Past Paper Questions: Linear Motion Under a Variable Force

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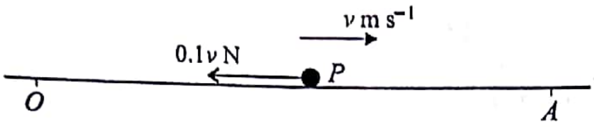

A particle P of mass 0.4 kg travels on a horizontal surface along the line OA in the direction from O to A. Air resistance of magnitude $0.1v$ N opposes the motion, where $v\,\mathrm{m\,s^{-1}}$ is the speed of P at time t s after it passes through the fixed point O (see diagram). The speed of P at O is $2\,\mathrm{m\,s^{-1}}$.

(i) Assume that the horizontal surface is smooth. Show that $\frac{dv}{dx} = -\frac{1}{4}$, where x m is the distance of P from O at time t s, and hence find the distance from O at which the speed of P is zero. [4]

(ii) Assume instead that the horizontal surface is not smooth and that the coefficient of friction between P and the surface is $\frac{3}{40}$.

(a) Show that $4\frac{dv}{dt} = -(v+3)$. [3]

(b) Hence find the value of t for which the speed of P is zero. [3]

i)

$$F = ma$$

$$-0.1v = 0.4v\frac{dv}{dx}$$

$$\frac{-0.1v}{0.4v} = \frac{dv}{dx}$$

$$\therefore \frac{dv}{dx} = \frac{-1}{4}.$$

$$\int_2^0 dv = \int_0^x \frac{-1}{4}\,dx$$

since when $x=0, v=2$
and when $x=?, v=0$

$$[v]_2^0 = \left[\frac{-1}{4}x\right]_0^x$$

$$0-2 = \frac{-x}{4} - \left(\frac{-0}{4}\right)$$

$$-2 = \frac{-x}{4}$$

$$\Rightarrow x = 8m.$$

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ii) a) $R(\uparrow)$: $R = 0.4g \quad \therefore f = \mu R = \frac{3}{40}(0.4g) = \frac{3}{10}$.

$R(\rightarrow)$:

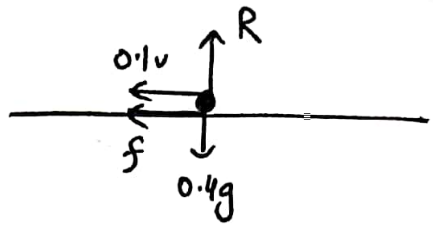
$$F_{net} = ma$$

$$-0.1v - \frac{3}{10} = 0.4\frac{dv}{dt}$$

$$\frac{-v-3}{10} = \frac{4}{10}\frac{dv}{dt}$$

$$-(v+3) = 4\frac{dv}{dt}$$

(shown).



b) $$\int_0^t \frac{-1}{4}\,dt = \int_2^0 \frac{1}{v+3}\,dv$$

since when $t=0, v=2$
and when $t=?, v=0$

$$\left[\frac{-1}{4}t\right]_0^t = \left[\ln|v+3|\right]_2^0$$

$$\frac{-t}{4} - \left(\frac{-0}{4}\right) = \ln 3 - \ln 5$$

$$\frac{-t}{4} = \ln\left(\frac{3}{5}\right)$$

$$t = -4\ln\left(\frac{3}{5}\right)$$

$$t = 2.043\text{ s} \approx 2.04\text{ s}.$$

2 A particle of mass 0.2 kg moves in a straight line on a smooth horizontal surface. When its displacement from a fixed point on the surface is x m, its velocity is $v\,\mathrm{m\,s^{-1}}$. The motion is opposed by a force of magnitude $\frac{1}{3v}$ N.

(i) Show that $3v^2\frac{dv}{dx} = -5$. [3]

(ii) Find the value of v when $x = 7.4$, given that $v = 4$ when $x = 0$. [4]

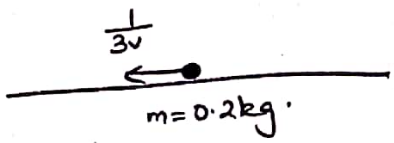
i)

$$F = ma$$

$$\frac{-1}{3v} = 0.2\,v\,\frac{dv}{dx}$$

$$\frac{-1}{3(0.2)}\cancel{dx} = v^2\frac{dv}{dx}$$

$$-5\cancel{dx} = 3v^2\frac{dv}{dx}. \quad \text{(Shown)}$$



ii)

$$\int_0^{7.4} -5\,dx = \int_4^{v} 3v^2\,dv$$

since when $x=0$, $v=4$
and when $x=7.4$, $v=?$

$$\left[-5x\right]_0^{7.4} = \left[\frac{3v^3}{3}\right]_4^{v}$$

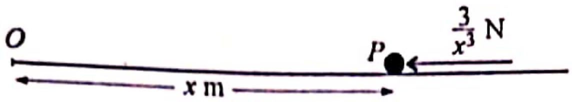
$$-5(7.4) - (-5\times 0) = v^3 - 4^3$$

$$-37 = v^3 - 64$$

$$v^3 = 27$$

$$v = 3\,\mathrm{ms^{-1}}.$$

3



A particle P of mass 0.6 kg moves in a straight line on a smooth horizontal surface. A force of magnitude $\frac{3}{x^3}$ newtons acts on the particle in the direction from P to O, where O is a fixed point of the surface and x m is the distance OP (see diagram). The particle P is released from rest at the point where $x = 10$. Find the speed of P when $x = 2.5$. [7]

$$F = ma$$

$$\frac{-3}{x^3} = 0.6\, v\frac{dv}{dx}$$

$$\frac{-3}{0.6x^3} = v\frac{dv}{dx}$$

$$\int_{10}^{2.5} -5x^{-3}\,dx = \int_0^v v\,dv$$

since when $x = 10$, $v = 0$
and when $x = 2.5$, $v = ?$

$$\left[\frac{-5x^{-2}}{-2}\right]_{10}^{2.5} = \left[\frac{v^2}{2}\right]_0^v$$

$$\left[\frac{5}{2x^2}\right]_{10}^{2.5} = \frac{v^2}{2} - 0$$

$$\frac{5}{2\times 2.5^2} - \frac{5}{2\times 10^2} = \frac{v^2}{2}$$

$$0.375 = \frac{v^2}{2}$$

$$v^2 = 0.75$$

$$\Rightarrow v = \frac{\sqrt{3}}{2}\ \text{ms}^{-1}$$

4 The acceleration of a particle moving in a straight line is $(x - 2.4)\ \mathrm{m\,s^{-2}}$ when its displacement from a fixed point O of the line is x m. The velocity of the particle is $v\ \mathrm{m\,s^{-1}}$, and it is given that $v = 2.5$ when $x = 0$. Find

(i) an expression for v in terms of x, **[5]**

(ii) the minimum value of v. **[2]**

i)

$$a = x - 2.4$$

$$v\frac{dv}{dx} = x - 2.4$$

$$\int_{2.5}^{v} v\,dv = \int_{0}^{x} x - 2.4\,dx$$

since when $x = 0$, $v = 2.5$
and when $x = x$, $v = v$

$$\left[\frac{v^2}{2}\right]_{2.5}^{v} = \left[\frac{x^2}{2} - 2.4x\right]_0^x$$

$$\frac{v^2}{2} - \frac{2.5^2}{2} = \left(\frac{x^2}{2} - 2.4x\right) - \left(\frac{0^2}{2} - 2.4(0)\right)$$

$$\frac{v^2}{2} - 3.125 = \frac{x^2}{2} - 2.4x$$

$$\frac{v^2}{2} = \frac{x^2}{2} - 2.4x + 3.125$$

$$v^2 = x^2 - 4.8x + 6.25$$

$$v = \sqrt{x^2 - 4.8x + 6.25}$$

ii) For minimum value of $\cancel{v}$, $v\frac{dv}{dx} = 0 \Rightarrow x - 2.4 = 0$
$\Rightarrow x = 2.4$

Then when $x = 2.4$,

$$v = \sqrt{2.4^2 - 4.8(2.4) + 6.25} = 0.7\,\mathrm{ms^{-1}}$$

i.e. Minimum value of $v = 0.7\,\mathrm{ms^{-1}}$.

5 An object of mass 0.4 kg is projected vertically upwards from the ground, with an initial speed of $16\,\mathrm{m\,s^{-1}}$. A resisting force of magnitude $0.1v$ newtons acts on the object during its ascent, where $v\,\mathrm{m\,s^{-1}}$ is the speed of the object at time t s after it starts to move.

(i) Show that $\dfrac{dv}{dt} = -0.25(v + 40)$. [2]

(ii) Find the value of t at the instant that the object reaches its maximum height. [5]

i)

$$F_{net} = ma$$

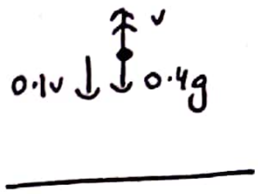
$$-(0.1v + 0.4g) = 0.4 \times \frac{dv}{dt}$$

$$-\left(\frac{v}{10} + 4\right) = 0.4\frac{dv}{dt}$$

$$\frac{-1}{0.4}\left(\frac{v+40}{10}\right) = \frac{dv}{dt}$$

$$-0.25(v+40) = \frac{dv}{dt}$$

(Shown)



ii) For maximum height, $\underline{v = 0}$.

$$\int_0^t -0.25\,dt = \int_{16}^{0} \frac{1}{v+40}\,dv$$

since when $t = 0$, $v = 16$

and when $t = t$, $v = 0$.

$$\left[-0.25t\right]_0^t = \left[\ln|v+40|\right]_{16}^{0}$$

$$-0.25t - (-0.25 \times 0) = \ln|40| - \ln|16+40|$$

$$-0.25t = \ln\left(\frac{40}{56}\right)$$

$$t = \frac{\ln(40/56)}{-0.25}$$

$$t = 1.345 \approx 1.35\,\mathrm{s}.$$

6 A particle starts from rest at O and travels in a straight line. Its acceleration is $(3 - 2x)\,\mathrm{m\,s^{-2}}$, where x m is the displacement of the particle from O.

(i) Find the value of x for which the velocity of the particle reaches its maximum value. [1]

(ii) Find this maximum velocity. [4]

i) For maximum velocity, $a = 0$

$$\Rightarrow 3-2x = 0$$

$$x = 1.5\,\text{m}.$$

ii)

$$a = 3-2x$$

$$v\frac{dv}{dx} = 3-2x$$

$$\int_0^v v\,dv = \int_0^{1.5} 3-2x\,dx$$

since when $x=0$, $v=0$
and when $x=1.5$, $v=v_{max}$?

$$\left[\frac{v^2}{2}\right]_0^v = \left[3x - x^2\right]_0^{1.5}$$

$$\frac{v^2}{2} - \frac{0^2}{2} = (3\times1.5 - 1.5^2) - (3\times0 - 0^2)$$

$$\frac{v^2}{2} = \frac{9}{4}$$

$$v^2 = \frac{9}{2}$$

$$v = \sqrt{\frac{9}{2}} = 2.121\,\mathrm{ms^{-1}} \approx 2.12\,\mathrm{ms^{-1}}$$

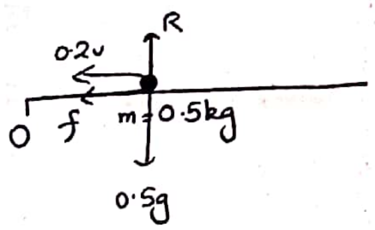
7 A particle P of mass 0.5 kg moves on a horizontal surface along the straight line OA, in the direction from O to A. The coefficient of friction between P and the surface is 0.08. Air resistance of magnitude $0.2v$ N opposes the motion, where v m s^{-1} is the speed of P at time t s. The particle passes through O with speed 4 m s^{-1} when $t = 0$.

(i) Show that $2.5\frac{dv}{dt} = -(v+2)$ and hence find the value of t when $v = 0$. [7]

(ii) Show that $\frac{dx}{dt} = 6e^{-0.4t} - 2$, where x m is the displacement of P from O at time t s, and hence find the distance OP when $v = 0$. [5]

i) $R(\uparrow)$: $R = 0.5g = 5\text{N}.$

$R(\rightarrow)$: $-(f + 0.2v) = ma$ using $F_{net} = ma$



$$-(0.08\times5 + 0.2v) = 0.5\frac{dv}{dt}$$

$$-0.4 - 0.2v = 0.5\frac{dv}{dt}$$

$$5(-0.4-0.2v) = 5(0.5)\frac{dv}{dt}$$

$$-2 - v = 2.5\frac{dv}{dt}$$

$$\therefore\ 2.5\frac{dv}{dt} = -(v+2)\ \text{(Shown)}.$$

$$\int_4^v \frac{1}{v+2}\,dv = \int_0^t \frac{-1}{2.5}\,dt$$

since when $t=0$, $v=4$
and when $t=t$, $v=v$

$$\Big[\ln|v+2|\Big]_4^v = \Big[-0.4t\Big]_0^t$$

$$\ln|v+2| - \ln|6| = -0.4t - 0$$

$$\ln\left|\frac{v+2}{6}\right| = -0.4t$$

$$\frac{v+2}{6} = e^{-0.4t}$$

$$v = 6e^{-0.4t} - 2$$

When $v = 0$,

$$0 = 6e^{-0.4t} - 2 \Rightarrow e^{-0.4t} = \frac{1}{3}$$

$$-0.4t = \ln\left(\frac{1}{3}\right)$$

$$t = \frac{\ln\left(\frac{1}{3}\right)}{-0.4} = 2.746 \approx 2.75\text{s}.$$

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ii)

$$v = \frac{dx}{dt} = 6e^{-0.4t} - 2.$$

~~When~~ $\int_0^x dx = \int_0^t 6e^{-0.4t} - 2 \, dt$

$$[x]_0^x = \left[\frac{6e^{-0.4t}}{-0.4} - 2t\right]_0^t$$

$$x - 0 = \left[-15e^{-0.4t} - 2t\right]_0^t$$

$$x = \left(-15e^{-0.4t} - 2t\right) - \left(-15e^{-0.4\times 0} - 2\times 0\right)$$

$$x = -15e^{-0.4t} - 2t + 15.$$

$\therefore$ When $v=0$, $t = 2.746s \Rightarrow x = -15e^{-0.4\times 2.746} - 2\times 2.746 + 15$

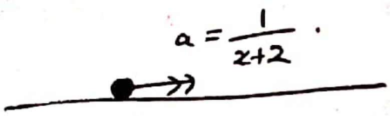
$$x = 4.506m \approx 4.51m.$$

8 A particle P starts from a fixed point O and moves in a straight line. When the displacement of P from O is x m, its velocity is $v\,\mathrm{m\,s^{-1}}$ and its acceleration is $\dfrac{1}{x+2}\,\mathrm{m\,s^{-2}}$.

(i) Given that $v = 2$ when $x = 0$, use integration to show that $v^2 = 2\ln\left(\frac{1}{2}x+1\right)+4$. [4]

(ii) Find the value of v when the acceleration of P is $\frac{1}{4}\,\mathrm{m\,s^{-2}}$. [2]

i)



$$a = v\frac{dv}{dx}$$

$$\int_0^x \frac{1}{x+2}\,dx = \int_2^v v\,dv$$

since when $x=0, v=2$
and when $x=x, v=v$

$$\Big[\ln|x+2|\Big]_0^x = \left[\frac{v^2}{2}\right]_2^v$$

$$\ln|x+2| - \ln 2 = \frac{v^2}{2} - \frac{2^2}{2}$$

$$\ln\left|\frac{x+2}{2}\right| = \frac{v^2}{2} - 2$$

$$2\ln\left|\frac{x+2}{2}\right| + 4 = v^2$$

$$\therefore\ v^2 = 2\ln\left|\frac{1}{2}x+1\right| + 4 \text{ (shown).}$$

ii)

$$a = \frac{1}{x+2} = \frac{1}{4}$$

$$\Rightarrow x+2 = 4$$

$$x = 2.$$

$$\therefore\ v^2 = 2\ln\left|\frac{1}{2}\times 2+1\right| + 4$$

$$v^2 = 2\ln|2| + 4$$

$$v = \sqrt{2\ln 2 + 4}$$

$$v = 2.320\,\mathrm{ms^{-1}} \approx 2.32\,\mathrm{ms^{-1}}.$$

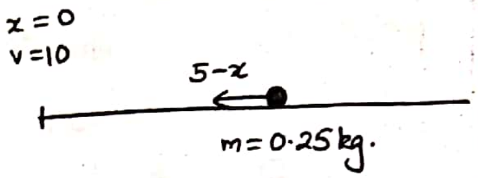
9 A particle P of mass 0.25 kg moves in a straight line on a smooth horizontal surface. P starts at the point O with speed $10\,\mathrm{m\,s^{-1}}$ and moves towards a fixed point A on the line. At time t s the displacement of P from O is x m and the velocity of P is $v\,\mathrm{m\,s^{-1}}$. A resistive force of magnitude $(5-x)$ N acts on P in the direction towards O.

