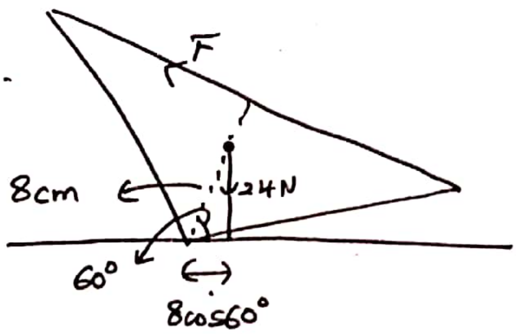
$$W = 37.26\text{ N} \approx 37.3\text{N}$$

40



A uniform lamina ABC in the shape of an isosceles triangle has weight 24 N. The perpendicular distance from A to BC is 12 cm. The lamina rests in a vertical plane in equilibrium, with the vertex A in contact with a horizontal surface. Angle $BAC = 100°$ and AB makes an angle of 10° with the horizontal. Equilibrium is maintained by a force of magnitude F N acting along BC (see diagram). Show that $F = 8$. [3]

$AG = \frac{2}{3}(12) = 8$ cm where G is the centre of mass of $\triangle ABC$.

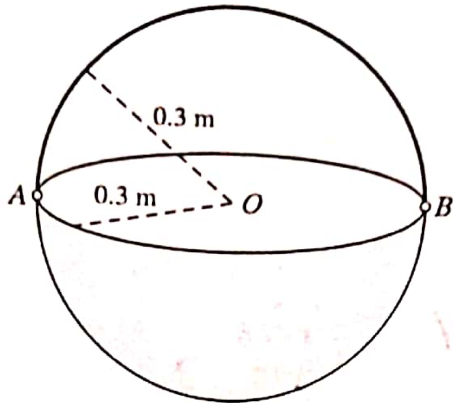


Taking moments about A:

$$24 \times 8\cos 60° = F \times 12$$

$$\frac{24 \times 8\cos 60°}{12} = F$$

$$\Rightarrow F = 8\text{ N (shown)}$$



The diagram shows a container which consists of a bowl of weight 14 N and a handle of weight 8 N. The bowl of the container is in the form of a uniform hemispherical shell with centre O and radius 0.3 m. The handle is in the form of a uniform semicircular arc of radius 0.3 m and is freely hinged to the bowl at A and B, where AB is a diameter of the bowl.

(i) Calculate the distance of the centre of mass of the container from O for the position indicated in the diagram, where the handle is perpendicular to the rim of the bowl. [3]

(ii) Show that the distance of the centre of mass of the container from O when the handle lies on the rim of the bowl is 0.118 m, correct to 3 significant figures. [5]

In the case when the handle lies on the rim of the bowl, the container rests in equilibrium with the curved surface of the bowl on a horizontal table.

(iii) Find the angle which the plane containing the rim of the bowl makes with the horizontal. [2]

i)

Object	Mass (kg)	Distance of centre of mass from O (m)
Bowl	1.4	$\frac{1}{2}(0.3) = 0.15$
Handle	0.8	$\dfrac{0.3\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \dfrac{3}{5\pi}$
Container	2.2	$\bar{x}$

Taking moments about OB:

$$2.2\bar{x} = 1.4 \times 0.15 + 0.8 \times \frac{-3}{5\pi}$$

$$\bar{x} = 0.02601 \approx 0.0260 \text{ m}$$

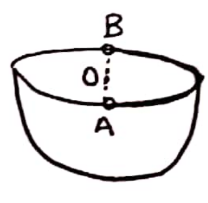
$\Rightarrow$ Distance of centre of mass of container from O = 0.0260m.

ii) Treating O as the origin, when handle AB is on rim,

Centre of mass of handle $= \left(\frac{3}{5\pi}, 0\right)$

Centre of mass of bowl $= (0, 0.15)$.

Centre of mass of container $= (\bar{x}, \bar{y})$.



Taking moments about y-axis:

$$2.2\bar{x} = 0.8 \times \frac{3}{5\pi} + 1.4 \times 0 \Rightarrow \bar{x} = 0.06945$$

Taking moments about x-axis:

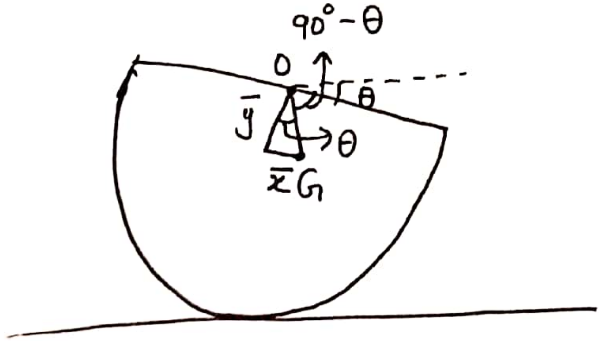
$$2.2\bar{y} = 0.8 \times 0 + 1.4 \times 0.15$$

$$\bar{y} = 0.09545$$

$\Rightarrow$ Distance of centre of mass of container from O

$$= \sqrt{(\bar{x})^2 + (\bar{y})^2} = \sqrt{0.06945^2 + 0.09545^2} = 0.1180 \approx 0.118\,m.$$

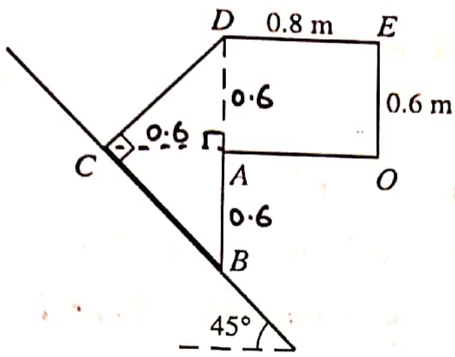
iii)



In the triangle, G (centre of mass of the container) lies vertically below O. ~~as~~

$$\tan\theta = \frac{\bar{x}}{\bar{y}} = \frac{0.06945}{0.09545} \Rightarrow \theta = 36.03° \approx 36.0°$$

$\Rightarrow$ Angle between the plane containing the rim and the horizontal $= \theta = 36.0°$.



The diagram shows the cross-section $OABCDE$ through the centre of mass of a uniform prism on a rough inclined plane. The portion $ADEO$ is a rectangle in which $AD = OE = 0.6$ m and $DE = AO = 0.8$ m; the portion BCD is an isosceles triangle in which angle BCD is a right angle, and A is the mid-point of BD. The plane is inclined at 45° to the horizontal, BC lies along a line of greatest slope of the plane and DE is horizontal.

(i) Calculate the distance of the centre of mass of the prism from BD. [3]

The weight of the prism is 21 N, and it is held in equilibrium by a horizontal force of magnitude P N acting along ED.

(ii) (a) Find the smallest value of P for which the prism does not topple. [2]

(b) It is given that the prism is about to slip for this smallest value of P. Calculate the coefficient of friction between the prism and the plane. [3]

The value of P is gradually increased until the prism ceases to be in equilibrium.

(iii) Show that the prism topples before it begins to slide, stating the value of P at which equilibrium is broken. [5]

i)

Shape	Mass	Distance of centre of mass from BD
Rectangle OADE	$0.6 \times 0.8 \times \rho = 0.48\rho$	$\frac{0.8}{2} = 0.4$
Δ BCD	$\frac{1}{2} \times 0.6 \times 1.2 \times \rho = 0.36\rho$	$\cancel{0.6} - \frac{0.6}{3} = -0.2$
Composite figure	$0.48\rho + 0.36\rho = 0.84\rho$	$\bar{x}$

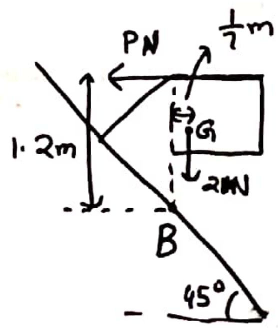
Taking moments about BD: $0.84\rho \times \bar{x} = 0.48\rho \times 0.4 + 0.36\rho(-0.2)$

$\bar{x} = \frac{1}{7} = 0.143$ m.

ii) a) Taking moments about P:

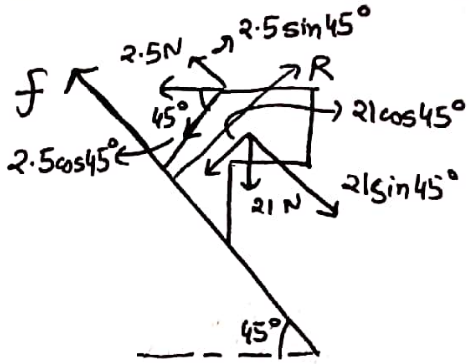
$21 \times \frac{1}{7} = P \times 1.2$

$P = 2.5$ N



b) $R(\parallel)$: $f + 2.5\sin 45° = 21 \sin 45°$

$f = 18.5 \sin 45°$.



$R(\perp)$: $R = 2.5\cos 45° + 21\cos 45°$

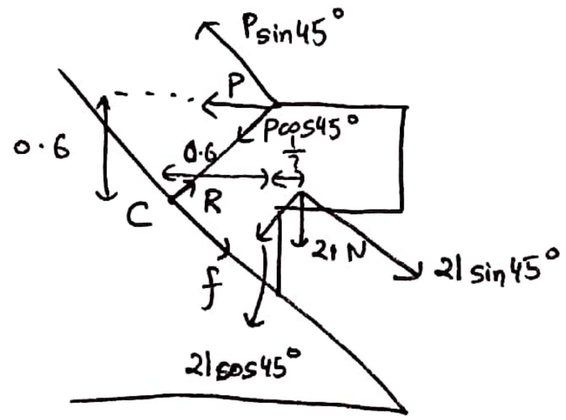
$R = 23.5\cos 45°$.

Since prism is about to slip, $f = \mu R$

$18.5\sin 45° = \mu\,(23.5\cos 45°)$

$\mu = 0.7872 \approx 0.787$

iii)



Taking moments about C: $P \times 0.6 = 21 \times \left(0.6 + \frac{1}{7}\right)$

$P = 26\,N$.

i.e. Prism begins toppling when $P = 26\,N$.

$R(\parallel)$: $P\sin 45° = f + 21\sin 45°$

with $P = 26 \Rightarrow 26\sin 45° = f + 21\sin 45° \Rightarrow f = 5\sin 45° = 3.535$

But $f_{lim} = \mu R = 0.7872\,(P\cos 45° + 21\cos 45°)$

$= 0.7872\,(26\cos 45° + 21\cos 45°) = 26.16$

Then $f < f_{lim}$ i.e. no sliding takes place.

Equilibrium is broken by toppling about C when $P = 26N$.

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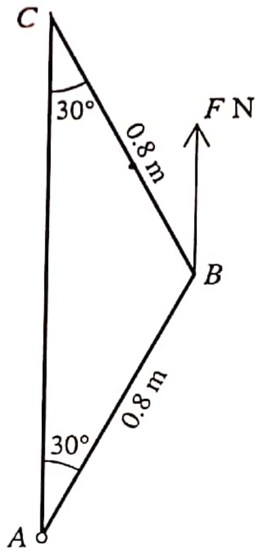
43 A triangular frame ABC consists of two uniform rigid rods each of length 0.8 m and weight 3 N, and a longer uniform rod of weight 4 N. The triangular frame has $AB = BC$, and angle BAC = angle BCA = 30°.

(i) Calculate the distance of the centre of mass of the frame from AC. [3]



The vertex A of the frame is attached to a smooth hinge at a fixed point. The frame is held in equilibrium with AC vertical by a vertical force of magnitude F N applied to the frame at B (see diagram).

(ii) Calculate F, and state the magnitude and direction of the force acting on the frame at the hinge. [3]

i)

Rod	Weight(N)	Distance of centre of mass from AC (m)
AC	4	0
BC	3	$0.4 \sin 30° = 0.2$
AB	3	$0.4 \sin 30° = 0.2$
Frame	10	$\bar{x}$

Taking moments about AC: $10\bar{x} = 4(0) + 3(0.2) + 3(0.2)$

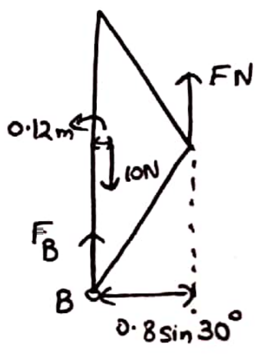
$\bar{x} = 0.12$ m

i.e. Distance of centre of mass from AC = 0.12m.

ii) Taking moments about B:

$10 \times 0.12 = F \times 0.8 \sin 30°$

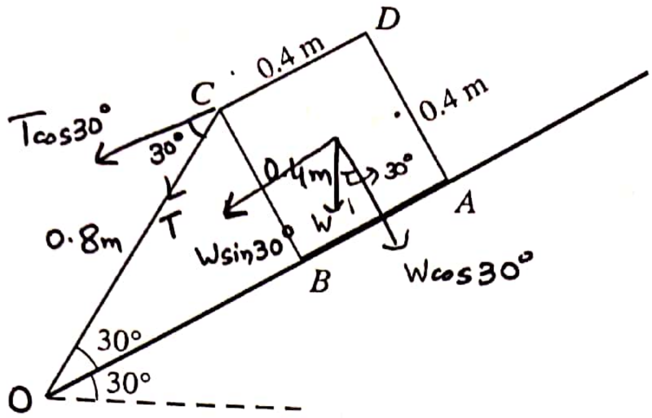
$F = 3$ N.



$R(\uparrow)$: $F_B + 3 = 10$

$F_B = 7$ N.

i.e. A 7N force acts on the frame at the hinge in the upward direction.



A uniform solid cube with edges of length 0.4 m rests in equilibrium on a rough plane inclined at an angle of 30° to the horizontal. *ABCD* is a cross-section through the centre of mass of the cube, with *AB* along a line of greatest slope. *B* lies below the level of *A*. One end of a light elastic string with modulus of elasticity 12 N and natural length 0.4 m is attached to *C*. The other end of the string is attached to a point below the level of *B* on the same line of greatest slope, such that the string makes an angle of 30° with the plane (see diagram). The cube is on the point of toppling. Find

(i) the tension in the string, [3]

(ii) the weight of the cube. [4]

i) In ΔOBC, $\sin 30^\circ = \frac{BC}{OC} \Rightarrow \frac{1}{2} = \frac{0.4}{OC} \Rightarrow OC = 0.8\text{m}$.

$$T = \frac{\lambda x}{l} = \frac{12 \times (0.8 - 0.4)}{0.4} = 12\,\text{N}.$$

ii) Taking moments about B :

$$T\cos 30^\circ \times 0.4 + W\sin 30^\circ \times 0.2 = W\cos 30^\circ \times 0.2$$

$$12\cos 30^\circ \times 0.4 + W \times 0.1 = W \times 0.1\sqrt{3}$$

$$\frac{12\sqrt{3}}{5} + 0.1W = 0.1\sqrt{3}\,W$$

$$\frac{12\sqrt{3}}{5(0.1\sqrt{3} - 0.1)} = W$$

$$W = 56.78\,\text{N} \approx 56.8\,\text{N}.$$

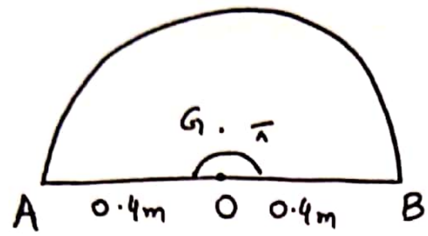
45 A uniform semicircular lamina has diameter AB of length 0.8 m.

(i) Find the distance of the centre of mass of the lamina from AB. [2]

The lamina rests in a vertical plane, with the point B of the lamina in contact with a rough horizontal surface and with A vertically above B. Equilibrium is maintained by a force of magnitude 6 N in the plane of the lamina, applied to the lamina at A and acting at an angle of 20° below the horizontal.

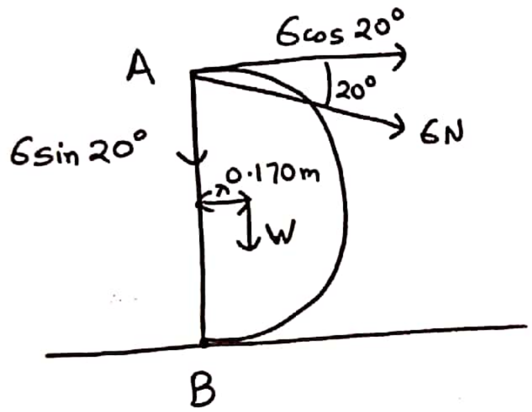
(ii) Calculate the mass of the lamina. [3]

i)



$$OG = \frac{2r\sin\alpha}{3\alpha} = \frac{2 \times 0.4 \times \sin\left(\frac{\pi}{2}\right)}{3\left(\frac{\pi}{2}\right)} = \frac{8}{15\pi} = 0.1697 \approx 0.170\ m.$$

ii)



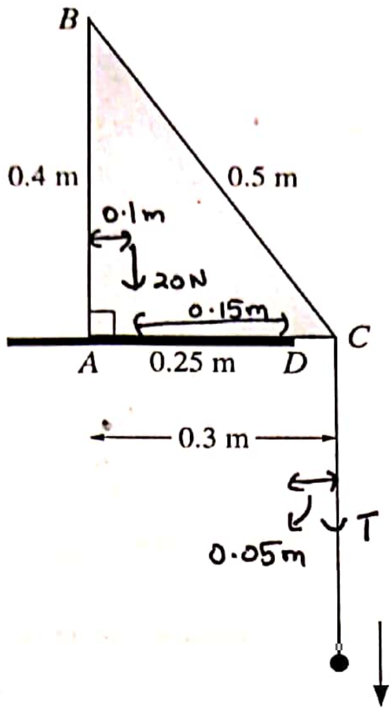
Taking moments about B :

$$6\cos 20° \times 0.8 = W \times 0.170$$

$$W = 26.56\ N$$

$$mg = 26.56$$

$$m = \frac{26.56}{10} = 2.656\ kg \approx 2.66\ kg$$



A uniform triangular prism of weight 20 N rests on a horizontal table. ABC is the cross-section through the centre of mass of the prism, where $BC = 0.5$ m, $AB = 0.4$ m, $AC = 0.3$ m and angle $BAC = 90°$. The vertical plane ABC is perpendicular to the edge of the table. The point D on AC is at the edge of the table, and $AD = 0.25$ m. One end of a light elastic string of natural length 0.6 m and modulus of elasticity 48 N is attached to C and a particle of mass 2.5 kg is attached to the other end of the string. The particle is released from rest at C and falls vertically (see diagram).

(i) Show that the tension in the string is 60 N at the instant when the prism topples. [3]

(ii) Calculate the speed of the particle at the instant when the prism topples. [5]

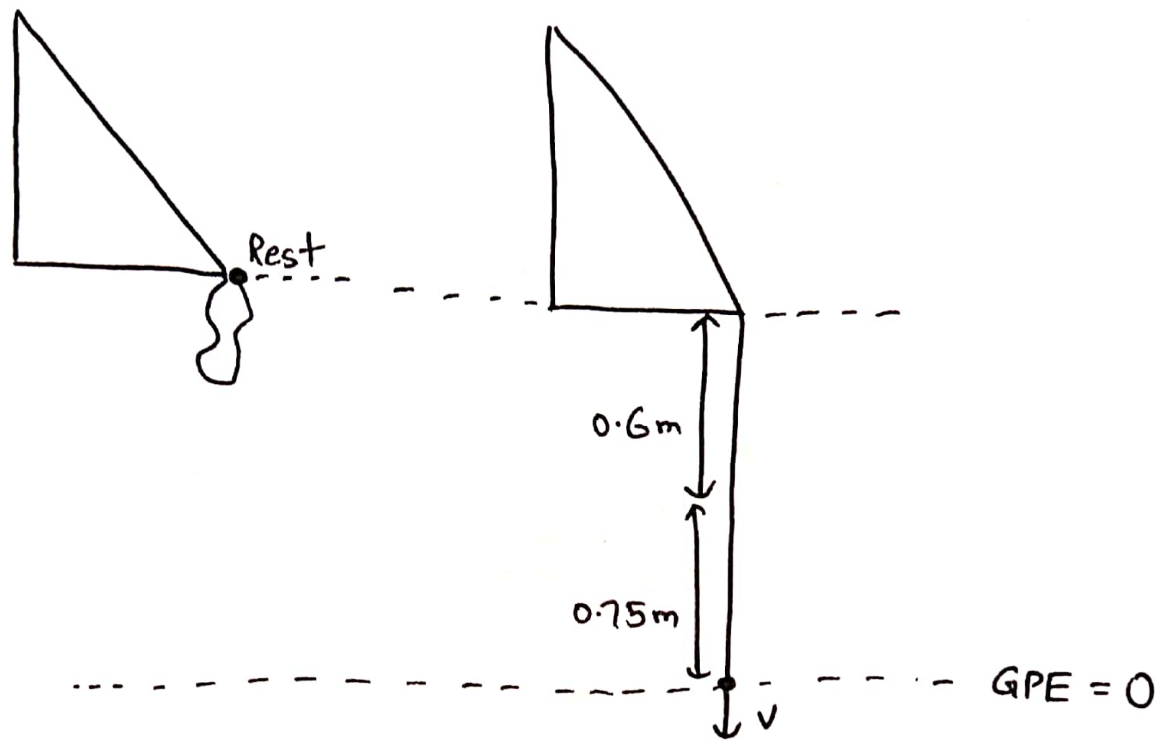
i) The cente of mass of the prism is a distance $0{\cdot}25 - 0{\cdot}1 = 0{\cdot}15$m from D.
When prism is about to topple, normal reaction force acts through D.
Taking moment about D, $20 \times 0{\cdot}15 = T \times 0{\cdot}05$

$T = 60$ N (Shown).

ii) When tension is 60 N, $T = \dfrac{\lambda x}{L}$

$$60 = \frac{48 \times x}{0{\cdot}6}$$

$$x = 0{\cdot}75\text{m}.$$



$$(KE + EPE + GPE)_{initial} = (KE + EPE + GPE)_{final}$$

$$0 + 0 + 2.5 \times 10 \times (1.35) = \frac{1}{2} \times 2.5 \times v^2 + \frac{48 \times 0.75^2}{2 \times 0.6} + 0$$

$$33.75 = 1.25 v^2 + 22.5$$

$$v^2 = 9$$

$$v = 3 m/s.$$

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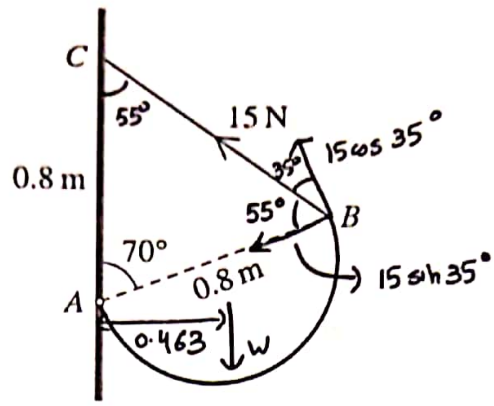


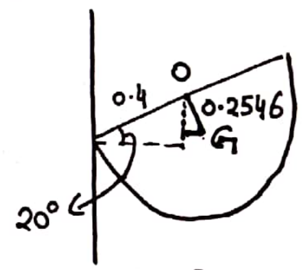
A uniform wire has the shape of a semicircular arc, with diameter AB of length 0.8 m. The wire is attached to a vertical wall by a smooth hinge at A. The wire is held in equilibrium with AB inclined at 70° to the upward vertical by a light string attached to B. The other end of the string is attached to the point C on the wall 0.8 m vertically above A. The tension in the string is 15 N (see diagram).

(i) Show that the horizontal distance of the centre of mass of the wire from the wall is 0.463 m, correct to 3 significant figures. [3]

(ii) Calculate the weight of the wire. [2]

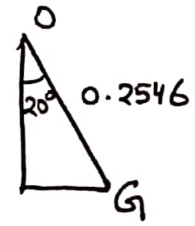
i) Let, O be centre of arc AB and G be centre of mass of AB.

$$OG = \frac{r \sin \alpha}{\alpha} = \frac{0.4 \times \sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = 0.2546\text{ m}$$



Horizontal distance of G from wall

$= 0.4 \cos 20° + 0.2546 \sin 20° = 0.4629 \approx 0.463\text{ m}.$



ii) Taking moments about A:

$$W \times 0.463\ \cancel{m} = 15 \cos 35° \times 0.8$$

$$W = 21.23\text{ N} \approx 21.2\text{ N}.$$

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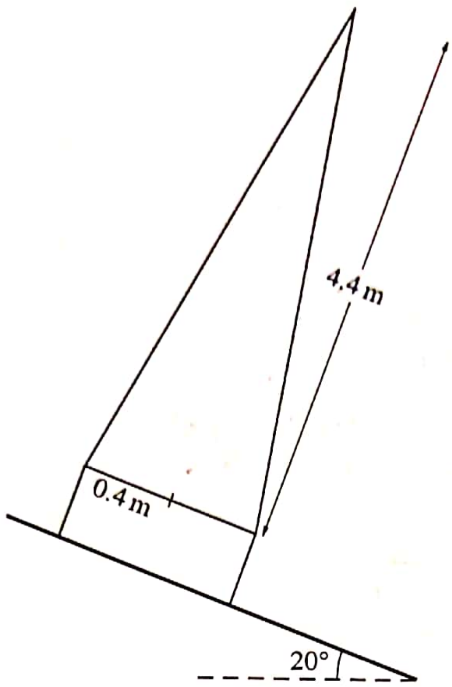


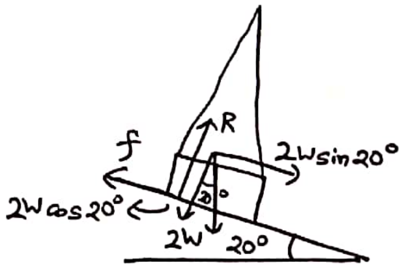
A uniform solid cone has base radius 0.4 m and height 4.4 m. A uniform solid cylinder has radius 0.4 m and weight equal to the weight of the cone. An object is formed by attaching the cylinder to the cone so that the base of the cone and a circular face of the cylinder are in contact and their circumferences coincide. The object rests in equilibrium with its circular base on a plane inclined at an angle of 20° to the horizontal (see diagram).

(i) Calculate the least possible value of the coefficient of friction between the plane and the object. [2]

(ii) Calculate the greatest possible height of the cylinder. [4]

i) R(//): $f = 2W\sin 20°$

R(⊥): $R = 2W\cos 20°$



Since object rests in equilibrium, $f \le \mu R$

$\Rightarrow 2W\sin 20° \le \mu \times 2W\cos 20°$

$\tan 20° \le \mu$

∴ Least possible value of $\mu = \tan 20° = 0.3639 \approx 0.364$.

ii) For greatest possible height of cylinder, the solid is about to topple and the line of action of weight passes through A.

Let, h be height of cylinder.

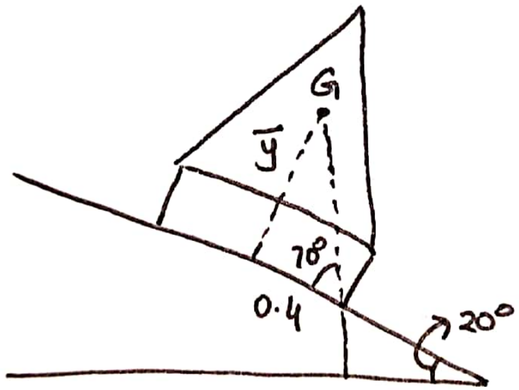
Shape	Weight	Distance of centre of mass from plane
Cylinder	W	$\frac{h}{2}$
Cone	W	$h + \frac{4.4}{4} = h + 1.1$
Composite figure	2W	$\bar{y}$

Taking moments about plane :

$$W \times \frac{h}{2} + W \times (h + 1.1) = 2W \times \bar{y}$$

$$0.5h + h + 1.1 = 2\bar{y}$$

$$0.75h + 0.55 = \bar{y}$$



$$\tan 70^\circ = \frac{\bar{y}}{0.4}$$

$$\bar{y} = 0.4 \tan 70^\circ$$

$$0.75h + 0.55 = 0.4 \tan 70^\circ$$

$$h = 0.7319 \simeq 0.732\text{m}.$$

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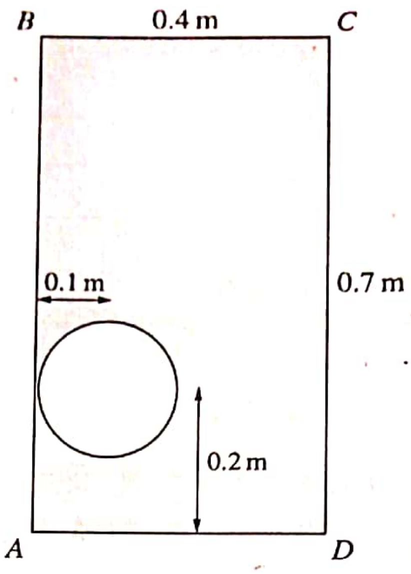


A uniform object is made by drilling a cylindrical hole through a rectangular block. The axis of the cylindrical hole is perpendicular to the cross-section $ABCD$ through the centre of mass of the object. $AB = CD = 0.7$ m, $BC = AD = 0.4$ m, and the centre of the hole is 0.1 m from AB and 0.2 m from AD (see diagram). The hole has a cross-section of area $0.03\ \mathrm{m}^2$.

(i) Show that the distance of the centre of mass of the object from AB is 0.212 m, and calculate the distance of the centre of mass from AD. [4]

The object has weight 70 N and is placed on a rough horizontal surface, with AD in contact with the surface. A vertically upwards force of magnitude F N acts on the object at C. The object is on the point of toppling.

(ii) Find the value of F. [2]

The force acting at C is removed, and the object is placed on a rough plane inclined at an angle $\theta°$ to the horizontal. AD lies along a line of greatest slope, with A higher than D. The plane is sufficiently rough to prevent sliding, and the object does not topple.

(iii) Find the greatest possible value of θ. [2]

i) Let, ρ = Mass per unit area.

Shape	Mass	Coordinates of centre of mass with A as origin
Rectangle ABCD	$0.4 \times 0.7 \times \rho = 0.28\rho$	$(0.2, 0.35)$
Circle	0.03ρ	$(0.1, 0.2)$
Composite figure	0.25ρ	$(\bar{x}, \bar{y})$

Taking moments about AB: $0.25\rho \times \bar{x} = 0.28\rho \times 0.2 - 0.03\rho \times 0.1$

$\bar{x} = 0.212$ m.

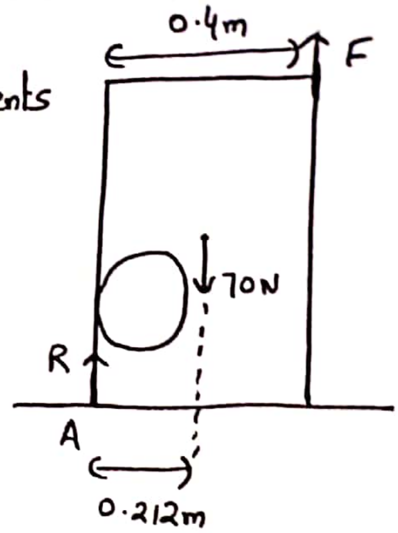
Taking moments about AD: $0.25\rho \times \bar{y} = 0.28\rho \times 0.35 - 0.03\rho \times 0.2$

$\bar{y} = 0.368$ m.

i.e. Distance of center of mass from AB = 0.212 m (shown).

Distance of center of mass from AD = 0.368 m

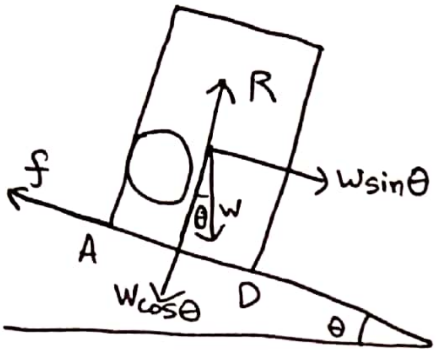
ii) Since object is on point of toppling, taking moments about A:



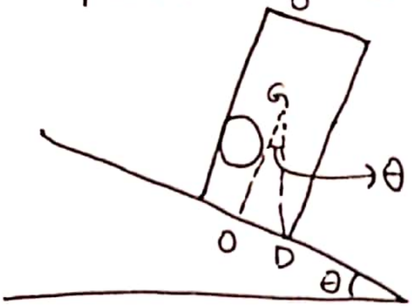
$$70 \times 0.212 = F \times 0.4$$

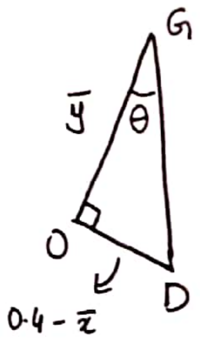
$$F = 37.1\ N.$$

iii)



~~R~~ Since object does not topple, θ can be increased so that line of action of weight W passes through D. At that point, R passes through D.





$$\tan\theta = \frac{0.4 - \bar{x}}{\bar{y}} \Rightarrow \tan\theta = \frac{0.4 - 0.212}{0.368}$$

$$\theta = 27.06^\circ = 27.1^\circ$$

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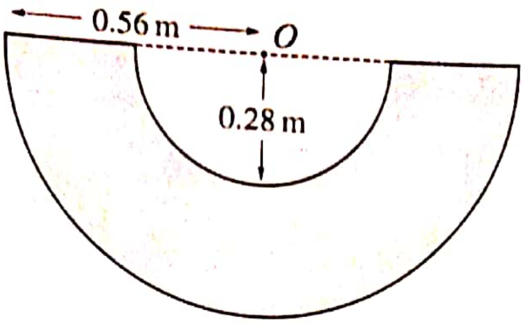


An object is made from a uniform solid hemisphere of radius 0.56 m and centre O by removing a hemisphere of radius 0.28 m and centre O. The diagram shows a cross-section through O of the object.

(i) Calculate the distance of the centre of mass of the object from O. [4]

[The volume of a hemisphere is $\frac{2}{3}\pi r^3$.]

Figure	Mass	Distance of centre of mass from O
Larger hemisphere	$\frac{2}{3}\pi(0.56)^3 \times \rho = \frac{5488\pi\rho}{46875}$	$\frac{3}{8} \times 0.56 = 0.21$
Smaller hemisphere	$\frac{2}{3}\pi(0.28)^3 \times \rho = \frac{686\pi\rho}{46875}$	$\frac{3}{8} \times 0.28 = 0.105$.
Composite figure	$\frac{4802\pi\rho}{46875}$	$\bar{x}$

Taking moments about O:

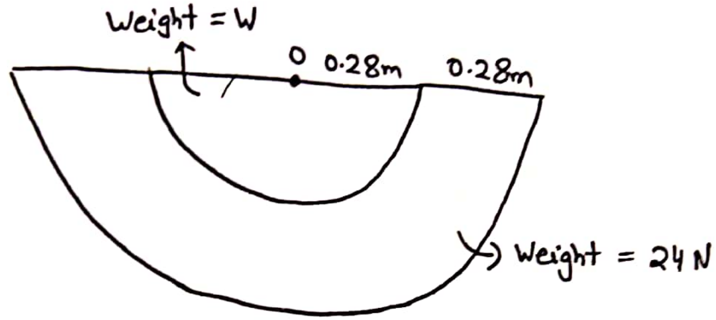
$$\frac{4802\pi\rho}{46875} \times \bar{x} = \frac{5488\pi\rho}{46875} \times 0.21 - \frac{686\pi\rho}{46875} \times 0.105$$

$$\bar{x} = 0.225 \text{ m}.$$

i.e. Distance of centre of mass from O = 0.225 m.

The object has weight 24 N. A uniform hemisphere H of radius 0.28 m is placed in the hollow part of the object to create a non-uniform hemisphere with centre O. The centre of mass of the non-uniform hemisphere is 0.15 m from O.

(ii) Calculate the weight of H. [3]



Shape	Mass	Distance of centre of mass from O
Object from (i)	$\frac{24}{10} = 2.4$	0.225
Hemisphere H	$\frac{W}{10} = 0.1W$	$\frac{3}{8} \times 0.28 = 0.105$
Non-uniform hemisphere	$2.4 + 0.1W$	0.15

Taking moments about O:

$$2.4 \times 0.225 + 0.1W \times 0.105 = (2.4 + 0.1W) \times 0.15$$

$$0.54 + 0.0105W = 0.36 + 0.015W$$

$$0.18 = 0.0045W$$

$$W = 40N.$$

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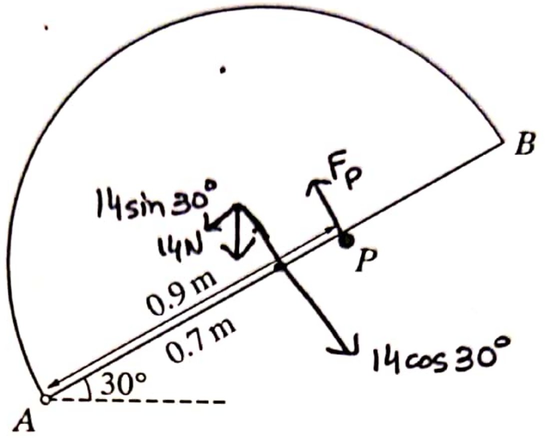

A uniform semicircular lamina of radius 0.7 m and weight 14 N has diameter AB. The lamina is in a vertical plane with A freely pivoted at a fixed point. The straight edge AB rests against a small smooth peg P above the level of A. The angle between AB and the horizontal is 30° and $AP = 0.9$ m (see diagram).

(i) Show that the magnitude of the force exerted by the peg on the lamina is 7.12 N, correct to 3 significant figures. [4]

Let, O be midpoint of AB and G be center of mass of lamina.

$$OG = \frac{2r\sin\alpha}{3\alpha} = \frac{2\times 0.7\times \sin\left(\frac{\pi}{2}\right)}{3\left(\frac{\pi}{2}\right)} = \frac{14}{15\pi} = 0.2970$$

Taking moments about A :

$$14\cos 30°\times 0.7 = F_P \times 0.9 + 14\sin 30° \times 0.2970$$

$$F_P = 7.119\,\text{N} = 7.12\,\text{N}.$$

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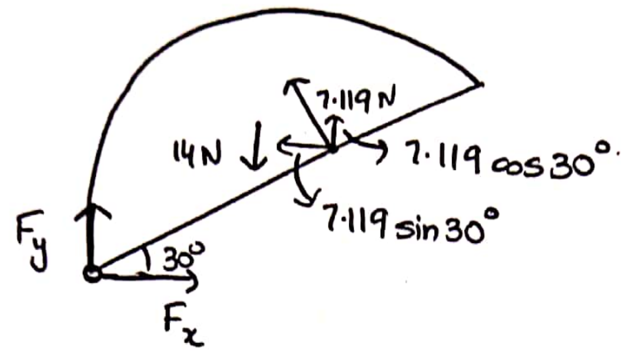
(ii) Find the angle with the horizontal of the force exerted by the pivot on the lamina at A. [3]



$R(\rightarrow)$:

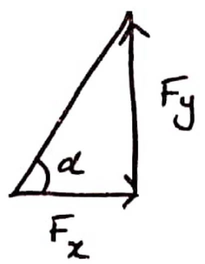
$$F_x = 7.119 \sin 30°$$

$$F_x = 3.5596 \approx 3.560 \text{ N}.$$

$R(\uparrow)$:

$$F_y + 7.119 \cos 30° = 14$$

$$F_y = 7.8344 \approx 7.834 \text{ N}.$$

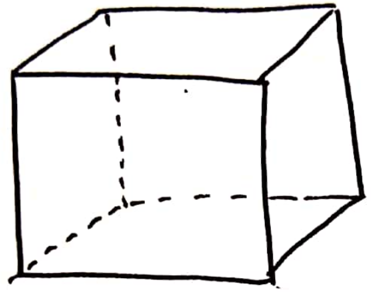


$$\tan\alpha = \frac{F_y}{F_x} \Rightarrow \alpha = \tan^{-1}\left(\frac{7.834}{3.560}\right) = 65.56° \approx 65.6°$$

∴ Angle with horizontal of the force exerted by pivot on lamina at A $= 65.6°$

52 An open box in the shape of a cube with edges of length 0.2 m is placed with its base horizontal and its four sides vertical. The four sides and base are uniform laminas, each with weight 3 N.

(i) Calculate the height of the centre of mass of the box above its base. [3]



Lamina	Mass	Distance of centre of mass from base
Base	0·3	0
Vertical faces	1·2 $= 4 \times 0{\cdot}3$	$\frac{0{\cdot}2}{2} = 0{\cdot}1$
Open box	$0{\cdot}3 + 1{\cdot}2 = 1{\cdot}5$	$\bar{y}$

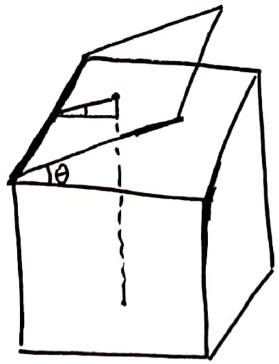
Taking moments about base :

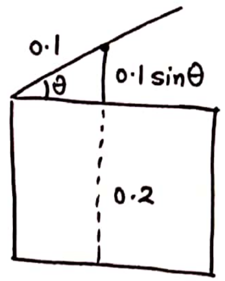
$$0{\cdot}3 \times 0 + 1{\cdot}2 \times 0{\cdot}1 = 1{\cdot}5 \times \bar{y}$$

$$\bar{y} = 0{\cdot}08 \text{ m.}$$

The box is now fitted with a thin uniform square lid of weight 3 N and with edges of length 0.2 m. The lid is attached to the box by a hinge of length 0.2 m and weight 2 N. The lid of the box is held partly open.

(ii) Find the angle which the lid makes with the horizontal when the centre of mass of the box (including the lid and hinge) is 0.12 m above the base of the box. [4]





Shape	Mass	Distance of centre of mass from base
Open box	1·5	0·08
Open lid	0·3	$0\cdot2+0\cdot1\sin\theta$
Hinge	0·2	0·2
Composite	2·0	0·12

Taking moments about base :

$$2\cdot0 \times 0\cdot12 = 1\cdot5 \times 0\cdot08 + 0\cdot3\,(0\cdot2+0\cdot1\sin\theta) + 0\cdot2\times0\cdot2$$

$$0\cdot24 = 0\cdot12 + 0\cdot06 + 0\cdot03\sin\theta + 0\cdot04$$

$$0\cdot02 = 0\cdot03\sin\theta$$

$$\sin\theta = \frac{2}{3}$$

$$\theta = 41\cdot81^\circ = 41\cdot8^\circ$$

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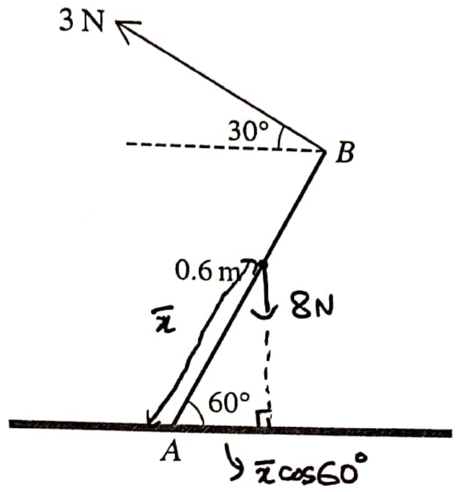


The end A of a non-uniform rod AB of length 0.6 m and weight 8 N rests on a rough horizontal plane, with AB inclined at 60° to the horizontal. Equilibrium is maintained by a force of magnitude 3 N applied to the rod at B. This force acts at 30° above the horizontal in the vertical plane containing the rod (see diagram).

(i) Find the distance of the centre of mass of the rod from A. [2]

Taking moments about A:

$$3 \times 0{\cdot}6 = 8 \times \bar{x} \cos 60°$$

$$\bar{x} = \frac{3 \times 0{\cdot}6}{8 \cos 60°}$$

$$\bar{x} = 0{\cdot}45$$

i.e. distance of centre of mass of rod from A = 0.45m.

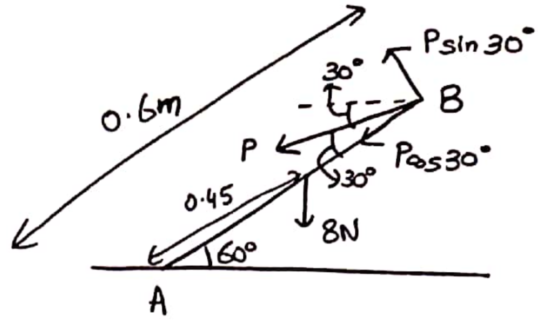
The 3 N force is removed, and the rod is held in equilibrium by a force of magnitude P N applied at B, acting in the vertical plane containing the rod, at an angle of 30° below the horizontal.

(ii) Calculate P. [2]

Taking moments about A:

$$P\sin 30^\circ \times 0.6 = 8 \times 0.45 \times \cos 60^\circ$$

$$P = 6\text{N}.$$



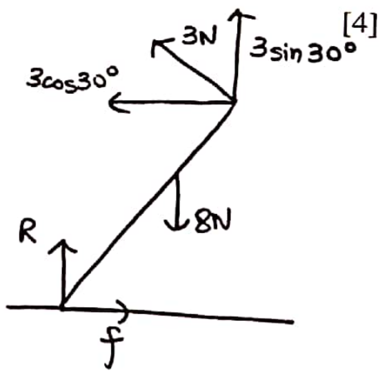
In one of the two situations described, the rod AB is in limiting equilibrium.

(iii) Find the coefficient of friction at A. [4]

$R(\rightarrow)$: $f = 3\cos 30^\circ = \frac{3\sqrt{3}}{2}$

$R(\uparrow)$: $R + 3\sin 30^\circ = 8 \Rightarrow R = 6.5$

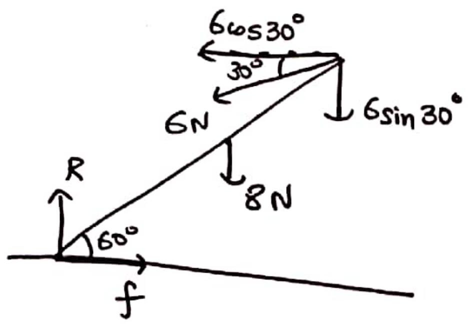
$\Rightarrow \mu = \frac{f}{R} = \frac{\frac{3\sqrt{3}}{2}}{6.5} = 0.3997 \approx 0.400$



$R(\rightarrow)$: $f = 6\cos 30^\circ = 3\sqrt{3}$

$R(\uparrow)$: $R = 8 + 6\sin 30^\circ = 11$ N.

$\Rightarrow \mu = \frac{f}{R} = \frac{3\sqrt{3}}{11} = 0.4723 \approx 0.472$



Then rod is in limiting equilibrium in (ii).

$\Rightarrow$ Coefficient of friction at A = 0.472.

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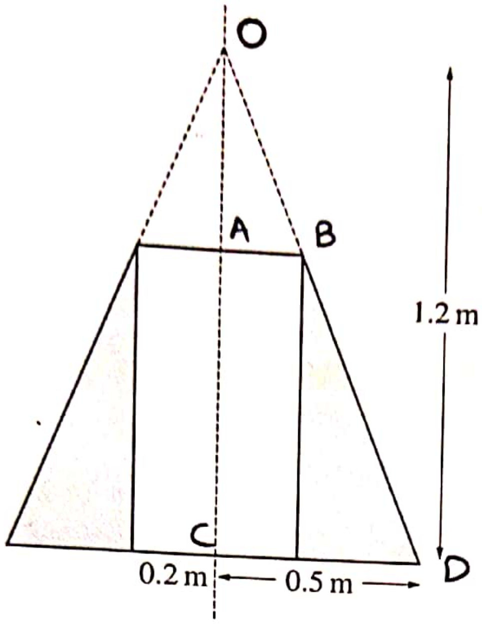

A uniform solid cone has height 1.2 m and base radius 0.5 m. A uniform object is made by drilling a cylindrical hole of radius 0.2 m through the cone along the axis of symmetry (see diagram).

(i) Show that the height of the object is 0.72 m and that the volume of the cone removed by the drilling is $0.0352\pi \text{ m}^3$. [4]

[The volume of a cone is $\frac{1}{3}\pi r^2 h$.]

$\Delta OAB \sim \Delta OCD$

$\Rightarrow \frac{OA}{OC} = \frac{AB}{CD} \Rightarrow \frac{OA}{1{\cdot}2} = \frac{0{\cdot}2}{0{\cdot}5} \Rightarrow OA = 0{\cdot}48.$

$\Rightarrow AC = OC - OA = 1{\cdot}2 - 0{\cdot}48 = 0{\cdot}72 \text{ m}.$

i.e. Height of object $= 0{\cdot}72$ m.

Volume removed from cone = Volume of cylinder + Volume of smaller cone

$= \pi \times 0{\cdot}2^2 \times 0{\cdot}72 + \frac{\pi \times 0{\cdot}2^2 \times 0{\cdot}48}{3}$

$= 0{\cdot}0352\pi \text{ cm}^3$. (Shown).

(ii) Find the distance of the centre of mass of the object from its base. [6]

Figure	Mass	Distance of centre of mass from base
Larger cone	$\frac{\pi \times 0.5^2 \times 1.2}{3} \times \rho = 0.1\pi\rho$	$\frac{1.2}{4} = 0.3$
Cylinder	$\pi \times 0.2^2 \times 0.72 \times \rho = 0.0288\pi\rho$	$\frac{0.72}{2} = 0.36$
Smaller cone	$\frac{\pi \times 0.2^2 \times 0.48\rho}{3} = 0.0064\pi\rho$	$0.72 + \frac{0.48}{4} = 0.84.$
Object remaining	$0.0648\pi\rho$	$\bar{y}.$

Taking moments about base:

$$0.0648\pi\rho \times \bar{y} = 0.1\pi\rho \times 0.3 - 0.0288\pi\rho \times 0.36 - 0.0064\pi\rho \times 0.84$$

$$\bar{y} = 0.22.$$

i.e. Distance of centre of mass from base $= 0.22\text{m}$

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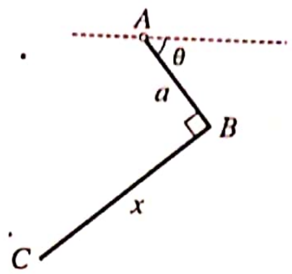
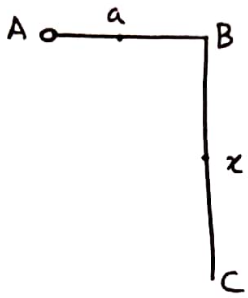


ABC is an object made from a uniform wire consisting of two straight portions AB and BC, in which $AB = a$, $BC = x$ and angle $ABC = 90°$. When the object is freely suspended from A and in equilibrium, the angle between AB and the horizontal is θ (see diagram).

(i) Show that $x^2 \tan\theta - 2ax - a^2 = 0$. [3]



Rod	Mass	~~Distance~~ Coordinates of centre of mass with A as origin.
AB	$a\rho$	$\left(\frac{a}{2}, 0\right)$
BC	$x\rho$	$\left(a, +\frac{x}{2}\right)$
Composite ABC	$(a+x)\rho$	$(\bar{x}, \bar{y})$

Taking moments about a vertical line through A:

$$a\rho \times \frac{a}{2} + x\rho \times a = (a+x)\rho \times \bar{x} \Rightarrow \frac{a^2}{2} + xa = (a+x)\bar{x}$$

$$\bar{x} = \frac{a^2 + 2xa}{2(a+x)}$$

Taking moments about AB:

$$a\rho \times 0 + x\rho \times \frac{x}{2} = (a+x)\rho \times \bar{y}$$

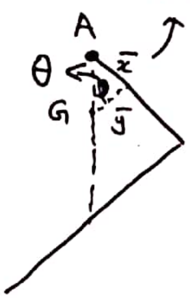
$$\bar{y} = \frac{x^2}{2(a+x)}$$

$$\tan\theta = \frac{\bar{x}}{\bar{y}} = \frac{\frac{a^2+2xa}{2(a+x)}}{\frac{x^2}{2(a+x)}} = \frac{a^2+2xa}{x^2}$$

$$\Rightarrow x^2\tan\theta = a^2 + 2xa$$

$$x^2\tan\theta - 2ax - a^2 = 0 \quad \text{(shown).}$$

When in equilibrium, centre of mass lies vertically below A.



(ii) Given that $\tan\theta = 1.25$, calculate the length of the wire in terms of a. [2]

$\tan\theta = 1\cdot 25 \Rightarrow \; x^2(1\cdot 25) - 2ax - a^2 = 0$

$$\frac{5x^2}{4} - 2ax - a^2 = 0$$

$$5x^2 - 8ax - 4a^2 = 0$$

$$5x^2 - 10ax + 2ax - 4a^2 = 0$$

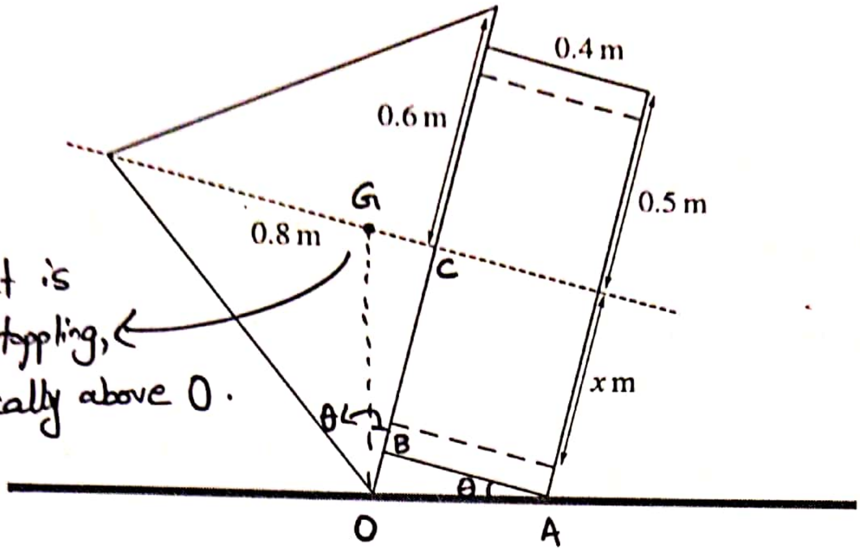
$$(5x + 2a)(x - 2a) = 0$$

$$x = \frac{-2a}{5}, \; 2a.$$

Ignore $x = -\frac{2a}{5}$ since $x > 0$.

$\therefore \; x = 2a \Rightarrow$ length of wire $= a + x = a + 2a = 3a.$

Since object is on point of toppling, G is vertically above O.

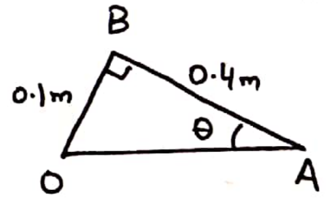


A uniform object is made by joining a solid cone of height 0.8 m and base radius 0.6 m and a cylinder. The cylinder has length 0.4 m and radius 0.5 m. The cylinder has a cylindrical hole of length 0.4 m and radius x m drilled through it along the axis of symmetry. A plane face of the cylinder is attached to the base of the cone so that the object has an axis of symmetry perpendicular to its base and passing through the vertex of the cone. The object is placed with points on the base of the cone and the base of the cylinder in contact with a horizontal surface (see diagram). The object is on the point of toppling.

(i) Show that the centre of mass of the object is 0.15 m from the base of the cone. [3]

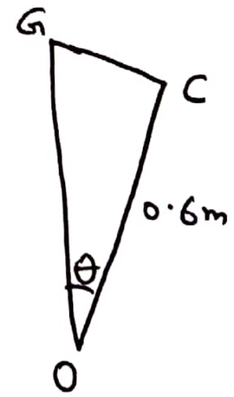
In ΔAOB, $OB = 0.6 - 0.5 = 0.1$ m, $AB = 0.4$ m

$$\tan\theta = \frac{0.1}{0.4} = \frac{1}{4}$$



In ΔOCG, $OC = 0.6$m, $CG = ?$, $C\hat{O}G = \theta$

$$\tan\theta = \frac{CG}{OC} \Rightarrow \frac{1}{4} = \frac{CG}{0.6} \Rightarrow CG = 0.15\text{ m}.$$



$\Rightarrow$ Distance of centre of mass of object from base of cone $= 0.15$m

(ii) Find x. [4]

[The volume of a cone is $\frac{1}{3}\pi r^2 h$.]

Figure	Mass	Distance of centre of mass from base of cone
Cylinder	$\pi(0.5)^2 \times 0.4 \times \rho = 0.1\pi\rho$	$\frac{-0.4}{2} = -0.2$
Cylinderical hole	$\pi(\cancel{0.4})^2 \times 0.4 \times \rho = \cancel{0.064\pi\rho}$ $0.4x^2\pi\rho$	$\frac{-0.4}{2} = -0.2$
Cone	$\frac{\pi(0.6)^2 \times 0.8\rho}{3} = 0.096\pi\rho$	$\frac{0.8}{4} = 0.2.$
Remaining solid	$0.1\pi\rho + 0.096\pi\rho - 0.4x^2\pi\rho$ $= (0.196 - 0.4x^2)\pi\rho$	0.15

Taking moment about base:

$$(0.196 - 0.4x^2) \times 0.15\pi\rho = 0.1\pi\rho \times -0.2 - 0.4x^2\pi\rho \times -0.2$$
$$+ 0.096\pi\rho \times 0.2$$

$$0.0294 - 0.06x^2 = -0.02 + 0.08x^2 + 0.0192$$

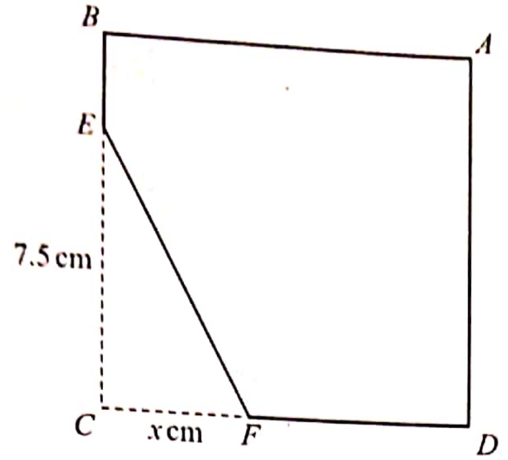
$$\cancel{0.02x^2 =}$$

$$0.0294 - 0.06x^2 = -0.02 + 0.08x^2 + 0.0192$$

$$-0.14x^2 = -0.0302$$

$$x^2 = 0.2157$$

$$x = 0.4644 \approx 0.464.$$



A uniform square lamina $ABCD$ has sides of length 10 cm. The point E is on BC with $EC = 7.5$ cm, and the point F is on DC with $CF = x$ cm. The triangle EFC is removed from $ABCD$ (see diagram). The centre of mass of the resulting shape $ABEFD$ is a distance $\bar{x}$ cm from CB and a distance $\bar{y}$ cm from CD.

(a) Show that $\bar{x} = \dfrac{400 - x^2}{80 - 3x}$ and find a corresponding expression for $\bar{y}$. [4]

Let, ρ be mass per unit area, let C be origin.

Mass of square ABCD $= 10 \times 10 \times \rho = 100\rho$

Mass of $\Delta ECF = \frac{1}{2} \times 7.5 \times x \times \rho = 3.75x\rho$

Mass of remaining shape $= (100 - 3.75x)\rho$.