(i) Form a differential equation in v and x. By solving this differential equation, show that $v = 10 - 2x$. [6]

(ii) Find x in terms of t, and hence show that the particle is always less than 5 m from O. [5]

i)

$$F = ma$$



$$-(5-x) = 0.25\, v\frac{dv}{dx}$$

$$\frac{-(5-x)}{0.25} = v\frac{dv}{dx}$$

$$4x - 20 = v\frac{dv}{dx}$$

$$\int_0^x 4x-20\;dx = \int_{10}^{v} v\,dv$$

since when $x=0, v=10$
and when $x=x, v=v$

$$\left[\frac{4x^2}{2} - 20x\right]_0^x = \left[\frac{v^2}{2}\right]_{10}^{v}$$

$$\left[2x^2 - 20x\right]_0^x = \left[\frac{v^2}{2}\right]_{10}^{v}$$

$$(2x^2 - 20x) - 0 = \frac{v^2}{2} - 50$$

$$2x^2 - 20x + 50 = \frac{v^2}{2}$$

$$4x^2 - 40x + 100 = v^2$$

$$(2x-10)^2 = v^2$$

$$\therefore v = \pm(2x - 10)$$

When $x = 0, v = 10$ $\therefore$ Use $v = -(2x-10)$ i.e. $v = 10 - 2x$.

ii)

$$v = \frac{dx}{dt} = 10 - 2x$$

$$\int_0^x \frac{1}{10-2x}\,dx = \int_0^t dt$$

since when $t=0, x=0$
and when $t=t, x=x$

$$\left[\frac{\ln|10-2x|}{-2}\right]_0^x = \left[t\right]_0^t$$

$$\frac{\ln|10-2x|}{-2} - \left[\frac{\ln|10-2(0)|}{-2}\right] = t - 0$$

$$\frac{\ln 10}{2} + \frac{\ln|10-2x|}{-2} = t$$

$$\Rightarrow \ln\left|\frac{10}{10-2x}\right| = +2t$$

$$\frac{10}{10-2x} = e^{+2t}$$

$$10e^{-2t} = 10-2x$$

$$2x = 10 - 10e^{-2t}$$

$$x = 5 - 5e^{-2t}$$

$$x = 5\left[1-e^{-2t}\right]$$

For $t > 0$, $\quad 0 < e^{-2t} < 1$

$$-1 < -e^{-2t} < 0$$

$$0 < 1-e^{-2t} < 1$$

$$0 < 5\left(1-e^{-2t}\right) < 5$$

$\therefore \quad 0 < x < 5$ for all $t > 0$

i.e. Particle is always less than 5m from O.

10 A particle P of mass 0.5 kg moves in a straight line on a smooth horizontal surface. At time t s, the displacement of P from a fixed point on the line is x m and the velocity of P is $v\,\mathrm{m\,s^{-1}}$. It is given that when $t = 0$, $x = 0$ and $v = 9$. The motion of P is opposed by a force of magnitude $3\sqrt{v}$ N.

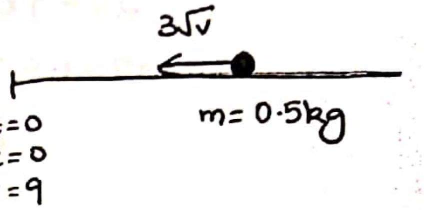
(i) By solving an appropriate differential equation, show that $v = (27 - 9x)^{\frac{2}{3}}$. [5]

(ii) Calculate the value of x when $t = 0.5$. [4]

i)

$$F_{net} = ma$$

$$-3\sqrt{v} = 0.5 \times v\frac{dv}{dx}$$



$$-6\sqrt{v} = v\frac{dv}{dx}$$

$$\int_0^x -6\,dx = \int_9^v \frac{v}{\sqrt{v}}\,dv$$

since when $x = 0, v = 9$
and when $x = x, v = v$.

$$\int_0^x -6\,dx = \int_9^v \sqrt{v}\,dv$$

$$\left[-6x\right]_0^x = \left[\frac{v^{3/2}}{3/2}\right]_9^v$$

$$-6x - (-6\times 0) = \frac{2v^{3/2}}{3} - \cancel{0} - \frac{2\times 9^{3/2}}{3}$$

$$\cancel{54}\ -6x = \frac{2v^{3/2}}{3} - 18$$

$$18 - 6x = \frac{2v^{3/2}}{3}$$

$$\frac{3(18-6x)}{2} = v^{3/2}$$

$$27 - 9x = v^{3/2}$$

$$\Rightarrow\ v = (27-9x)^{2/3}.$$

ii)

$$v = \frac{dx}{dt} = (27-9x)^{2/3}$$

$$\int_0^x (27-9x)^{-2/3}\,dx = \int_0^{0.5} dt$$

since when $x = 0, t = 0$
and when $x = ?, t = 0.5$

$$\left[\frac{(27-9x)^{\frac{1}{3}}}{\frac{1}{3}\times -9}\right]_0^x = \left[t\right]_0^{0.5}$$

$$\left[\frac{(27-9x)^{\frac{1}{3}}}{-3}\right]_0^x = 0.5-0$$

$$\frac{(27-9\times \cancel{x})^{\frac{1}{3}}-(27-9\times 0)^{\frac{1}{3}}}{-3} = 0.5$$

$$(27-9x)^{1/3} - 3 = -1.5$$

$$(27-9x)^{1/3} = 1.5$$

$$27-9x = 1.5^3$$

$$27-1.5^3 = 9x$$

$$x = \frac{27-1.5^3}{9}$$

$$x = 2.625\,\text{m.}$$

11 A particle P starts from rest at a point O and travels in a straight line. The acceleration of P is $(15-6x)\ \mathrm{m\,s^{-2}}$, where x m is the displacement of P from O.

(i) Find the value of x for which P reaches its maximum velocity, and calculate this maximum velocity. [5]

(ii) Calculate the acceleration of P when it is at instantaneous rest and $x > 0$. [2]

i) For maximum velocity, $a = 0$

$$\Rightarrow 15 - 6x = 0$$

$$x = \frac{15}{6} = 2.5\text{ m.}$$

$$a = 15 - 6x$$

$$v\frac{dv}{dx} = 15 - 6x$$

$$\int_0^v v\,dv = \int_0^x 15 - 6x\,dx$$

since when $x = 0, v = 0$ and when $x = x, v = v$

$$\left[\frac{v^2}{2}\right]_0^v = \left[15x - 3x^2\right]_0^x$$

$$\frac{v^2}{2} - \frac{0^2}{2} = \left(15x - 3x^2\right) - \left(15(0) - 3(0)^2\right)$$

$$\frac{v^2}{2} = 15x - 3x^2$$

$$v^2 = 30x - 6x^2$$

$$v = \sqrt{30x - 6x^2}.$$

When $x = 2.5$, $v_{max} = \sqrt{30(2.5) - 6\times 2.5^2} = 6.123 \approx 6.12\ \mathrm{ms^{-1}}$.

ii) When P is at instantaneous rest,

$$v = 0 \Rightarrow \sqrt{30x - 6x^2} = 0$$

$$6x(5 - x) = 0$$

$$\therefore x = 0 \text{ or } x = 5.$$

(Ignore)

$\therefore$ When $x = 5$, $a = 15 - 6(5) = -15\ \mathrm{ms^{-2}}$.

12 A particle P of mass 0.4 kg moves in a straight line on a horizontal surface and has velocity $v\,\mathrm{m\,s^{-1}}$ at time t s. A horizontal force of magnitude $k\sqrt{v}$ N opposes the motion of P. When $t = 0$, $v = 9$ and when $t = 2$, $v = 4$.

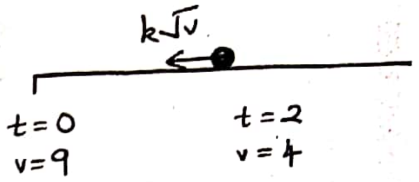
(i) Express $\frac{dv}{dt}$ in terms of k and v, and hence show that $v = \frac{1}{4}(t - 6)^2$. [5]

(ii) Find the distance travelled by P in the first 3 seconds of its motion. [4]

i)

$$F = ma$$

$$-k\sqrt{v} = 0.4\frac{dv}{dt}$$



$$\frac{-k\sqrt{v}}{0.4} = \frac{dv}{dt}$$

$$-2.5k\sqrt{v} = \frac{dv}{dt}$$

$$\int -2.5k\,dt = \int \frac{1}{\sqrt{v}}\,dv$$

$$-2.5kt + c = 2\sqrt{v}$$

When $t = 0$, $v = 9$

$$\Rightarrow \quad -2.5k(0) + c = 2\sqrt{9}$$

$$c = 6$$

$$\therefore \quad -2.5kt + 6 = 2\sqrt{v}$$

when $t = 2$, $v = 4$

$$\Rightarrow -2.5k(2) + 6 = 2\sqrt{4}$$

$$-5k + 6 = 4$$

$$k = \frac{-2}{-5} = 0.4$$

Then $\quad -2.5(0.4)t + 6 = 2\sqrt{v}$

$$6 - t = 2\sqrt{v}$$

$$(t-6)^2 = (-2\sqrt{v})^2$$

$$(t-6)^2 = 4v$$

$$\Rightarrow \quad v = \frac{1}{4}(t-6)^2.$$

ii)

$$v = \frac{dx}{dt} = \frac{1}{4}(t-6)^2$$

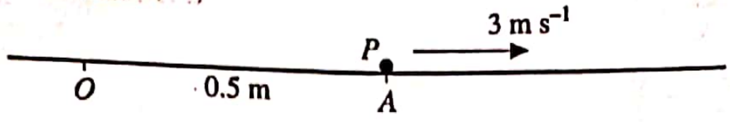
$$\Rightarrow \quad x = \int_0^3 \frac{1}{4}(t-6)^2 \, dt$$

$$x = \frac{1}{4}\left[\frac{(t-6)^3}{3}\right]_0^3$$

$$= \frac{1}{12}\left[(3-6)^3 - (0-6)^3\right]$$

$$= \frac{1}{12}\left[(-3)^3 - (-6)^3\right]$$

$$x = \frac{1}{12}\left[189\right] = 15.75$$



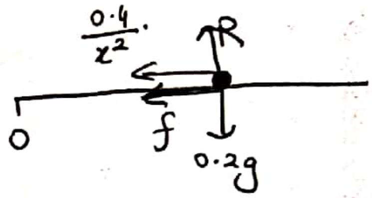
O and A are fixed points on a horizontal surface, with $OA = 0.5$ m. A particle P of mass 0.2 kg is projected horizontally with speed 3 m s^{-1} from A in the direction OA and moves in a straight line (see diagram). At time t s after projection, the velocity of P is v m s^{-1} and its displacement from O is x m. The coefficient of friction between the surface and P is 0.5, and a force of magnitude $\frac{0.4}{x^2}$ N acts on P in the direction PO.

(i) Show that, while the particle is in motion, $v\frac{dv}{dx} = -\left(5 + \frac{2}{x^2}\right)$. [2]

(ii) Calculate the distance travelled by P before it comes to rest, and show that P does not subsequently move. [7]

i)

$$F_{net} = ma$$



$$-\left(f + \frac{0.4}{x^2}\right) = 0.2\, v\frac{dv}{dx}$$

$$\frac{-\left(0.5 \times 0.2g + \frac{0.4}{x^2}\right)}{0.2} = v\frac{dv}{dx}$$

$$-\left(5 + \frac{2}{x^2}\right) = v\frac{dv}{dx}$$

(Shown)

ii)

$$\int_{0.5}^{x} -\left(5 + \frac{2}{x^2}\right) dx = \int_{3}^{0} v\, dv$$

Since when ~~x = 0~~ $x = 0.5$, $v = 3$
and when $x = x$, $v = 0$.

$$-\left[\left(5x - \frac{2}{x}\right)\right]_{0.5}^{x} = \left[\frac{v^2}{2}\right]_3^0$$

$$-\left[\left(5x - \frac{2}{x}\right) - \left(5 \times 0.5 - \frac{2}{0.5}\right)\right] = \frac{0}{2} - \frac{9}{2}$$

$$-\left[5x - \frac{2}{x} + \frac{3}{2}\right] = \frac{-9}{2}$$

$$-5x + \frac{2}{x} - \frac{3}{2} = \frac{-9}{2}$$

$$\frac{2}{x} - 5x + 3 = 0 \Rightarrow 2 - 5x^2 + 3x = 0$$

$$5x^2 - 3x - 2 = 0$$

$$5x^2 - 5x + 2x - 2 = 0$$

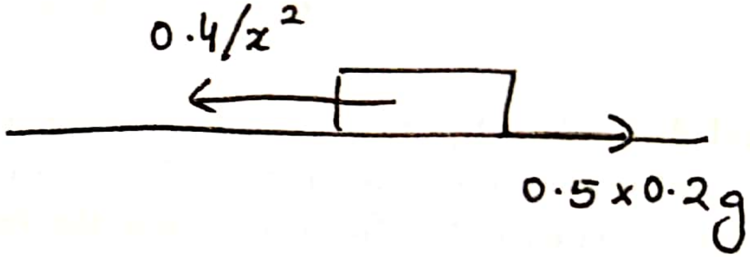
$$(5x + 2)(x - 1) = 0$$

$$x = \frac{-2}{5} \text{ (Ignore)}. \quad x = 1$$

$\therefore$ Distance travelled by P $= 1 - 0.5 = 0.5\,m$.

For motion after coming to rest at $x = 1\,m$,

$$F_{net} = \frac{0.4}{x^2} - f$$



But not $f_{lim} = 0.5 \times 0.2g = 1\,N$.

Therefore, to cause the block to start moving,

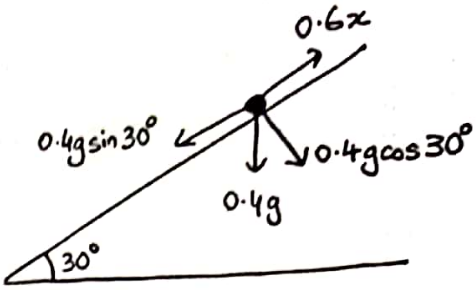
$$\frac{0.4}{x^2} > 1 \quad \text{but at } x = 1, \ \frac{0.4}{x^2} = \frac{0.4}{1^2} = 0.4\,N$$

$\therefore$ The force towards O is less than the maximum frictional force and so P does not subsequently move.

14 A particle P of mass 0.4 kg is released from rest at the top of a smooth plane inclined at 30° to the horizontal. The motion of P down the slope is opposed by a force of magnitude $0.6x$ N, where x m is the distance P has travelled down the slope. P comes to rest before reaching the foot of the slope. Calculate

(i) the greatest speed of P during its motion, [7]

(ii) the distance travelled by P during its motion. [2]



i)
$$F_{net} = ma$$
$$0.4g\sin 30° - 0.6x = 0.4\, v\frac{dv}{dx}$$
$$\frac{0.4g\sin 30° - 0.6x}{0.4} = v\frac{dv}{dx}$$
$$5 - 1.5x = v\frac{dv}{dx}$$
$$\int_0^x 5 - 1.5x\, dx = \int_0^v v\, dv$$ as when $x=0, v=0$ and when $x = x, v = v$.
$$\left[5x - \frac{1.5x^2}{2}\right]_0^x = \left[\frac{v^2}{2}\right]_0^v$$
$$\left(5x - \frac{1.5x^2}{2}\right) - \left(5\times0 - \frac{1.5\times0^2}{2}\right) = \frac{v^2}{2} - \frac{0^2}{2}$$
$$5x - 0.75x^2 = \frac{v^2}{2}$$
$$\boxed{10x - 1.5x^2 = v^2}$$

For greatest speed, $a = v\frac{dv}{dx} = 0 \Rightarrow 5 - 1.5x = 0$
$$x = \frac{5}{1.5} = \frac{10}{3}.$$

Then $v^2 = 10\left(\frac{10}{3}\right) - 1.5\left(\frac{10}{3}\right)^2$
$$v = \sqrt{\frac{100}{3} - \frac{150}{9}} = \sqrt{\frac{150}{9}} \approx 4.082\ ms^{-1} \approx 4.08 ms^{-1}.$$

ii) When $v = 0 \Rightarrow 10x - 1.5x^2 = 0^2$
$$x(10 - 1.5x) = 0$$
$x = 0$ or $10 - 1.5x = 0$
$$x = \frac{10}{1.5} = \frac{20}{3}$$

Therefore P comes to rest at the foot of the slope when $x = 6\frac{2}{3}$ m.
Then distance travelled by P during its motion $= 6\frac{2}{3}$ m.

15 A particle P of mass 0.25 kg moves in a straight line on a smooth horizontal surface. At time t s the velocity of P is $v\,\mathrm{m\,s^{-1}}$. A variable force of magnitude $3t$ N opposes the motion of P.

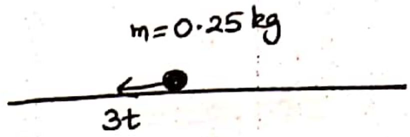
(i) Given that P comes to rest when $t = 3$, find v when $t = 0$. [4]

(ii) Calculate the distance travelled by P in the interval $0 \leqslant t \leqslant 3$. [3]

i)

$$F = ma$$

$$-3t = 0.25\frac{dv}{dt}$$



$$-12t = \frac{dv}{dt}$$

$$\int_3^t -12t\,dt = \int_0^v dv$$ as when $t=3, v=0$ and when $t=t, v=v$

$$\left[\frac{-12t^2}{2}\right]_3^t = \left[v\right]_0^v$$

$$\left[-6t^2\right]_3^t = v - 0$$

$$-6t^2 - (-6\times 3^2) = v$$

$$54 - 6t^2 = v$$

when $t = 0$, $v = 54 - 6(0)^2 = 54\ \mathrm{ms^{-1}}$

ii)

$$v = \frac{dx}{dt} = 54 - 6t^2.$$

$$x = \int_0^3 54 - 6t^2\,dt$$

$$x = \left[54t - \frac{6t^3}{3}\right]_0^3$$

$$= \left[54t - 2t^3\right]_0^3$$

$$= (54\times 3 - 2\times 3^3) - (54\times 0 - 2\times 0^3)$$

$$x = 108\,\mathrm{m}.$$

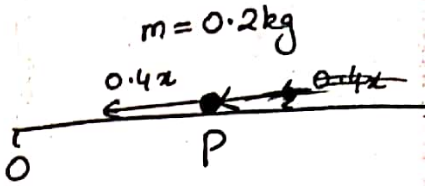
$\therefore$ Distance travelled in interval $0 \leq t \leq 3 = 108\,\mathrm{m}.$

16 A particle P of mass 0.2 kg is projected horizontally from a fixed point O, and moves in a straight line on a smooth horizontal surface. A force of magnitude $0.4x$ N acts on P in the direction PO, where x m is the displacement of P from O.