Coordinates ~~Distance~~ of centre of mass of square ABCD : $(5, 5)$

Coordinates of centre of mass of ΔCEF : $\left(\frac{0+x+0}{3}, \frac{0+0+7.5}{3}\right) = \left(\frac{x}{3}, 2.5\right)$

Taking moments about CB:

$$100\rho \times 5 - 3.75x\rho \times \frac{x}{3} = (100 - 3.75x)\rho \times \bar{x}$$

$$500 - 1.25x^2 = (100 - 3.75x)\bar{x}$$

$$\bar{x} = \frac{500 - 1.25x^2}{100 - 3.75x} = \frac{\frac{2000 - 5x^2}{4}}{\frac{400 - 15x}{4}}$$

$$\bar{x} = \frac{5(400 - x^2)}{5(80 - 3x)} = \frac{400 - x^2}{80 - 3x} \text{ (Shown).}$$

Taking moments about CD:

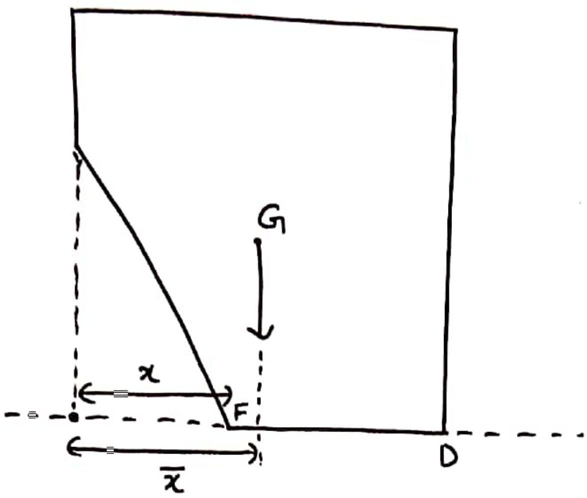
$$100\rho \times 5 - 3.75x\rho \times 2.5 = (100 - 3.75x)\rho \times \bar{y}$$

$$500 - 9.375x = (100 - 3.75x)\bar{y}$$

$$\bar{y} = \frac{500 - 9.375x}{100 - 3.75x} = \frac{4000 - 75x}{800 - 30x} = \frac{800 - 15x}{160 - 6x}.$$

The shape $ABEFD$ is in equilibrium in a vertical plane with the edge DF resting on a smooth horizontal surface.

(b) Find the greatest possible value of x, giving your answer in the form $a+b\sqrt{2}$, where a and b are constants to be determined. [3]



Equilibrium is maintained if line of action of weight lies within edge DF.

Then we require $\quad \bar{x} \geq x$

$$\frac{400-x^2}{80-3x} \geq x$$

$$400-x^2 \geq 80x-3x^2$$

$$2x^2-80x+400 \geq 0$$

$$x^2-40x+200 \geq 0$$

Critical values: $\quad x^2-40x+200=0$

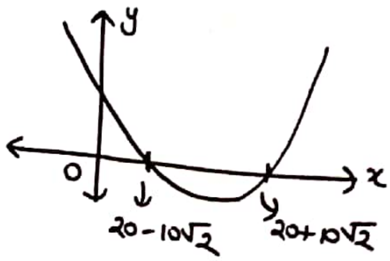
$$x = \frac{-(-40) \pm \sqrt{(-40)^2-4(1)(200)}}{2(1)} = \frac{40 \pm \sqrt{800}}{2}$$

$$x = 20 \pm 10\sqrt{2}$$

Reject $x = 20+10\sqrt{2}$ since $x<10$ (length of square is 10 cm).

$$x \leq 20-10\sqrt{2}.$$

$\Rightarrow$ Greatest possible value of $x = 20-10\sqrt{2}$ cm.



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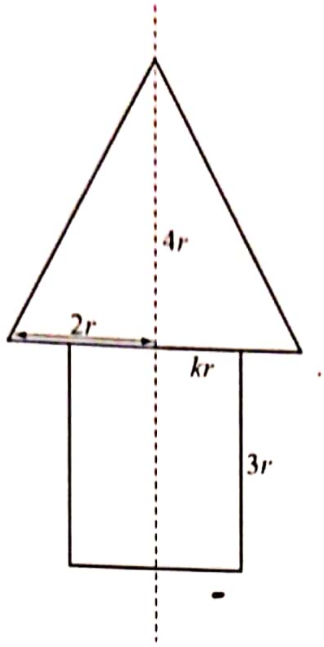


A uniform solid circular cone, of vertical height $4r$ and radius $2r$, is attached to a uniform solid cylinder, of height $3r$ and radius kr, where k is a constant less than 2. The base of the cone is joined to one of the circular faces of the cylinder so that the axes of symmetry of the two solids coincide (see diagram). The cone and the cylinder are made of the same material.

(a) Show that the distance of the centre of mass of the combined solid from the vertex of the cone is $\dfrac{(99k^2+96)r}{18k^2+32}$. [4]

Let, ρ = mass per unit volume.

Volume of cone $= \dfrac{\pi(2r)^2(4r)}{3} = \dfrac{16\pi r^3}{3} \Rightarrow$ Mass of cone $= \dfrac{16\pi r^3\rho}{3}$.

Mass of cylinder $= \pi(kr)^2(3r)\rho = 3k^2\pi r^3\rho$.

Mass of composite solid $= \left(\dfrac{16}{3} + 3k^2\right)\pi r^3\rho$

Distance of centre of mass of cylinder from ~~base~~ vertex $= 4r + \dfrac{1}{2}(3r) = \dfrac{3r}{2} + 4r = \dfrac{11r}{2}$.

" " " " " " cone " " $= \cancel{3r+}\dfrac{3}{4}(4r) = 3r$.

Taking moments about vertex:

$$\frac{16\pi r^3\rho}{3} \times 3r + 3k^2\pi r^3\rho \times \frac{11r}{2} = \left(\frac{16}{3} + 3k^2\right)\pi r^3\rho \times \bar{y}$$

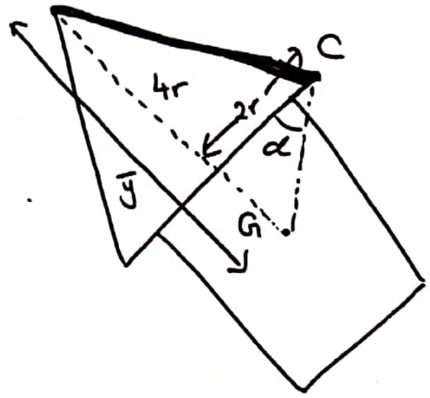
$$16r + \frac{33k^2r}{2} = \left(\frac{16+9k^2}{3}\right)\bar{y}$$

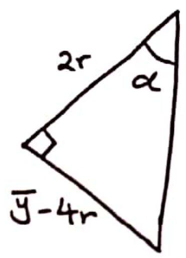
$$\frac{32r + 33k^2r}{2} = \left(\frac{16+9k^2}{3}\right)\bar{y}$$

$$\Rightarrow \bar{y} = \frac{3(32+33k^2)r}{2(16+9k^2)} = \frac{(99k^2+96)r}{18k^2+32}.$$

The point C is on the circumference of the base of the cone. When the combined solid is freely suspended from C and hanging in equilibrium, the diameter through C makes an angle α with the downward vertical, where $\tan\alpha = \frac{1}{8}$.

(b) Given that the centre of mass of the combined solid is within the cylinder, find the value of k. [4]





$$\tan\alpha = \frac{\bar{y}-4r}{2r}$$

$$\frac{1}{8} = \frac{\bar{y}-4r}{2r} \Rightarrow \bar{y}-4r = \frac{r}{4} \Rightarrow \bar{y} = \frac{17r}{4}$$

$$\Rightarrow \frac{(99k^2+96)r}{18k^2+32} = \frac{17r}{4}$$

$$396k^2+384 = 306k^2+544$$

$$90k^2 = 160$$

$$k^2 = \frac{160}{90} = \frac{16}{9}$$

$$k = \frac{4}{3}$$

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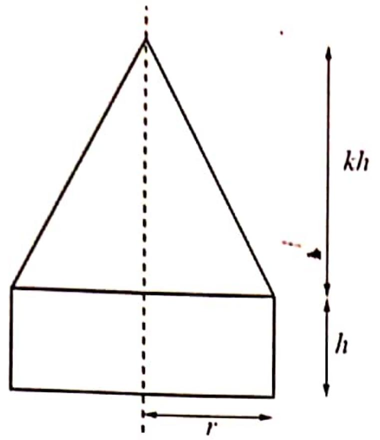


A uniform solid circular cone has vertical height kh and radius r. A uniform solid cylinder has height h and radius r. The base of the cone is joined to one of the circular faces of the cylinder so that the axes of symmetry of the two solids coincide (see diagram, which shows a cross-section). The cone and the cylinder are made of the same material.

(a) Show that the distance of the centre of mass of the combined solid from the base of the cylinder is $\dfrac{h(k^2+4k+6)}{4(3+k)}$. [4]

Let, ρ be mass per unit volume.

Mass of cylinder $= \pi r^2 h \rho$.

Mass of cone $= \dfrac{\pi r^2 (kh)}{3}\rho = \dfrac{k\pi r^2 h\rho}{3}$.

Mass of composite figure $= \pi r^2 h\rho + \dfrac{k\pi r^2 h\rho}{3} = \left(1+\dfrac{k}{3}\right)\pi r^2 h\rho = \left(\dfrac{3+k}{3}\right)\pi r^2 h\rho$.

Distance of centre of mass of cylinder from base $= \dfrac{1}{2}h$.

" " " " " " cone " " $= h + \dfrac{1}{4}kh = \left(1+\dfrac{1}{4}k\right)h = \left(\dfrac{4+k}{4}\right)h$.

Taking moments about base:

$$\pi r^2 h\rho \times \frac{h}{2} + \frac{k\pi r^2 h\rho}{3} \times \left(\frac{4+k}{4}\right)h = \left(\frac{3+k}{3}\right)\pi r^2 h\rho \times \bar{x}$$

$$\frac{h}{2} + \frac{k}{3}\left(\frac{4+k}{4}\right)h = \left(\frac{3+k}{3}\right)\bar{x}$$

$$h\left[\frac{1}{2} + \frac{4k+k^2}{12}\right] = \left(\frac{3+k}{3}\right)\bar{x}$$

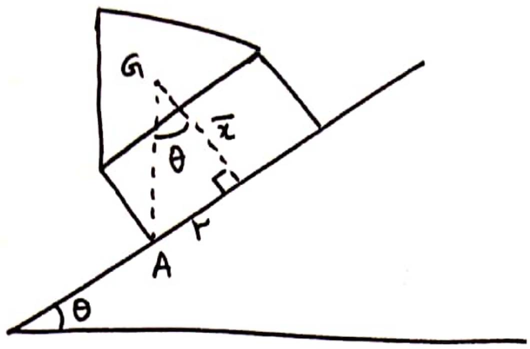
$$h\left[\frac{6+4k+k^2}{12}\right] = \left(\frac{3+k}{3}\right)\bar{x}$$

$$\frac{3h(6+4k+k^2)}{12(3+k)} = \bar{x}$$

$$\Rightarrow \bar{x} = \frac{h(k^2+4k+6)}{4(3+k)}.$$

The solid is placed on a plane that is inclined to the horizontal at an angle θ. The base of the cylinder is in contact with the plane. The plane is sufficiently rough to prevent sliding. It is given that $3h = 2r$ and that the solid is on the point of toppling when $\tan\theta = \frac{4}{3}$.

(b) Find the value of k. [3]



$$\tan\theta = \frac{r}{\bar{x}}$$

$$\frac{4}{3} = \frac{r}{\frac{h(k^2+4k+6)}{4(3+k)}}$$

$$\frac{4}{3} = \frac{4r(3+k)}{h(k^2+4k+6)}$$

$$4h(k^2+4k+6) = 12r(3+k)$$

$$h(k^2+4k+6) = 3r(3+k).$$

Since $2r = 3h \Rightarrow h = \frac{2r}{3}$

$$\Rightarrow \frac{2r}{3}(k^2+4k+6) = 3r(3+k)$$

$$2k^2+8k+12 = 9(3+k)$$

$$2k^2+8k+12 = 27+9k$$

$$2k^2-k-15 = 0$$

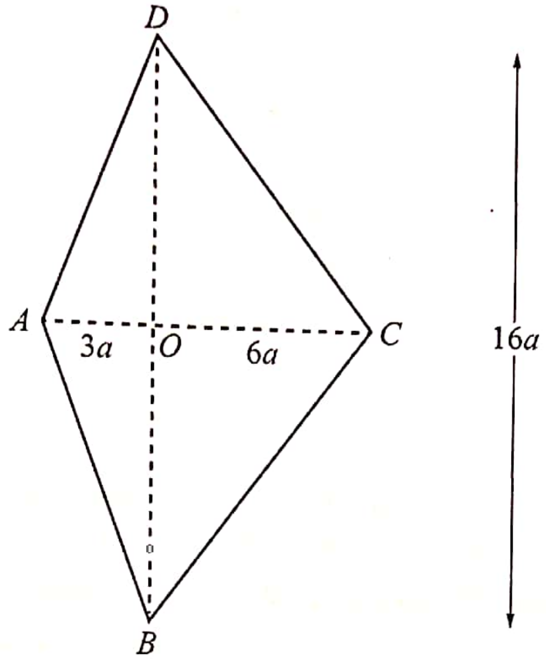
$$2k^2-6k+5k-15 = 0$$

$$(2k+5)(k-3) = 0$$

$$\Rightarrow k = \frac{-5}{2} \text{ or } k = 3.$$

Ignore $k = \frac{-5}{2}$ since k is a positive constant.

$$\Rightarrow k = 3.$$



A uniform lamina $ABCD$ consists of two isosceles triangles ABD and BCD. The diagonals of $ABCD$ meet at the point O. The length of AO is $3a$, the length of OC is $6a$ and the length of BD is $16a$ (see diagram).

Find the distance of the centre of mass of the lamina from DB. [3]

Let, ρ = Mass per unit area.

Area of $\Delta ABD = \frac{1}{2} \times 3a \times 16a = 24a^2 \Rightarrow$ Mass of $\Delta ABD = 24a^2\rho$

Area of $\Delta BCD = \frac{1}{2} \times 6a \times 16a = 48a^2 \Rightarrow$ Mass of $\Delta BCD = 48a^2\rho$.

Total mass $= 24a^2\rho + 48a^2\rho = 72a^2\rho$.

Distance of centre of mass of ΔABD from $BD = \frac{1}{3}(0+0+(-3a)) = -a$.

" " " " " " ΔBCD from $BD = \frac{1}{3}(0+0+6a) = 2a$.

Taking moments about BD:

$$48a^2\rho \times 2a + 24a^2\rho \times (-a) = 72a^2\rho \times \bar{x}$$

$$96a - 24a = 72\bar{x}$$

$$\bar{x} = \frac{72a}{72} = a$$

$\therefore$ Distance of centre of mass of lamina from DB $= a$.

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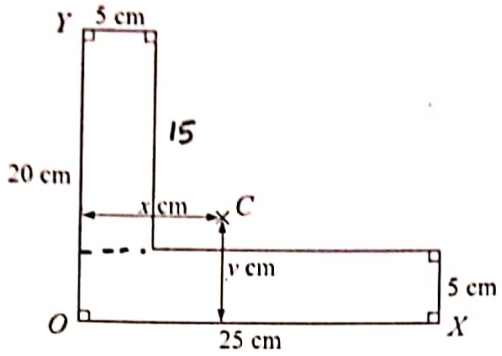


Fig. 1

Fig. 1 shows the cross section through the centre of mass C of a uniform L-shaped prism. C is x cm from OY and y cm from OX. Find the values of x and y. [4]

Shape	Mass	Coordinates of center of mass
Small rectangle	$15(5)\rho = 75\rho$	$(2.5,\ 12.5)$
Large rectangle	$25(5)\rho = 125\rho$	$(12.5,\ 2.5)$
Composite figure	200ρ	$(\bar{x},\ \bar{y})$.

Taking moments about OY: $75\rho \times 2.5 + 125\rho \times 12.5 = 200\rho \times \bar{x}$

$\bar{x} = 8.75$

Taking moments about OX: $75\rho \times 12.5 + 125\rho \times 2.5 = 200\rho \times \bar{y}$

$\bar{y} = 6.25$.

$x = 8.75,\ y = 6.25$.

(ii)

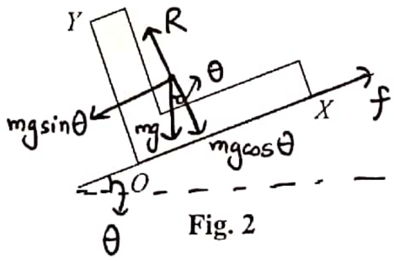


Fig. 2

The prism is placed on a rough plane with OX in contact with the plane. The plane is tilted from the horizontal so that OX lies along a line of greatest slope, as shown in Fig. 2. When the angle of inclination of the plane is sufficiently great the prism starts to slide (without toppling). Show that the coefficient of friction between the prism and the plane is less than $\frac{7}{5}$. [4]

$R(\parallel)$: $f = mg\sin\theta$.

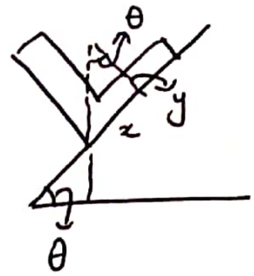
$R(\perp)$: $R = mg\cos\theta$

Since prism starts to slide, friction is limiting $\Rightarrow f_{lim} = \mu R \Rightarrow mg\sin\theta = \mu(mg\cos\theta)$

$\Rightarrow \mu = \tan\theta$.

For maximum value of μ, we increase θ until prism is on verge of toppling.
When on point of toppling, centre of mass G is vertically above O.

$$\tan\theta = \frac{x}{y} = \frac{8.75}{6.25} = \frac{7}{5}.$$



i.e. Maximum value of $\tan\theta$ is $\frac{7}{5}$ such that toppling doesn't occur.

$$\Rightarrow \mu = \tan\theta < \frac{7}{5}$$

$$\text{i.e. } \mu < \frac{7}{5}.$$

(iii)

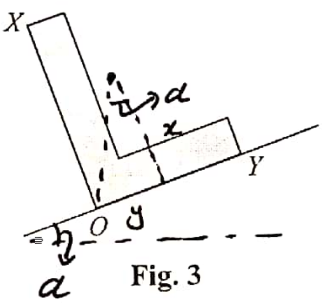


Fig. 3

The prism is now placed on a rough plane with OY in contact with the plane. The plane is tilted from the horizontal so that OY lies along a line of greatest slope, as shown in Fig. 3. When the angle of inclination of the plane is sufficiently great the prism starts to topple (without sliding). Find the least possible value of the coefficient of friction between the prism and the plane. [3]

When prism is about to topple, the centre of mass G is vertically above O.

Then $\tan\alpha = \frac{y}{x} \Rightarrow \tan\alpha = \frac{6.25}{8.75} = \frac{5}{7}.$

Since the prism does not slide,

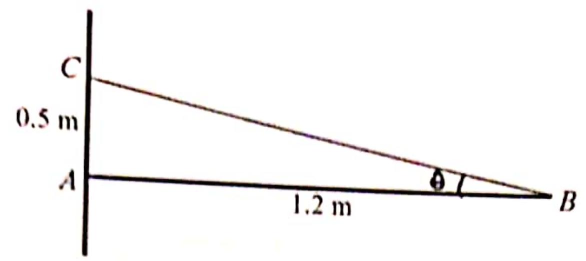
$$f \leq \mu R$$

$$mg\sin\alpha \leq \mu\,(mg\cos\alpha)$$

$$\tan\alpha \leq \mu$$

$$\Rightarrow \mu \geq \frac{5}{7}.$$

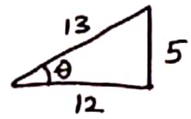
$\Rightarrow$ Least value of coefficient of friction $= \frac{5}{7}$.



$\tan\theta = \frac{0.5}{1.2} = \frac{5}{12}$

$\cos\theta = \frac{12}{13}$

$\sin\theta = \frac{5}{13}$.

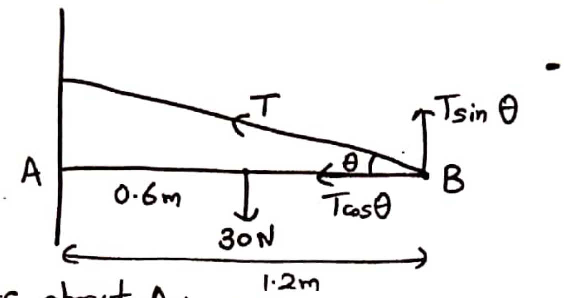


A uniform rod AB of length 1.2 m and weight 30 N is in equilibrium with the end A in contact with a vertical wall. AB is held at right angles to the wall by a light inextensible string. The string has one end attached to the rod at B and the other end attached to a point C of the wall. The point C is 0.5 m vertically above A (see diagram). Find

(i) the tension in the string, [3]

(ii) the horizontal and vertical components of the force exerted on the rod by the wall at A. [3]

i)

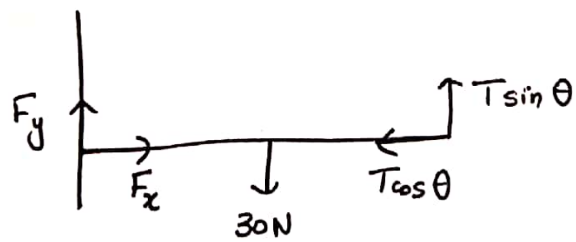


Taking moments about A:

$$T\sin\theta \times 1.2 = 30 \times 0.6$$

$$T = \frac{30 \times 0.6}{\frac{5}{13} \times 1.2} = 39\text{N}.$$

ii)



R(→): $F_x = T\cos\theta = 39\left(\frac{12}{13}\right) = 36$ N.

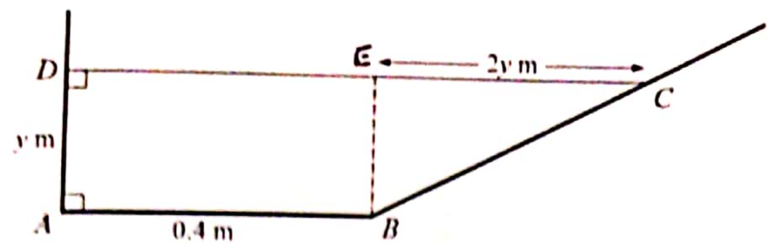
R(↑): $F_y + T\sin\theta = 30$

$$F_y + 39\left(\frac{5}{13}\right) = 30$$

$$F_y = 15\text{ N}.$$

i.e. Horizontal component of force exerted by wall at A = 36 N

Vertical component of force exerted by wall at A = 15 N.



A light container has a vertical cross-section in the form of a trapezium. The container rests on a horizontal surface. Grain is poured into the container to a depth of y m. As shown in the diagram, the cross-section $ABCD$ of the grain is such that $AB = 0.4$ m and $DC = (0.4 + 2y)$ m.

(i) When $y = 0.3$, find the vertical height of the centre of mass of the grain above the base of the container. [5]

(ii) Find the value of y for which the container is about to topple. [5]

i)

Shape	Mass	Distance of centre of mass from AB
Rectangle ABED	$0.4(0.3)\rho = 0.12\rho$	$\frac{0.3}{2} = 0.15$
Triangle BCE	$\frac{1}{2}(0.3)(0.6)\rho = 0.09\rho$	$\frac{1}{3}(0+0.3+0.3) = 0.2$
Composite	0.21ρ	$\bar{y}$

Taking moments about AB:

$$0.12\rho \times 0.15 + 0.09\rho \times 0.2 = 0.21\rho \times \bar{y}$$

$$0.1714 = \bar{y}$$

i.e. Vertical height of centre of mass above base of container = 0.171 m.

ii)

Shape	Mass	Distance of centre of mass from AD
Rectangle	$0.4y\rho$ ~~0.12ρ~~	$\frac{0.4}{2} = 0.2$
Triangle	$\frac{1}{2}y(2y)\rho = y^2\rho$	$\frac{1}{3}(0+0+2y)+0.4 = 0.4+\frac{2y}{3}$
Composite	$(y^2+0.4y)\rho$	$\bar{x}$

Taking moments about AD:

$$(y^2+0.4y)\rho \times \bar{x} = y^2\rho\left(0.4+\frac{2y}{3}\right) + 0.4y\rho \times 0.2$$

$$\bar{x} = \frac{0.4y^2 + \frac{2y^3}{3} + 0.08y}{y^2+0.4y}$$

Container is about to topple if $\bar{x} = 0.4$

$$0.4 = \frac{0.4y^2 + \frac{2y^3}{3} + 0.08y}{y^2+0.4y}$$

$$0.4y^2 + 0.16y = 0.4y^2 + \frac{2y^3}{3} + 0.08y$$

$$0 = \frac{2y^3}{3} + 0.08y - 0.16y$$

$$0 = \frac{2y^3}{3} - 0.08y$$

$$0 = y\left(\frac{2y^2}{3} - 0.08\right)$$

Either, $y = 0$ (Ignore) or $\frac{2y^2}{3} - 0.08 = 0$

$$y^2 = \frac{0.24}{2}$$

$$y^2 = 0.12$$

$$y = \frac{\sqrt{3}}{5} \simeq 0.346$$

i.e. Container is about to topple when $y \simeq 0.346$.

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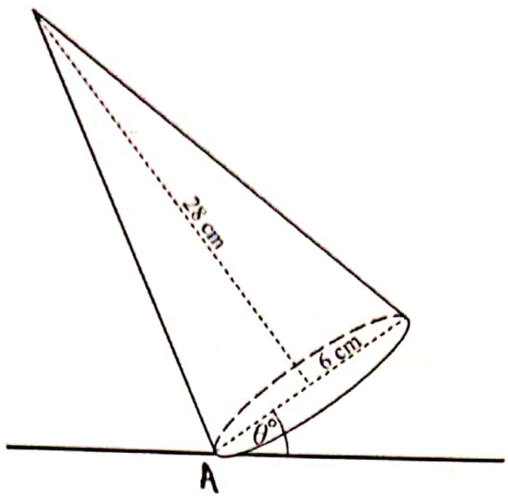
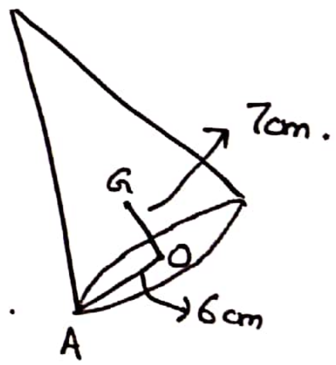


A uniform solid cone has vertical height 28 cm and base radius 6 cm. The cone is held with a point of the circumference of its base in contact with a horizontal table, and with the base making an angle of $\theta°$ with the horizontal (see diagram). When the cone is released, it moves towards the equilibrium position in which its base is in contact with the table. Show that $\theta < 40.6$, correct to 1 decimal place. [3]

Let, O be centre of circular base and G be centre of mass of cone.

$$OG = \frac{1}{4}(28) = 7 \text{ cm}$$



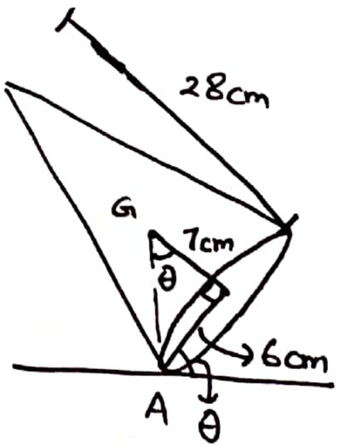
Since cone falls back on its circular base, the centre of mass lies such that the line of action of weight lies within the base of the cone.

~~If~~ If cone were about to toppling, G is vertically above point A.

$$\tan\theta = \frac{6}{7} \Rightarrow \theta = 40.6°$$

ie. cone is about to topple if $\theta = 40.6°$

Since cone doesn't topple, $\theta < 40.6°$. (shown).



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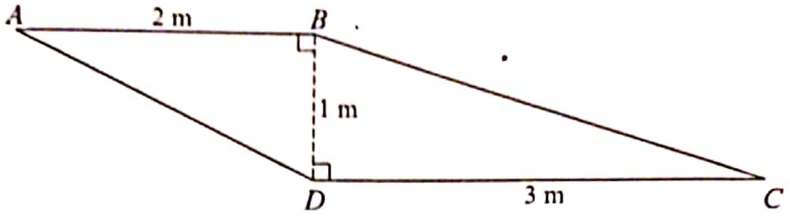


A uniform lamina $ABCD$ is in the form of a trapezium in which AB and DC are parallel and have lengths 2 m and 3 m respectively. BD is perpendicular to the parallel sides and has length 1 m (see diagram).

(i) Find the distance of the centre of mass of the lamina from BD. [3]

The lamina has weight W N and is in equilibrium, suspended by a vertical string attached to the lamina at B. The lamina rests on a vertical support at C. The lamina is in a vertical plane with AB and DC horizontal.

(ii) Find, in terms of W, the tension in the string and the magnitude of the force exerted on the lamina at C. [3]

i)

Shape	Mass	Distance of centre of mass from BD
Triangle ABD	$\frac{1}{2}(2)(1)\rho = \rho$	$\frac{1}{3}(0+0+(-2)) = \frac{-2}{3}$.
Triangle BCD	$\frac{1}{2}(3)(1)\rho = 1.5\rho$	$\frac{1}{3}(0+0+3) = 1$.
Composite figure	2.5ρ	$\bar{x}$

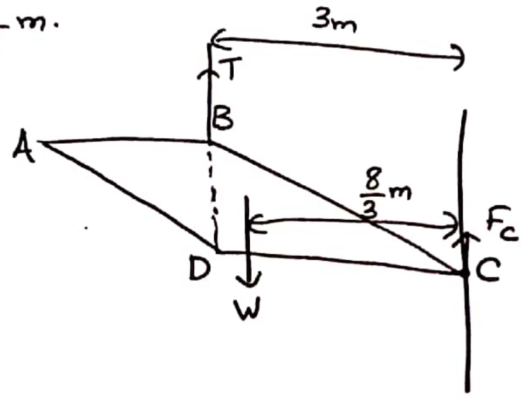
Taking moments about BD :

$$2.5\rho \times \bar{x} = \rho \times \frac{-2}{3} + 1.5\rho \times 1$$

$$\bar{x} = \frac{1}{3}\text{ m}$$

$\Rightarrow$ Distance of centre of mass from $BD = \frac{1}{3}$ m.

ii) Taking moments about C:



$$T \times 3 = W \times \frac{8}{3}$$

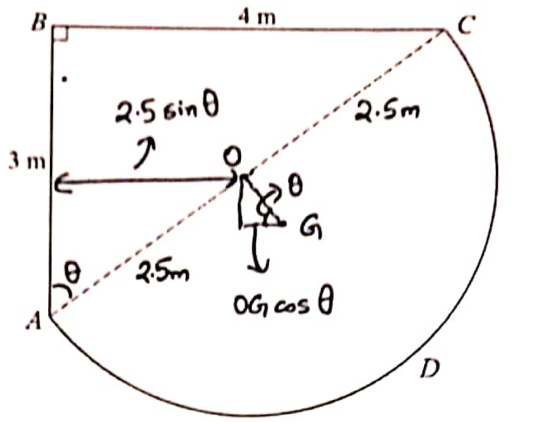
$$T = \frac{8}{9}W.$$

$R(\uparrow)$: $F_c + T = W$

$$F_c + \frac{8W}{9} = W$$

$$F_c = \frac{W}{9}.$$

i.e. Tension in the string $= \frac{8W}{9}$, Force on lamina at $C = \frac{W}{9}$.



$AC^2 = 3^2 + 4^2$

$AC = 5$

$\Rightarrow r = 2.5.$

$\tan\theta = \frac{4}{3}$

$\sin\theta = \frac{4}{5}$

$\cos\theta = \frac{3}{5}.$

A large uniform lamina is in the shape of a right-angled triangle ABC, with hypotenuse AC, joined to a semicircle ADC with diameter AC. The sides AB and BC have lengths 3 m and 4 m respectively, as shown in the diagram.

(i) Show that the distance from AB of the centre of mass of the semicircular part ADC of the lamina is $\left(2 + \frac{2}{\pi}\right)$ m. [3]

(ii) Show that the distance from AB of the centre of mass of the complete lamina is 2.14 m, correct to 3 significant figures. [5]

i) Let, O be midpoint of AC. Let, G be centre of mass of semicircular lamina.

$$OG = \frac{2r\sin\alpha}{3\alpha} = \frac{2(2.5)\sin\left(\frac{\pi}{2}\right)}{3\left(\frac{\pi}{2}\right)} = \frac{5}{\frac{3\pi}{2}} = \frac{10}{3\pi}.$$

Horizontal distance from G to AB

$$= 2.5\sin\theta + OG\cos\theta = 2.5\left(\frac{4}{5}\right) + \frac{10}{3\pi}\left(\frac{3}{5}\right) = \left(2 + \frac{2}{\pi}\right)\text{m}.$$

ii)

Shape	Mass	Distance of centre of mass from AB
Semicircular sector ADC	$\frac{\pi(2.5)^2}{2} \times \rho = \frac{25\pi\rho}{8}$	$2 + \frac{2}{\pi}$
ΔABC	$\frac{1}{2} \times 3 \times 4 \times \rho = 6\rho$	$\frac{1}{3}(0+0+4) = \frac{4}{3}$
Composite figure	$\left(\frac{25\pi}{8} + 6\right)\rho$	$\bar{x}$

Taking moments about AB:

$$\left(\frac{25\pi}{8} + 6\right)\rho \times \bar{x} = \frac{25\pi\rho}{8} \times \left(2 + \frac{2}{\pi}\right) + 6\rho \times \frac{4}{3}$$

$$\bar{x} = \frac{6.25\pi + 6.25 + 8}{\left(\frac{25\pi}{8} + 6\right)} = 2.142 \approx 2.14\text{m}$$

∴ Distance of centre of mass of composite lamina from AB ≈ 2.14m

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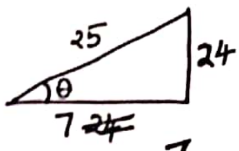
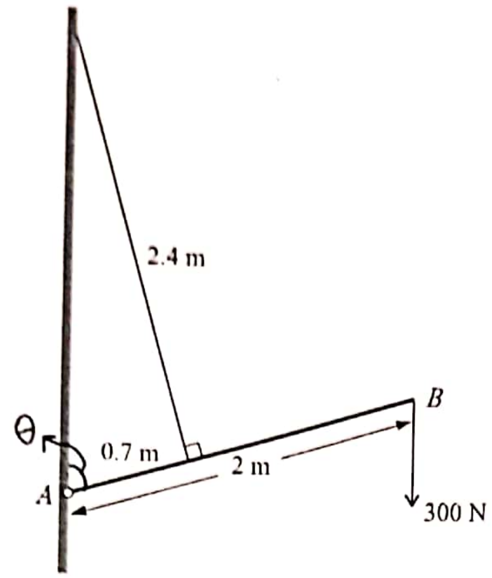
$\tan\theta = \frac{2.4}{0.7} = \frac{24}{7}$



$\sin\theta = \frac{24}{25}$, $\cos\theta = \frac{7}{25}$.



A uniform beam AB has length 2 m and mass 10 kg. The beam is hinged at A to a fixed point on a vertical wall, and is held in a fixed position by a light inextensible string of length 2.4 m. One end of the string is attached to the beam at a point 0.7 m from A. The other end of the string is attached to the wall at a point vertically above the hinge. The string is at right angles to AB. The beam carries a load of weight 300 N at B (see diagram).

(i) Find the tension in the string. [4]

The components of the force exerted by the hinge on the beam are X N horizontally away from the wall and Y N vertically downwards.

(ii) Find the values of X and Y. [3]

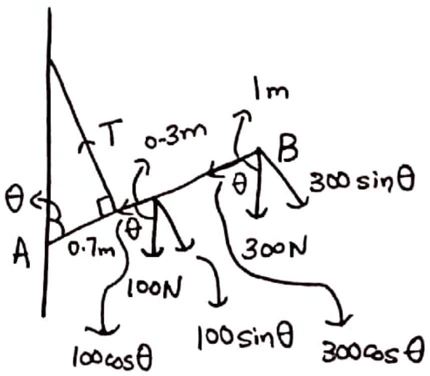
i) Taking moments about A:

$$T \times 0.7 = 100\sin\theta \times 1 + 300\sin\theta \times 2$$

$$0.7T = 100\left(\frac{24}{25}\right) \times 1 + 300\left(\frac{24}{25}\right) \times 2$$

$$0.7T = 672$$

$$T = 960\text{N}$$



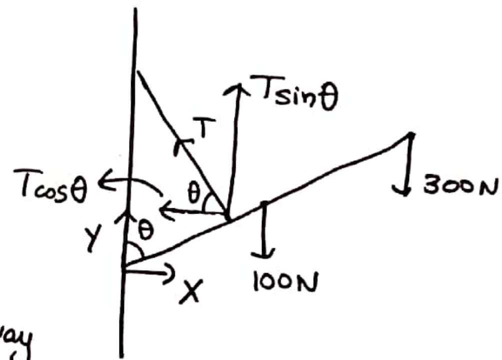
ii) $R(\rightarrow)$: $X = T\cos\theta = 960\left(\frac{7}{25}\right) = 268.8\text{N}$

$R(\uparrow)$: $Y + T\sin\theta = 100 + 300$

$$Y + 960\left(\frac{24}{25}\right) = 400$$

$$Y = -521.6\text{ N}.$$

i.e. Force on beam at hinge is 268.8N away from wall and 521.6N vertically downwards.



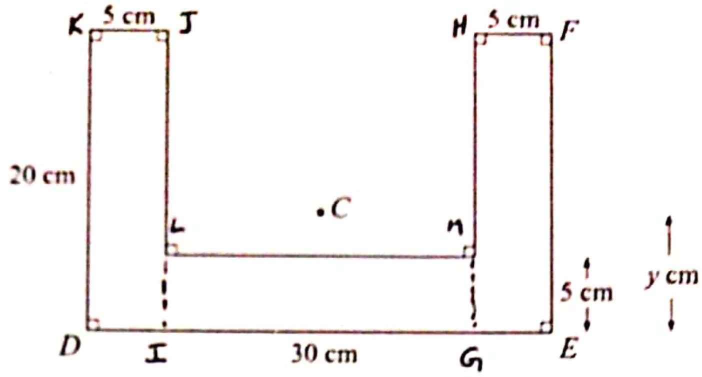


Fig. 1

Fig. 1 shows the cross-section of a uniform solid. The cross-section has the shape and dimensions shown. The centre of mass C of the solid lies in the plane of this cross-section. The distance of C from DE is y cm.

(i) Find the value of y. [3]

Shape	Mass	Distance of centre of mass from DE
Rectangle DKJI	$20(5)\ell = 100\ell$	$\frac{1}{2} \times 20 = 10$
Rectangle GILM	$20(5)\ell = 100\ell$	$\frac{1}{2} \times 5 = 2.5$
Rectangle GEFH	$5(20)\ell = 100\ell$	$\frac{1}{2} \times 20 = 10$
Composite figure	300ℓ	$\bar{y}$

Taking ~~moments~~ moments about DE: $300\ell \times \bar{y} = 100\ell \times 10 + 100\ell \times 2.5 + 100\ell \times 10$

$$\bar{y} = 7.5 \text{ cm}$$

$$\Rightarrow y = 7.5 \text{cm}.$$

The solid is placed on a rough plane. The coefficient of friction between the solid and the plane is μ. The plane is tilted so that EF lies along a line of greatest slope.

(ii)

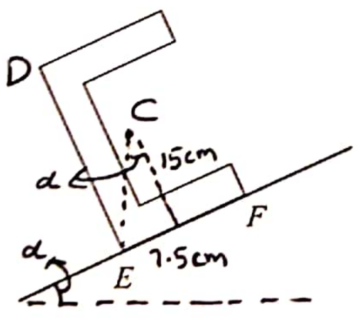


Fig. 2

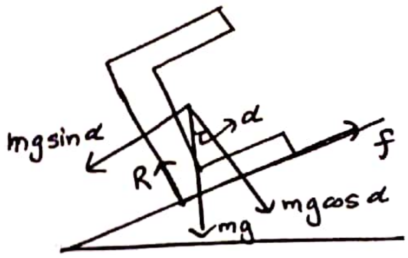
The solid is placed so that F is higher up the plane than E (see Fig. 2). When the angle of inclination is sufficiently great the solid starts to topple (without sliding). Show that $\mu > \frac{1}{2}$. [3]

When solid starts to topple, C is vertically above E.

$$\tan\alpha = \frac{7.5}{15} = \frac{1}{2}.$$

$R(\parallel): f = mg\sin\alpha$

$R(\perp): R = mg\cos\alpha.$



Since solid does not slide,

$$f < \mu R$$

$$mg\sin\alpha < \mu(mg\cos\alpha)$$

$$\tan\alpha < \mu$$

$\Rightarrow \mu > \tan\alpha \Rightarrow \mu > \frac{1}{2}$ (Shown).

(iii)

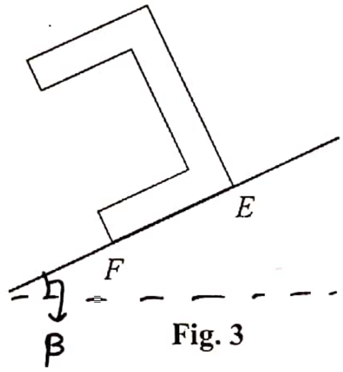


Fig. 3

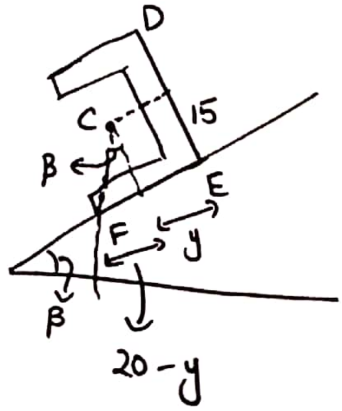
The solid is now placed so that E is higher up the plane than F (see Fig. 3). When the angle of inclination is sufficiently great the solid starts to slide (without toppling). Show that $\mu < \frac{5}{6}$. [3]

Since object starts to slide, friction is limiting.

$$f = \mu R$$

$$mg\sin\beta = \mu(mg\cos\beta)$$

$$\mu = \tan\beta.$$



For maximum value of μ, we increase β until solid is about to topple. Then C is vertically above F.

$$\tan\beta = \frac{20-y}{15} = \frac{20-7.5}{15} = \frac{12.5}{15} = \frac{5}{6}.$$

Since object does not topple, $\tan\beta < \frac{5}{6} \Rightarrow \mu < \frac{5}{6}$.

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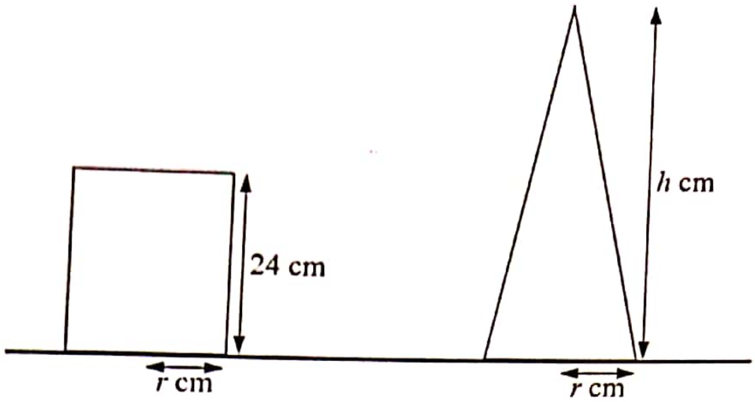
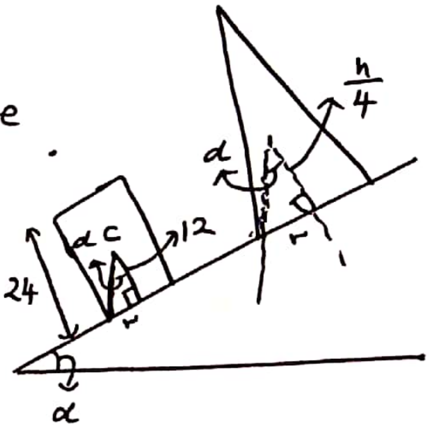


A uniform solid cylinder has height 24 cm and radius r cm. A uniform solid cone has base radius r cm and height h cm. The cylinder and the cone are both placed with their axes vertical on a rough horizontal plane (see diagram, which shows cross-sections of the solids). The plane is slowly tilted and both solids remain in equilibrium until the angle of inclination of the plane reaches $\alpha°$, when both solids topple simultaneously.

(i) Find the value of h. [2]

(ii) Given that $r = 10$, find the value of α. [2]



i) Let, C be centre of mass of cylinder and G be centre of mass of cone.

For ~~cone~~ cylinder, $\tan \alpha = \frac{r}{12} \rightarrow (1)$

For cone, $\tan \alpha = \frac{r}{\frac{h}{4}} \rightarrow (2)$

Then (1) = (2) gives: $\frac{r}{12} = \frac{r}{\frac{h}{4}}$

$\frac{h}{4} = 12 \Rightarrow h = 48$ cm.

ii) Substituting $r = 10$ in (1) gives:

$\tan \alpha = \frac{10}{12} = \frac{5}{6}$

$\alpha = \tan^{-1}\left(\frac{5}{6}\right) = 39.80° \approx 39.8°$

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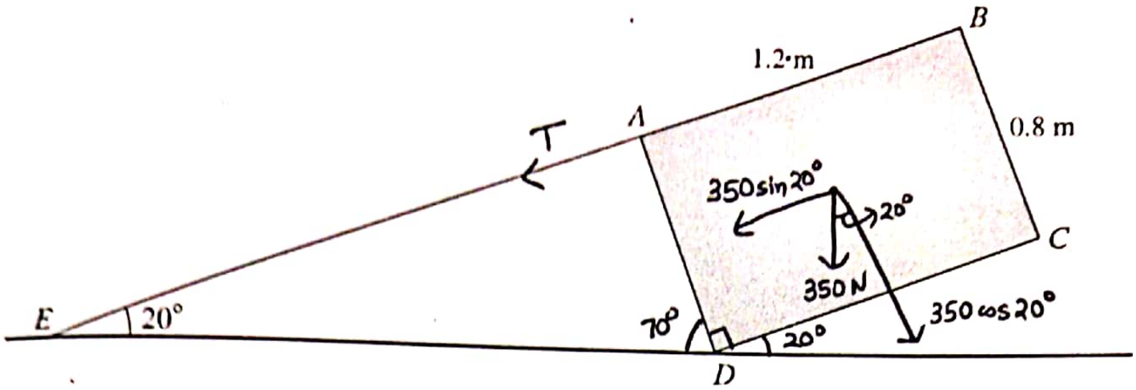


ABCD is a central cross-section of a uniform rectangular block of mass 35 kg. The lengths of *AB* and *BC* are 1.2 m and 0.8 m respectively. The block is held in equilibrium by a rope, one end of which is attached to the point *E* of a rough horizontal floor. The other end of the rope is attached to the block at *A*. The rope is in the same vertical plane as *ABCD*, and *EAB* is a straight line making an angle of 20° with the horizontal (see diagram).

(i) Show that the tension in the rope is 187 N, correct to the nearest whole number. [5]

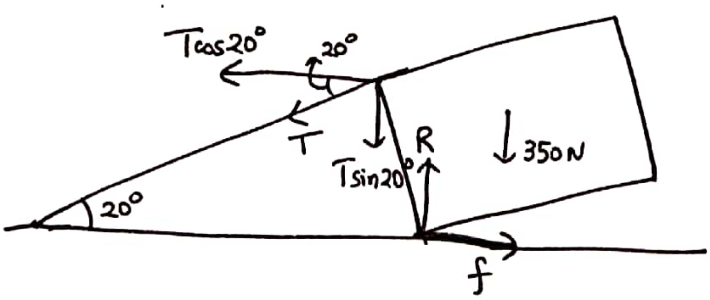
(ii) The block is on the point of slipping. Find the coefficient of friction between the block and the floor. [4]

i) Taking moments about D:

$$T \times 0.8 + 350 \sin 20^\circ \times 0.4 = 350 \cos 20^\circ \times 0.6$$

$$T = 186.81 \text{ N} \approx 187 \text{ N}.$$

ii)



$R(\parallel)$: $f = T\cos 20^\circ = 175.549 \text{ N}$

$R(\perp)$: $R = T\sin 20^\circ + 350 = 413.895 \text{ N}.$

Since block is on the point of slipping, friction is limiting

$$\Rightarrow f = \mu R \Rightarrow 175.549 = \mu \times 413.895$$

$$\mu = 0.4241 \approx 0.424$$

i.e. coefficient of friction between block and floor $= 0.424$.

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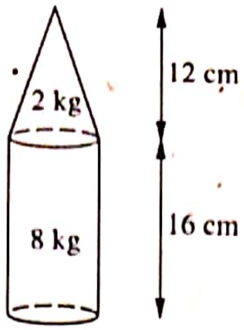


Fig. 1

A uniform solid cylinder has mass 8 kg and height 16 cm. A uniform solid cone, whose base radius is the same as the radius of the cylinder, has mass 2 kg and height 12 cm. A composite solid is formed by joining the cylinder and cone so that the base of the cone coincides with one end of the cylinder (see Fig. 1).

(i) Show that the centre of mass of the composite solid is 10.2 cm from its base. [3]

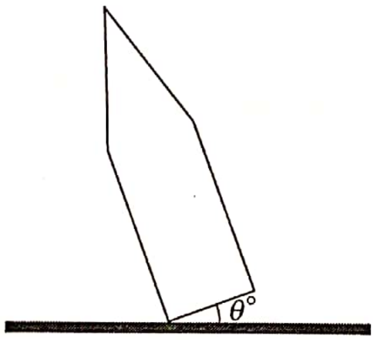


Fig. 2

The composite solid is held with a point on the circumference of its base in contact with a horizontal table. The base makes an angle $\theta°$ with the table (see Fig. 2, which shows a cross-section). When the cone is released it moves towards the equilibrium position in which its base is in contact with the table.

(ii) Given that the radius of the base is 4 cm, find the greatest possible value of θ, correct to 1 decimal place. [3]

i)

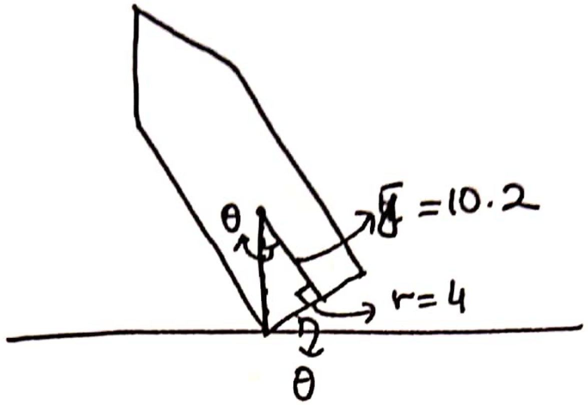
Shape	Mass (kg)	Distance of centre of mass from base (cm)
Cylinder	8	8
Cone	2	$16 + \frac{12}{4} = 19$
Composite	10	$\bar{y}$

Taking moments about base :

$$8 \times 8 + 2 \times 19 = 10\bar{y}$$

$$\bar{y} = 10.2 \text{ cm.}$$

ii)



Maximum value of θ occurs when centre of mass is vertically above point of contact.

$$\tan\theta = \frac{r}{\bar{y}} = \frac{4}{10.2}$$

$$\theta = \tan^{-1}\left(\frac{4}{10.2}\right) \approx 21.4°$$

i.e. Greatest possible value of $\theta = 21.4°$

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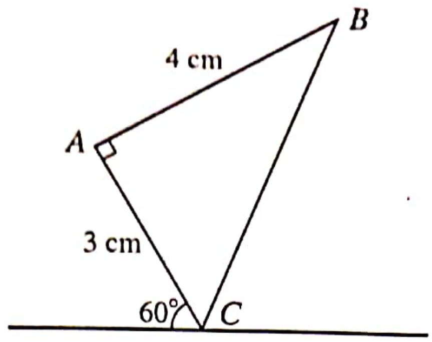
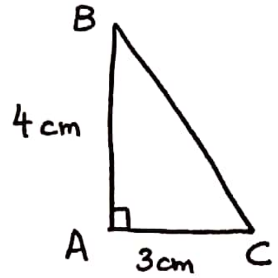


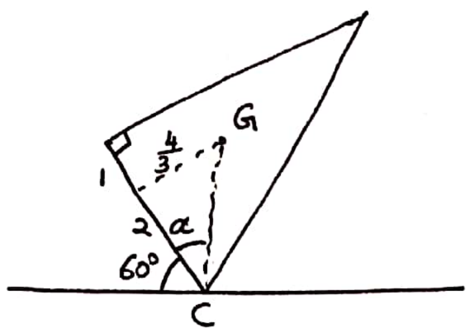
A uniform prism has a cross-section in the form of a triangle ABC which is right-angled at A. The sides AB and AC have lengths 4 cm and 3 cm respectively. The prism is held with the edge containing C in contact with a horizontal surface and with AC making an angle of 60° with the horizontal (see diagram). The prism is now released. Determine whether it falls on the face containing AC or the face containing BC. [4]



Let, G be centre of mass of ΔABC

$\Rightarrow$ Distance of G from AB $= \frac{1}{3}(3) = 1$ cm

$\Rightarrow$ Distance of G from AC $= \frac{1}{3}(4) = \frac{4}{3}$ cm.

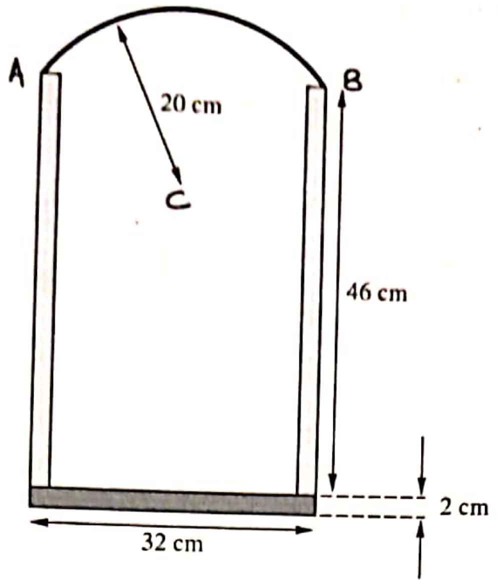


$$\tan\alpha = \frac{\frac{4}{3}}{2} = \frac{2}{3} \Rightarrow \alpha = \tan^{-1}\left(\frac{2}{3}\right) = 33.69°.$$

$\therefore$ Angle between horizontal and CG $= 60° + 33.69° = 93.69° > 90°$

$\Rightarrow$ Line of action of weight lies to the right of point of contact C

$\Rightarrow$ Prism falls on face containing BC.



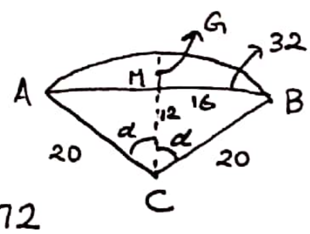
A bucket that consists of three parts stands on horizontal ground. The base is in the form of a uniform circular disc of diameter 32 cm and thickness 2 cm. The body is in the form of a uniform hollow cylinder of outer diameter 32 cm and height 46 cm. The handle is in a vertical plane, attached at opposite ends of an outer diameter at the top of the cylinder. The handle is in the form of a uniform circular arc of radius 20 cm. The diagram shows the cross-section of the bucket in the plane of the handle.

(i) Show that the centre of mass of the handle is 53.25 cm above the ground, correct to 4 significant figures. [3]

The weights of the base, body and handle are 50 N, 100 N and 25 N respectively.

(ii) Find the height of the centre of mass of the bucket above the ground. [2]

i) In ΔCMB, $MB = \frac{AB}{2} = \frac{32}{2} = 16$



$MC = \sqrt{20^2 - 16^2} = 12.$

$\sin\alpha = \frac{16}{20} = \frac{4}{5}. \Rightarrow \alpha = \sin^{-1}\left(\frac{4}{5}\right) = 0.9272$

$CG = \frac{r\sin\alpha}{\alpha} = \frac{20 \times \frac{4}{5}}{0.9272} = 17.254$

Distance of G from floor $= 2 + 46 - 12 + 17.254 = 53.254 \approx 53.25$ cm

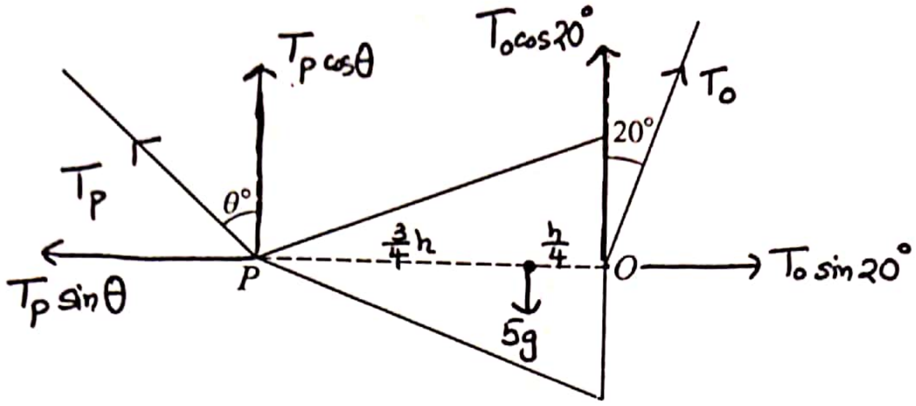
ii)

Shape	Mass (kg)	Distance of centre of mass from ground (cm)
Base	$\frac{50}{10} = 5$	$\frac{2}{2} = 1$
Body	$\frac{100}{10} = 10$	$2 + \frac{46}{2} = 25$
Handle	$\frac{25}{10} = 2.5$	53.25
Composite bucket	17.5	$\bar{y}$

Taking moments about the ground:

$$17.5\bar{y} = 5 \times 1 + 10 \times 25 + 2.5 \times 53.25$$

$$\bar{y} = 22.178 \simeq 22.2 \text{ cm.}$$



P is the vertex of a uniform solid cone of mass 5 kg, and *O* is the centre of its base. Strings are attached to the cone at *P* and at *O*. The cone hangs in equilibrium with *PO* horizontal and the strings taut. The strings attached at *P* and *O* make angles of $\theta°$ and 20°, respectively, with the vertical (see diagram, which shows a cross-section).