(i) Given that P comes to instantaneous rest when $x = 2.5$, find the initial kinetic energy of P. [4]

(ii) Find the value of x on the first occasion when the speed of P is $2\,\mathrm{m\,s^{-1}}$. [2]

i) $F = ma$



$$-0.4x = 0.2\, v\frac{dv}{dx}$$

$$-2x = v\frac{dv}{dx}$$

$$\int_0^{2.5} -2x\, dx = \int_v^0 v\, dv$$

as when $x = 2.5$, $v = 0$
and when $x = 0$, $v = ?$

$$\left[\frac{-2x^2}{2}\right]_0^{2.5} = \left[\frac{v^2}{2}\right]_v^0$$

$$\left[-x^2\right]_0^{2.5} = \frac{0^2}{2} - \frac{v^2}{2}$$

$$(-2.5^2) - 0^2 = \frac{-v^2}{2}$$

$$-2.5^2 = \frac{-v^2}{2}$$

$$v = \sqrt{12.5}$$

$\therefore$ Initial kinetic energy $= \frac{1}{2} \times 0.2 \times \left(\sqrt{12.5}\right)^2 = 1.25\,\mathrm{J}$.

ii)

$$-2x = v\frac{dv}{dx}$$

$$\int_{2.5}^{x} -2x\, dx = \int_0^2 v\, dv$$

since when $x = 2.5$, $v = 0$
and when $x = x$, $v = 2$

$$\left[-x^2\right]_{2.5}^{x} = \left[\frac{v^2}{2}\right]_0^2$$

$$-x^2 - (-2.5^2) = \frac{2^2}{2} - \frac{0^2}{2}$$

$$2.5^2 - x^2 = 2$$

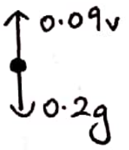
$$x = \sqrt{2.5^2 - 2} = 2.061\,\mathrm{m} \approx 2.06\,\mathrm{m}.$$

17 A particle of mass 0.2 kg is projected vertically downwards with initial speed $4\,\text{m s}^{-1}$. A resisting force of magnitude $0.09v$ N acts vertically upwards on the particle during its descent, where $v\,\text{m s}^{-1}$ is the downwards velocity of the particle at time t s after being set in motion.

(i) Show that the acceleration of the particle is $(10 - 0.45v)\,\text{m s}^{-2}$. [1]

(ii) Find v when $t = 1.5$. [5]

i)



$$F_{net} = ma$$

$$0.2g - 0.09v = 0.2a$$

$$\frac{0.2g - 0.09v}{0.2} = a$$

$$\Rightarrow a = 10 - 0.45v.$$

ii)

$$\frac{dv}{dt} = 10 - 0.45v$$

$$\int_4^v \frac{1}{10-0.45v}\,dv = \int_0^{1.5} dt$$

as when $t=0$, $v=4$
and when $t=1.5$, $v=?$

$$\left[\frac{\ln|10-0.45v|}{-0.45}\right]_4^v = [t]_0^{1.5}$$

$$\frac{1}{-0.45}\left[\ln|10-0.45v| - \ln|8.2|\right] = 1.5 - 0$$

$$\ln\left|\frac{10-0.45v}{8.2}\right| = -0.675$$

$$\frac{10-0.45v}{8.2} = e^{-0.675}$$

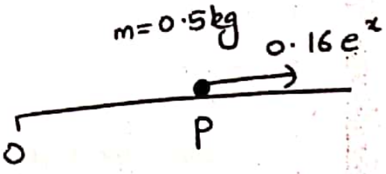
$$v = \frac{8.2e^{-0.675} - 10}{-0.45}$$

$$v = 12.94\ \text{ms}^{-1} \approx 12.9\ \text{ms}^{-1}.$$

18 A particle P of mass 0.5 kg moves in a straight line on a smooth horizontal surface. The velocity of P is $v\,\mathrm{m\,s^{-1}}$ when the displacement of P from O is x m. A single horizontal force of magnitude $0.16e^{x}$ N acts on P in the direction OP. The velocity of P when it is at O is $0.8\,\mathrm{m\,s^{-1}}$.

(i) Show that $v = 0.8e^{\frac{1}{2}x}$. [6]

(ii) Find the time taken by P to travel 1.4 m from O. [4]



i)

$$F = ma$$

$$0.16e^{x} = 0.5\,v\frac{dv}{dx}$$

$$\int_0^x \frac{0.16e^{x}}{0.5}\,dx = \int_{0.8}^{v} v\,dv$$

as when $x=0, v=0.8$
and when $x=x, v=v$.

$$\left[0.32e^{x}\right]_0^x = \left[\frac{v^2}{2}\right]_{0.8}^{v}$$

$$0.32e^{x} - 0.32e^{0} = \frac{v^2}{2} - \frac{0.8^2}{2}$$

$$0.32e^{x} - 0.32 = \frac{v^2}{2} - 0.32$$

$$\Rightarrow 0.32e^{x} = \frac{v^2}{2}$$

$$0.64e^{x} = v^2$$

$$\Rightarrow v = 0.8e^{\frac{x}{2}}.$$

ii)

$$v = \frac{dx}{dt} = 0.8e^{x/2}$$

$$\Rightarrow \int_0^{1.4} e^{-x/2}\,dx = \int_0^t 0.8\,dt$$

as when $t=0, x=0$
and when $t=?, x=1.4$

$$\left[\frac{e^{-x/2}}{-1/2}\right]_0^{1.4} = \left[0.8t\right]_0^t$$

$$\left[-2e^{-x/2}\right]_0^{1.4} = 0.8t$$

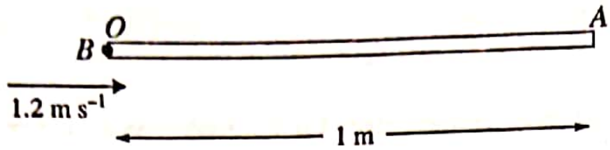
$$-2e^{-1.4/2} - \left(-2e^{-0/2}\right) = 0.8t$$

$$\frac{-2e^{-0.7} + 2}{0.8} = t$$

$$\therefore t = 1.258 \approx 1.26\,\mathrm{s}.$$

19 A small ball B of mass 0.2 kg moves in a narrow fixed smooth cylindrical tube OA of length 1 m, closed at the end A. When the ball has displacement x m from O, it has velocity $v\,\mathrm{m\,s^{-1}}$ in the direction OA and experiences a resisting force of magnitude $\frac{k}{1-x}$ N.

(i)



The tube is fixed in a horizontal position and B is projected from O towards A with velocity $1.2\,\mathrm{m\,s^{-1}}$ (see diagram). Given that B comes to instantaneous rest after travelling 0.55 m, show that $k = 0.1803$, correct to 4 significant figures. [6]

(ii) The tube is now fixed in a vertical position with O above A. The ball B is released from rest at O. Calculate the speed of B after it has descended 0.1 m. [4]

i)

$$F = ma$$

$$\frac{-k}{1-x} = 0.2\, v \frac{dv}{dx}$$

$$\int_0^{0.55} \frac{-k}{0.2(1-x)}\, dx = \int_{1.2}^{0} v\, dv$$

as when $x = 0$, $v = 1.2\,\mathrm{ms^{-1}}$
and $x = 0.55$, $v = 0$

$$\left[-5k \times \frac{\ln(1-x)}{-1}\right]_0^{0.55} = \left[\frac{v^2}{2}\right]_{1.2}^{0}$$

$$5k\left[\ln(1-x)\right]_0^{0.55} = \frac{0^2}{2} - \frac{1.2^2}{2}$$

$$5k\left[\ln(1-0.55) - \ln(1)\right] = -0.72$$

$$k = \frac{-0.72}{5\ln(0.45)} = 0.18033 \approx 0.1803.$$

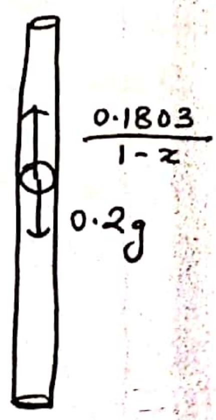
ii)

$$F_{net} = ma$$

$$0.2g - \frac{0.1803}{1-x} = 0.2\, v\frac{dv}{dx}$$

$$\frac{0.2g}{0.2} - \frac{0.1803}{0.2(1-x)} = v\frac{dv}{dx}$$

$$10 - \frac{0.9017}{1-x} = v\frac{dv}{dx}$$



$$\int_0^{0.1} 10 - \frac{0.9017}{1-x}\, dx = \int_0^{v} v\, dv$$

as when $x = 0$, $v = 0$
and when $x = 0.1$, $v = ?$

$$\left[10x - \frac{0.9017 \ln|1-x|}{(-1)}\right]_0^{0.1} = \left[\frac{v^2}{2}\right]_0^2$$

$$\left[10x + 0.9017 \ln|1-x|\right]_0^{0.1} = \frac{v^2}{2} - \frac{0^2}{2}$$

$$\Rightarrow \left[10 \times 0.1 + 0.9017 \ln|1-0.1|\right] - \left[10 \times 0 + 0.9017 \times \ln(1)\right] = \frac{v^2}{2}$$

$$0.9049 = \frac{v^2}{2}$$

$$\Rightarrow v = 1.345\,ms^{-1}$$

$$v \approx 1.35\,ms^{-1}$$

20 A particle P of mass 0.6 kg is released from rest at a point above ground level and falls vertically. The motion of P is opposed by a force of magnitude $3v$ N, where $v\,\mathrm{m\,s^{-1}}$ is the speed of P. Immediately before P reaches the ground, $v = 1.95$.

(i) Calculate the time after its release when P reaches the ground. [5]

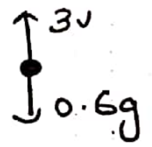
P is now projected horizontally with speed $1.95\,\mathrm{m\,s^{-1}}$ across a smooth horizontal surface. The motion of P is again opposed by a force of magnitude $3v$ N, where $v\,\mathrm{m\,s^{-1}}$ is the speed of P.

(ii) Calculate the distance P travels after projection before coming to rest. [3]

i)

$$F_{net} = ma$$

$$0.6g - 3v = 0.6 \times \frac{dv}{dt}$$



$$\frac{0.6g - 3v}{0.6} = \frac{dv}{dt}$$

$$10 - 5v = \frac{dv}{dt}$$

$$\Rightarrow \int_0^{t} dt = \int_0^{1.95} \frac{1}{10-5v}\,dv$$

as when $t = 0, v = 0$
and when $t = t, v = 1.95$.

$$[t]_0^t = \left[\frac{\ln|10-5v|}{-5}\right]_0^{1.95}$$

$$t - 0 = \frac{\ln|10 - 5\times 1.95|}{-5} - \frac{\ln|10-5(0)|}{-5}$$

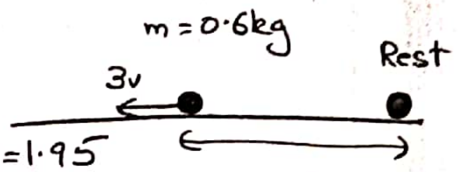
$$t = \frac{\ln 10 - \ln 0.25}{5} = \frac{\ln 40}{5} = 0.7377\text{ s}$$

$$\therefore t = 0.738\text{ s.}$$

II)

$$F = ma$$

$$-3v = 0.6\, v \frac{dv}{dx}$$



$$\int_0^{x} dx = \int_{1.95}^{0} \frac{0.6v}{-3v}\,dv$$

as when $x = 0, v = 1.95$
and when $x = ?, v = 0$ (so at rest)

$$\int_0^{x} dx = \int_{1.95}^{0} -0.2\,dv$$

$$x - 0 = -0.2[v]_{1.95}^{0}$$

$$x = -0.2[0 - 1.95] = 0.39\text{ m.}$$

21 A small block B of mass 0.2 kg is placed at a fixed point O on a smooth horizontal surface. A horizontal force of magnitude 0.42 N is applied to B. At time t s after the force is first applied, the velocity of B away from O is v m s^{-1}.

(i) Find the value of v when $t = 1$. [2]

For $t > 1$ an additional force, of magnitude $0.32t$ N and directed towards O, is applied to B. The force of magnitude 0.42 N continues to act as before.

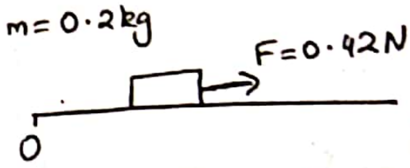
(ii) Find the value of v when $t = 2$. [3]

For $t > 2$ a third force, of magnitude $0.06t^2$ N and directed away from O, is applied to B. The other two forces continue to act as before.

(iii) Show that the velocity of B is the same when $t = 2$ and when $t = 3$. [3]

i)

$$F = ma$$
$$0.42 = 0.2 \times a$$
$$2.1 \text{ ms}^{-2} = a$$



Using $\quad v = u + at$

$$v = 0 + 2.1(1) = 2.1 \text{ ms}^{-1}.$$

ii)

$$F_{net} = ma$$

$$0.42 - 0.32t = 0.2\frac{dv}{dt}$$

$$\frac{0.42 - 0.32t}{0.2} = \frac{dv}{dt}$$

$$2.1 - 1.6t = \frac{dv}{dt}$$

$$\therefore \int_1^2 2.1 - 1.6t \; dt = \int_{2.1}^{v} dv$$

as when $t = 1, v = 2.1$
and when $t = 2, v = ?$

$$\left[2.1t - \frac{1.6t^2}{2}\right]_1^2 = v - 2.1$$

$$\left[2.1t - 0.8t^2\right]_1^2 = v - 2.1$$

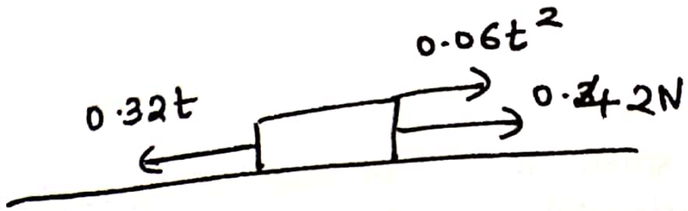
$$\left[2.1(2) - 0.8(2)^2\right] - \left[2.1(1) - 0.8 \times 1^2\right] = v - 2.1$$

$$-0.3 = v - 2.1$$

$$\Rightarrow v = 1.8 \text{ ms}^{-1}.$$

iii)

$$F_{net} = ma$$



$$0.06t^2 + 0.42 - 0.32t = 0.2\frac{dv}{dt}$$

$$0.3t^2 + 2.1 - 1.6t = \frac{dv}{dt}.$$

$$\int_2^3 0.3t^2 + 2.1 - 1.6t \, dt = \int_{1.8}^{v} dv$$

as when $t = 2, v = 1.8$
and when $t = 3, v = ?$

$$\left[\frac{0.3t^3}{3} + 2.1t - \frac{1.6t^2}{2}\right]_2^3 = \left[v\right]_{1.8}^{v}$$

$$\left[0.1t^3 + 2.1t - 0.8t^2\right]_2^3 = v - 1.8$$

$$(0.1\times3^3 + 2.1\times3 - 0.8\times3^2) - (0.1\times2^3 + 2.1\times2 - 0.8\times2^2) = v - 1.8$$

$$0 = v - 1.8$$

$$\Rightarrow v = 1.8\,ms^{-1}.$$

The velocity when $t = 3s$ $= 1.8\,ms^{-1}$ = Velocity when $t = 2s$.

22 A particle P of mass 0.1 kg moves with decreasing speed in a straight line on a smooth horizontal surface. A horizontal resisting force of magnitude $0.2e^{-x}$ N acts on P, where x m is the displacement of P from a fixed point O on the line. The velocity of P is v m s^{-1} when its displacement from O is x m.

(i) Show that

$$v\frac{dv}{dx} = ke^{-x},$$

where k is a constant to be found. [2]

P passes through O with velocity 2.2 m s^{-1}.

(ii) Calculate the value of x at the instant when the velocity of P is 2 m s^{-1}. [4]

(iii) Show that the speed of P does not fall below 0.917 m s^{-1}, correct to 3 significant figures. [2]

i)

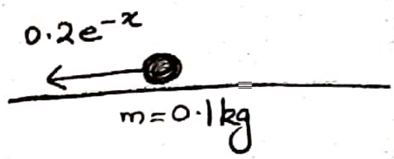
$$F = ma$$

$$-0.2e^{-x} = 0.1\, v\frac{dv}{dx}$$

$$\frac{-0.2e^{-x}}{0.1} = v\frac{dv}{dx}$$

$$-2e^{-x} = v\frac{dv}{dx}$$

$$\therefore k = -2.$$



ii)

$$\int_0^x -2e^{-x}\,dx = \int_{2.2}^{2} v\,dv$$

sinch when $x=0, v=2.2$
as when $x=?, v=2$.

$$\left[\frac{-2e^{-x}}{-1}\right]_0^x = \left[\frac{v^2}{2}\right]_{2.2}^{2}$$

$$2e^{-x} - 2e^{-0} = \frac{2^2}{2} - \frac{2.2^2}{2}$$

$$2e^{-x} - 2 = -0.42$$

$$e^{-x} = 0.79$$

$$x = -\ln 0.79$$

$$x = 0.2357 \approx 0.236.$$

iii)

$$\int_0^x -2e^{-x}\,dx = \int_{2.2}^{v} v\,dv$$

since when $x=0, v=2.2$
and when $x=x, v=v$.

$$\left[2e^{-x}\right]_0^x = \left[\frac{v^2}{2}\right]_{2.2}^{v}$$

$$2e^{-x} - 2e^{-0} = \frac{v^2}{2} - \frac{2.2^2}{2}$$

$$2e^{-x} - 2 = \frac{v^2}{2} - 2.42$$

$$2e^{-x} + 0.42 = \frac{v^2}{2}$$

$$\therefore \quad v^2 = 4e^{-x} + 0.84$$

As $x \mapsto \infty$, $e^{-x} \mapsto 0$ $\therefore$ $4e^{-x} + 0.84 \mapsto 0.84$

$$\therefore \quad v^2 = 0.84$$

$$v = \sqrt{0.84}$$

$$v = 0.9165 \approx 0.917\,ms^{-1}.$$

23 A force of magnitude $0.4t$ N, applied at an angle of 30° above the horizontal, acts on a particle P, where t s is the time since the force starts to act. P is at rest on rough horizontal ground when $t = 0$. The mass of P is 0.2 kg and the coefficient of friction between P and the ground is μ.

(i) Given that P is about to slip when $t = 2$, find μ and the value of t for the instant when P loses contact with the ground. [5]

(ii) While P is moving on the ground, it has velocity v m s^{-1} at time t s. Show that

$$\frac{dv}{dt} = 2.165t - 4.330,$$

where the coefficients are correct to 4 significant figures. [3]

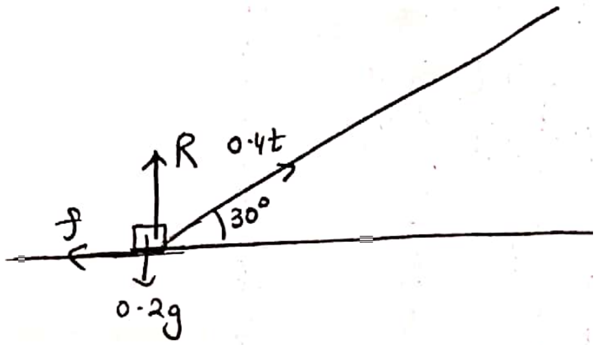
(iii) Calculate the speed of P when it loses contact with the ground. [4]

i) $R(\uparrow)$: When $t = 2$,

$$0.2g = R + 0.4(2)\sin 30^\circ$$

$$2 = R + 0.4$$

$$\Rightarrow R = 1.6$$



$R(\rightarrow)$: $f = 0.4 \times 2 \times \cos 30^\circ$

$$f = \frac{2\sqrt{3}}{5}.$$

$$\therefore \mu = \frac{f}{R} = \frac{\frac{2\sqrt{3}}{5}}{1.6} = 0.4330 \approx \underline{0.433}.$$

When contact is lost with the ground, $R = 0$

$R(\uparrow)$: $\therefore 0.4t \sin 30^\circ = 0.2g$

$$t = \frac{0.2g}{0.4 \sin 30^\circ} = 10\text{s}.$$

ii) $R(\uparrow)$: $\cancel{F = ma}$

$$R + 0.4t \sin 30^\circ = 0.2g$$

$$R = 0.2g - 0.4t \sin 30^\circ$$

$$R = 0.2g - 0.2t$$

$$R = 2 - 0.2t$$

$$\therefore f = \mu R = 0.4330(2 - 0.2t)$$

Then $R(\rightarrow)$: $F = ma$

$$0.4t\cos 30^\circ - f = 0.2 \times \frac{dv}{dt}$$

$$\frac{0.4t\cos 30^\circ - 0.4330(2 - 0.2t)}{0.2} = \frac{dv}{dt}$$

$$\Rightarrow \frac{dv}{dt} = 2.165t - 4.330.$$

(Shown).

iii)

$$\int_0^v dv = \int_2^{10} 2.165t - \cancel{0}\ 4.330\ dt$$

since when $t = 2, v = 0$

and when $t = 10, v = ?$

$$[v]_0^v = \left[\frac{2.165t^2}{2} - 4.330t\right]_2^{10}$$

$$v - 0 = \left(\frac{2.165 \times 10^2}{2} - 4.330 \times 10\right) - \left(\frac{2.165 \times 2^2}{2} - 4.330 \times 2\right)$$

$$v = 69.28\ ms^{-1} \approx 69.3\ ms^{-1}.$$

24 A cyclist and her bicycle have a total mass of 60 kg. The cyclist rides in a horizontal straight line, and exerts a constant force in the direction of motion of 150 N. The motion is opposed by a resistance of magnitude $12v$ N, where $v\,\mathrm{m\,s^{-1}}$ is the cyclist's speed at time t s after passing through a fixed point A.

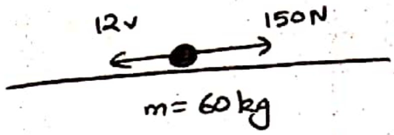
(i) Show that $5\dfrac{dv}{dt} = 12.5 - v$. [2]

(ii) Given that the cyclist passes through A with speed $11.5\,\mathrm{m\,s^{-1}}$, solve this differential equation to show that $v = 12.5 - e^{-0.2t}$. [4]

(iii) Express the displacement of the cyclist from A in terms of t. [3]

i)

$$F_{net} = ma$$



$$150 - 12v = 60\frac{dv}{dt}$$

$$12(12.5 - v) = 60\frac{dv}{dt}$$

$$12.5 - v = 5\frac{dv}{dt} \quad \text{(Shown)}.$$

ii)

$$\int_0^t \frac{1}{5}\,dt = \int_{11.5}^{v} \frac{1}{12.5-v}\,dv$$

since when $t = 0$, $v = 11.5$
and when $t = t$, $v = v$.

$$\left[\frac{t}{5}\right]_0^t = \left[\frac{\ln|12.5-v|}{-1}\right]_{11.5}^{v}$$

$$\frac{t}{5} - \frac{0}{5} = -\ln|12.5-v| - [-\ln|12.5-11.5|]$$

$$\frac{t}{5} = -\ln|12.5-v| + \ln|1|$$

$$\frac{-t}{5} = \ln|12.5-v|$$

$$e^{-0.2t} = 12.5 - v$$

$$v = 12.5 - e^{-0.2t}.$$

iii)

$$\frac{dx}{dt} = 12.5 - e^{-0.2t}.$$

$$\int_0^x dx = \int_0^t 12.5 - e^{-0.2t}\,dt$$

as when $t = 0$, $x = 0$
and when $t = t$, $x = x$.

$$[x]_0^x = \left[12.5t - \frac{e^{-0.2t}}{-0.2}\right]_0^t$$

$$x - 0 = \left[12.5t + 5e^{-0.2t}\right]_0^t$$

$$x = \left(12.5t + 5e^{-0.2t}\right) - \left(12.5 \times 0 + 5e^{-0.2 \times 0}\right)$$

$$x = 12.5t + 5e^{-0.2t} - 5.$$

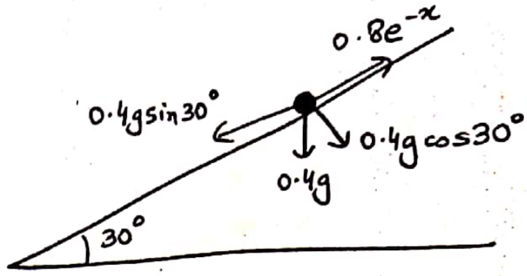
25 A particle P of mass 0.4 kg is released from rest at a point O on a smooth plane inclined at 30° to the horizontal. When the displacement of P from O is x m down the plane, the velocity of P is $v\,\mathrm{m\,s^{-1}}$. A force of magnitude $0.8e^{-x}$ N acts on P up the plane along the line of greatest slope through O.

(i) Show that $v\dfrac{dv}{dx} = 5 - 2e^{-x}$. [2]

(ii) Find v when $x = 0.6$. [4]

i)

$$F_{net} = ma$$



$$0.4g\sin 30^\circ - 0.8e^{-x} = 0.4\,v\frac{dv}{dx}$$

$$2 - 0.8e^{-x} = 0.4\,v\frac{dv}{dx}$$

$$\frac{2-0.8e^{-x}}{0.4} = v\frac{dv}{dx}$$

$$5 - 2e^{-x} = v\frac{dv}{dx}$$

ii)

$$\int_0^{0.6} 5 - 2e^{-x}\,dx = \int_0^{v} v\,dv$$

since when $x = 0$, $v = 0$
and when $x = 0.6$, $v = ?$

$$\left[5x - \frac{2e^{-x}}{-1}\right]_0^{0.6} = \left[\frac{v^2}{2}\right]_0^{v}$$

$$\left(5(0.6) + 2e^{-0.6}\right) - \left(5\times 0 + 2e^{-0}\right) = \frac{v^2}{2} - \frac{0^2}{2}$$

$$3 + \frac{2}{e^{0.6}} - 2 = \frac{v^2}{2}$$

$$2\left(1 + \frac{2}{e^{0.6}}\right) = v^2$$

$$\sqrt{2\left(1 + \frac{2}{e^{0.6}}\right)} = v$$

$$\therefore\ v = 2.048\,\mathrm{ms^{-1}} \simeq 2.05\,\mathrm{ms^{-1}}.$$

26 A particle P of mass 0.4 kg is placed at rest at a point A on a rough horizontal surface. A horizontal force, directed away from A and with magnitude $0.6t$ N, acts on P, where t s is the time after P is placed at A. The coefficient of friction between P and the surface is 0.3, and P has displacement from A of x m at time t s.

(i) Show that P starts to move when $t = 2$. Show also that when P is in motion it has acceleration $(1.5t - 3)\,\mathrm{m\,s^{-2}}$. [3]

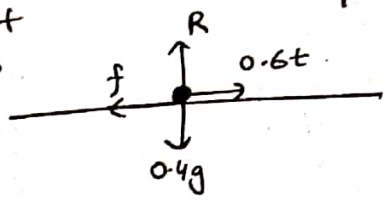
(ii) Express the velocity of P in terms of t, for $t \geqslant 2$. [4]

(iii) Express x in terms of t, for $t \geqslant 2$. [3]

i) $f = \mu R = 0.3 \times 0.4g = 1.2.$

$\mu = 0.3.$

For P to start moving, resultant force should be at least zero



$\therefore\ 0.6t = 1.2$

$t = \frac{1.2}{0.6} = 2s.$

For $t \geqslant 2$,

$F_{net} = ma$

$0.6t - 1.2 = 0.4a$

$\frac{0.6t - 1.2}{0.4} = a$

$1.5t - 3 = a$

(Shown).

ii) $a = \frac{dv}{dt} = 1.5t - 3$

$\int_0^v dv = \int_2^t 1.5t - 3\ dt$ Since when $t = 2, v = 0$ and when $t = t, v = v$.

$v - 0 = \left[\frac{1.5t^2}{2} - 3t\right]_2^t$

$v = (0.75t^2 - 3t) - (0.75 \times 2^2 - 3 \times 2)$

$v = 0.75t^2 - 3t - (-3)$

$v = 0.75t^2 - 3t + 3.$

iii) $v = \frac{dx}{dt} = 0.75t^2 - 3t + 3$

$\int_0^x dx = \int_2^t 0.75t^2 - 3t + 3\ dt$ Since when $t = 2, x = 0$ and when $t = t, x = x$.

$(x - 0) = \left[\frac{0.75t^3}{3} - \frac{3t^2}{2} + 3t\right]_2^t$

$$x = \left[0.25t^3 - \frac{3t^2}{2} + 3t\right]_2^t$$

$$x = \left(0.25t^3 - \frac{3t^2}{2} + 3t\right) - \left(0.25 \times 2^3 - \frac{3 \times 2^2}{2} + 3 \times 2\right)$$

$$x = 0.25t^3 - \frac{3t^2}{2} + 3t - (2)$$

$$\therefore \quad x = 0.25t^3 - \frac{3t^2}{2} + 3t - 2.$$

$$\therefore \quad x = 0.25t^3 - 1.5t^2 + 3t - 2.$$

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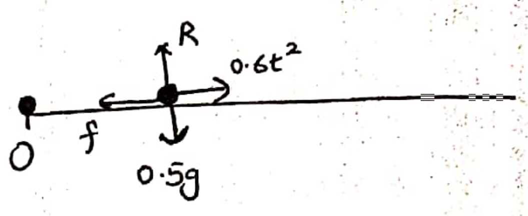
27 A particle P of mass 0.5 kg is at rest at a point O on a rough horizontal surface. At time $t = 0$, where t is in seconds, a horizontal force acting in a fixed direction is applied to P. At time t s the magnitude of the force is $0.6t^2$ N and the velocity of P away from O is $v\,\mathrm{m\,s^{-1}}$. It is given that P remains at rest at O until $t = 0.5$.

(i) Calculate the coefficient of friction between P and the surface, and show that

$$\frac{dv}{dt} = 1.2t^2 - 0.3 \quad \text{for } t > 0.5. \qquad [3]$$

At $t = 0.5$s, friction is limiting:



$R(\rightarrow)$: $\quad f = 0.6t^2$

$\mu R = 0.6 \times 0.5^2$

$\mu(0.5 \times g) = 0.6 \times 0.5^2$

$\mu = \dfrac{0.6 \times 0.5^2}{0.5 \times g} = 0.03.$

For $t > 0.5$, $\quad F_{net} = ma$

$0.6t^2 - f = 0.5 \times \dfrac{dv}{dt}$

$\dfrac{0.6t^2 - 0.03 \times 5}{0.5} = \dfrac{dv}{dt}$

$1.2t^2 - 0.3 = \dfrac{dv}{dt}$

(ii) Express v in terms of t for $t > 0.5$. [3]

$\displaystyle\int_{0.5}^{t} 1.2t^2 - 0.3\, dt = \int_0^v dv$ $\quad$ since when $t = 0.5$, $v = 0$ and when $t = t$, $v = v$.

$\left[\dfrac{1.2t^3}{3} - 0.3t\right]_{0.5}^{t} = v - 0$

$(0.4t^3 - 0.3t) - (0.4 \times 0.5^3 - 0.3 \times 0.5) = v$

$0.4t^3 - 0.3t - (-0.1) = v$

$\therefore \quad \underline{v = 0.4t^3 - 0.3t + 0.1}.$

(iii) Find the displacement of P from O when $t = 1.2$. [3]

$$v = \frac{dx}{dt} = 0.4t^3 - 0.3t + 0.1$$

$$\Rightarrow \int_0^x dx = \int_{0.5}^{1.2} 0.4t^3 - 0.3t + 0.1 \, dt$$

as when $t = 0.5, v = 0$

and when $t = 1.2, v = ?$

$$x - 0 = \left[\frac{0.4t^4}{4} - \frac{0.3t^2}{2} + 0.1t\right]_{0.5}^{1.2}$$

$$x = \left[0.1t^4 - 0.15t^2 + 0.1t\right]_{0.5}^{1.2}$$

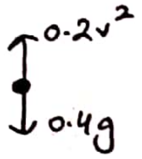
$$x = (0.1 \times 1.2^4 - 0.15 \times 1.2^2 + 0.1 \times 1.2) - (0.1 \times 0.5^4 - 0.15 \times 0.5^2 + 0.1 \times 0.5)$$

$$x = 0.09261 \approx 0.0926 \text{ m.}$$

28 A small object of mass 0.4 kg is released from rest at a point 8 m above the ground. The object descends vertically and when its downwards displacement from its initial position is x m the object has velocity $v\,\mathrm{m\,s^{-1}}$. While the object is moving, a force of magnitude $0.2v^2$ N opposes the motion.

(i) Show that $v\dfrac{dv}{dx} = 10 - 0.5v^2$. [2]

$$F_{net} = ma$$



$$0.4g - 0.2v^2 = 0.4\, v\frac{dv}{dx}$$

$$\frac{4 - 0.2v^2}{0.4} = v\frac{dv}{dx}$$

$$10 - 0.5v^2 = v\frac{dv}{dx}$$

(ii) Express v in terms of x. [4]

$$\int_0^x dx = \int_0^v \frac{v}{10-0.5v^2}\,dv$$

since when $x = 0, v = 0$
and when $x = x, v = v$

$$\int_0^x dx = -\int_0^v \frac{-v}{10-0.5v^2}\,dv$$

$$x - 0 = -\left[\ln|10-0.5v^2|\right]_0^v$$

$$x = -\left[\ln|10-0.5v^2| - \ln|10-0.5\times 0^2|\right]$$

$$x = -\left[\ln|10-0.5v^2| - \ln(10)\right]$$

$$x = \ln\left|\frac{10}{10-0.5v^2}\right|$$

$$e^x = \frac{10}{10-0.5v^2}$$

$$10 - 0.5v^2 = 10e^{-x}$$

$$10 - 10e^{-x} = 0.5v^2$$

$$10\left(1-e^{-x}\right) = 0.5v^2$$

$$\frac{10\left(1-e^{-x}\right)}{0.5} = v^2$$

$$20\left(1-e^{-x}\right) = v^2$$

$$\therefore \quad v = \sqrt{20\left(1-e^{-x}\right)}.$$

(iii) Find the increase in the value of v during the final 4 m of the descent of the object. [2]

Increase in value of v

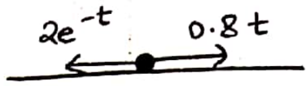
$$= v(8) - v(4)$$

$$= \sqrt{20\left(1-e^{-8}\right)} - \sqrt{20\left(1-e^{-4}\right)}$$

$$= 0.0404 \text{ ms}^{-1}.$$

29 A particle P of mass 0.4 kg is projected horizontally along a smooth horizontal plane from a point O. At time t s after projection the velocity of P is $v\,\mathrm{m\,s^{-1}}$. A force of magnitude $0.8t$ N directed away from O acts on P and a force of magnitude $2e^{-t}$ N opposes the motion of P.