(i) By taking moments about *P* for the cone, find the tension in the string attached at *O*. [4]

(ii) Find the value of θ and the tension in the string attached at *P*. [6]

i) Taking moments about P:

$$5g \times \frac{3h}{4} = T_o \cos 20° \times h$$

$$T_o = \frac{5g \times 3h}{4h \cos 20°} = 39.90 \text{ N} \approx 39.9 \text{ N}.$$

ii) $R(\rightarrow)$:

$$T_p \sin\theta = T_o \sin 20°$$

$$T_p \sin\theta = 39.90 \sin 20° = 13.648 \quad \rightarrow (1)$$

$R(\uparrow)$:

$$T_p \cos\theta + T_o \cos 20° = 5g$$

$$T_p \cos\theta = 5g - 39.90 \cos 20° = 12.5 \quad \rightarrow (2)$$

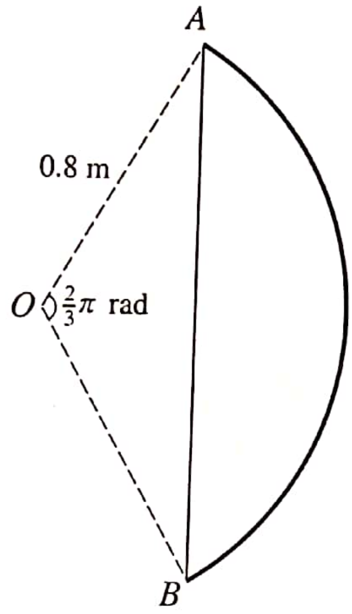
Dividing (1) by (2): $\frac{T_p \sin\theta}{T_p \cos\theta} = \frac{13.648}{12.5} \Rightarrow \tan\theta = 1.0919$

$$\theta = 47.51° \approx 47.5°$$

$$\Rightarrow T_p \sin 47.51° = 13.648$$

$$T_p = 18.50 \text{ N} \approx 18.5 \text{ N}.$$

75



A bow consists of a uniform curved portion AB of mass 1.4 kg, and a uniform taut string of mass m kg which joins A and B. The curved portion AB is an arc of a circle centre O and radius 0.8 m. Angle AOB is $\frac{2}{3}\pi$ radians (see diagram). The centre of mass of the bow (including the string) is 0.65 m from O. Calculate m. [6]

Shape	Mass	Distance of centre of mass from O
Arc AB	1.4	$\frac{r\sin\alpha}{\alpha} = \frac{0.8\sin\left(\frac{\pi}{3}\right)}{\frac{\pi}{3}} = \frac{6\sqrt{3}}{5\pi}$
String AB	m	$0.8\cos\left(\frac{\pi}{3}\right) = 0.4$
Composite figure	m + 1.4	0.65

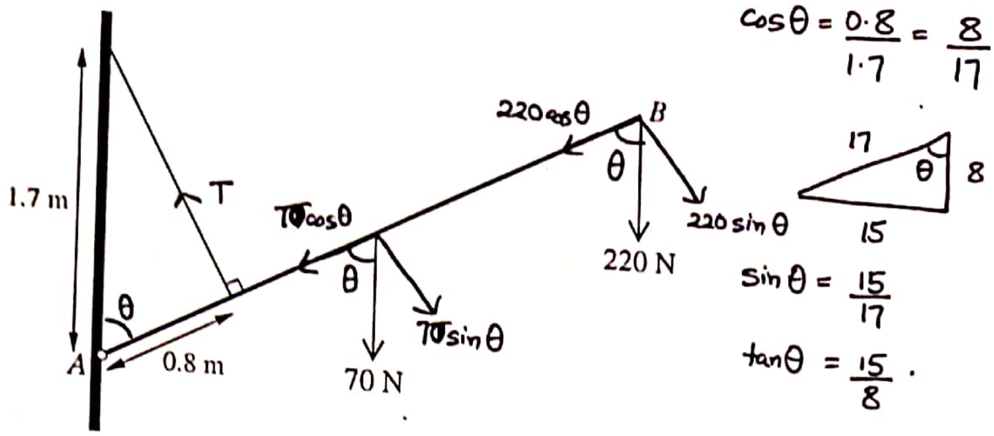
Taking moments about O:

$$(m+1.4)(0.65) = 1.4 \times \frac{6\sqrt{3}}{5\pi} + m \times 0.4$$

$$0.65m + 0.91 = 0.4m + 0.9262$$

$$0.25m = 0.0162$$

$$m = 0.06493 = 0.0649$$



A uniform beam AB has length 2 m and weight 70 N. The beam is hinged at A to a fixed point on a vertical wall, and is held in equilibrium by a light inextensible rope. One end of the rope is attached to the wall at a point 1.7 m vertically above the hinge. The other end of the rope is attached to the beam at a point 0.8 m from A. The rope is at right angles to AB. The beam carries a load of weight 220 N at B (see diagram).

(i) Find the tension in the rope. [3]

(ii) Find the direction of the force exerted on the beam at A. [4]

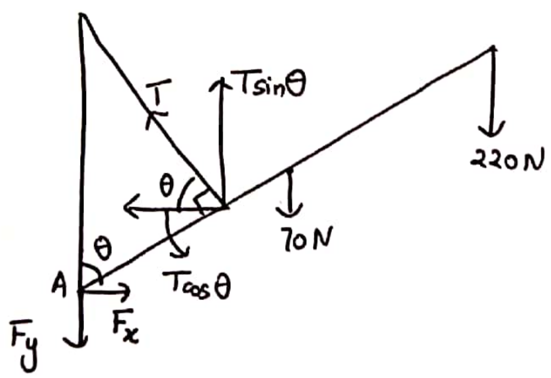
i) Taking moments about A:

$$T \times 0.8 = 70\sin\theta \times 1 + 220\sin\theta \times 2$$

$$0.8T = 70\left(\frac{15}{17}\right) + 220\left(\frac{15}{17}\right) \times 2$$

$$T = 562.5 \text{ N}.$$

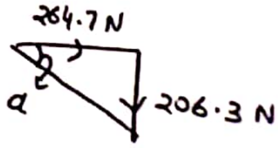
ii)



R(→): $F_x = T\cos\theta = 562.5 \times \left(\frac{8}{17}\right) = 264.7 \text{ N}$

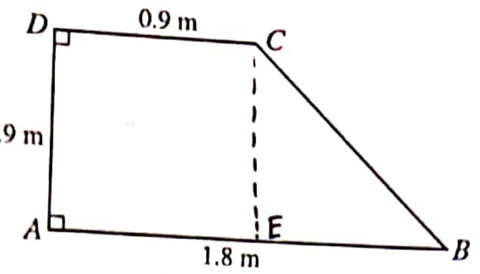
R(↑): $F_y + 70 + 220 = T\sin\theta \Rightarrow F_y = 562.5 \times \frac{15}{17} - 70 - 220 = 206.3 \text{ N}$

$$\tan\alpha = \frac{F_y}{F_x} \Rightarrow \alpha = \tan^{-1}\left(\frac{206.3}{264.7}\right) = 37.9°$$



⇒ Direction of force exerted on beam at A: 37.9° below horizontal.

17



$ABCD$ is a uniform lamina with $AB = 1.8$ m, $AD = DC = 0.9$ m, and AD perpendicular to AB and DC (see diagram).

(i) Find the distance of the centre of mass of the lamina from AB and the distance from AD. [4]

The lamina is freely suspended at A and hangs in equilibrium.

(ii) Calculate the angle between AB and the vertical. [2]

i)

Shape	Mass	Coordinates of centre of mass using A as origin
Square ADCE	$0.9^2\rho = 0.81\rho$	$\left(\frac{0.9}{2}, \frac{0.9}{2}\right) = (0.45, 0.45)$
ΔBCE	$\frac{1}{2} \times 0.9 \times 0.9\rho = 0.405\rho$	$\left[\frac{0.9+0.9+1.8}{3}, \frac{0+0+0.9}{3}\right] = (1.2, 0.3)$
Lamina ABCD	1.215ρ	$(\bar{x}, \bar{y})$.

Taking moments about AD:

$$0.81\rho \times 0.45 + 0.405\rho \times 1.2 = 1.215\rho \times \bar{x}$$
$$\bar{x} = 0.7$$

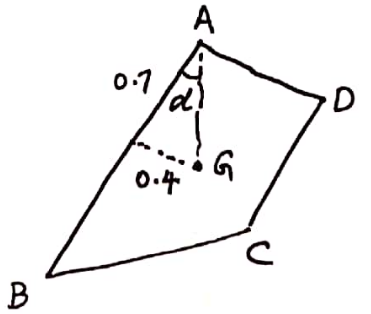
Taking moments about AB:

$$0.81\rho \times 0.45 + 0.405\rho \times 0.3 = 1.215\rho \times \bar{y}$$
$$\bar{y} = 0.4$$

Distance of centre of mass from AD = 0.7 m

Distance of centre of mass from AB = 0.4 m.

ii)



When suspended freely, G lies vertically below A.

$\tan\alpha = \frac{0.4}{0.7} \Rightarrow \alpha = \tan^{-1}\left(\frac{0.4}{0.7}\right) = 29.74° \approx 29.7°$

$\Rightarrow$ Angle between AB and the vertical $= 29.7°$.

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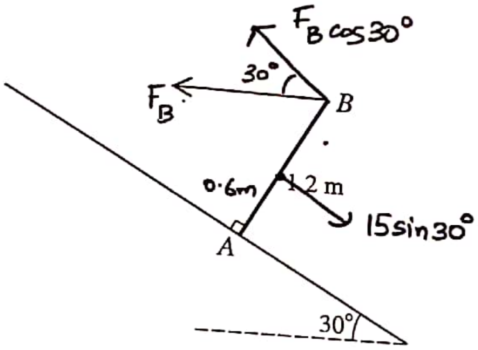
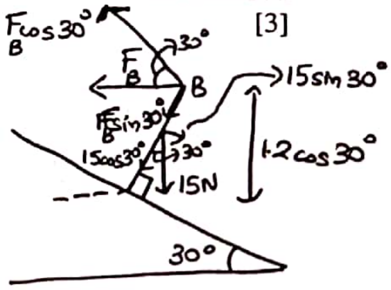


A uniform rod AB has weight 15 N and length 1.2 m. The end A of the rod is in contact with a rough plane inclined at 30° to the horizontal, and the rod is perpendicular to the plane. The rod is held in equilibrium in this position by means of a horizontal force applied at B, acting in the vertical plane containing the rod (see diagram).

(i) Show that the magnitude of the force applied at B is 4.33 N, correct to 3 significant figures. [3]

(ii) Find the magnitude of the frictional force exerted by the plane on the rod. [2]

(iii) Given that the rod is in limiting equilibrium, calculate the coefficient of friction between the rod and the plane. [3]



i) Taking moments about A:

$$F_B\cos30° \times 1.2 = 15\sin30° \times 0.6$$

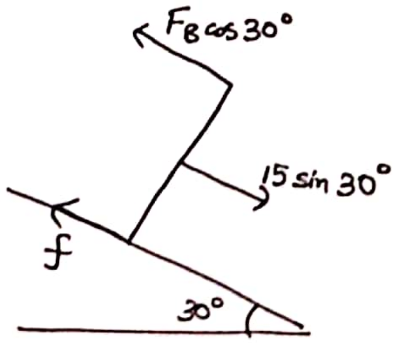
$$F_B = \frac{5\sqrt{3}}{2} = 4.330\text{ N}$$

$$F_B = 4.33\text{ N}.$$

ii) R(//): $f + F_B\cos30° = 15\sin30°$

$$f = 15\sin30° - 4.33\cos30°$$

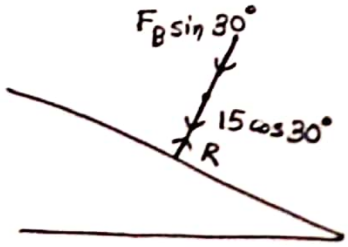
$$f = \frac{15}{4} = 3.75\text{ N}.$$



iii) R(⊥): $R = F_B\sin30° + 15\cos30°$

$$R = \frac{5\sqrt{3}}{2}\left(\frac{1}{2}\right) + 15\left(\frac{\sqrt{3}}{2}\right)$$

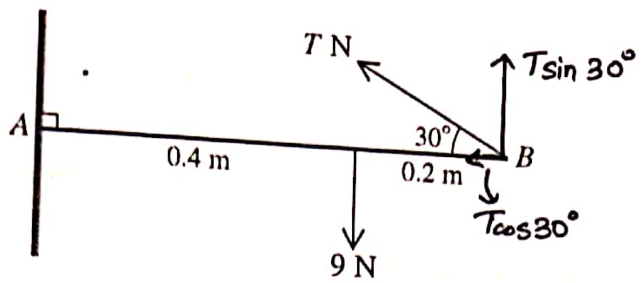
$$R = \frac{35\sqrt{3}}{4}$$



Since friction is limiting, $f = \mu R$

$$\frac{15}{4} = \mu \times \frac{35\sqrt{3}}{4} \Rightarrow \mu = \frac{15\times4}{4\times35\sqrt{3}} = 0.2474$$

$\Rightarrow$ Coefficient of friction $= 0.247$.



A non-uniform rod AB, of length 0.6 m and weight 9 N, has its centre of mass 0.4 m from A. The end A of the rod is in contact with a rough vertical wall. The rod is held in equilibrium, perpendicular to the wall, by means of a light string attached to B. The string is inclined at 30° to the horizontal. The tension in the string is T N (see diagram).

(i) Calculate T. [2]

(ii) Find the least possible value of the coefficient of friction at A. [3]

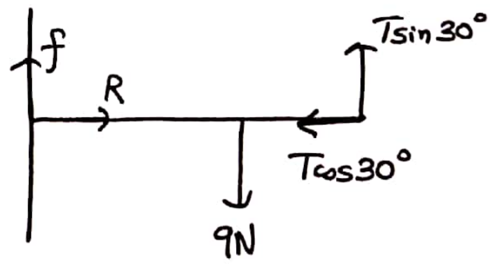
i) Taking moments about A:

$$T\sin 30° \times 0.6 = 9 \times 0.4$$

$$T = \frac{9 \times 0.4}{0.6 \sin 30°} = 12\,\text{N}.$$

ii) $R(\rightarrow)$: $R = T\cos 30°$

$$R = 12\cos 30° = 6\sqrt{3}.$$



$R(\uparrow)$: $f + T\sin 30° = 9$

$$f + 12\sin 30° = 9$$

$$f = 3$$

Since rod is in equilibrium,

$$f \leq \mu R$$

$$3 \leq \mu(6\sqrt{3})$$

$$\frac{3}{6\sqrt{3}} \leq \mu$$

$$\Rightarrow \mu \geq \frac{1}{2\sqrt{3}} \Rightarrow \mu \geq 0.2886$$

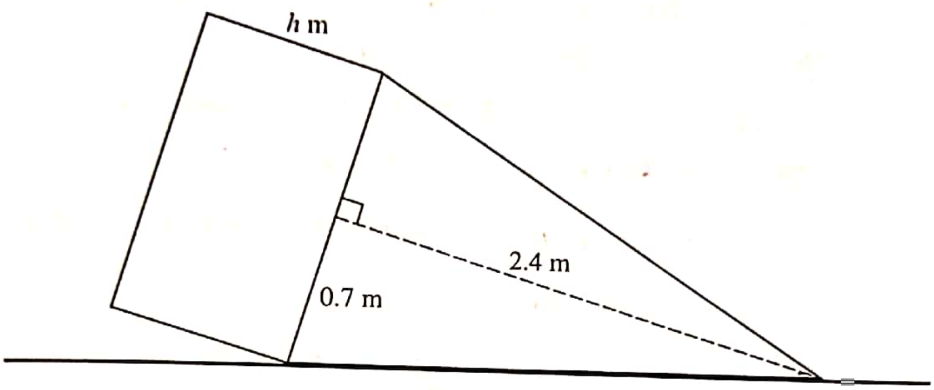
i.e. least value of coefficient of friction = 0.289.

80 A uniform solid cylinder has radius 0.7 m and height h m. A uniform solid cone has base radius 0.7 m and height 2.4 m. The cylinder and the cone both rest in equilibrium each with a circular face in contact with a horizontal plane. The plane is now tilted so that its inclination to the horizontal, $\theta°$, is increased gradually until the cone is about to topple.

(i) Find the value of θ at which the cone is about to topple. [2]

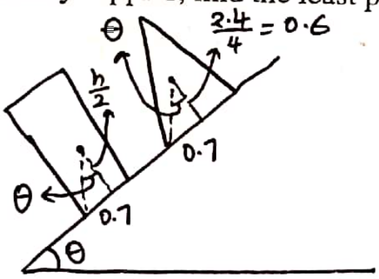
(ii) Given that the cylinder does not topple, find the greatest possible value of h. [2]

The plane is returned to a horizontal position, and the cone is fixed to one end of the cylinder so that the plane faces coincide. It is given that the weight of the cylinder is three times the weight of the cone. The curved surface of the cone is placed on the horizontal plane (see diagram).



(iii) Given that the solid immediately topples, find the least possible value of h. [5]

i)



When cone is about to topple, $\tan\theta = \frac{0.7}{0.6} \Rightarrow \theta = \tan^{-1}\left(\frac{7}{6}\right) = 49.39° \approx 49.4°$

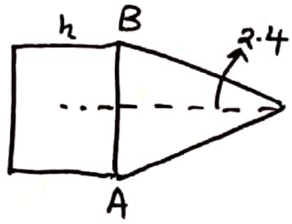
ii) If cylinder does not topple, $\tan\theta = \frac{0.7}{\frac{h}{2}} \Rightarrow \frac{7}{6} = \frac{2 \times 0.7}{h}$

$h = 1.2$

Greatest possible value of $h = 1.2$

iii)

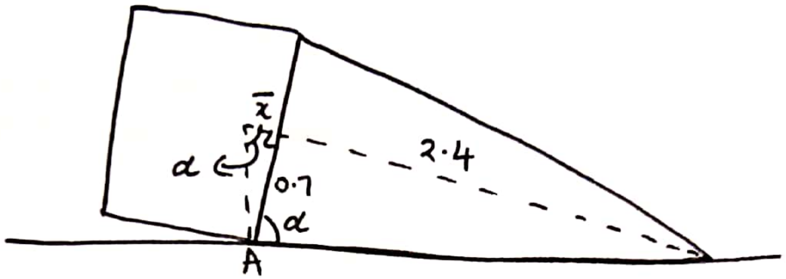
Shape	Mass	Distance of centre of mass from AB
Cone	m	$\frac{2.4}{4} = 0.6$
Cylinder	$3m$	$\frac{h}{2} = 0.5h$
Composite figure	$4m$	$\bar{x}$



Taking moments about AB :

$4m \times \bar{x} = m \times -0.6 + 3m(+0.5h) \Rightarrow \bar{x} = \frac{-0.6 + 1.5h}{4}$

iii) When solid is about to topple, centre of ~~pass~~ mass is vertically above point of contact A



$$\tan\alpha = \frac{2.4}{0.7} = \frac{24}{7}$$

$$\tan\alpha = \frac{0.7}{\bar{x}} \Rightarrow \frac{24}{7} = \frac{0.7}{\frac{-0.6+1.5h}{4}}$$

$$\frac{24}{7} = \frac{0.7 \times 4}{-0.6+1.5h}$$

$$24(-0.6+1.5h) = 7 \times 0.7 \times 4$$

$$-14.4 + 36h = 19.6$$

$$36h = 34$$

$$h = \frac{17}{18} = 0.9444$$

i.e. Least value of $h = 0.944$

81 An object is made from two identical uniform rods AB and BC each of length 0.6 m and weight 7 N. The rods are rigidly joined to each other at B and angle $ABC = 90°$.

(i) Calculate the distance of the centre of mass of the object from B. [1]

Shape	Mass	Coordinates of centre of mass with B as origin.
AB	0.7	$(0, 0.3\cancel{0})$
BC	0.7	$(0.3\cancel{0}, 0)$.
Composite	1.4	$(\bar{x}, \bar{y})$.

Taking moments about AB: $0.7 \times 0 + 0.7 \times 0.3\cancel{0} = 1.4\,\bar{x}$

$\bar{x} = 0.15$

Taking moments about BC: $0.7 \times 0.3\cancel{0} + 0.7 \times 0 = 1.4 \times \bar{y}$

$\bar{y} = 0.15$

$\Rightarrow BG = \cancel{\sqrt{0.175^2 + 0.175^2}} = \sqrt{0.15^2 + 0.15^2} = 0.2121$

$\therefore$ Distance of centre of mass from B = 0.212 m.

The object is freely suspended at A and a force of magnitude F N is applied to the rod BC at C. The object is in equilibrium with AB inclined at 45° to the horizontal.

(ii) (a)

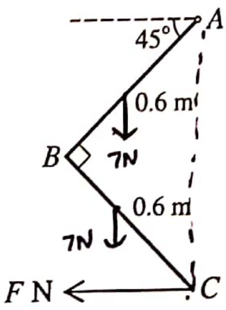


Fig. 1

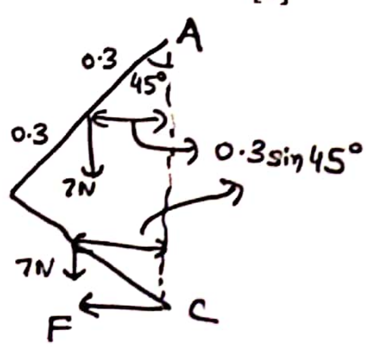
Calculate F given that the force acts horizontally as shown in Fig. 1. [2]

$AC = \sqrt{0.6^2 + 0.6^2} = \frac{3\sqrt{2}}{5} = 0.6\sqrt{2}$.

Taking moments about A:

$7 \times 0.3 \sin 45° + 7 \times 0.3 \sin 45° = F \times 0.6\sqrt{2}$

$3.5\text{N} = F$



(b)

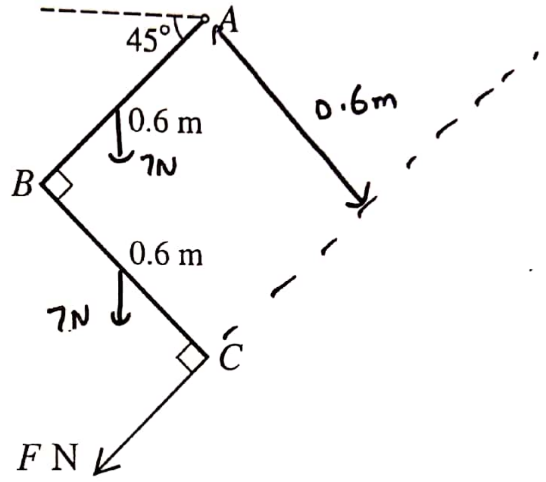


Fig. 2

Calculate F given instead that the force acts perpendicular to the rod as shown in Fig. 2.

[2]

Taking moments about A:

$$7 \times 0.3 \sin 45^\circ + 7 \times 0.3 \sin 45^\circ = F \times 0.6$$

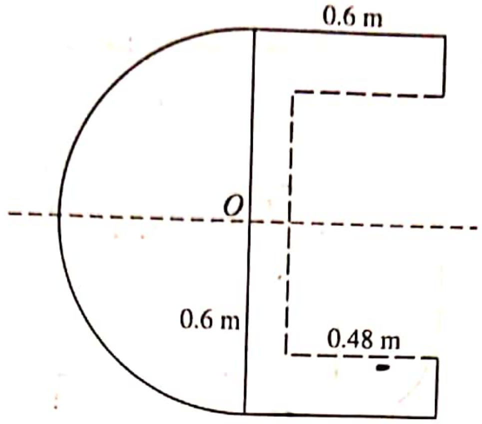
$$F = 4.949 \approx 4.95\text{ N}.$$

82 A uniform solid consists of a hemisphere with centre O and radius 0.6 m joined to a cylinder of radius 0.6 m and height 0.6 m. The plane face of the hemisphere coincides with one of the plane faces of the cylinder.

(i) Calculate the distance of the centre of mass of the solid from O. [4]

[The volume of a hemisphere of radius r is $\frac{2}{3}\pi r^3$.]

(ii)

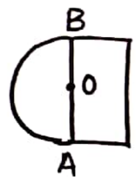


A cylindrical hole, of length 0.48 m, starting at the plane face of the solid, is made along the axis of symmetry (see diagram). The resulting solid has its centre of mass at O. Show that the area of the cross-section of the hole is $\frac{3}{16}\pi\ \text{m}^2$. [4]

(iii) It is possible to increase the length of the cylindrical hole so that the solid still has its centre of mass at O. State the increase in the length of the hole. [1]

i)

Figure	Mass	Center of mass of solid from O.
Hemisphere	$\frac{2\pi(0.6)^3}{3}\rho = \frac{18\pi\rho}{125}$	$\frac{3r}{8} = \frac{3\times0.6}{8} = 0.225$
Cylinder	$\pi(0.6)^2(0.6)\rho = \frac{27\pi\rho}{125}$	$\frac{0.6}{2} = 0.3$
Composite figure	$\frac{9\pi\rho}{25}$	$\bar{z}$



Taking moments about AB :

$$\frac{9\pi\rho}{25} \times \bar{z} = \frac{18\pi\rho}{125} \times -0.225 + \frac{27\pi\rho}{125} \times 0.3$$

$$\bar{z} = 0.09\ \text{m}.$$

i.e. Distance of centre of mass of solid from O = 0.09 m.

ii) Let, Area of cross-section of hole = A.

Figure	Mass	Centre of mass of solid from O
Hemisphere	$\frac{18\pi\rho}{125}$	0.225
Cylinder	$\frac{27\pi\rho}{125}$	0.3
Hole	$A\times0.48\rho$	$0.6 - \frac{0.48}{2} = 0.36$
Remaining solid	$\frac{9\pi\rho}{25} - A\times0.48\rho$	0

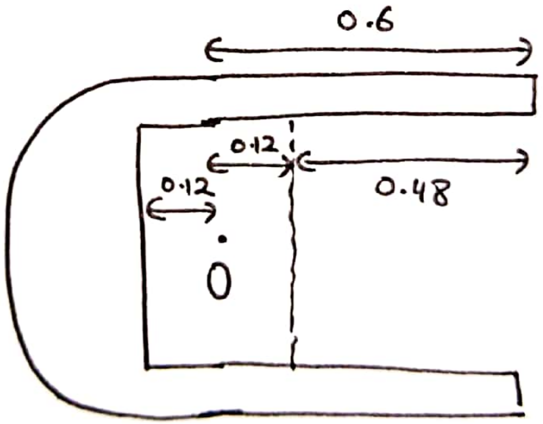
Taking moments about AB:

$$\left(\frac{9\pi\ell}{25} - A\times 0.48\ell\right)0 = \frac{18\pi\ell}{125}\times -0.225 + \frac{27\pi\ell}{125}\times 0.3 - 0.48A\ell\times 0.36$$

$$0 = -0.0324\pi + 0.0648\pi - 0.1728A$$

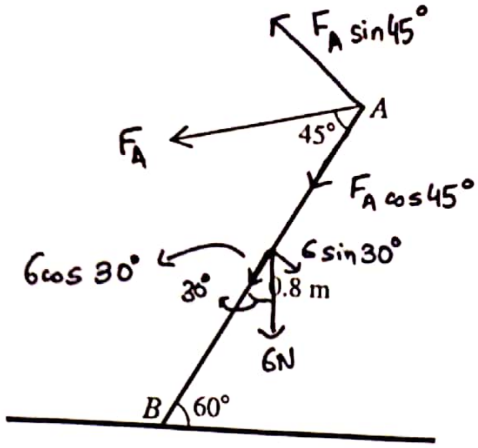
$$A = \frac{3}{16}\pi$$

iii) The cylindrical hole can be extended so that the same amount is taken away from the sphere as from the cylinder.



Increase in length of cylindrical hole $= 0.12 + 0.12 = 0.24$ m.

8



A uniform rod AB has weight 6 N and length 0.8 m. The rod rests in limiting equilibrium with B in contact with a rough horizontal surface and AB inclined at 60° to the horizontal. Equilibrium is maintained by a force, in the vertical plane containing AB, acting at A at an angle of 45° to AB (see diagram). Calculate

(i) the magnitude of the force applied at A, [3]

(ii) the least possible value of the coefficient of friction at B. [4]

i) Taking moments about B :

$$F_A \sin 45° \times 0.8 = 6\sin 30° \times 0.4$$

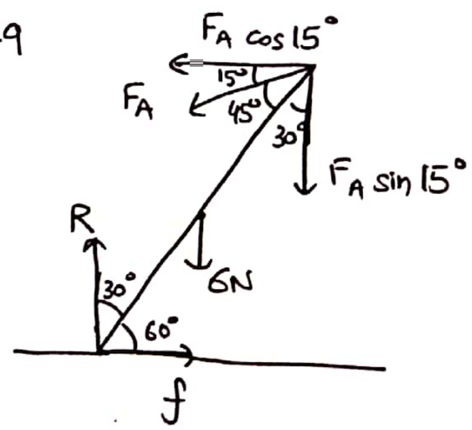
$$F_A = \frac{6\sin 30° \times 0.4}{0.8 \times \sin 45°} = \frac{3\sqrt{2}}{2} \text{ N}$$

$$F_A = 2.12 \text{ N}$$

ii) $R(\rightarrow)$: $f = F_A \cos 15° = \frac{3\sqrt{2}}{2} \cos 15° = 2.049$

$R(\uparrow)$: $R = 6 + F_A \sin 15°$

$$R = 6 + \frac{3\sqrt{2}}{2} \sin 15° = 6.549$$

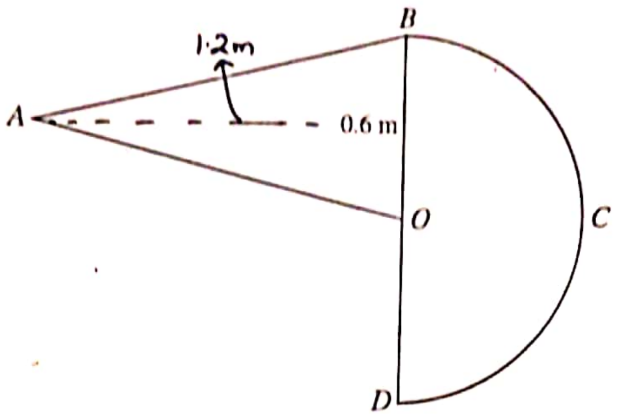


Since friction is limiting,

$$f = \mu R$$

$$2.049 = \mu \times 6.549$$

$$\mu = 0.3128 \approx 0.313 .$$



Area of $\Delta = \frac{1}{2} \times b \times h$

$0.36 = \frac{1}{2} \times 0.6 \times h$

$h = 1.2$

A uniform lamina $OABCD$ consists of a semicircle BCD with centre O and radius 0.6 m and an isosceles triangle OAB, joined along OB (see diagram). The triangle has area $0.36\,\text{m}^2$ and $AB = AO$.

(i) Show that the centre of mass of the lamina lies on OB. [4]

(ii) Calculate the distance of the centre of mass of the lamina from O. [4]

i)

Shape	Mass	Distance of centre of mass from OB
ΔAOB	0.36ρ	$\frac{1.2}{3} = 0.4$
Semicircle BCD	$\frac{\pi(0.6)^2\rho}{2} = \frac{0.36\pi\rho}{2}$	$\frac{2 \times 0.6 \times \sin\left(\frac{\pi}{2}\right)}{\frac{3\pi}{2}} = \frac{4}{5\pi}$
Composite	$\left(0.36 + \frac{0.36\pi}{2}\right)\rho$	$\bar{x}$

Taking moments about OB:

$$0.36\rho \times 0.4 + \frac{0.36\pi\rho}{2} \times \frac{-4}{5\pi} = \left(0.36 + \frac{0.36\pi}{2}\right)\rho \times \bar{x}$$

$$0 = \left(0.36 + \frac{0.36\pi}{2}\right)\bar{x}$$

$$\bar{x} = 0$$

i.e. Distance of centre of mass of lamina from OB = 0

$\Rightarrow$ Centre of mass of lamina lies on OB.

ii)

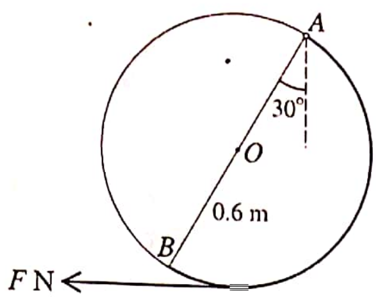
Shape	Mass	Distance of centre of mass from line OC
ΔAOB	0.36ρ	0.3
Semicircle BCD	$\cancel{\pi\rho}$ $0.18\pi\rho$	0
Composite	$(0.36 + 0.18\pi)\rho$	$\bar{y}$

Taking moments about line OC:

$$0.36\rho \times 0.3 + 0.18\pi\rho \times 0 = (0.36 + 0.18\pi)\rho\,\bar{y}$$

$$\bar{y} = 0.1166 = 0.117$$

i.e. Distance of centre of mass from O = 0.117 m.



A circular object is formed from a uniform semicircular lamina of weight 12 N and a uniform semicircular arc of weight 8 N. The lamina and the arc both have centre O and radius 0.6 m and are joined at the ends of their common diameter AB. The object is freely pivoted to a fixed point at A with AB inclined at 30° to the vertical. The object is in equilibrium acted on by a horizontal force of magnitude F N applied at the lowest point of the object, and acting in the plane of the object (see diagram).

(i) Show that the centre of mass of the object is at O. [3]

(ii) Calculate F. [3]

i)

Shape	Mass	Distance of centre of mass from O
Semicircular sector	1·2	$\dfrac{2\times0.6\times\sin\left(\frac{\pi}{2}\right)}{\frac{3\pi}{2}} = \dfrac{4}{5\pi}$
Semircircular arc	0·8	$\dfrac{0.6\times\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \dfrac{6}{5\pi}$
Composite figure	2.0	$\bar{x}$

Taking moments about AB :

$$2.0\,\bar{x} = 1.2\times\frac{4}{5\pi} + 0.8\times\frac{-6}{5\pi}$$

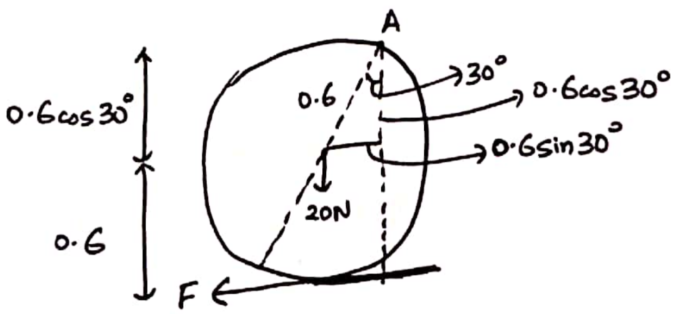
$$2.0\,\bar{x} = 0$$

$$\bar{x} = 0$$

i.e. distance of centre of mass of object from O = 0 m

$\Rightarrow$ Centre of mass of object is at O.

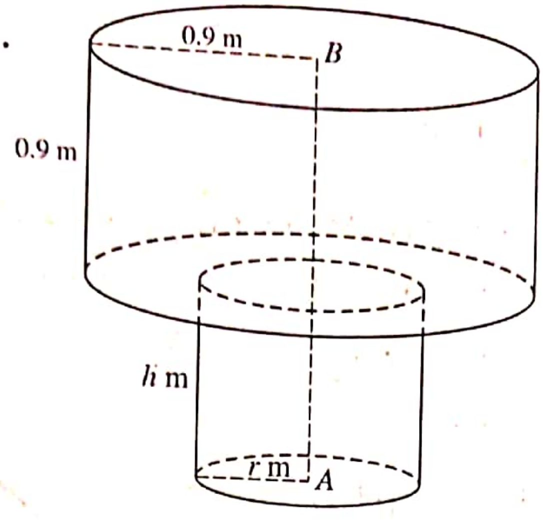
ii)



Taking moments about A :

$$20\times0.6\sin30° = F\times(0.6+0.6\cos30°)$$

$$F = 5.358\text{ N} \approx 5.36\text{ N}.$$



A cylinder of height 0.9 m and radius 0.9 m is placed symmetrically on top of a cylinder of height h m and radius r m, where $r < 0.9$, with plane faces in contact and axes in the same vertical line AB, where A and B are centres of plane faces of the cylinders (see diagram). Both cylinders are uniform and made of the same material. The lower cylinder is gradually tilted and when the axis of symmetry is inclined at 45° to the horizontal the upper cylinder is on the point of toppling without sliding.

(i) Find r. [2]

The upper cylinder is now fixed to the lower cylinder to create a uniform object.

(ii) Show that the centre of mass of the object is

$$\frac{25h^2 + 180h + 81}{50h + 180} \text{ m}$$

from A. [3]

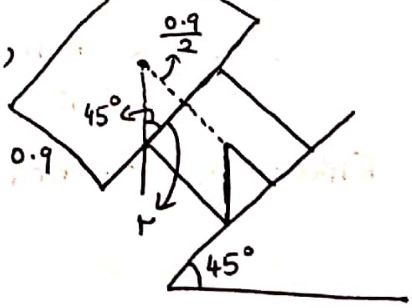
The object is placed with the plane face containing A in contact with a rough plane inclined at $\alpha°$ to the horizontal, where $\tan \alpha = 0.5$. The object is on the point of toppling without sliding.

(iii) Calculate h. [3]

when upper cylinder is about to topple,

i) $\tan 45° = \dfrac{\left(\frac{0.9}{2}\right)}{r} = \dfrac{0.45}{r}$

$r = \dfrac{0.45}{\tan 45°} = 0.45$ m.



ii)

Figure	Mass	Distance of centre of mass from A
Small Cylinder	$\pi(0.45)^2 h\rho = 0.2025\pi h\rho$	$\frac{h}{2}$
Large cylinder	$\pi(0.9)^2 \times 0.9\rho = 0.729\pi\rho$	$h + \frac{0.9}{2}$
Composite	$(0.2025h + 0.729)\pi\rho$	$\bar{x}$

Taking moments about base of small cylinder:

$$(0.2025h + 0.729)\pi\rho \times \bar{x} = 0.2025\pi h\rho \times \frac{h}{2} + 0.729\pi\rho \times \left(h + \frac{0.9}{2}\right)$$

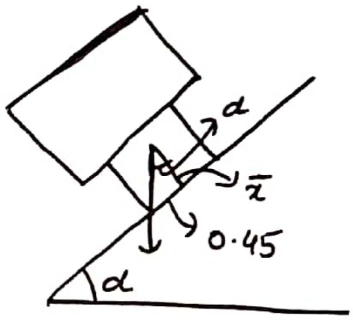
$$\bar{x} = \frac{0.10125h^2 + 0.729h + 0.32805}{0.2025h + 0.729}$$

$$\bar{x} = \frac{\frac{81}{800}h^2 + \frac{729h}{1000} + \frac{6561}{20000}}{\frac{81}{400}h + \frac{729}{1000}} = \frac{\frac{2025h^2 + 14580h + 6561}{20000}}{\frac{405h + 1458}{2000}}$$

$$\bar{x} = \frac{2000(2025h^2 + 14580h + 6561)}{20000(405h + 1458)} = \frac{81(25h^2 + 180h + 81)}{10 \times 81(5h + 18)}$$

$$\bar{x} = \frac{25h^2 + 180h + 81}{50h + 180} \quad \text{(Shown)}.$$

iii) Since object is on point of toppling, centre of mass is vertically above point of contact.



$$\tan\alpha = \frac{0.45}{\bar{x}}$$

$$\frac{1}{2} = \frac{0.45}{\bar{x}}$$

$$\bar{x} = 0.9$$

$$\frac{25h^2 + 180h + 81}{50h + 180} = \frac{9}{10}$$

$$250h^2 + 1800h + 810 = 450h + 1620$$

$$250h^2 + 1350h - 810 = 0$$

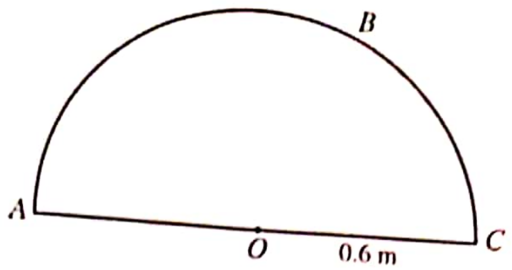
$$25h^2 + 135h - 81 = 0$$

$$h = \frac{-135 \pm \sqrt{135^2 - 4(25)(-81)}}{2(25)}$$

$$h = \frac{-27 \pm 9\sqrt{13}}{10}$$

Either, $h = 0.5449$ or $h = -5.944$ (Ignore)

$\Rightarrow h = 0.545.$



A uniform frame consists of a semicircular arc ABC of radius 0.6 m together with its diameter AOC, where O is the centre of the semicircle (see diagram).

(i) Calculate the distance of the centre of mass of the frame from O. [4]

The frame is freely suspended at A and hangs in equilibrium.

(ii) Calculate the angle between AC and the vertical. [2]

i)

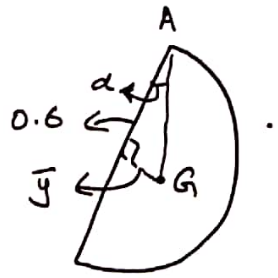
Shape	Mass	Distance of centre of mass from O
Semicircular arc	$\pi(0.6)\rho = 0.6\pi\rho$	$\dfrac{0.6\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \dfrac{6}{5\pi}$
Segment AC	1.2ρ	0
Composite figure	$(0.6\pi+1.2)\rho$	$\bar{y}$

Taking moments about AC:

$$0.6\pi\rho \times \frac{6}{5\pi} + 1.2\rho \times 0 = (0.6\pi + 1.2)\rho \times \bar{y}$$

$$\bar{y} = 0.2333\text{m} \approx 0.233\text{m}.$$

ii)



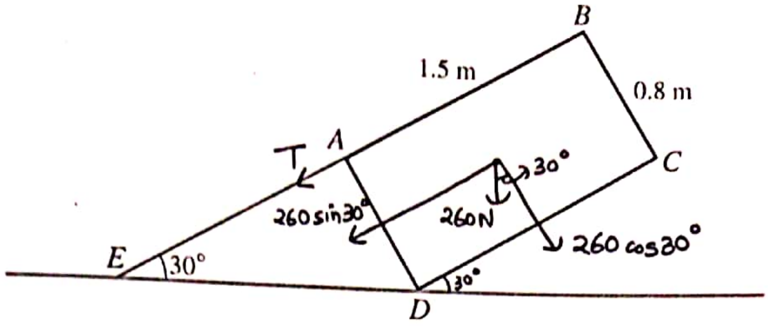
When suspended freely, centre of mass lies vertically below A.

$$\tan\alpha = \frac{\bar{y}}{0.6}$$

$$\tan\alpha = \frac{0.2333}{0.6}$$

$$\alpha = 21.3^\circ$$

i.e. Angle between AC and vertical $= 21.3^\circ$.



$ABCD$ is the cross-section through the centre of mass of a uniform rectangular block of weight 260 N. The lengths AB and BC are 1.5 m and 0.8 m respectively. The block rests in equilibrium with the point D on a rough horizontal floor. Equilibrium is maintained by a light rope attached to the point A on the block and the point E on the floor. The points E, A and B lie in a straight line inclined at 30° to the horizontal (see diagram).

(i) By taking moments about D, show that the tension in the rope is 146 N, correct to 3 significant figures. [5]

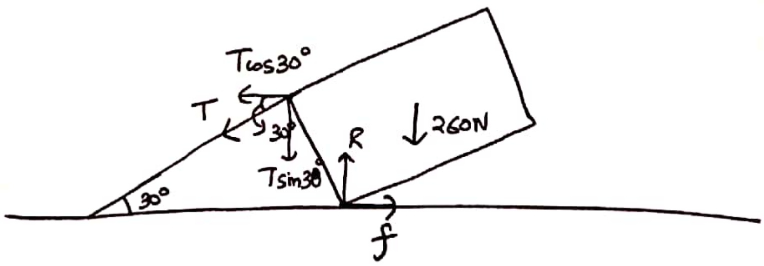
(ii) Given that the block is in limiting equilibrium, calculate the coefficient of friction between the block and the floor. [4]

i) Taking moments about D,

$$T \times 0.8 + 260 \sin 30° \times 0.4 = 260 \cos 30° \times 0.75$$

$$T = 146.09\text{N} \approx 146\text{ N}$$

ii)



$R(\rightarrow)$: $T\cos 30° = f \Rightarrow f = 146.09 \cos 30° = 126.52\text{ N}$

$R(\uparrow)$: $R = T\sin 30° + 260 \Rightarrow R = 146.09 \sin 30° + 260 = 333.04\text{ N}$

Since friction is limiting, $f = \mu R$

$$126.52 = \mu(333.04)$$

$$\mu = 0.3798$$

$$\mu \approx 0.380$$

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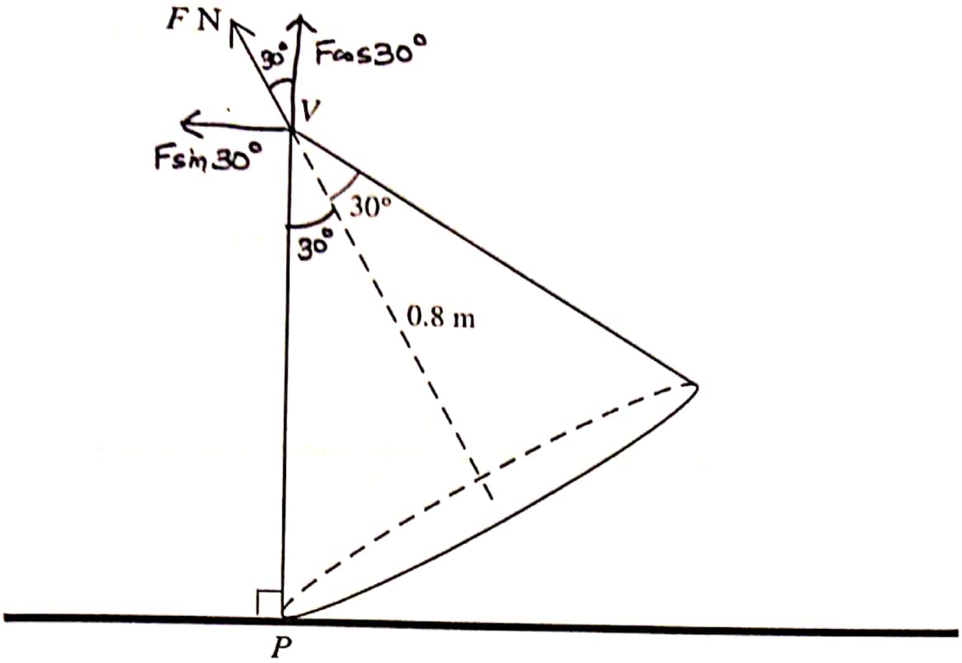
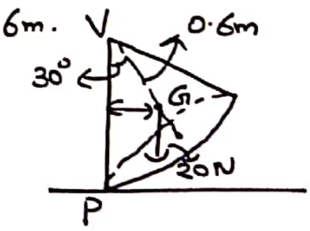


A uniform solid cone with height 0.8 m and semi-vertical angle 30° has weight 20 N. The cone rests in equilibrium with a single point P of its base in contact with a rough horizontal surface, and its vertex V vertically above P. Equilibrium is maintained by a force of magnitude F N acting along the axis of symmetry of the cone and applied to V (see diagram).

(i) Show that the moment of the weight of the cone about P is 6 N m. [2]

(ii) Hence find F. [2]

i) Let, G be centre of mass of cone $\Rightarrow VG = \frac{3}{4}(0.8) = 0.6\text{m}$.



Moment of weight about P $= 20 \times 0.6 \sin 30°$

$= 6$ Nm.

ii) Taking moment about P:

$F \sin 30° \times VP = 6$

$F \sin 30° \times \frac{8\sqrt{3}}{15} = 6$

$F = \frac{6 \times 15}{8\sqrt{3} \times \sin 30°}$

$F = \frac{15\sqrt{3}}{2} \approx 12.99 = 13.0$ N.

$\cancel{\sin}\ \cos 30° = \frac{0.8}{VP}$

$VP = \frac{0.8}{\cos 30°} = \frac{8\sqrt{3}}{15}$

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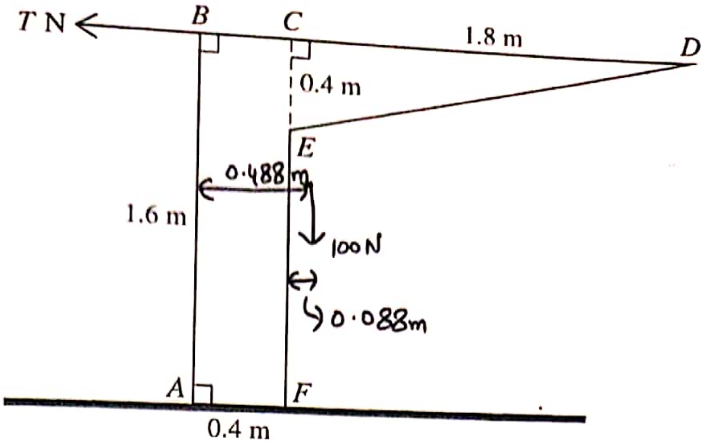

$ABCDEF$ is the cross-section through the centre of mass of a uniform solid prism. $ABCF$ is a rectangle in which $AB = CF = 1.6$ m, and $BC = AF = 0.4$ m. CDE is a triangle in which $CD = 1.8$ m, $CE = 0.4$ m, and angle $DCE = 90°$. The prism stands on a rough horizontal surface. A horizontal force of magnitude T N acts at B in the direction CB (see diagram). The prism is in equilibrium.

(i) Show that the distance of the centre of mass of the prism from AB is 0.488 m. [4]

(ii) Given that the weight of the prism is 100 N, find the greatest and least possible values of T. [3]

i)

Shape	Mass	Distance of center of mass from AB.
Rectangle ABCF	$1.6(0.4)\rho = 0.64\rho$	0.2
Triangle CDE	$\frac{1}{2}\times0.4\times1.8\rho = 0.36\rho$	$\frac{0.4+0.4+2.2}{3} = 1$
Composite figure	ρ	$\bar{x}$

Taking moments about AB:

$$\rho\bar{x} = 0.64\rho \times 0.2 + 0.36\rho \times 1$$

$$\bar{x} \simeq 0.488\text{m}$$

i.e. Distance of centre of mass from AB = 0.488m.

ii) Since prism is in equilibrium, it does not topple.

Taking moments about A,

$$T \times 1.6 = 100 \times 0.488$$

$$T = 30.5\text{N}$$

Then, maximum value of $T = 30.5$N.

Taking moments about F,

$$T \times 1.6 = 100 \times (0.488 - 0.4)$$

$$T = 5.5\text{N}$$

Then, minimum value of $T = 5.5$N.

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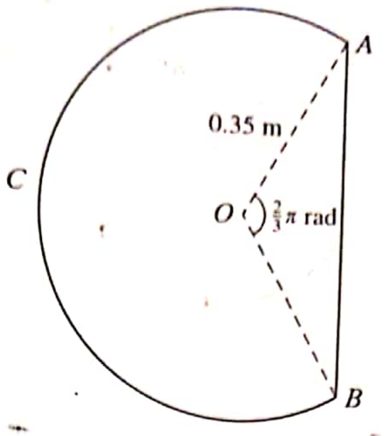


A uniform lamina ABC is in the form of a major segment of a circle with centre O and radius 0.35 m. The straight edge of the lamina is AB, and angle $AOB = \frac{2}{3}\pi$ radians (see diagram).

(i) Show that the centre of mass of the lamina is 0.0600 m from O, correct to 3 significant figures. [6]

The weight of the lamina is 14 N. It is placed on a rough horizontal surface with A vertically above B and the lowest point of the arc BC in contact with the surface. The lamina is held in equilibrium in a vertical plane by a force of magnitude F N acting at A.

(ii) Find F in each of the following cases:

(a) the force of magnitude F N acts along AB; [2]

(b) the force of magnitude F N acts along the tangent to the circular arc at A. [3]

i)

Shape	Mass	Distance of centre of mass from O
Sector OACB	$\frac{0.35^2 \times \frac{4\pi}{3}}{2} \times \rho = \frac{49\pi\rho}{600}$	$\frac{2 \times 0.35 \times \sin\left(\frac{2\pi}{3}\right)}{3 \times \frac{2\pi}{3}} = \frac{7\sqrt{3}}{40\pi}$
Δ AOB	$\frac{1}{2} \times 0.35^2 \times \sin\left(\frac{2\pi}{3}\right) \times \rho = \frac{49\sqrt{3}\rho}{1600}$	$\frac{0 + 0.35\cos\left(\frac{\pi}{3}\right) \times 2}{3} = \frac{7}{60}$
Composite figure	$\frac{49\pi\rho}{600} + \frac{49\sqrt{3}\rho}{1600}$	$\bar{x}$

Taking moment about a vertical line through O:

$$\frac{49\pi\rho}{600} \times \frac{7\sqrt{3}}{40\pi} + \frac{49\sqrt{3}\rho}{1600} \times \frac{-7}{60} = \left(\frac{49\pi\rho}{600} + \frac{49\sqrt{3}\rho}{1600}\right)\bar{x}$$

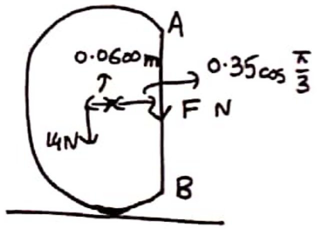
$$\bar{x} = 0.05996 \approx 0.0600$$

i.e. distance of centre of mass of lamina from $O = 0.0600$ m (Shown).

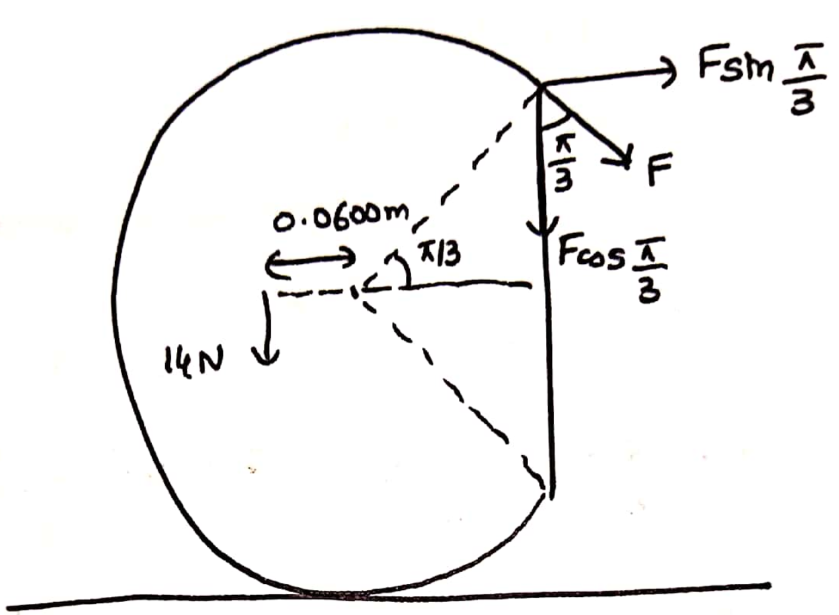
ii) Taking moments about point of contact:

$$14 \times 0.0600 = F \times 0.35\cos\left(\frac{\pi}{3}\right)$$

$$F = 4.797\text{ N} \approx 4.80\text{ N}.$$



b)



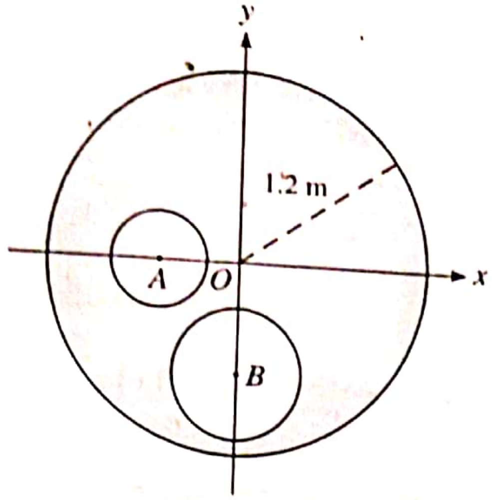
Taking moments about point of contact:

$$14 \times 0.0600 = F\left(\sin\frac{\pi}{3}\right) \times \left[0.35\sin\left(\frac{\pi}{3}\right) + 0.35\right] + F\cos\left(\frac{\pi}{3}\right)\left[0.35\cos\left(\frac{\pi}{3}\right)\right]$$

$$0.84 = 0.5656F + 0.0875F$$

$$0.84 = 0.6531F$$

$$F = 1.286\,N \approx 1.29\,N.$$



A uniform circular disc has centre O and radius 1.2 m. The centre of the disc is at the origin of x- and y-axes. Two circular holes with centres at A and B are made in the disc (see diagram). The point A is on the negative x-axis with $OA = 0.5$ m. The point B is on the negative y-axis with $OB = 0.7$ m. The hole with centre A has radius 0.3 m and the hole with centre B has radius 0.4 m. Find the distance of the centre of mass of the object from

(i) the x-axis, [4]

(ii) the y-axis. [3]

The object can rotate freely in a vertical plane about a horizontal axis through O.

(iii) Calculate the angle which OA makes with the vertical when the object rests in equilibrium. [2]

i)

Shape	Mass	Coordinates of centre of mass
Circle centred at O	$\pi(1.2)^2\rho = 1.44\pi\rho$	$(0,0)$
Circle centred at A	$\pi(0.3)^2\rho = 0.09\pi\rho$	$(-0.5,0)$
Circle centred at B	$\pi(0.4)^2\rho = 0.16\pi\rho$	$(0,-0.7)$
Remaining figure	$1.19\pi\rho$	$(\bar{x}, \bar{y})$

ii) Taking moments about y-axis:

$$1.19\pi\rho \times \bar{x} = 1.44\pi\rho \times 0 - 0.09\pi\rho \times -0.5 - 0.16\pi\rho \times 0$$

$$\bar{x} = 0.03781\text{m} \approx 0.0378\text{m}$$

i) Taking moments about x-axis:

$$1.19\pi\rho \times \bar{y} = 1.44\pi\rho \times 0 - 0.09\pi\rho \times 0 - 0.16\pi\rho \times -0.7$$

$$\bar{y} = 0.09411\text{ m} \approx 0.0941\text{ m.}$$

Distance of centre of mass from x-axis = 0.0941 m.

Distance of centre of mass from y-axis = 0.0378m.

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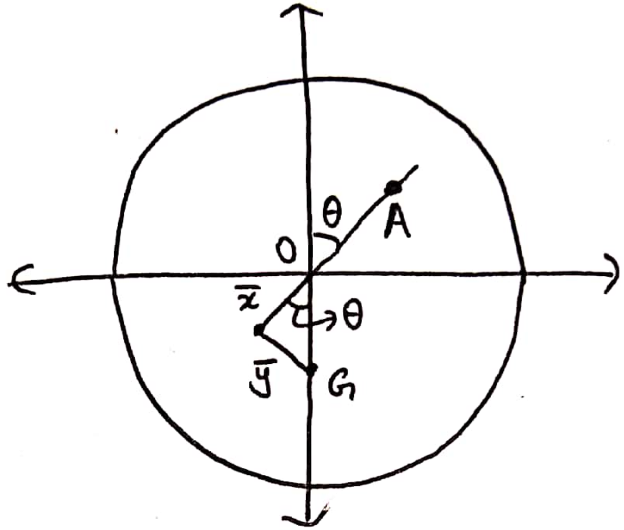
iii) When object rests in equilibrium, centre of mass G lies vertically below O.



$$\tan\theta = \frac{\bar{y}}{\bar{x}} = \frac{0.0941}{0.0378}$$

$$\theta = 68.11^\circ \approx 68.1^\circ$$

i.e. Angle between OA and vertical = 68.1°.

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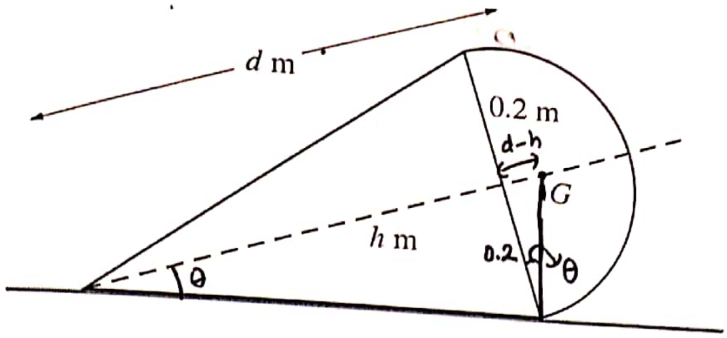


An object is formed by joining a hemispherical shell of radius 0.2 m and a solid cone with base radius 0.2 m and height h m along their circumferences. The centre of mass, G, of the object is d m from the vertex of the cone on the axis of symmetry of the object. The object rests in equilibrium on a horizontal plane, with the curved surface of the cone in contact with the plane (see diagram). The object is on the point of toppling.