(i) Show that $\frac{dv}{dt} = 2t - 5e^{-t}$. [2]



$$F_{net} = ma$$

$$0.8t - 2e^{-t} = 0.4\frac{dv}{dt}$$

$$\frac{0.8t - 2e^{-t}}{0.4} = \frac{dv}{dt}$$

$$\Rightarrow \frac{dv}{dt} = 2t - 5e^{-t}.$$

(ii) Given that $v = 8$ when $t = 1$, express v in terms of t. [3]

$$\int_8^v dv = \int_1^t 2t - 5e^{-t}\,dt$$ since when $t = 1, v = 8$ and when $t = t, v = v$.

$$v - 8 = \left(\frac{2t^2}{2} - \frac{5e^{-t}}{-1}\right) - \left(\frac{2\times 1^2}{2} - \frac{5e^{-1}}{-1}\right)$$

$$v - 8 = \frac{2t^2}{2} - \frac{5e^{-t}}{-1} - \left(\frac{2}{2} + \frac{5}{e}\right) = t^2 + \frac{5}{e^t} - 1 - \frac{5}{e}$$

$$v = \frac{2t^2}{2} + \frac{5}{e^t} + \cancel{7 -} \frac{5}{e}.$$

$$v = t^2 + \frac{5}{e^t} + 7 - \frac{5}{e}$$

(iii) Find the speed of projection of P. [2]

When $t=0$,

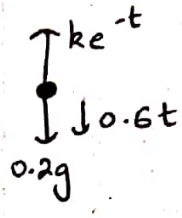
$$v = 0^2 + \frac{5}{e^0} + 7 - \frac{5}{e} = 10 \cdot 16 \, ms^{-1} \approx 10 \cdot 2 \, ms^{-1}.$$

30 A particle P of mass 0.2 kg is released from rest at a point O above horizontal ground. At time t s after its release the velocity of P is v m s^{-1} downwards. A vertically downwards force of magnitude $0.6t$ N acts on P. A vertically upwards force of magnitude ke^{-t} N, where k is a constant, also acts on P.

(i) Show that $\frac{dv}{dt} = 10 - 5ke^{-t} + 3t$. [2]

$F_{net} = ma$

$0.2g + 0.6t - ke^{-t} = 0.2\frac{dv}{dt}$



$\frac{0.2g + 0.6t - ke^{-t}}{0.2} = \frac{dv}{dt}$

$\Rightarrow \frac{dv}{dt} = 10 - 3t - 5ke^{-t}.$

(ii) Find the greatest value of k for which P does not initially move upwards. [3]

For P to not move upwards initially,

$\frac{dv}{dt} = 0$ when $t = 0$

$\therefore \quad 10 - 3(0) - 5ke^{-0} = 0$

$10 - 5k = 0$

$10 = 5k$

$\Rightarrow k = 2.$

(iii) Given that $k = 1$, and that P strikes the ground when $t = 2$, find the height of O above the ground. [5]

$$\frac{dv}{dt} = 10 + 3t - 5(1)e^{-t}$$

$$\frac{dv}{dt} = 10 + 3t - 5e^{-t}$$

$$\int_0^v dv = \int_0^t 10 + 3t - 5e^{-t}$$

since when $t = 0, v = 0$
and when $t = t, v = t$

$$[v]_0^v = \left[10t + \frac{3t^2}{2} + 5e^{-t}\right]_0^t$$

$$v - 0 = \left(10t + \frac{3t^2}{2} + 5e^{-t}\right) - \left(10 \times 0 + \frac{3 \times 0^2}{2} + 5e^{-0}\right)$$

$$v = 10t + \frac{3t^2}{2} + 5e^{-t} - 5.$$

$$\frac{dx}{dt} = 10t + \frac{3t^2}{2} + 5e^{-t} - 5$$

$$\int_0^x dx = \int_0^2 10t + \frac{3t^2}{2} + 5e^{-t} - 5\,dt$$

since when $t = 0, x = 0$
and when $t = 2, x = ?$

$$[x]_0^x = \left[\frac{10t^2}{2} + \frac{3t^3}{6} - 5e^{-t} - 5t\right]_0^2$$

$$x - 0 = \left[5t^2 + \frac{t^3}{2} - 5e^{-t} - 5t\right]_0^2$$

$$x = \left[5(2)^2 + \frac{2^3}{2} - 5e^{-2} - 5(2)\right] - \left[5 \times 0^2 + \frac{0^3}{2} - 5e^{-0} - 5(0)\right]$$

$$x = 18.32 \approx 18.3\text{m}.$$

31 A particle P is moving along a straight line with acceleration $3ku - kv$ where v is its velocity at time t, u is its initial velocity and k is a constant. The velocity and acceleration of P are both in the direction of increasing displacement from the initial position.

(a) Find the time taken for P to achieve a velocity of $2u$. [3]

$$a = 3ku - kv$$

$$\frac{dv}{dt} = 3ku - kv$$

$$\int_u^{2u} \frac{1}{3ku - kv}\, dv = \int_0^t dt$$

Since when $t=0$, $v=u$.
and when $t=t$, $v=2u$

$$\left[\frac{\ln|3ku-kv|}{-k}\right]_u^{2u} = (t-0)$$

$$\frac{-1}{k}\left[\ln|3ku-2ku| - \ln|3ku-ku|\right] = t$$

$$\frac{-1}{k}\left[\ln(ku) - \ln(2ku)\right] = t$$

$$\frac{-1}{k}\ln\left(\frac{ku}{2ku}\right) = t$$

$$\frac{-1}{k}\ln\left(\frac{1}{2}\right) = t$$

$$\frac{-1}{k}(-\ln 2) = t$$

$$\therefore t = \frac{\ln 2}{k}.$$

(b) Find an expression for the displacement of P from its initial position when its velocity is $2u$. [5]

$$a = 3ku - kv$$

$$v\frac{dv}{dx} = 3ku - kv$$

$$\int \frac{v}{3ku-kv}\,dv = \int dx$$

$$\int \frac{v}{k(3u-v)}\,dv = \int dx$$

$$\frac{1}{k}\int \frac{v}{3u-v}\,dv = \int dx.$$

$$\frac{1}{k}\int -1+\frac{3u}{3u-v}\,dv = \int dx$$

since when $x=0, v=u$
and when $x=x, v=2u$

$$\begin{array}{r|l} & -1 \\ \hline -v+3u & v \\ & v-3u \\ & -\quad + \\ \hline & 3u \end{array}$$

$$\frac{1}{k}\int_u^{2u} -1+\frac{3u}{3u-v}\,dv = \int_0^x dx$$

$$\frac{1}{k}\left[-v + \frac{3u\ln|3u-v|}{-1}\right]_u^{2u} = x - 0$$

$$\frac{1}{k}\left[-v - 3u\ln|3u-v|\right]_u^{2u} = x$$

$$\therefore\ x = \frac{1}{k}\left[\left(-2u-3u\ln|3u-2u|\right)-\left(-u-3u\ln|3u-u|\right)\right]$$

$$= \frac{1}{k}\left[\left(-2u-3u\ln u\right)-\left(-u-3u\ln 2u\right)\right]$$

$$= \frac{1}{k}\left[-2u-3u\ln u+u+3u\ln 2u\right]$$

$$= \frac{1}{k}\left[-2u+u+3u\ln\left(\frac{2u}{u}\right)\right]$$

$$x = \frac{1}{k}\left[-u+3u\ln 2\right] = \frac{u}{k}\left[3\ln 2-1\right]$$

32 A particle Q of mass m kg falls from rest under gravity. The motion of Q is resisted by a force of magnitude mkv N, where $v\,\mathrm{m\,s^{-1}}$ is the speed of Q at time t s and k is a positive constant.

Find an expression for v in terms of g, k and t. [6]

$$F_{net} = ma$$

$$mg - mkv = m\frac{dv}{dt}$$

$$g - kv = \frac{dv}{dt}$$

$$\int_0^t dt = \int_0^v \frac{1}{g-kv}\,dv$$

since when $t=0, v=0$
and when $t=t, v=v$

$$\left[t\right]_0^t = \left[\frac{\ln|g-kv|}{-k}\right]_0^v$$

$$t - 0 = \frac{-1}{k}\left[\ln|g-kv| - \ln|g-k(0)|\right]$$

$$t = \frac{-1}{k}\ln\left|\frac{g-kv}{g}\right|$$

$$-kt = \ln\left|\frac{g-kv}{g}\right|$$

$$e^{-kt} = \frac{g-kv}{g}$$

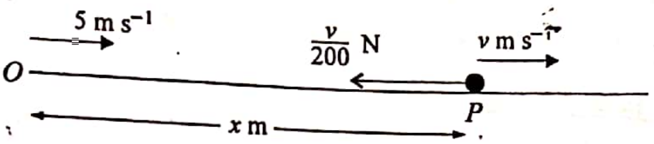
$$ge^{-kt} = g - kv$$

$$kv = g - ge^{-kt}$$

$$kv = g(1-e^{-kt})$$

$$v = \frac{g}{k}(1-e^{-kt}).$$

33



A particle P of mass $\frac{1}{10}$ kg travels in a straight line on a smooth horizontal surface. It passes through the fixed point O with velocity $5\,\mathrm{m\,s^{-1}}$ at time $t = 0$. After t seconds its displacement from O is x m and its velocity is $v\,\mathrm{m\,s^{-1}}$. P is subject to a single force of magnitude $\frac{v}{200}$ N which acts in a direction opposite to the motion (see diagram).

(i) Find an expression for v in terms of x. [4]

(ii) Find an expression for x in terms of t. [5]

(iii) Show that the value of x is less than 100 for all values of t. [1]

i)

$$F = ma$$

$$\frac{-v}{200} = \frac{1}{10} v \frac{dv}{dx}$$

$$\frac{-10v}{200v} = \frac{dv}{dx}$$

$$\frac{-1}{20} = \frac{dv}{dx}$$

$$\int_0^x \frac{-1}{20}\,dx = \int_5^v dv$$

since when $x = 0$, $v = 5$
and when $x = x$, $v = v$

$$\left[\frac{-x}{20}\right]_0^x = \left[v\right]_5^v$$

$$\frac{-x}{20} - \left(\frac{-0}{20}\right) = v - 5$$

$$\frac{-x}{20} = v - 5$$

$$\therefore\ v = 5 - \frac{x}{20}.$$

ii)

$$v = \frac{dx}{dt} = 5 - \frac{x}{20}$$

$$\frac{dx}{dt} = \frac{100 - x}{20}$$

$$\int_0^x \frac{1}{100 - x}\,dx = \int_0^t \frac{1}{20}\,dt$$

since when $t = 0$, $x = 0$
and when $t = t$, $x = x$

$$\left[\frac{\ln|100 - x|}{-1}\right]_0^x = \left[\frac{t}{20}\right]_0^t$$

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$$\left[-\ln|100-x|\right]-\left[-\ln|100-0|\right] = \frac{t}{20} - \frac{0}{20}$$

$$\ln(100) - \ln|100-x| = \frac{t}{20}$$

$$\ln\left|\frac{100}{100-x}\right| = \frac{t}{20}$$

$$\therefore \frac{100}{100-x} = e^{t/20}$$

$$100e^{-t/20} = 100-x$$

$$x = 100 - 100e^{-t/20}$$

$$x = 100\left(1-e^{-t/20}\right)$$

vi) As $t \mapsto \infty$, $e^{-t/20} \mapsto 0$ and for $t > 0$

$$\therefore \quad 0 < e^{-t/20} \leq 1$$

$$0 > -e^{-t/20} \geq -1$$

$$1 > 1-e^{-t/20} \geq 0$$

$$100 > 100\left(1-e^{-t/20}\right) \geq 0$$

Then $\quad 0 \leq x < 100$

$\therefore$ Value of x is less than 100 for all values of x.

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34 A car of mass 1000 kg is moving on a straight horizontal road. The driving force of the car is $\frac{28\,000}{v}$ N and the resistance to motion is $4v$ N, where v m s^{-1} is the speed of the car t seconds after it passes a fixed point on the road.

(i) Show that $\frac{dv}{dt} = \frac{7000 - v^2}{250v}$. [2]

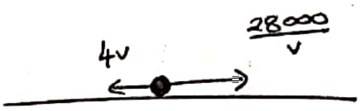
The car passes points A and B with speeds 10 m s^{-1} and 40 m s^{-1} respectively.

(ii) Find the time taken for the car to travel from A to B. [4]

i)

$$F_{net} = ma$$

$$\frac{28000}{v} - 4v = 1000\frac{dv}{dt}$$



$$\frac{28000 - 4v^2}{1000v} = \frac{dv}{dt}$$

Then $\frac{dv}{dt} = \frac{4(7000-v^2)}{1000v} = \frac{7000-v^2}{250v}$

(Shown).

ii) Let, time when car travels through A be t_A, time when car travels through B be t_B.

$$\Rightarrow \int \frac{250v}{7000-v^2}\,dv = \int dt$$

$$-125\int_{10}^{40} \frac{-2v}{7000-v^2}\,dv = \int_{t_A}^{t_B} dt$$

Since when $t = t_A$, $v = 10$

when $t = t_B$, $v = 40$

$$-125\left[\ln|7000-v^2|\right]_{10}^{40} = \left[t\right]_{t_A}^{t_B}$$

$$-125\left[\ln(7000-40^2) - \ln(7000-10^2)\right] = t_B - t_A$$

$$30.64\text{s} = t_B - t_A$$

$\therefore$ Time taken to travel from A to B ≈ 30.6s.

35 A particle of mass 0.25 kg moves in a straight line on a smooth horizontal surface. A variable resisting force acts on the particle. At time t s the displacement of the particle from a point on the line is x m, and its velocity is $(8 - 2x)$ m s^{-1}. It is given that $x = 0$ when $t = 0$.

(i) Find the acceleration of the particle in terms of x, and hence find the magnitude of the resisting force when $x = 1$. [3]

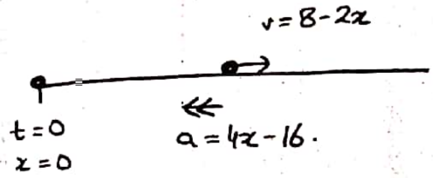
(ii) Find an expression for x in terms of t. [6]

(iii) Show that the particle is always less than 4 m from its initial position. [2]

i) $a = v\dfrac{dv}{dx}$

$= (8-2x)(-2)$

$a = 4x - 16$



When $x = 1$, $a = 4(1) - 16 = -12\,ms^{-2}$.

$\therefore$ $F = ma$

$F = 0.25(-12) = -3N$

Then resisting force has magnitude 3N.

ii) $v = \dfrac{dx}{dt} = 8 - 2x$

$\displaystyle\int_0^x \frac{1}{8-2x}\,dx = \int_0^t dt$ since when $t=0$, $x=0$ and when $t=t$, $x=x$.

$\left[\dfrac{\ln|8-2x|}{-2}\right]_0^x = t - 0$

$\dfrac{\ln|8-2x|}{-2} - \dfrac{\ln|8-2(0)|}{-2} = t$

$-\dfrac{\ln|8-2x|}{2} + \dfrac{\ln 8}{2} = t$

$\ln\left|\dfrac{8}{8-2x}\right| = 2t$

$\dfrac{8}{8-2x} = e^{2t}$

$8e^{-2t} = 8 - 2x$

$2x = 8(1-e^{-2t})$

$x = 4(1-e^{-2t})$.

iii) As $t \to \infty$, $e^{-2t} \to 0$ $\therefore$ $1 - e^{-2t} \to 1 \Rightarrow 4(1-e^{-2t}) \to 4$

i.e. $x \to 4$

Then $0 \le x < 4$.

$\therefore$ Particle is always less than 4m from its initial position.

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36 A cyclist starts from rest at a point O and travels along a straight path. At time t s after starting, the displacement of the cyclist from O is x m, and the acceleration of the cyclist is a m s^{-2}, where $a = 0.6x^{0.2}$.

(i) Find an expression for the velocity v m s^{-1} of the cyclist in terms of x. [4]

(ii) Show that $t = 2.5x^{0.4}$. [3]

(iii) Find the distance travelled by the cyclist in the first 10 s of the journey. [2]

i)
$$a = 0.6x^{0.2}$$

$$\therefore v\frac{dv}{dx} = 0.6x^{0.2}$$

$$\int_0^v v\,dv = \int_0^x 0.6x^{0.2}\,dx$$

Since when $x=0$, $v=0$
and when $x=x$, $v=v$

$$\left[\frac{v^2}{2}\right]_0^v = \left[\frac{0.6x^{1.2}}{1.2}\right]_0^x$$

$$\frac{v^2}{2} - \frac{0^2}{2} = \frac{0.6x^{1.2}}{1.2} - \frac{0.6(0)^{1.2}}{1.2}$$

$$\frac{v^2}{2} = \frac{x^{1.2}}{2}$$

$$v^2 = x^{1.2}$$

$$v = x^{0.6}$$

ii)
$$\frac{dx}{dt} = x^{0.6}$$

$$\int_0^x x^{-0.6}\,dx = \int_0^t dt$$

Since when $t=0$, $x=0$
and when $t=t$, $x=x$.

$$\left[\frac{x^{0.4}}{0.4}\right]_0^x = \left[t\right]_0^t$$

$$\frac{x^{0.4}}{0.4} - 0 = t - 0$$

$$\frac{5x^{0.4}}{2} = t$$

$$\Rightarrow t = 2.5x^{0.4}.$$

iii) When $t = 10$,
$$10 = 2.5x^{0.4}$$
$$4 = x^{0.4}$$
$$4 = x^{\frac{2}{5}}$$
$$\Rightarrow x = 4^{\frac{5}{2}} = 32\text{ m}.$$

37 A particle of mass 0.4 kg is released from rest and falls vertically. A resisting force of magnitude $0.08v$ N acts upwards on the particle during its descent, where v m s^{-1} is the velocity of the particle at time t s after its release.

(i) Show that the acceleration of the particle is $(10 - 0.2v)$ m s^{-2}. [2]

(ii) Find the velocity of the particle when $t = 15$. [5]

i)

$$F_{net} = ma$$

$$0.4g - 0.08v = 0.4a$$

$$\frac{4 - 0.08v}{0.4} = a$$

$$10 - 0.2v = a$$

ii)

$$a = \frac{dv}{dt} = 10 - 0.2v$$

$$\int_0^v \frac{1}{10 - 0.2v}\,dv = \int_0^{15} dt$$

Since when $t = 0, v = 0$

and when $t = 15, v = ?$

$$\left[\frac{\ln|10 - 0.2v|}{-0.2}\right]_0^v = [t]_0^{15}$$

$$\frac{\ln|10 - 0.2v|}{-0.2} - \left[\frac{\ln|10 - 0.2(0)|}{-0.2}\right] = 15 - 0$$

$$\frac{\ln|10 - 0.2v|}{-0.2} + \frac{\ln 10}{0.2} = 15$$

$$\frac{\ln\left|\frac{10}{10 - 0.2v}\right|}{0.2} = 15$$

$$\ln \frac{10}{10 - 0.2v} = 3$$

$$\frac{10}{10 - 0.2v} = e^3$$

$$10e^{-3} = 10 - 0.2v$$

$$0.2v = 10 - 10e^{-3}$$

$$v = \frac{10 - 10e^{-3}}{0.2}$$

$$v = 47.51 \text{ ms}^{-1} \approx 47.5 \text{ ms}^{-1}.$$

38 A particle P of mass 0.5 kg moves along the x-axis on a horizontal surface. When the displacement of P from the origin O is x m the velocity of P is $v\,\mathrm{m\,s^{-1}}$ in the positive x-direction. Two horizontal forces act on P; one force has magnitude $(1 + 0.3x^2)$ N and acts in the positive x-direction, and the other force has magnitude $8e^{-x}$ N and acts in the negative x-direction.

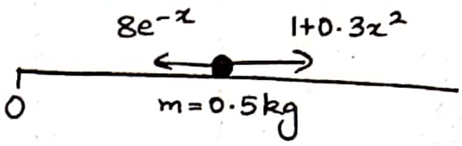
(i) Show that $v\dfrac{dv}{dx} = 2 + 0.6x^2 - 16e^{-x}$. [2]

(ii) The velocity of P as it passes through O is $6\,\mathrm{m\,s^{-1}}$. Find the velocity of P when $x = 3$. [5]

i)

$$F_{net} = ma$$

$$1+0.3x^2-8e^{-x} = 0.5\, v\frac{dv}{dx}$$



$$\frac{(1+0.3x^2-8e^{-x})}{0.5} = v\frac{dv}{dx}$$

$$2+0.6x^2-16e^{-x} = v\frac{dv}{dx}.$$

(Shown).

ii)

$$\int_0^3 2+0.6x^2-16e^{-x}\,dx = \int_6^v v\,dv$$

since when $x=0, v = 6\,\mathrm{ms^{-1}}$
and when $x = 3, v = ?$

$$\left[2x+\frac{0.6x^3}{3}-\frac{16e^{-x}}{-1}\right]_0^3 = \left[\frac{v^2}{2}\right]_6^v$$

$$\left[2x+0.2x^3+16e^{-x}\right]_0^3 = \frac{v^2}{2}-18$$

$$\left(2(3)+0.2\times3^3+16e^{-3}\right)-\left(2\times0+0.2\times0^3+16e^{-0}\right) = \frac{v^2}{2}-18$$

$$5.328 = v$$

$$\Rightarrow v \approx 5.33\,\mathrm{ms^{-1}}.$$

39 A particle P of mass 0.1 kg is projected vertically upwards from a point O with speed $20\,\mathrm{m\,s^{-1}}$. Air resistance of magnitude $0.1v$ N opposes the motion, where $v\,\mathrm{m\,s^{-1}}$ is the speed of P at time t s after projection.

(i) Show that, while P is moving upwards, $\dfrac{1}{v+10}\dfrac{dv}{dt} = -1$. [2]

(ii) Hence find an expression for v in terms of t, and explain why it is valid only for $0 \leqslant t \leqslant \ln 3$. [6]

(iii) Find the initial acceleration of P. [2]

i)

$$F_{net} = ma$$

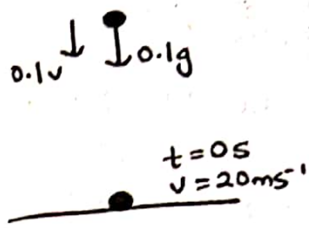
$$-(0.1g + 0.1v) = 0.1\frac{dv}{dt}$$

$$\frac{-0.1g - 0.1v}{0.1} = \frac{dv}{dt}$$

$$-10 - v = \frac{dv}{dt}$$

$$-(v+10) = \frac{dv}{dt}$$

$$-1 = \frac{1}{v+10}\frac{dv}{dt}$$



ii)

since when $t = 0$, $v = 20\,\mathrm{ms^{-1}}$;
$t = t$, $v = v$.

$$\int_0^t -1\,dt = \int_{20}^{v} \frac{1}{v+10}\,dv$$

$$\left[-t\right]_0^t = \left[\ln|v+10|\right]_{20}^{v}$$

$$-t - (-0) = \ln|v+10| - \ln 30$$

$$-t = \ln\left|\frac{v+10}{30}\right|$$

$$e^{-t} = \frac{v+10}{30}$$

$$\therefore\ v + 10 = 30e^{-t}$$

$$v = 30e^{-t} - 10.$$

This expression is valid only for upward motion. When particle reaches greatest height, $v = 0$

$$\Rightarrow 30e^{-t} - 10 = 0 \quad \Rightarrow e^{-t} = \frac{1}{3}$$

$$-t = \ln\left(\frac{1}{3}\right)$$

$$-t = -\ln 3$$

$$t = \ln 3.$$

$$\therefore\ v = 30e^{-t} - 10 \text{ for } 0 \leq t \leq \ln 3.$$

iii)

$$\frac{1}{v+10}\frac{dv}{dt} = -1$$

$$\therefore \frac{dv}{dt} = -(v+10)$$

$$\text{i.e. } a = -(v+10)$$

Initially, $v = 20$ $\therefore a = -(20+10) = -30 \text{ ms}^{-2}$.

40 A particle P of mass 0.3 kg is projected vertically upwards from the ground with an initial speed of $20\,\text{m s}^{-1}$. When P is at height x m above the ground, its upward speed is $v\,\text{m s}^{-1}$. It is given that

$$3v - 90\ln(v+30) + x = A,$$

where A is a constant.

(i) Differentiate this equation with respect to x and hence show that the acceleration of the particle is $-\frac{1}{3}(v+30)\,\text{m s}^{-2}$. [3]

(ii) Find, in terms of v, the resisting force acting on the particle. [2]

(iii) Find the time taken for P to reach its maximum height. [5]

i)

$$3v - 90\ln(v+30) + x = A$$

$$\frac{d}{dx}\left[3v - 90\ln(v+30) + x\right] = \frac{d}{dx}[A]$$

$$3\frac{dv}{dx} - 90\left[\frac{1}{v+30}\right]\frac{dv}{dx} + 1 = 0$$

$$\frac{dv}{dx}\left(3 - \frac{90}{v+30}\right) = -1$$

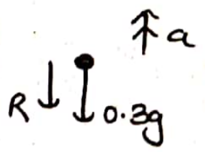
$$\frac{dv}{dx}\left[\frac{3v+90-90}{v+30}\right] = -1$$

$$\Rightarrow \frac{dv}{dx} = \frac{-(v+30)}{3v}$$

$$\therefore a = v\frac{dv}{dx} = v \times \frac{(-(v+30))}{3v} = \frac{-(v+30)}{3}.$$

i.e. $a = \frac{-1}{3}(v+30)$.

ii)



$$F_{net} = ma$$

$$-(R + 0.3g) = 0.3 \times \frac{-1}{3}(v+30)$$

$$R + 0.3g = 0.1(v+30)$$

$$R + 3 = 0.1v + 3$$

$$R = 0.1v.$$

iii)

$$a = \frac{dv}{dt} = \frac{-1}{3}(v+30).$$

$$\int_{20}^{0} \frac{1}{v+30}\,dv = \int_0^T \frac{-1}{3}\,dt$$

when $t = 0$, $v = 20$

when $t = T$, $v = 0$

$$\left[\ln|v+30|\right]_{20}^{0} = \frac{-1}{3}\left[t\right]_0^T$$

Since at maximum height, velocity $= 0\,\text{ms}^{-1}$.

$$\ln 30 - \ln 50 = \frac{-1}{3}[T]$$

$$\ln\left(\frac{50}{30}\right) = \frac{T}{3}$$

$$\therefore T = 3\ln\left(\frac{5}{3}\right) = 1.532 \approx 1.53s.$$

i.e. Time taken to reach maximum height $= 1.53s$.

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41 A cyclist and his bicycle have a total mass of 81 kg. The cyclist starts from rest and rides in a straight line. The cyclist exerts a constant force of 135 N and the motion is opposed by a resistance of magnitude $9v$ N, where $v\,\mathrm{m\,s^{-1}}$ is the cyclist's speed at time t s after starting.

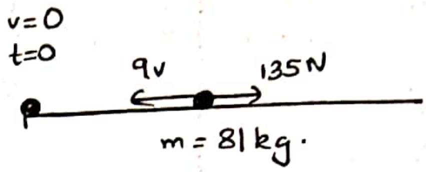
(i) Show that $\frac{9}{15-v}\frac{dv}{dt} = 1$. [2]

(ii) Solve this differential equation to show that $v = 15(1 - e^{-\frac{1}{9}t})$. [4]

(iii) Find the distance travelled by the cyclist in the first 9 s of the motion. [4]

i)

$$F_{net} = ma$$



$$135 - 9v = 81\frac{dv}{dt}$$

$$\frac{135-9v}{81} = \frac{dv}{dt}$$

$$\frac{15-v}{9} = \frac{dv}{dt}$$

$$1 = \frac{9}{15-v}\frac{dv}{dt} \text{ (Shown).}$$

ii)

$$\int_0^t \frac{1}{9}\,dt = \int_0^v \frac{1}{15-v}\,dv$$

since when $t=0$, $v=0$
and let when $t=t$, $v=v$.

$$\left[\frac{t}{9}\right]_0^t = \left[\frac{\ln|15-v|}{-1}\right]_0^v$$

$$\frac{t}{9} - \frac{0}{9} = -\ln|15-v| - (-\ln 15)$$

$$\frac{t}{9} = -\ln(15-v) + \ln 15$$

$$\frac{t}{9} = \ln\left(\frac{15}{15-v}\right)$$

$$\therefore e^{t/9} = \frac{15}{15-v}$$

$$15 - v = 15e^{-t/9}$$

$$15 - 15e^{-t/9} = v$$

$$\therefore v = 15(1-e^{-t/9}).$$

(Shown).

iii)

$$\frac{dx}{dt} = 15\left(1 - e^{-t/9}\right)$$

$$\int_0^X \frac{1}{15}\,dx = \int_0^9 1 - e^{-t/9}\,dt$$

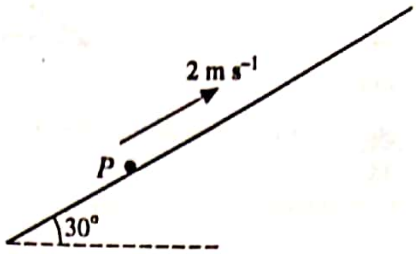
since when $t = 0$, $x = 0$
and when $t = 9$, $x = X$

$$\left[\frac{x}{15}\right]_0^X = \left[t - \frac{e^{-t/9}}{-1/9}\right]_0^9$$

$$\frac{X}{15} - \frac{0}{15} = \left(9 + 9e^{-9/9}\right) - \left(0 + 9e^{-0/9}\right)$$

$$\frac{X}{15} = 9 + \frac{9}{e} - 9$$

$$X = 49.66 \approx 49.7\,\text{m}.$$

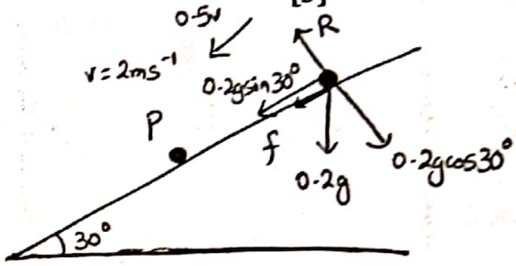


A particle P of mass 0.2 kg is projected with velocity 2 m s^{-1} upwards along a line of greatest slope on a plane inclined at 30° to the horizontal (see diagram). Air resistance of magnitude $0.5v$ N opposes the motion of P, where $v\text{ m s}^{-1}$ is the velocity of P at time t s after projection. The coefficient of friction between P and the plane is $\frac{1}{2\sqrt{3}}$. The particle P reaches a position of instantaneous rest when $t = T$.

(i) Show that, while P is moving up the plane, $\frac{dv}{dt} = -2.5(3 + v)$. [3]

(ii) Calculate T. [4]

(iii) Calculate the speed of P when $t = 2T$. [5]



i) R (⊥ to plane): $R = 0.2g\cos 30° = \sqrt{3}$.

$f = \mu R = \frac{1}{2\sqrt{3}} \times \sqrt{3} = \frac{1}{2}$.

R (∥ to plane): $-(f + 0.2g\sin 30° + 0.5v) = ma$

$-\left(\frac{1}{2} + 1 + 0.5v\right) = 0.2\frac{dv}{dt}$

$-\left(\frac{3}{2} + 0.5v\right) = 0.2\frac{dv}{dt}$

$\frac{-1.5}{0.2} - \frac{0.5v}{0.2} = \frac{dv}{dt}$

$-7.5 - 2.5v = \frac{dv}{dt}$

$-2.5(3+v) = \frac{dv}{dt}$

ii) $\int_0^T -2.5\, dt = \int_2^0 \frac{1}{3+v}\, dv$ Since when $t=0, v=2$ and when $t=T, v=0$.

$[-2.5t]_0^T = [\ln|3+v|]_2^0$

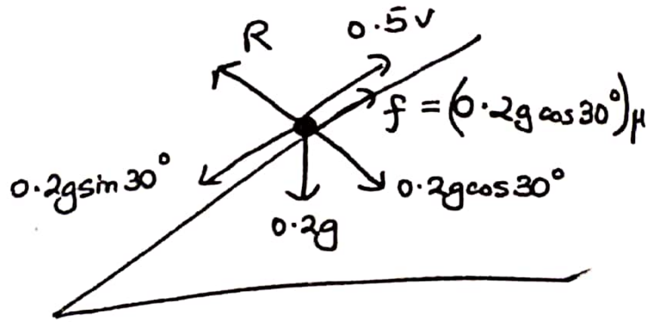
$-2.5T - (-2.5(0)) = \ln 3 - \ln 5$

$-2.5T = \ln\left(\frac{3}{5}\right)$

$T = \frac{\ln\left(\frac{3}{5}\right)}{-2.5} = 0.204$ s.

iii) R ($\perp$ to plane): $R = 0.2g\cos 30^\circ = \sqrt{3}$.

$$f = \mu R = \frac{1}{2\sqrt{3}} \times \sqrt{3} = \frac{1}{2}.$$



R (// to plane): $F_{net} = ma$

$$0.2g\sin 30^\circ - \frac{1}{2} - 0.5v = 0.2\frac{dv}{dt}$$

$$1 - \frac{1}{2} - 0.5v = 0.2\frac{dv}{dt}$$

$$\frac{0.5(1-v)}{0.2} = \frac{dv}{dt}$$

$$\int_0^{0.204} 2.5\,dt = \int_0^V \frac{1}{1-v}\,dv$$

since when $t = 0$, $v = 0$
and when $t = T = 0.204$, $v = V$

$$2.5\left[t\right]_0^{0.204} = \left[\frac{\ln|1-v|}{-1}\right]_0^V$$

$$2.5(0.204 - 0) = -\ln|1-V| - (-\ln|1-0|)$$

$$0.5108 = -\ln|1-V|$$

$$e^{-0.5108} = 1 - V$$

$$\frac{3}{5} = 1 - V$$

$$\therefore V = \frac{2}{5}\,ms^{-1} = 0.4\,ms^{-1}.$$

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43 A ball of mass 0.05 kg is released from rest at a height h m above the ground. At time t s after its release, the downward velocity of the ball is $v\,\mathrm{m\,s^{-1}}$. Air resistance opposes the motion of the ball with a force of magnitude $0.01v$ N.