(i) Show that $d = h + \dfrac{0.04}{h}$. [3]

(ii) It is given that the cone is uniform and of weight 4 N, and that the hemispherical shell is uniform and of weight W N. Given also that $h = 0.8$, find W. [6]

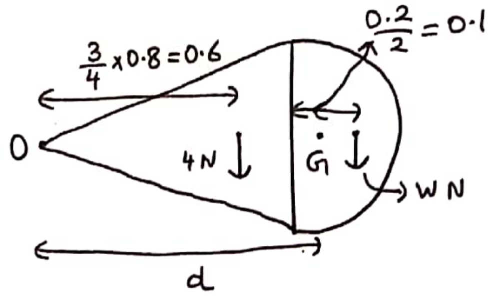
i) $\tan\theta = \dfrac{0.2}{h} \to (1)$ and $\tan\theta = \dfrac{d-h}{0.2} \to (2)$

Setting (1) = (2) gives: $\dfrac{0.2}{h} = \dfrac{d-h}{0.2}$

$\dfrac{0.04}{h} = d\cancel{h} - h$

$\Rightarrow \quad d = h + \dfrac{0.04}{h}$ (shown).

ii)



$= 0.8 + \dfrac{0.04}{0.8} = 0.85$

Shape	Mass	Distance of centre of mass from O.
Cone	0.4	$\frac{3}{4} \times 0.8 = 0.6$
Hemisphere	$\frac{W}{10} = 0.1W$	$0.8 + \frac{0.2}{2} = 0.9$
Composite figure	$0.4 + 0.1W$	$d = 0.85$

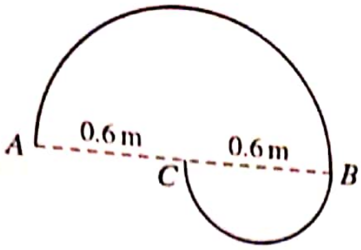
Taking moments about a vertical line through O:

$0.4 \times 0.6 + 0.1W \times 0.9 = (0.4 + 0.1W) \times 0.85$

$$0.24 + 0.09W = 0.34 + 0.085W$$

$$0.005W = 0.1$$

$$W = 20N$$



A uniform wire is bent to form an object which has a semicircular arc with diameter AB of length 1.2 m, with a smaller semicircular arc with diameter BC of length 0.6 m. The end C of the smaller arc is at the centre of the larger arc (see diagram). The two semicircular arcs of the wire are in the same plane.

(i) Show that the distance of the centre of mass of the object from the line ACB is 0.191 m, correct to 3 significant figures. [3]

The object is freely suspended at A and hangs in equilibrium.

(ii) Find the angle between ACB and the vertical. [4]

i)

Shape	Mass	Distance of centre of mass from ACB
Arc AB	$\pi(0.6)\rho = 0.6\pi\rho$	$\frac{0.6\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \frac{6}{5\pi}$
Arc BC	$\pi(0.3)\rho = 0.3\pi\rho$	$\frac{0.3\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \frac{3}{5\pi}$
Composite figure	$0.9\pi\rho$	$\bar{y}$

Taking moments about ACB:

$$0.6\pi\rho \times \frac{6}{5\pi} + 0.3\pi\rho \times -\frac{3}{5\pi} = 0.9\pi\rho \times \bar{y}$$

$$\bar{y} = 0.1909 \approx 0.191\text{ m}.$$

ii)

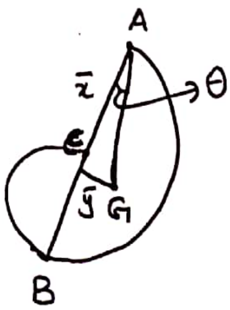
Shape	Mass	Distance of centre of mass from vertical line through A
Arc AB	$0.6\pi\rho$	0.6
Arc BC	$0.3\pi\rho$	0.9
Composite figure	$0.9\pi\rho$	$\bar{x}$

Taking moments about a vertical line through A:

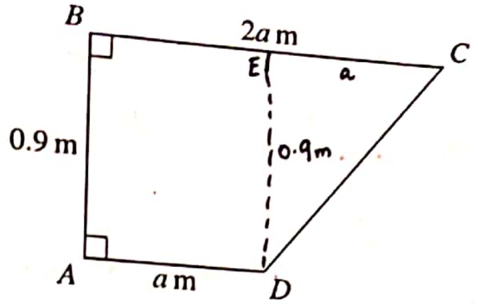
$$0.6\pi\rho \times 0.6 + 0.3\pi\rho \times 0.9 = 0.9\pi\rho \times \bar{x}$$

$$\bar{x} = 0.7$$

$$\tan\theta = \frac{\bar{y}}{\bar{x}} = \frac{0.1909}{0.7} \Rightarrow \theta = 15.25° \approx 15.3°$$



i.e. Angle between ACB and vertical $= 15.3°$.



The diagram shows the cross-section $ABCD$ through the centre of mass of a uniform solid prism. $AB = 0.9$ m, $BC = 2a$ m, $AD = a$ m and angle ABC = angle $BAD = 90°$.

(i) Calculate the distance of the centre of mass of the prism from AD. [2]

(ii) Express the distance of the centre of mass of the prism from AB in terms of a. [2]

The prism has weight 18 N and rests in equilibrium on a rough horizontal surface, with AD in contact with the surface. A horizontal force of magnitude 6 N is applied to the prism. This force acts through the centre of mass in the direction BC.

(iii) Given that the prism is on the point of toppling, calculate a. [3]

i)

Shape	Mass	Distance of centre of mass from AD
Rectangle ABED	$a(0.9)\ell$ $= 0.9a\ell$	$\frac{0.9}{2} = 0.45$
Δ CED	$\frac{1}{2} \times a \times 0.9\ell$ $= 0.45a\ell$	$\frac{0+0.9+0.9}{3} = 0.6$
Composite figure	$(0.9a + 0.45a)\ell$ $= 1.35a\ell$	$\bar{y}$

Taking moments about AD:

$$0.9a\ell \times 0.45 + 0.45a\ell \times 0.6 = 1.35a\ell \times \bar{y}$$

$$\bar{y} = 0.5$$

$\Rightarrow$ Distance of centre of mass from AD = 0.5 m.

ii)

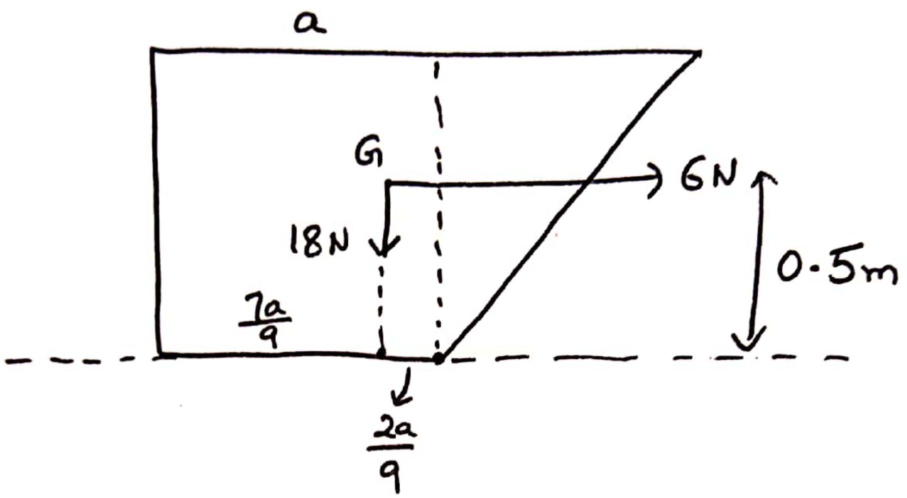
Shape	Mass	Distance of centre of mass from AB
Rectangle ABED	$0.9a\ell$	$\frac{a}{2}$
Triangle CED	$0.45a\ell$	$\frac{a+a+2a}{3} = \frac{4a}{3}$
Composite figure	$1.35a\ell$	$\bar{x}$

Taking moments about AB: $0.9a\ell \times \frac{a}{2} + 0.45a\ell \times \frac{4a}{3} = 1.35a\ell \times \bar{x}$

$$\bar{x} = \frac{7a}{9}.$$

$\Rightarrow$ Distance of center of mass from AB $= \frac{7a}{9}$.

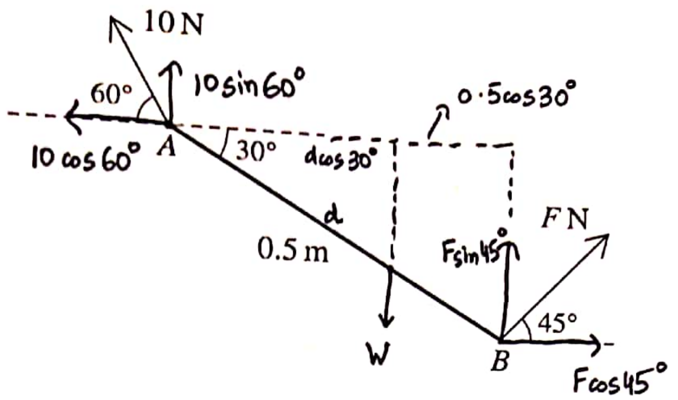
iii)



Since prism is about to topple, taking moments about point of contact:

$$18 \times \frac{2a}{9} = 6 \times 0.5$$

$$a = \frac{3}{4} = 0.75\,m$$



A non-uniform rod AB of length 0.5 m is freely hinged to a fixed point at A. The rod is in equilibrium at an angle of 30° with the horizontal with B below the level of A. Equilibrium is maintained by a force of magnitude F N applied at B acting at 45° above the horizontal in the vertical plane containing AB. The force exerted by the hinge on the rod has magnitude 10 N and acts at an angle of 60° above the horizontal (see diagram).

(i) By resolving horizontally and vertically, calculate F and the weight of the rod. [4]

(ii) Find the distance of the centre of mass of the rod from A. [3]

i) $R(\rightarrow)$:

$$F\cos 45^\circ = 10\cos 60^\circ$$

$$F = \frac{10\cos 60^\circ}{\cos 45^\circ} = 5\sqrt{2} \approx 7.07\text{N}$$

$R(\uparrow)$:

$$10\sin 60^\circ + F\sin 45^\circ = W$$

$$10\left(\frac{\sqrt{3}}{2}\right) + 5\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = W$$

$$W = 5 + 5\sqrt{3} \approx 13.7\text{N}.$$

ii) Taking moments about A:

$$W \times d\cos 30^\circ = F\sin 45^\circ \times 0.5\cos 30^\circ + F\cos 45^\circ \times 0.5\sin 30^\circ$$

$$d(5+5\sqrt{3})\cos 30^\circ = (5\sqrt{2})\left(\frac{1}{\sqrt{2}}\right)\times\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) + 5\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)\times\frac{1}{2}\times\frac{1}{2}$$

$$d = \frac{\sqrt{3}}{6}\text{m} \approx 0.289\text{m}.$$

i.e. distance of centre of mass from A = 0.289 m.

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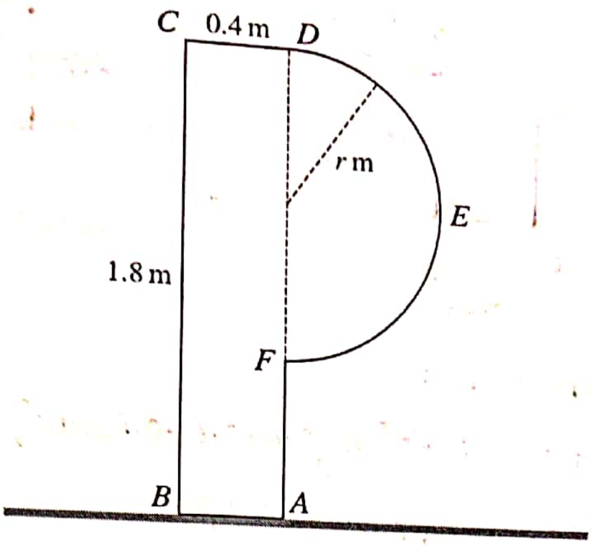

The diagram shows the cross-section $ABCDEF$ through the centre of mass of a uniform prism which rests with AB on rough horizontal ground. $ABCD$ is a rectangle with $AB = CD = 0.4$ m and $BC = AD = 1.8$ m. The other part of the cross-section is a semicircle with diameter DF and radius r m.

(i) Given that the prism is on the point of toppling, show that $r = 0.6$. [3]

A force of magnitude P N is applied to the prism, acting at 60° to the upwards vertical along a tangent to the semicircle at a point between D and E. The prism has weight 15 N and is in equilibrium on the point of toppling about B.

(ii) Show that $P = 3.26$, correct to 3 significant figures. [4]

(iii) Find the smallest possible value of the coefficient of friction between the prism and the ground. [2]

i)

Shape	Mass	Distance of center of mass from AD
Rectangle ABCD	$0.4 \times 1.8 \times \rho$ $= 0.72\rho$	$\frac{0.4}{2} = 0.2$
Sector DEF	$\cancel{\frac{2r\sin\left(\frac{\pi}{2}\right)}{3\left(\frac{\pi}{2}\right)} = \frac{4r}{3}}$ $\frac{\pi r^2}{2}\rho$	$\frac{2r\sin\left(\frac{\pi}{2}\right)}{3\left(\frac{\pi}{2}\right)} = \frac{4r}{3\pi}$
Composite figure	$\left(0.72 + \frac{\pi r^2}{2}\right)\rho$	0

Taking moments about AD:

$$0.72\rho \times 0.2 + \frac{\pi r^2 \rho}{2} \times \frac{-4r}{3\pi} = \left(0.72 + \frac{\pi r^2}{2}\right)\rho \times 0$$

$$0.144 = \frac{2}{3}r^3$$

$$r^3 = 0.216 \Rightarrow r = 0.6$$

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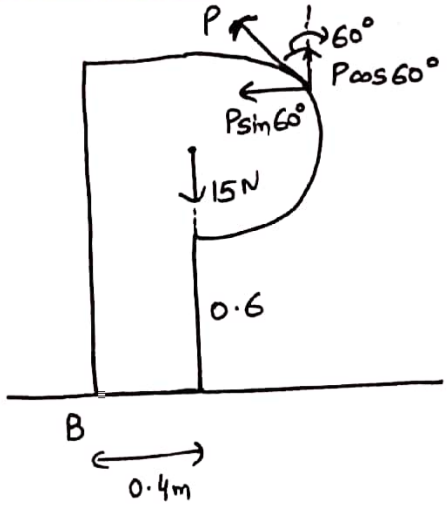
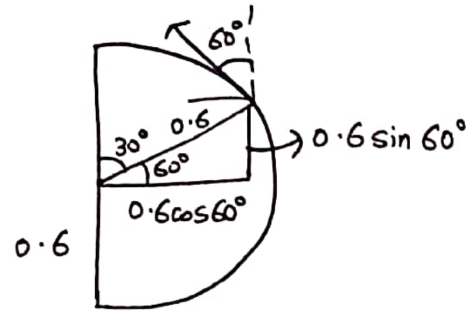
ii)





Taking moments about B :

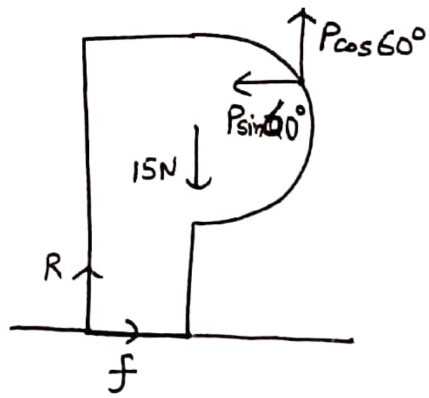
$$15 \times 0.4 = P\sin 60° \times (0.6 + 0.6 + 0.6\sin 60°) + P\cos 60° \, (0.4 + 0.6\cos 60°)$$

$$6 = 1.489P + 0.35P$$

$$P = 3.262 \approx 3.26 \text{ N} \cdot \text{ (Shown)}.$$

iii) $R(\uparrow)$: $R + P\cos 60° = 15$

$$R = 13.368 \text{ N}$$



$R(\rightarrow)$: $f = P\sin 60° = \cancel{115}\ 2.825$

Since object does not slip,

$$f \leq \mu R$$

$$2.825 \leq \mu \times 13.368$$

$$\Rightarrow \mu \geq 0.2113$$

$\Rightarrow$ Minimum value of $\mu = 0.211$

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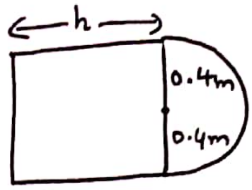
98 A solid object consists of a uniform hemisphere of radius 0.4 m attached to a uniform cylinder of radius 0.4 m so that the circumferences of their circular faces coincide. The hemisphere and cylinder each have weight 20 N. The centre of mass of the object lies at the centre O of their common circular face.

(i) Show that the height of the cylinder is 0.3 m. [2]



Shape	Mass	Distance of centre of mass from common face
Hemisphere	2	$\frac{3\times 0.4}{8} = 0.15\,m$
Cylinder	2	$\frac{h}{2}$
Composite figure	4	0

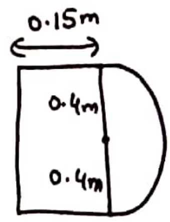
Taking moments about common face:

$$4\times 0 = 2\times 0.15 + 2\times\left(\frac{-h}{2}\right)$$

$$h = 0.3\,m \text{ (Shown)}.$$

A new object is made by cutting the cylinder in half and removing the half not attached to the hemisphere. The cut is perpendicular to the axis of symmetry, so the new object consists of a hemisphere and a cylinder half the height of the original cylinder.

(ii) Find the distance of the centre of mass of the new object from O. [4]



Shape	Mass	Distance of centre of mass from O
Hemisphere	2	$\frac{3\times 0.4}{8} = 0.15$
Cylinder	1	$\frac{0.15}{2} = 0.075$
Composite figure	3	$\bar{x}$

Taking moments about common faces:

$$2\times 0.15 + 1\times -0.075 = 3\bar{x}$$

$$\bar{x} = 0.075$$

i.e. Distance of centre of mass of new object from O $= 0.075\,m$

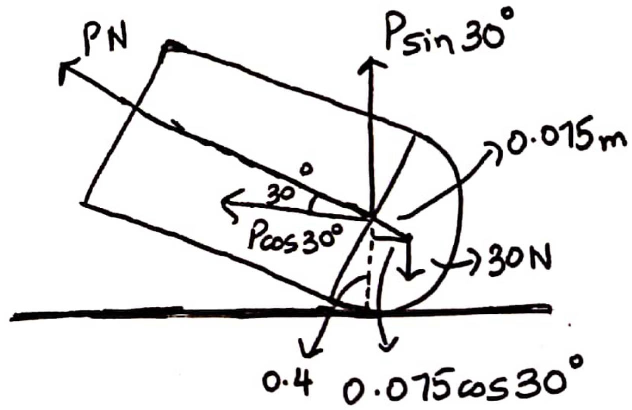
The new object is placed with its hemispherical part on a rough horizontal surface. The new object is held in equilibrium by a force of magnitude P N acting along its axis of symmetry, which is inclined at 30° to the horizontal.

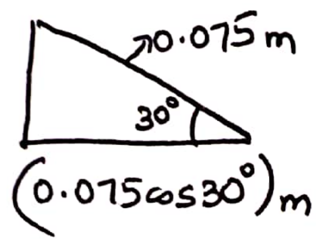
(iii) Find P. [3]

Taking moments about point of contact,

$$P\cos 30° \times 0.4 = 30 \times 0.075 \cos 30°$$

$$P = 5.625\text{ N}.$$





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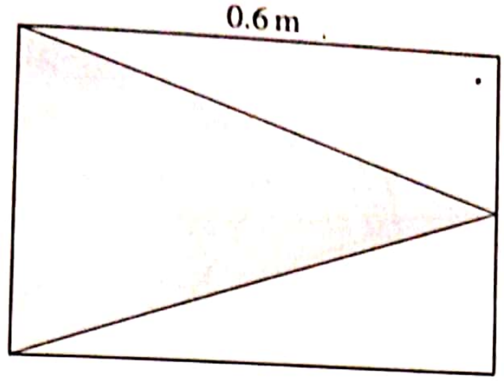


A uniform solid cone has height 0.6 m and base radius 0.2 m. A uniform hollow cylinder, open at both ends, has the same dimensions. An object is made by putting the cone inside the cylinder so that the base of the cone coincides with one end of the cylinder (see diagram, which shows a cross-section). The total weight of the object is 60 N and its centre of mass is 0.25 m from the base of the cone. Calculate the weight of the cone. [3]

Let, weight of cone $= W$ N

Since total weight $= 60$ N $\Rightarrow$ Weight of hollow cylinder $= 60 - W$.

From base of cone,

distance of centre of mass of cone $= \frac{1}{4}(0.6) = 0.15$ m

distance of centre of mass of cylinder $= 0.3$ m.

distance of centre of mass of composite object $= 0.25$ m.

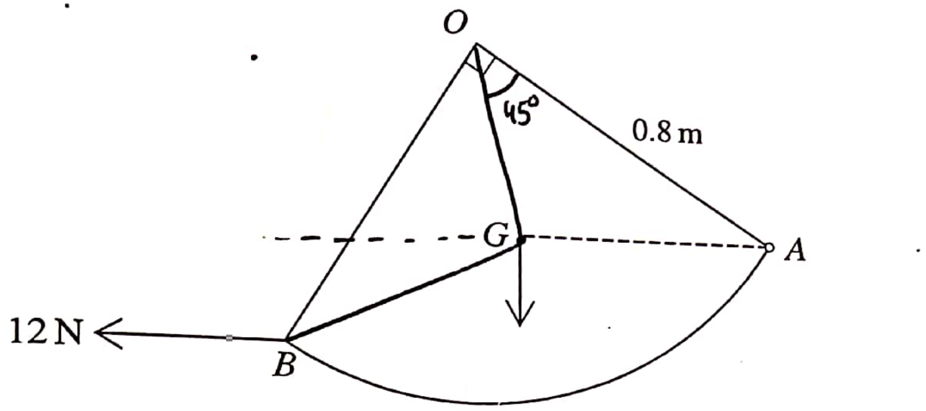
Taking moments about base of cone:

$$60(0.25) = W \times 0.15 + (60 - W)(0.3)$$

$$15 = 0.15W + 18 - 0.3W$$

$$-3 = -0.15W$$

$$W = 20 \text{ N}$$



OAB is a uniform lamina in the shape of a quadrant of a circle with centre O and radius 0.8 m which has its centre of mass at G. The lamina is smoothly hinged at A to a fixed point and is free to rotate in a vertical plane. A horizontal force of magnitude 12 N acting in the plane of the lamina is applied to the lamina at B. The lamina is in equilibrium with AG horizontal (see diagram).

(i) Calculate the length AG. [3]

$$OG = \frac{2r\sin\alpha}{3\alpha} = \frac{2(0.8)\sin\left(\frac{\pi}{4}\right)}{3\left(\frac{\pi}{4}\right)} = \frac{\frac{8}{5}\left(\frac{\sqrt{2}}{2}\right)}{\frac{3\pi}{4}} = \frac{\frac{4\sqrt{2}}{5}}{\frac{3\pi}{4}} = \frac{4\sqrt{2}\times 4}{5\times 3\pi}$$

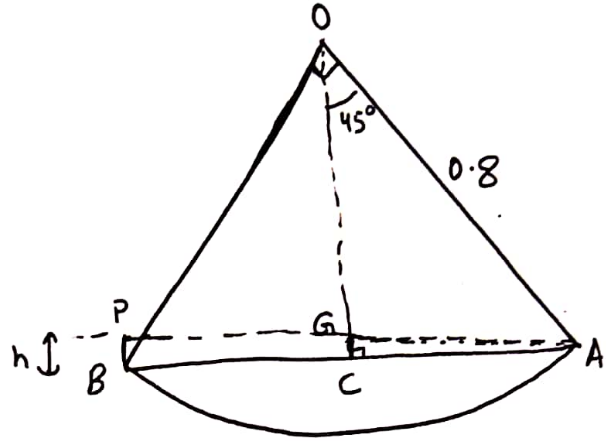
$$OG = 0.48016$$

In ΔAOG, $AG^2 = OG^2 + OA^2 - 2(OG)(OA)\cos A\hat{O}G$

$$= 0.48016^2 + 0.8^2 - 2(0.48016)(0.8)\cos 45^\circ$$

$$AG = 0.5721 \text{ m} \approx 0.572 \text{ m}.$$

(ii) Find the weight of the lamina. [5]



In ΔOAC, $AC = 0.8 \sin 45° = 0.4\sqrt{2}$ and $OC = 0.8 \sin 45° = 0.4\sqrt{2}$ m.

$\Rightarrow GC = OC - OG = 0.4\sqrt{2} - \dfrac{16\sqrt{2}}{15\pi} = 0.08551$ m.

$\tan G\hat{A}C = \dfrac{GC}{AG} \Rightarrow \tan G\hat{A}C = \dfrac{0.08551}{0.5721} \Rightarrow G\hat{A}C = 8.50°$

AB is a chord with $O\hat{C}A = 90° \Rightarrow AB = 2(AC) = 2(0.4\sqrt{2}) = 0.8\sqrt{2}$

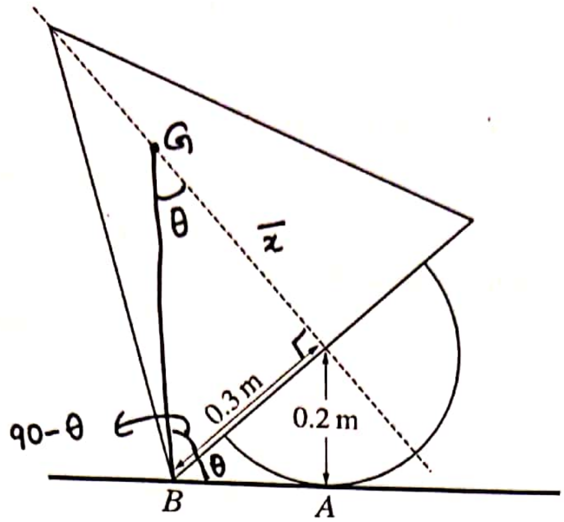
In ΔABP, $\sin \hat{A} = \dfrac{h}{AB} \Rightarrow \sin 8.50° \times 0.8\sqrt{2} = h$

$h = 0.1672$ m

Taking moments about A:

$W \times 0.5721 = 12 \times 0.1672$

$W = 3.51$ N



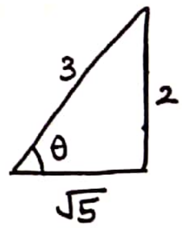
A uniform object is made by attaching the base of a solid hemisphere to the base of a solid cone so that the object has an axis of symmetry. The base of the cone has radius 0.3 m, and the hemisphere has radius 0.2 m. The object is placed on a horizontal plane with a point A on the curved surface of the hemisphere and a point B on the circumference of the cone in contact with the plane (see diagram).

(i) Given that the object is on the point of toppling about B, find the distance of the centre of mass of the object from the base of the cone. [3]

When object is on point of toppling about B, G is vertically above B.

$$\sin\theta = \frac{0.2}{0.3} = \frac{2}{3}$$

$$\Rightarrow \tan\theta = \frac{2}{\sqrt{5}}.$$



In Δ formed within cone,

$$\tan\theta = \frac{0.3}{\bar{x}}$$

$$\frac{2}{\sqrt{5}} = \frac{0.3}{\bar{x}}$$

$$\bar{x} = \frac{0.3 \times \sqrt{5}}{2} = 0.3354$$

$$\bar{x} \approx 0.335\text{ m}$$

⇒ Distance of centre of mass from base of cone = 0.335 m.

(ii) Given instead that the object is on the point of toppling about A, calculate the height of the cone. [3]

[The volume of a cone is $\frac{1}{3}\pi r^2 h$. The volume of a hemisphere is $\frac{2}{3}\pi r^3$.]

Let, ρ = Mass per unit volume.

Mass of cone $= \frac{1}{3}\pi r_c^2 h \rho$

Mass of hemisphere $= \frac{2}{3}\pi r_H^3 \rho$.

$\Rightarrow$ Total mass $= \frac{1}{3}\pi r_c^2 h\rho + \frac{2}{3}\pi r_H^3 \rho$

From base of cone,

Distance of centre of mass of cone $= \frac{1}{4}h$

Distance of centre of mass of hemisphere $= \frac{3}{8}(0.2) = 0.075$

Distance of centre of mass of composite figure $= \bar{x}$.

Taking moments about base of cone:

$$\left(\frac{1}{3}\pi r_c^2 h \rho + \frac{2}{3}\pi r_H^3 \rho\right)\bar{x} = \frac{1}{3}\pi r_c^2 h\rho\left(\frac{1}{4}h\right) + \frac{2}{3}\pi r_H^3 \rho\,(-0.075)$$

Note $\bar{x} = 0$ since centre of mass is vertically above A and so lies on base of cone.

$$\Rightarrow \quad 0 = \frac{\pi r_c^2 h^2 \rho}{12} - \frac{\pi r_H^3 \rho}{20}$$

$$\frac{\cancel{\pi} r_H^3 \cancel{\rho}}{20} = \frac{\cancel{\pi} r_c^2 h^2 \cancel{\rho}}{12}$$

$$\frac{(0.2)^3}{20} = \frac{(0.3)^2 \times h^2}{12}$$

$$\Rightarrow h = 0.2309 \approx 0.231 \text{ m}$$

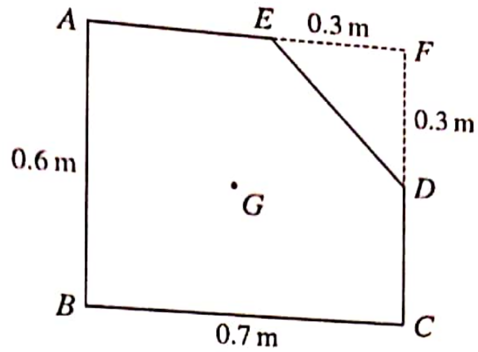


Fig. 1

Fig. 1 shows the cross-section $ABCDE$ through the centre of mass G of a uniform prism. The cross-section consists of a rectangle $ABCF$ from which a triangle DEF has been removed; $AB = 0.6$ m, $BC = 0.7$ m and $DF = EF = 0.3$ m.

(i) Show that the distance of G from BC is 0.276 m, and find the distance of G from AB. [5]

Let, ρ = Mass per unit area.