(i) Show that $\dfrac{dv}{dt} = 10 - 0.2v$. Hence find v in terms of t. [6]

(ii) Given that the ball reaches the ground when $t = 2$, calculate h. [4]

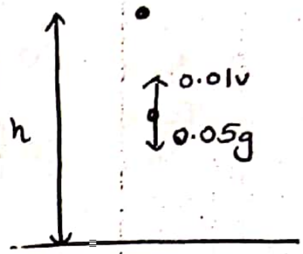
i)

$$F_{net} = ma$$

$$0.05g - 0.01v = 0.05\frac{dv}{dt}$$

$$\frac{0.05g - 0.01v}{0.05} = \frac{dv}{dt}$$

$$10 - 0.2v = \frac{dv}{dt} \quad \text{(shown).}$$



$$\int dt = \int \frac{1}{10-0.2v}\,dv$$

$$c + t = \frac{\ln|10-0.2v|}{-0.2}$$

$$c + t = -5\ln|10-0.2v|$$

When $t = 0$, $v = 0$

$$\therefore\ c + 0 = -5\ln|10 - 0.2\times 0|$$

$$c = -5\ln 10$$

Then $-5\ln 10 + t = -5\ln|10-0.2v|$

$$t = 5\ln 10 - 5\ln|10-0.2v|$$

$$t = 5\ln\left(\frac{10}{10-0.2v}\right)$$

$$\frac{t}{5} = \ln\left(\frac{10}{10-0.2v}\right)$$

$$e^{\frac{t}{5}} = \frac{10}{10-0.2v}$$

$$10 - 0.2v = 10e^{-t/5}$$

$$10 - 10e^{-t/5} = 0.2v$$

$$\frac{10(1-e^{-t/5})}{0.2} = v \quad \therefore\ v = 50(1-e^{-t/5}).$$

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ii) $v = \dfrac{dx}{dt} = 50\left(1-e^{-t/5}\right).$

$$\int_0^h dx = \int_0^2 50\left(1-e^{-t/5}\right) dt$$

when $t = 0, x = 0$

when $t = 2, x = h$

$$[x]_0^h = \left\{50\left[t - \frac{e^{-t/5}}{-1/5}\right]_0^2\right.$$

$$h - 0 = 50\left[t + 5e^{-t/5}\right]_0^2$$

$$h = 50\left[\left(2+5e^{-2/5}\right) - \left(0+5e^{-0/5}\right)\right]$$

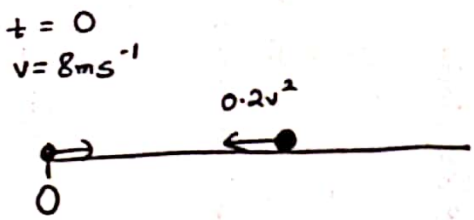
$$h = 50\left[0.3516\right] = 17.58 \approx 17.6\,m.$$

44 A particle P of mass 0.4 kg is projected horizontally with velocity 8 m s^{-1} from a point O on a smooth horizontal surface. The motion of P is opposed by a resisting force of magnitude $0.2v^2$ N, where $v\text{ m s}^{-1}$ is the velocity of P at time t s after projection.

(i) Show that $v = \dfrac{8}{1+4t}$. **[4]**

(ii) Calculate the distance OP when $t = 1.5$. **[4]**

i)



$$F = ma$$

$$-0.2v^2 = 0.4 \times \frac{dv}{dt}$$

$$\frac{-0.2v^2}{0.4} = \frac{dv}{dt}$$

$$\frac{-v^2}{2} = \frac{dv}{dt}$$

$$\int \frac{-1}{2}\,dt = \int \frac{1}{v^2}\,dv$$

$$\frac{-t}{2} + c = \frac{v^{-1}}{-1}$$

$$c - \frac{t}{2} = \frac{-1}{v}$$

when $t = 0$, $v = 8$

$$\therefore\ c - \frac{0}{2} = \frac{-1}{8}$$

$$c = \frac{-1}{8}.$$

Then $\dfrac{-1}{8} - \dfrac{t}{2} = \dfrac{-1}{v}$

$$\frac{-1-4t}{8} = \frac{-1}{v}$$

$$v = \frac{-8}{-1-4t}$$

$$v = \frac{8}{4t+1}$$

---

ii) $v = \dfrac{dx}{dt} = \dfrac{8}{4t+1}$

$$\int_0^X dx = \int_0^{1.5} \frac{8}{4t+1}\,dt$$

since when $t = 0$, $x = 0$

when $t = 1.5$, $x = X$

$$[x]_0^X = \left[\frac{8\ln(4t+1)}{4}\right]_0^{1.5}$$

$$X - 0 = \frac{8\ln 7}{4} - \frac{8\ln 1}{4}$$

$$X = \frac{8\ln 7}{4}.$$

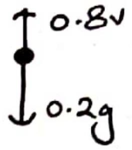
$$X = \underline{3.89\,m}.$$

45 A particle P of mass 0.2 kg is released from rest and falls vertically. At time t s after release P has speed v m s^{-1}. A resisting force of magnitude $0.8v$ N acts on P.

(i) Show that the acceleration of P is $(10 - 4v)$ m s^{-2}. [2]

(ii) Find the value of v when $t = 0.6$. [5]

i)



$$F_{net} = ma$$

$$0.2g - 0.8v = 0.2a$$

$$\frac{0.2g - 0.8v}{0.2} = a$$

$$10 - 4v = a \text{ (Shown).}$$

ii)

$$\frac{dv}{dt} = 10 - 4v$$

$$\int_0^V \frac{1}{10-4v}\,dv = \int_0^{0.6} dt$$

since when $t = 0, v = 0$
when $t = 0.6, v = V$

$$\left[\frac{\ln|10-4v|}{-4}\right]_0^V = [t]_0^{0.6}$$

$$\frac{\ln|10-4V|}{-4} - \frac{\ln|10|}{-4} = 0.6 - 0$$

$$\frac{\ln|10-4V|}{-4} + \frac{\ln 10}{4} = 0.6$$

~~$\ln 10(10-4V) =$~~

$$\ln\left|\frac{10}{10-4V}\right| = 2.4$$

$$\frac{10}{10-4V} = e^{2.4}$$

$$10e^{-2.4} = 10 - 4V$$

$$4V = 10 - 10e^{-2.4}$$

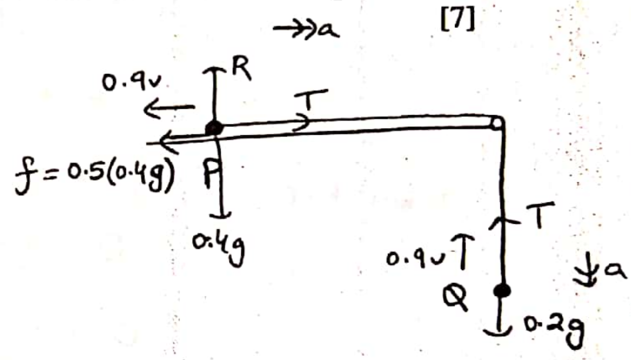
$$V = \frac{10 - 10e^{-2.4}}{4}$$

$$V = 2.273 \approx 2.27 \text{ ms}^{-1}.$$

Two particles P and Q, of masses 0.4 kg and 0.2 kg respectively, are attached to opposite ends of a light inextensible string. P is placed on a horizontal table and the string passes over a small smooth pulley at the edge of the table. The string is taut and the part of the string attached to Q is vertical (see diagram). The coefficient of friction between P and the table is 0.5. Q is projected vertically downwards with speed $5\,\mathrm{m\,s^{-1}}$, and at time t s after the instant of projection the speed of the particles is $v\,\mathrm{m\,s^{-1}}$. The motion of each particle is opposed by a resisting force of magnitude $0.9v$ N. The particle P does not reach the pulley.

(i) Show that $\dfrac{dv}{dt} = -3v$. [4]

(ii) Find the value of t when the particles have speed $2.5\,\mathrm{m\,s^{-1}}$ and the distance that each particle has travelled in this time. [7]



i) For P,

R(↑): $R = 0.4g$.

R(→): $F_{net} = ma$

$$T - 0.5(0.4g) - 0.9v = 0.4\frac{dv}{dt}$$

$$T - 2.0 - 0.9v = 0.4\frac{dv}{dt} \rightarrow (1)$$

For Q:

R(↑): $F_{net} = ma$

$$0.2g - T - 0.9v = 0.2\frac{dv}{dt}$$

$$2 - T - 0.9v = 0.2\frac{dv}{dt} \rightarrow (2)$$

Adding (1) and (2) gives:

$$T - 2.0 - 0.9v + 2 - T - 0.9v = 0.4\frac{dv}{dt} + 0.2\frac{dv}{dt}$$

$$-1.8v = 0.6\frac{dv}{dt}$$

$$-3v = \frac{dv}{dt}$$

(Shown).

ii) $\frac{dv}{dt} = -3v \Rightarrow \int \frac{dv}{v} = \int -3\,dt$

$$\ln|v| = -3t + c.$$

when $t = 0$, $v = 5$

$$\therefore \ln|5| = -3(0) + c$$

$$\ln|5| = c$$

Then $\ln|v| = -3t + \ln|5|$

$$\ln\left|\frac{v}{5}\right| = -3t$$

$$\frac{v}{5} = e^{-3t}$$

$$v = 5e^{-3t}.$$

When $v = 2.5$, $2.5 = 5e^{-3t} \Rightarrow e^{-3t} = \frac{1}{2} \Rightarrow t = \frac{\ln\left(\frac{1}{2}\right)}{-3} = 0.231\,s.$

$$v = \frac{dx}{dt} = 5e^{-3t}$$

$$x = \int 5e^{-3t}\,dt$$

$$x = \frac{5e^{-3t}}{-3} + c$$

When $t = 0$, $x = 0$

$$\therefore 0 = \frac{5e^{-3(0)}}{-3} + c$$

$$0 = \frac{-5}{3} + c$$

$$\therefore c = \frac{5}{3}.$$

Then $x = \frac{-5e^{-3t}}{3} + \frac{5}{3}.$

When $t = 0.231\,s$,

$$x = \frac{-5}{3}e^{-3(0.231)} + \frac{5}{3} = \frac{5}{6}\text{ m}.$$

i.e. Each particle has travelled $\frac{5}{6} = 0.833$m.

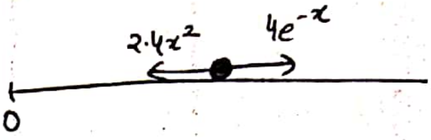
47 A particle P of mass 0.8 kg moves along the x-axis on a horizontal surface. When the displacement of P from the origin O is x m the velocity of P is $v\,\mathrm{m\,s^{-1}}$ in the positive x-direction. Two horizontal forces act on P. One force has magnitude $4e^{-x}$ N and acts in the positive x-direction. The other force has magnitude $2.4x^2$ N and acts in the negative x-direction.

(i) Show that $v\dfrac{dv}{dx} = 5e^{-x} - 3x^2$. [2]

(ii) The velocity of P as it passes through O is $6\,\mathrm{m\,s^{-1}}$. Find the velocity of P when $x = 2$. [5]

i)

$$F_{net} = ma$$



$$4e^{-x} - 2.4x^2 = 0.8\,v\frac{dv}{dx}$$

$$\frac{4e^{-x} - 2.4x^2}{0.8} = v\frac{dv}{dx}$$

$$5e^{-x} - 3x^2 = v\frac{dv}{dx} \text{ (Shown).}$$

ii)

$$\int_0^2 5e^{-x} - 3x^2\,dx = \int_6^V v\,dv$$

since when $x = 0$, $v = 6$
and when $x = 2$, $v = V$.

$$\left[-5e^{-x} - x^3\right]_0^2 = \left[\frac{v^2}{2}\right]_6^V$$

$$(-5e^{-2} - 2^3) - (-5e^{-0} - 0^3) = \frac{V^2}{2} - \frac{6^2}{2}$$

$$\left(\frac{-5}{e^2} - 8 + 5\right) = \frac{V^2}{2} - 18$$

$$\frac{-5}{e^2} - 8 + 5 + 18 = \frac{V^2}{2}$$

$$28.64 = V^2$$

$$\therefore V = 5.35\,\mathrm{ms^{-1}}.$$

i.e. Velocity of P when $x = 2$: $5.35\,\mathrm{ms^{-1}}$.

48 A small ball of mass m kg is projected vertically upwards with speed $14\,\mathrm{m\,s^{-1}}$. The ball has velocity $v\,\mathrm{m\,s^{-1}}$ upwards when it is x m above the point of projection. A resisting force of magnitude $0.02mv$ N acts on the ball during its upward motion.

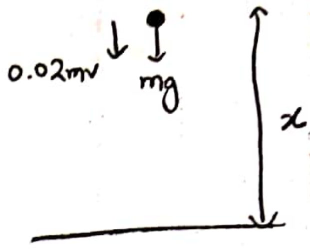
(i) Show that, while the ball is moving upwards, $\left(\frac{500}{v+500} - 1\right)\frac{dv}{dx} = 0.02.$ [3]

(ii) Find the greatest height of the ball above its point of projection. [3]

i)

$$F_{net} = ma$$

$$-(0.02mv + mg) = m v \frac{dv}{dx}$$



$$-0.02v - g = v\frac{dv}{dx}$$

$$\frac{-v}{50} - 10 = v\frac{dv}{dx}$$

$$\frac{-v-500}{50} = v\frac{dv}{dx}$$

$$1 = \frac{50v}{-v-500}\frac{dv}{dx}$$

$$1 = \frac{-50v}{v+500}\frac{dv}{dx}$$

$$\begin{array}{r} -50 \\ v+500 \enclose{longdiv}{-50v} \\ -50v - 25000 \\ + \quad + \\ \hline 25000 \end{array}$$

$$1 = \left(-50 + \frac{25000}{v+500}\right)\frac{dv}{dx}$$

$$1 = 50\left[-1 + \frac{500}{v+500}\right]\frac{dv}{dx}$$

$$\frac{1}{50} = \left[\frac{500}{v+500} - 1\right]\frac{dv}{dx}$$

$$0.02 = \left[\frac{500}{v+500} - 1\right]\frac{dv}{dx} \quad \text{(Shown)}$$

~~Then $\frac{dv}{dx} = 0.02\left(\frac{500}{v+500} - 1\right)$. (Shown).~~

ii)

$$\int_0^H 0.02\,dx = \int_{14}^0 \left(\frac{500}{v+500} - 1\right) dv$$

when $x = 0$, $v = 14$

when $x = H$, $v = 0$ since at greatest height, velocity $= 0\,\mathrm{ms^{-1}}$.

$$\left[0.02x\right]_0^H = \left[500\ln|v+500| - v\right]_{14}^0$$

$$0.02H - 0.02(0) = \left[500 \ln(0+500) - 0\right] - \left[500 \ln(14+500) - 14\right]$$

$$0.02H = 500 \ln 500 - 500 \ln(514) + 14$$

$$H = 9.620\,m \approx 9.62\,m$$

$\therefore$ Greatest height reached $= 9.62\,m$.

49 A particle P of mass 0.3 kg moves in a straight line on a smooth horizontal surface. P passes through a fixed point O of the line with velocity $8\,\mathrm{m\,s^{-1}}$. A force of magnitude $2x$ N acts on P in the direction PO, where x m is the displacement of P from O.

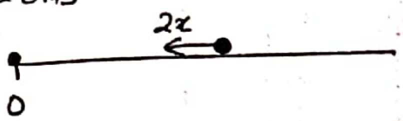
(i) Show that $v\dfrac{dv}{dx} = kx$ and state the value of the constant k. [2]

(ii) Find the value of x at the instant when P comes to instantaneous rest. [3]

i)

$m = 0.3\,kg$

$v = 8\,ms^{-1}$



$$F = ma$$

$$-2x = 0.3\, v\frac{dv}{dx}$$

$$\frac{-2x}{0.3} = v\frac{dv}{dx}$$

$$\frac{-20}{3}x = v\frac{dv}{dx}$$

$$\therefore\ k = \frac{-20}{3} = -6\frac{2}{3}.$$

ii)

$$\int_0^X \frac{-20}{3}x\,dx = \int_8^0 v\,dv$$

since when $x = 0$, $v = 8$

and when P is instantaneously at rest, $v = 0$, $x = ?$.

$$\frac{-20}{3}\left[\frac{x^2}{2}\right]_0^X = \left[\frac{v^2}{2}\right]_8^0$$

$$\frac{-20}{3}\left[\frac{X^2}{2} - 0\right] = \frac{0^2}{2} - \frac{8^2}{2}$$

$$\frac{-10}{3}X^2 = \frac{-64}{2}$$

$$X^2 = \frac{64 \times 3}{2 \times 10}$$

$$X = \sqrt{9.6}$$

$$X = 3.098\,m \simeq 3.10\,m$$

50 A particle P of mass 0.5 kg is projected vertically upwards from a point on a horizontal surface. A resisting force of magnitude $0.02v^2$ N acts on P, where v m s^{-1} is the upward velocity of P when it is a height of x m above the surface. The initial speed of P is 8 m s^{-1}.