Mass of rectangle ABCF $= 0.6(0.7)\rho = 0.42\rho$.

Mass of $\Delta DEF = \frac{1}{2}(0.3)(0.3)\rho = 0.045\rho$.

Mass of remaining lamina $= 0.42\rho - 0.045\rho = 0.375\rho$.

Using B as origin,

Centre of mass of ABCF $= (0.35, 0.3)$.

Centre of mass of $\Delta EFD = \left(\frac{0.4+0.7+0.7}{3}, \frac{0.3+0.6+0.6}{3}\right) = (0.6, 0.5)$

Centre of mass of remaining lamina $= (\bar{x}, \bar{y})$

Taking moments about AB:

$$0.375\rho \times \bar{x} = 0.42\rho \times 0.35 - 0.045\rho \times 0.6$$

$$\bar{x} = 0.32 \text{ m}$$

$\Rightarrow$ Distance of G from AB $= 0.32$ m.

Taking moments about BC:

$$0.375\rho \times \bar{y} = 0.42\rho \times 0.3 - 0.045\rho \times 0.5$$

$$\bar{y} = 0.276 \text{ m}$$

$\Rightarrow$ Distance of G from BC $= 0.276$ m (Shown).

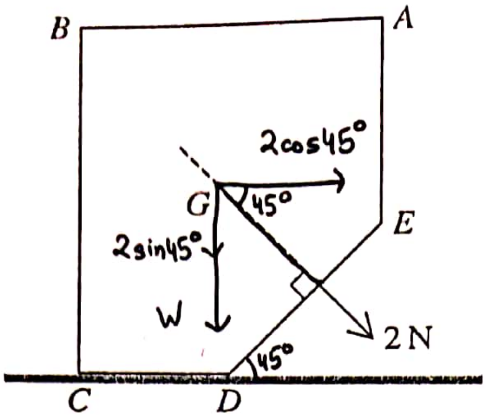
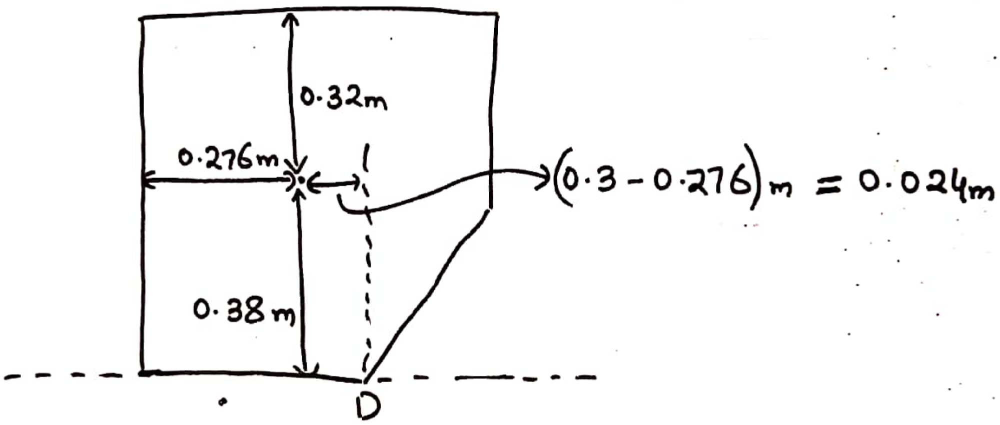


Fig. 2

The prism is placed with CD on a rough horizontal surface. A force of magnitude 2 N acting in the plane of the cross-section is applied to the prism. The line of action of the force passes through G and is perpendicular to DE (see Fig. 2). The prism is on the point of toppling about the edge through D.

(ii) Calculate the weight of the prism. [3]



Taking moments about D :

$$W \times 0.024 + 2\sin 45^\circ \times 0.024 = 2\cos 45^\circ \times 0.38$$

$$W = 20.97\ \text{N}$$

$$W \approx 21.0\ \text{N}$$

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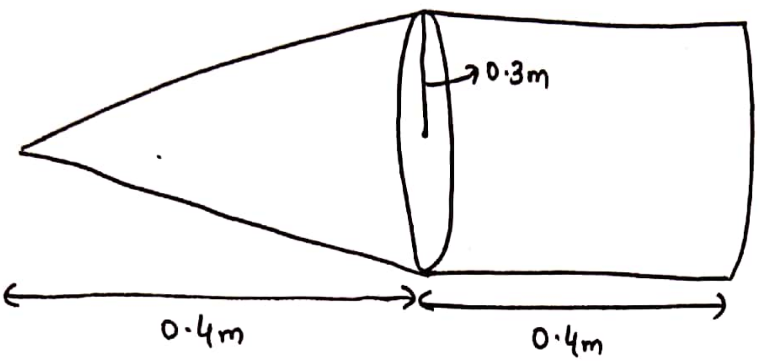
103 A uniform solid object is made by attaching a cone to a cylinder so that the circumferences of the base of the cone and a plane face of the cylinder coincide. The cone and the cylinder each have radius 0.3 m and height 0.4 m.

(i) Calculate the distance of the centre of mass of the object from the vertex of the cone. [4]

[The volume of a cone is $\frac{1}{3}\pi r^2 h$.]



Let, ρ = Mass per unit volume.

Mass of cone $= \frac{1}{3}\pi(0{\cdot}3)^2(0{\cdot}4) \times \rho = 0{\cdot}012\pi\rho$.

Mass of cylinder $= \pi(0{\cdot}3)^2(0{\cdot}4) \times \rho = 0{\cdot}036\pi\rho$.

$\Rightarrow$ Total mass of composite figure $= 0{\cdot}012\pi\rho + 0{\cdot}036\pi\rho = 0{\cdot}048\pi\rho$.

From the vertex of the cone,

distance of centre of mass of cone $= \frac{3}{4}(0{\cdot}4) = 0{\cdot}3$ m

distance of centre of mass of cylinder $= 0{\cdot}4 + \frac{0{\cdot}4}{2} = 0{\cdot}6$ m.

distance of centre of mass of composite figure $= \bar{x}$.

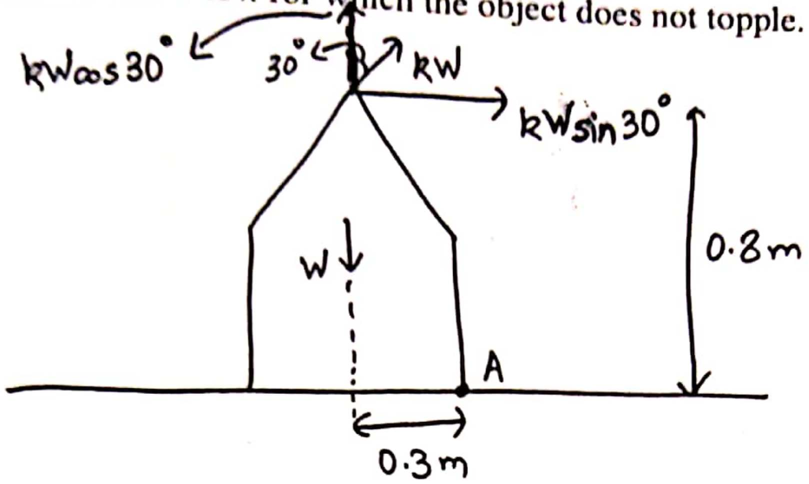
Taking moments about the vertex of the cone :

$$0{\cdot}048\pi\rho \times \bar{x} = 0{\cdot}012\pi\rho \times 0{\cdot}3 + 0{\cdot}036\pi\rho \times 0{\cdot}6$$

$$\bar{x} = 0{\cdot}525 \text{ m}.$$

The object has weight W N and is placed with its plane circular face on a rough horizontal surface. A force of magnitude kW N acting at 30° to the upward vertical is applied to the vertex of the cone. The object does not slip.

(ii) Find the greatest possible value of k for which the object does not topple. [3]



Taking moments about A:

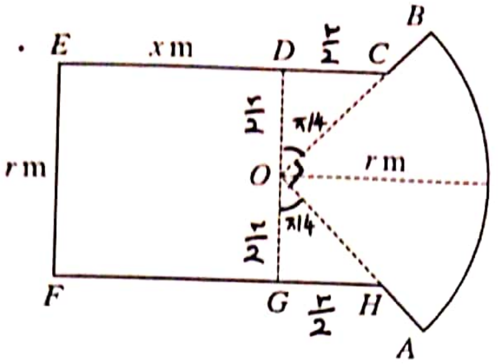
$$W \times 0.3 = kW\sin 30° \times 0.8 + kW\cos 30° \times 0.3$$

$$0.3W = 0.4kW + \frac{3\sqrt{3}}{20}kW$$

$$0.3 = 0.4k + \frac{3\sqrt{3}}{20}k = \left(\frac{2k}{5} + \frac{3\sqrt{3}k}{20}\right)$$

$$0.3 = \left(\frac{\cancel{0.}8 + 3\sqrt{3}}{20}\right)k$$

$$k = \frac{20(0.3)}{\cancel{0.}8 + 3\sqrt{3}} = 0.4546 \simeq 0.455$$



The diagram shows a uniform lamina $ABCDEFGH$. The lamina consists of a quarter-circle OAB of radius r m, a rectangle $DEFG$ and two isosceles right-angled triangles COD and GOH. The rectangle has $DG = EF = r$ m and $DE = FG = x$ m.

(i) Given that the centre of mass of the lamina is at O, express x in terms of r. [6]

Let, Mass per unit area $= \rho$.

Mass of rectangle $= xr\rho$.

Mass of $\Delta ODC = \frac{1}{2}\left(\frac{r}{2}\right)\left(\frac{r}{2}\right)\rho = \frac{r^2\rho}{8}$

Mass of $\Delta OGH = \frac{1}{2}\left(\frac{r}{2}\right)\left(\frac{r}{2}\right)\rho = \frac{r^2\rho}{8}$.

$\Rightarrow$ Total mass $= xr\rho + \frac{r^2\rho}{8} + \frac{r^2\rho}{8} + \frac{\pi r^2\rho}{4}$

Mass of sector $AOB = \frac{r^2\left(\frac{\pi}{2}\right)}{2}\rho = \frac{\pi r^2\rho}{4}$.

Distance of centre of mass of rectangle from $O = \frac{x}{2}$.

Distance of centre of mass of ΔODC from $O = \frac{1}{3}\left(0+0+\frac{r}{2}\right) = \frac{r}{6}$

Distance of centre of mass of ΔOGH from $O = \frac{1}{3}\left(0+0+\frac{r}{2}\right) = \frac{r}{6}$.

Distance of centre of mass of sector AOB from $O = \frac{2r\sin\left(\frac{\pi}{4}\right)}{3\left(\frac{\pi}{4}\right)} = \frac{2r\left(\frac{1}{\sqrt{2}}\right)}{\frac{3\pi}{4}} = \frac{4\sqrt{2}r}{3\pi}$.

Taking moments about DG:

$$\left(xr\rho + \frac{r^2\rho}{8} + \frac{r^2\rho}{8} + \frac{\pi r^2\rho}{4}\right)\bar{x} = \frac{r^2\rho}{8}\times\frac{r}{6} + \frac{r^2\rho}{8}\times\frac{r}{6} + \frac{\pi r^2\rho}{4}\times\frac{4\sqrt{2}r}{3\pi} - xr\rho\times\frac{x}{2}$$

$$0 = \frac{r^3\rho}{48} + \frac{r^3\rho}{48} + \frac{\sqrt{2}r^3\rho}{3} - \frac{x^2r\rho}{2}$$

$$0 = \frac{2r^2}{48} + \frac{\sqrt{2}r^2}{3} - \frac{x^2}{2}$$

$$0 = \frac{r^2}{24} + \frac{\sqrt{2}r^2}{3} - \frac{x^2}{2}$$

$$0 = \frac{\cancel{2}r^2 + 8\sqrt{2}r^2 - 12x^2}{24}$$

$$0 = r^2 + 8\sqrt{2}r^2 - 12x^2$$

$$12x^2 = r^2 + 8\sqrt{2}r^2$$

$$x^2 = \left(\frac{1+8\sqrt{2}}{12}\right)r^2$$

$$x = 1.012r \approx 1.01r$$

(ii) Given instead that the rectangle $DEFG$ is a square with edges of length r m, state with a reason whether the centre of mass of the lamina lies within the square or the quarter-circle. [1]

The square of length r will be smaller than the rectangle so the centre of mass of the lamina now lies within the quarter-circle.

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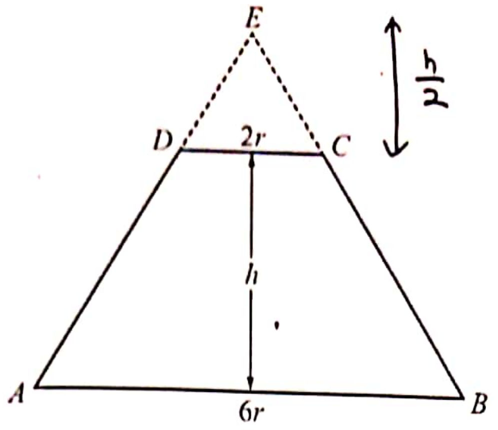


The diagram shows the cross-section $ABCD$ of a uniform solid object which is formed by removing a cone with cross-section DCE from the top of a larger cone with cross-section ABE. The perpendicular distance between AB and DC is h, the diameter AB is $6r$ and the diameter DC is $2r$.

(a) Find an expression, in terms of h, for the distance of the centre of mass of the solid object from AB. [4]

Let, height of smaller cone $= x$.

Using similarity: $\dfrac{x}{x+h} = \dfrac{2r}{6r} \Rightarrow \dfrac{x}{x+h} = \dfrac{1}{3} \Rightarrow 3x = x+h \Rightarrow 2x = h \Rightarrow x = \dfrac{h}{2}$.

Let, ρ = Mass per unit volume.

Mass of smaller cone $= \dfrac{1}{3}\pi(r)^2 \times \left(\dfrac{h}{2}\right) \times \rho = \dfrac{\pi r^2 h\rho}{6}$

Mass of larger cone $= \dfrac{1}{3}\pi(3r)^2\left(h+\dfrac{h}{2}\right) \times \rho = 3\pi r^2 \times \dfrac{3h}{2} \times \rho = \dfrac{9\pi r^2 h\rho}{2}$

Mass of remaining solid $= \dfrac{9\pi r^2 h\rho}{2} - \dfrac{\pi r^2 h\rho}{6} = \dfrac{13\pi r^2 h\rho}{3}$.

For smaller cone, distance of centre of mass from AB $= h + \dfrac{1}{4}\left(\dfrac{h}{2}\right) = \dfrac{9h}{8}$.

For larger cone, distance of centre of mass from AB $= \dfrac{1}{4}\left(h+\dfrac{h}{2}\right) = \dfrac{3h}{8}$

Taking moments about AB:

$$-\frac{\pi r^2 h\rho}{6} \times \frac{9h}{8} + \frac{9\pi r^2 h\rho}{2} \times \frac{3h}{8} = \frac{13\pi r^2 h\rho}{3} \times \bar{y}$$

$$\frac{-3h}{16} + \frac{27h}{16} = \frac{13}{3}\bar{y}$$

$$\frac{3h}{2} = \frac{13}{3}\bar{y}$$

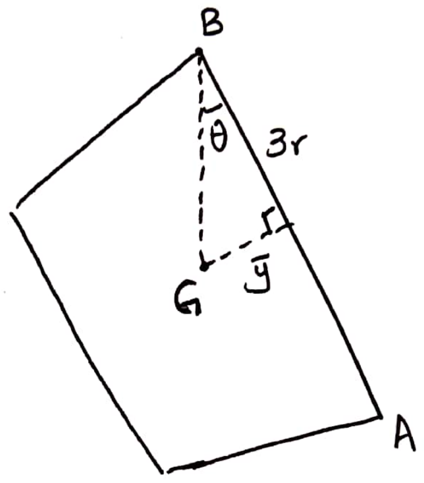
$$\bar{y} = \frac{9h}{26}$$

i.e. Distance of centre of mass of solid from AB $= \dfrac{9h}{26}$.

The object is freely suspended from the point B and hangs in equilibrium. The angle between AB and the downward vertical through B is θ.

(b) Given that $h = \frac{13}{4}r$, find the value of $\tan\theta$.

[2]

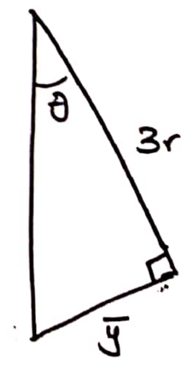


When object is suspended freely, centre of mass, G, is vertically below B.

Since $h = \frac{13r}{4}$, $\bar{y} = \frac{9}{26}\left(\frac{13r}{4}\right) = \frac{9r}{8}$.

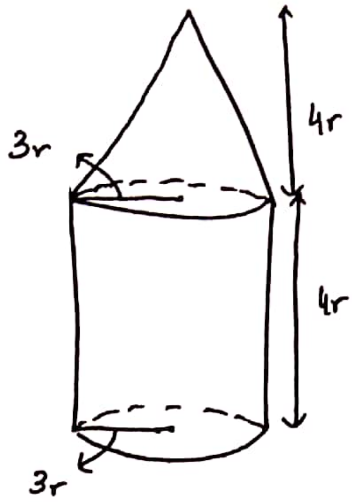
Then $\tan\theta = \frac{\bar{y}}{3r} = \frac{\frac{9r}{8}}{3r}$

$\tan\theta = \frac{3}{8}$.



106 An object consists of a uniform solid circular cone, of vertical height $4r$ and radius $3r$, and a uniform solid cylinder, of height $4r$ and radius $3r$. The circular base of the cone and one of the circular faces of the cylinder are joined together so that they coincide. The cone and the cylinder are made of the same material.

(a) Find the distance of the centre of mass of the object from the end of the cylinder that is not attached to the cone. [4]



Let, ρ be mass per unit volume.

Mass of cylinder $= \pi \times (3r)^2 \times 4r \times \rho = 36\pi r^3 \rho$

Mass of cone $= \dfrac{\pi(3r)^2 \times 4r}{3} \times \rho = 12\pi r^3 \rho$

Total mass $= 36\pi r^3 \rho + 12\pi r^3 \rho = 48\pi r^3 \rho$.

For cylinder, distance of centre of mass from base $= 2r$

For cone, distance of centre of mass from base $= 4r + \frac{1}{4}(4r) = 5r$.

Taking moments about base:

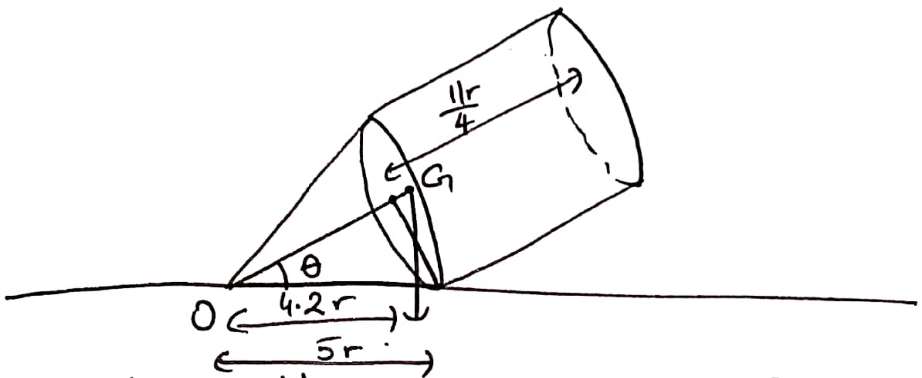
$$36\pi r^3 \rho \times 2r + 12\pi r^3 \rho \times 5r = 48\pi r^3 \rho \times \bar{y}$$

$$72r + 60r = 48\bar{y}$$

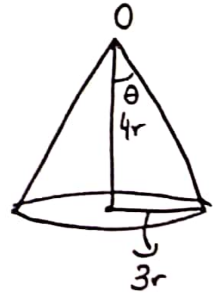
$$\bar{y} = \frac{11r}{4}.$$

$\Rightarrow$ Distance of centre of mass from base $= \dfrac{11r}{4}$.

(b) Show that the object can rest in equilibrium with the curved surface of the cone in contact with a horizontal surface. [3]



For object to rest in equilibrium, horizontal distance of centre of mass of composite figure is less than slant height of cone to ensure line of action of weight lies within base of composite figure.



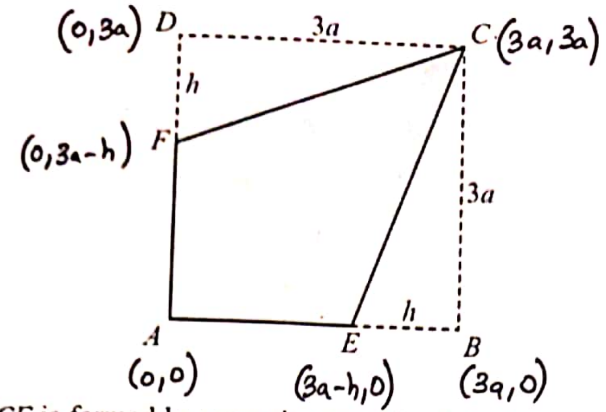
In cone, $l^2 = (4r)^2 + (3r)^2 \Rightarrow l^2 = 25r^2 = l = 5r.$

$\cos\theta = \frac{4r}{5r} = \frac{4}{5}.$

Horizontal distance from O to centre of mass

$= OG \times \cos\theta = \left(8r - \frac{11r}{4}\right)\cos\theta = \frac{21r}{4} \times \frac{4}{5} = \frac{21r}{5} = 4.2r$

Then since $4.2r < 5r$, line of action of weight lies within base of composite figure and so equilibrium is maintained.



A uniform lamina $AECF$ is formed by removing two identical triangles BCE and CDF from a square lamina $ABCD$. The square has side $3a$ and $EB = DF = h$ (see diagram).

(a) Find the distance of the centre of mass of the lamina $AECF$ from AD and from AB, giving your answers in terms of a and h. [5]

Let, ρ be mass per unit area.

Mass of square ABCD $= (3a)^2 \times \rho = 9a^2\rho$.

Mass of $\Delta BEC = \frac{1}{2} \times 3a \times h \times \rho = 1.5ah\rho$.

Mass of $\Delta CDF = \frac{1}{2} \times 3a \times h \times \rho = 1.5ah\rho$.

Mass of remaining lamina $= 9a^2\rho - 3ah\rho = 3a(3a-h)\rho$.

Let, A be the origin. Then centre of mass of square ABCD $= (1.5a, 1.5a)$

Centre of mass of $\Delta BEC = \left(\frac{3a+3a+3a-h}{3}, \frac{0+0+3a}{3}\right)$

$= \left(3a - \frac{h}{3}, a\right)$

Centre of mass of $\Delta CDF = \left(\frac{0+0+3a}{3}, \frac{3a-h+3a+3a}{3}\right)$

$= \left(a, 3a - \frac{h}{3}\right)$.

Taking moments about AB:

$$9a^2\rho \times 1.5a - 1.5ah\rho \times a - 1.5ah\rho \times \left(3a - \frac{h}{3}\right) = 3a(3a-h)\rho \times \bar{y}$$

$$13.5a^3 - 1.5a^2h - 4.5a^2h + 0.5ah^2 = 3a(3a-h) \times \bar{y}$$

$$13.5a^2 - 6ah + 0.5h^2 = 3(3a-h) \times \bar{y}$$

$$27a^2 - 12ah + h^2 = 6(3a-h) \times \bar{y}$$

$$\bar{y} = \frac{27a^2 - 12ah + h^2}{6(3a-h)}$$

$$\bar{y} = \frac{27a^2 - 9ah - 3ah + h^2}{6(3a-h)} = \frac{(9a-h)(3a-h)}{6(3a-h)} = \frac{9a-h}{6}.$$

Taking moments about AD:

$$9a^2\rho \times 1.5a - 1.5ah\rho \times \left(3a - \frac{h}{3}\right) - 1.5ah\rho \times a = 3a(3a-h)\rho \times \bar{x}$$

$$27a^2 - 12ah + h^2 = 6(3a-h)\bar{x}$$

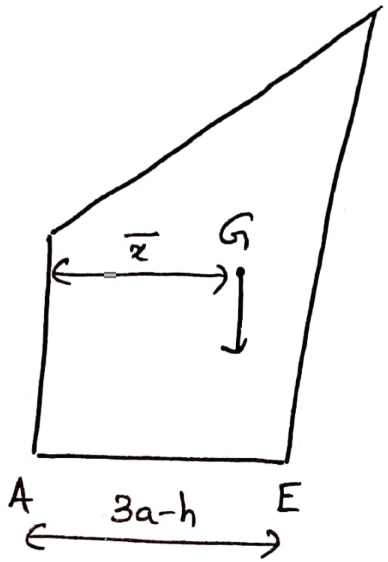
$$\bar{x} = \frac{27a^2 - 12ah + h^2}{6(3a-h)}$$

$$\bar{x} = \frac{9a-h}{6}$$

$\therefore$ Centre of mass: $\left(\frac{9a-h}{6}, \frac{9a-h}{6}\right)$.

The lamina $AECF$ is placed vertically on its edge AE on a horizontal plane.

(b) Find, in terms of a, the set of values of h for which the lamina remains in equilibrium. [3]



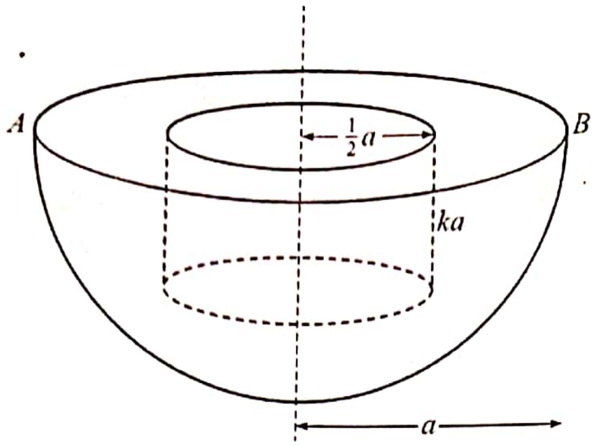
For lamina to remain in equilibrium, the line of action of weight should lie within the edge AE. Then we require $\bar{x} \leqslant AE$

$$\frac{9a-h}{6} \leqslant 3a-h$$

$$9a-h \leqslant 18a-6h$$

$$5h \leqslant 9a$$

$$h \leqslant \frac{9a}{5}.$$



An object is formed by removing a solid cylinder, of height ka and radius $\frac{1}{2}a$, from a uniform solid hemisphere of radius a. The axes of symmetry of the hemisphere and the cylinder coincide and one circular face of the cylinder coincides with the plane face of the hemisphere. AB is a diameter of the circular face of the hemisphere (see diagram).

(a) Show that the distance of the centre of mass of the object from AB is $\dfrac{3a(2-k^2)}{2(8-3k)}$. [4]

Let, Mass per unit volume $= \rho$.

Mass of hemisphere $= \frac{2}{3}\pi a^3 \times \rho = \frac{2}{3}\pi a^3 \rho$.

Mass of cylinder $= \pi\left(\frac{1}{2}a\right)^2 ka \times \rho = \dfrac{\pi k a^3 \rho}{4}$

Mass of remaining solid $= \frac{2}{3}\pi a^3\rho - \dfrac{\pi k a^3 \rho}{4} = \pi a^3 \rho\left(\frac{2}{3} - \frac{k}{4}\right) = \dfrac{\pi a^3 \rho(8-3k)}{12}$.

For hemisphere, distance of centre of mass from AB $= \frac{3}{8}a$.

For cylinder , " " " " " " " $= \dfrac{ka}{2}$.

Taking moments about AB:

$$\frac{2}{3}\pi a^3\rho \times \frac{3}{8}a - \frac{\pi k a^3 \rho}{4} \times \frac{ka}{2} = \frac{\pi a^3 \rho(8-3k)}{12} \times \bar{y}$$

$$\frac{a}{4} - \frac{k^2 a}{8} = \left(\frac{8-3k}{12}\right)\bar{y}$$

$$\frac{(2-k^2)a}{8} = \left(\frac{8-3k}{12}\right)\bar{y}$$

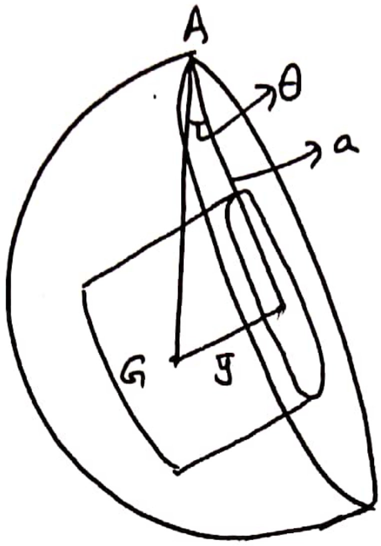
$$\bar{y} = \frac{12(2-k^2)a}{8(8-3k)}$$

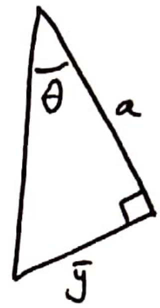
$$\bar{y} = \frac{3a(2-k^2)}{2(8-3k)}$$

$\therefore$ Distance of centre of mass from AB $= \dfrac{3a(2-k^2)}{2(8-3k)}$ (Shown).

When the object is freely suspended from the point A, the line AB makes an angle θ with the downward vertical, where $\tan\theta = \frac{7}{18}$.

(b) Find the possible values of k. [3]





when suspended from point A, centre of mass G is vertically below A.

$$\tan\theta = \frac{\bar{y}}{a}$$

$$\frac{7}{18} = \frac{3a(2-k^2)}{2(8-3k)} \div a$$

$$\frac{7}{18} = \frac{6-3k^2}{16-6k}$$

$$112 - 42k = 108 - 54k^2$$

$$54k^2 - 42k + 4 = 0$$

$$27k^2 - 21k + 2 = 0$$

$$27k^2 - 18k - 3k + 2 = 0$$

$$(9k-1)(3k-2) = 0$$

$$\Rightarrow k = \frac{1}{9} \quad \text{or} \quad k = \frac{2}{3}.$$