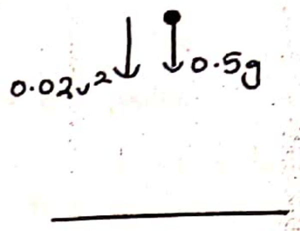
(i) Show that, while P is moving upwards, $v\dfrac{dv}{dx} = -10 - 0.04v^2$. [2]

(ii) Find the greatest height of P above the surface. [3]

(iii) Find the speed of P immediately before it strikes the surface after descending. [4]

i)

$$F_{net} = ma$$



$$-0.5g - 0.02v^2 = m v \frac{dv}{dx}$$

$$-0.5g - 0.02v^2 = 0.5 v \frac{dv}{dx}$$

$$\frac{-0.5g - 0.02v^2}{0.5} = v\frac{dv}{dx}$$

$$-g - 0.04v^2 = v\frac{dv}{dx}$$

$$\therefore v\frac{dv}{dx} = -10 - 0.04v^2. \quad \text{(Shown)}.$$

ii)

$$\int \frac{v}{-10-0.04v^2}\, dv = \int dx$$

$$\frac{1}{-0.08}\int \frac{-0.08v}{-10-0.04v^2}\, dv = \int dx$$

$$\frac{\ln|-10-0.04v^2|}{-0.08} = x + c$$

$$\frac{-25\ln|-10-0.04v^2|}{2} = x + c.$$

When $x = 0$, $v = 8$

$$\therefore \frac{-25\ln|-10-0.04\times 8^2|}{2} = 0 + c$$

$$\frac{-25\ln|-12.56|}{2} = c$$

Then $$\frac{-25\ln|-10-0.04v^2|}{2} = x - \frac{25\ln|-12.56|}{2}$$

For greatest height, $v = 0$

$$\Rightarrow \frac{-25\ln|-10-0.04\times 0^2|}{2} = x - \frac{25\ln|-12.56|}{2}$$

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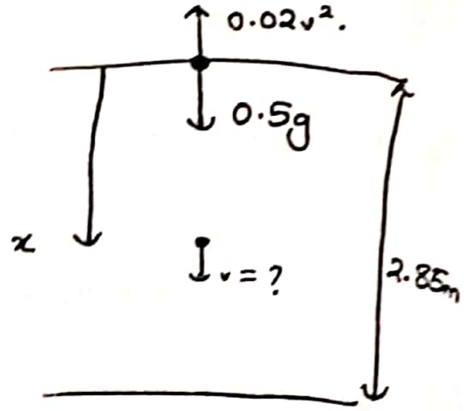
$$\Rightarrow x = 2.849\text{m} \approx 2.85\text{m}.$$

iii)

$$F_{net} = ma$$

$$0.5\times 10 - 0.02v^2 = 0.5\, v\frac{dv}{dx}$$

$$10 - 0.04v^2 = v\frac{dv}{dx}$$



$$\Rightarrow \int dx = \int \frac{v\,dv}{10-0.04v^2}$$

$$\int dx = \frac{1}{-0.08}\int \frac{-0.08v}{10-0.04v^2}\,dv$$

$$x + c = \frac{\ln|10-0.04v^2|}{-0.08}$$

When $x = 0,\ v = 0$

$$\therefore\ c = \frac{\ln|10|}{-0.08}$$

Then $$x - \frac{\ln 10}{0.08} = \frac{\ln|10-0.04v^2|}{-0.08}$$

When $x = 2.85\,m$,

$$2.85 - \frac{\ln 10}{0.08} = \frac{\ln|10-0.04v^2|}{-0.08}$$

$$2.074 = \ln|10-0.04v^2|$$

$$\Rightarrow 10 - 0.04v^2 = e^{2.074}$$

$$10 - 0.04v^2 = 7.9617$$

$$2.038 = 0.04v^2$$

$$50.95 = v^2$$

$$\Rightarrow v = 7.138\ ms^{-1}$$

$$\approx 7.14\ ms^{-1}.$$

51 A small block B of mass 0.25 kg is released from rest at a point O on a smooth horizontal surface. After its release the velocity of B is $v\,\mathrm{m\,s^{-1}}$ when its displacement is x m from O. The force acting on B has magnitude $(2 + 0.3x^2)$ N and is directed horizontally away from O.

(i) Show that $v\dfrac{dv}{dx} = 1.2x^2 + 8$. [2]

(ii) Find the velocity of B when $x = 1.5$. [3]

An extra force acts on B after $x = 1.5$. It is given that, when $x > 1.5$,

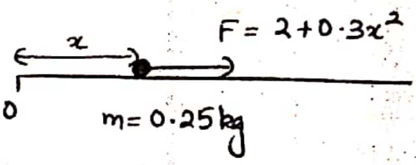
$$v\frac{dv}{dx} = 1.2x^2 + 6 - 3x.$$

(iii) Find the magnitude of this extra force and state the direction in which it acts. [2]

i)

$$F = ma$$

$$2 + 0.3x^2 = 0.25\, v\frac{dv}{dx}$$



$$\frac{2 + 0.3x^2}{0.25} = v\frac{dv}{dx}$$

$$8 + 1.2x^2 = v\frac{dv}{dx}$$

ii)

$$\int 8 + 1.2x^2\, dx = \int v\, dv$$

$$8x + \frac{1.2x^3}{3} + c = \frac{v^2}{2}$$

$$8x + 0.4x^3 + c = \frac{v^2}{2}$$

when $x = 0$, $v = 0$

$\therefore$ $c = 0$

$$\Rightarrow \frac{v^2}{2} = 8x + 0.4x^3.$$

when $x = 1.5$,

$$\frac{v^2}{2} = 8(1.5) + 0.4 \times 1.5^3$$

$$v^2 = 26.7$$

$$v = 5.167 \approx 5.17\ \mathrm{ms^{-1}}.$$

III) For $x > 1.5$, $v\frac{dv}{dx} = 1.2x^2 + 6 - 3x$

Then $F_{net} = mv\frac{dv}{dx} = 0.25(1.2x^2 + 6 - 3x)$

$$F_{net} = 0.3x^2 + 1.5 - 0.75x$$

Let, additional force be f:

Then $2 + 0.3x^2 - f = 0.3x^2 + 1.5 - 0.75x$

$$2 + 0.3x^2 - 0.3x^2 - 1.5 + 0.75x = f$$

$$0.5 + 0.75x = f$$

Therefore, additional force has magnitude $0.75x + 0.5$ and it acts towards O.

52 A particle P of mass 0.4 kg is released from rest at a point O on a smooth plane inclined at 30° to the horizontal. A force of magnitude $3e^{-t}$ N directed up a line of greatest slope acts on P, where t s is the time after release.

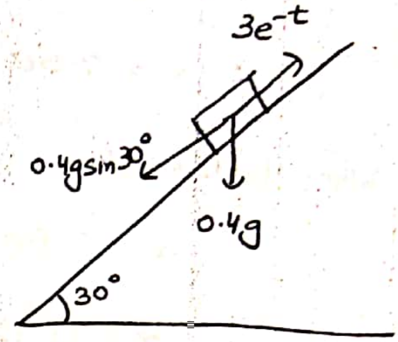
(i) Show that $\frac{dv}{dt} = 7.5e^{-t} - 5$, where $v\,\mathrm{m\,s^{-1}}$ is the velocity of P up the plane at time t s. [2]

(ii) Express v in terms of t. [3]

(iii) Find the distance of P from O when v has its maximum value. [3]

i)

$$F_{net} = ma$$



$$-[0.4g\sin 30° - 3e^{-t}] = 0.4\frac{dv}{dt}$$

$$\frac{-[0.4g\sin 30° - 3e^{-t}]}{0.4} = \frac{dv}{dt}$$

$$-5 + 7.5e^{-t} = \frac{dv}{dt}$$

$$\frac{dv}{dt} = 7.5e^{-t} - 5.$$

---

ii)

$$\int dv = \int 7.5e^{-t} - 5\, dt$$

$$v = \frac{7.5e^{-t}}{-1} - 5t + c$$

$$v = -7.5e^{-t} - 5t + c$$

when $t = 0$, $v = 0$

$$\therefore 0 = -7.5e^{-0} - 5(0) + c$$

$$0 = -7.5 + c$$

$$\therefore c = 7.5.$$

Then $v = -7.5e^{-t} - 5t + 7.5$

---

iii) For maximum velocity, $a = \frac{dv}{dt} = 0$

$$7.5e^{-t} - 5 = 0$$

$$e^{-t} = \frac{5}{7.5} = \frac{2}{3}$$

$$-t = \ln\left(\frac{2}{3}\right)$$

$$t = -\ln\left(\frac{2}{3}\right) = 0.4054.$$

$$\frac{dx}{dt} = -7.5e^{-t} - 5t + 7.5$$

$$\Rightarrow \quad x = \int -7.5e^{-t} - 5t + 7.5 \, dt$$

$$x = 7.5e^{-t} - \frac{5t^2}{2} + 7.5t + c$$

when $t = 0, x = 0$

$$0 = 7.5e^{-0} - \frac{5(0)^2}{2} + 7.5(0) + c$$

$$-7.5 = c$$

$$\therefore \quad x = 7.5e^{-t} - \frac{5t^2}{2} + 7.5t - 7.5$$

When $t = 0.4054$,

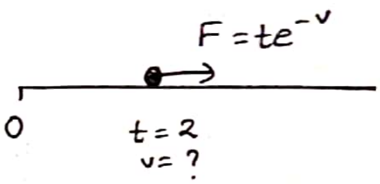
$$x = 7.5e^{-0.4054} - \frac{5(0.4054)^2}{2} + 7.5(0.4054) - 7.5$$

$$x = 0.1299 \approx 0.130 \text{ m}$$

53 A particle P of mass 0.2 kg is released from rest at a point O on a smooth horizontal surface. A horizontal force of magnitude te^{-v} N directed away from O acts on P, where v m s^{-1} is the velocity of P at time t s after release. Find the velocity of P when $t = 2$. [4]

$$F = ma$$

$$te^{-v} = 0.2\frac{dv}{dt}$$



$$\int t\,dt = \int 0.2e^{v}\,dv$$

$$t + c = 0.2e^{v}$$

when $t = 0$, $v = 0$

$$\therefore\ 0 + c = 0.2e^{0}$$

$$c = 0.2.$$

Then $t + 0.2 = 0.2e^{v}$

$$\frac{t+0.2}{0.2} = e^{v}$$

$$5t + 1 = e^{v}$$

when $t = 2$,

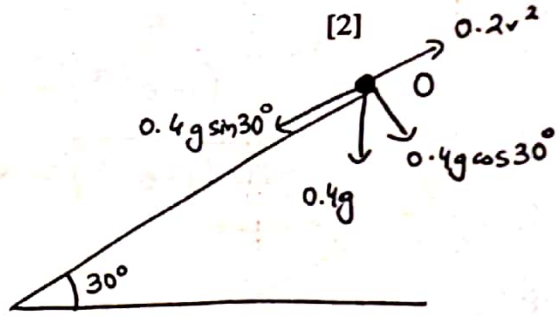
$$5(2) + 1 = e^{v}$$

$$11 = e^{v}$$

$$\therefore\ v = \ln(11)\ \text{ms}^{-1}.$$

54 A particle P of mass 0.4 kg is released from rest at a point O on a smooth plane inclined at 30° to the horizontal. P moves down the line of greatest slope through O. The velocity of P is $v\,\mathrm{m\,s^{-1}}$ when its displacement from O is x m. A retarding force of magnitude $0.2v^2$ N acts on P in the direction PO.

(i) Show that $v\dfrac{dv}{dx} = 5 - 0.5v^2$. [2]



$$F_{net} = ma$$

$$0.4g\sin 30° - 0.2v^2 = 0.4v\frac{dv}{dx}$$

$$\frac{0.4g\sin 30° - 0.2v^2}{0.4} = v\frac{dv}{dx}$$

$$5 - 0.5v^2 = v\frac{dv}{dx}.$$

(ii) Express v in terms of x. [4]

$$5 - 0.5v^2 = v\frac{dv}{dx}$$

$$\int dx = \int \frac{v}{5-0.5v^2}\,dv$$

$$\int dx = -1 \int \frac{-v}{5-0.5v^2}\,dv$$

$$x + c = -\ln|5-0.5v^2|$$

When $x = 0, v = 0$

$$\therefore \quad 0 + c = -\ln|5 - 0.5 \times 0^2|$$

$$c = -\ln 5$$

Then $x - \ln 5 = -\ln|5-0.5v^2|$

$$x = \ln 5 - \ln|5-0.5v^2|$$

$$x = \ln\left|\frac{5}{5-0.5v^2}\right|$$

$$e^{x} = \frac{5}{5 - 0.5v^2}$$

$$5 - 0.5v^2 = 5e^{-x}$$

$$5 - 5e^{-x} = 0.5v^2$$

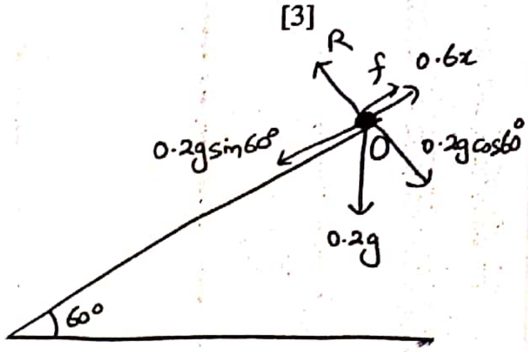
$$\frac{5 - 5e^{-x}}{0.5} = v^2$$

$$10\left(1 - e^{-x}\right) = v^2$$

$$\sqrt{10\left(1 - e^{-x}\right)} = v$$

55 A particle P of mass 0.2 kg is released from rest at a point O on a rough plane inclined at 60° to the horizontal, and travels down a line of greatest slope. The coefficient of friction between P and the plane is 0.3. A force of magnitude $0.6x$ N acts on P in the direction PO, where x m is the displacement of P from O.

(i) Show that $v\frac{dv}{dx} = 5\sqrt{3} - 1.5 - 3x$, where $v\,\text{m s}^{-1}$ is the velocity of P at a displacement x m from O. [3]



$f = \mu R = 0.3(0.2g\cos 60°) = \frac{3}{10}.$

$\therefore \quad F_{net} = ma$

$0.2g\sin 60° - 0.6x - 0.3(0.2g\cos 60°) = 0.2\,v\frac{dv}{dx}$

$\sqrt{3} - 0.6x - \frac{3}{10} = \frac{1}{5}v\frac{dv}{dx}$

$5\sqrt{3} - 3x - 1.5 = v\frac{dv}{dx}$

i.e. $v\frac{dv}{dx} = 5\sqrt{3} - 3x - 1.5.$

(ii) Find the value of x for which P reaches its maximum velocity, and calculate this maximum velocity. [4]

For maximum velocity, $a = 0 \Rightarrow v\frac{dv}{dx} = 0$

$5\sqrt{3} - 3x - 1.5 = 0$

$x = \frac{1.5 - 5\sqrt{3}}{-3} = 2.386 \approx 2.39\text{m}.$

$\int v\,dv = \int 5\sqrt{3} - 3x - 1.5\,dx$

$\frac{v^2}{2} = 5\sqrt{3}x - \frac{3x^2}{2} - 1.5x + c. \quad x = 0, v = 0 \therefore c = 0$

Substituting $x = 2.39$ gives:

$\frac{v^2}{2} = 5\sqrt{3}(2.39) - \frac{3(2.39)^2}{2} - 1.5(2.39)$

$v^2 = 17.08$

$v = 4.13\,\text{ms}^{-1}.$

(iii) Calculate the magnitude of the acceleration of P immediately after it has first come to instantaneous rest. [4]

When at instaneous rest, $v = 0$

$$\Rightarrow \frac{5\sqrt{3}}{1}x - \frac{3x^2}{2} - 1.5x = 0$$

$$x\left(5\sqrt{3} - \frac{3x}{2} - 1.5\right) = 0$$

$x = 0$ or $5\sqrt{3} - \frac{3x}{2} - 1.5 = 0$

$\therefore x = 4.773$

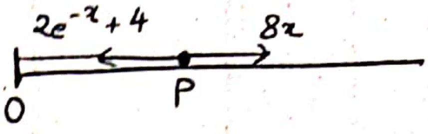
$\therefore a = 5\sqrt{3} - 3(4.773) - 1.5 = -7.158 \approx -7.16\,ms^{-2}$

$\therefore$ Magnitude of acceleration $= 7.16\,ms^{-2}$.

56 A particle P of mass 0.4 kg is projected horizontally along a smooth horizontal plane from a point O. After projection the velocity of P is $v\,\mathrm{m\,s^{-1}}$ and its displacement from O is x m. A force of magnitude $8x$ N directed away from O acts on P and a force of magnitude $(2e^{-x} + 4)$ N opposes the motion of P. One end of a light elastic string of natural length 0.5 m is attached to O and the other end of the string is attached to P.

(i) Show that $v\dfrac{dv}{dx} = 20x - 10 - 5e^{-x}$ before the elastic string becomes taut. [2]

$$F = ma$$



$$8x - (2e^{-x}+4) = 0.4\, v\frac{dv}{dx}$$

$$\frac{8x-(2e^{-x}+4)}{0.4} = v\frac{dv}{dx}$$

$$20x - (5e^{-x}+10) = v\frac{dv}{dx}$$

$$v\frac{dv}{dx} = 20x - 5e^{-x} + 10.$$

(Shown).

(ii) Given that the initial velocity of P is $6\,\mathrm{m\,s^{-1}}$, find v when the string first becomes taut. [3]

$$\int v\, dv = \int 20x - 5e^{-x} - 10\, dx$$

$$\frac{v^2}{2} = 10x^2 + 5e^{-x} - 10x + c$$

When $x = 0$, $v = 6$

$$\frac{6^2}{2} = 10(0)^2 + 5e^{-0} - 10(0) + c$$

$$18 = c + 5$$

$$\Rightarrow c = 13$$

Then $\dfrac{v^2}{2} = 10x^2 + 5e^{-x} - 10x + 13.$

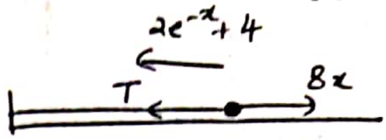
When string becomes taut, $x = 0.5$

$$\Rightarrow \frac{v^2}{2} = 10(0.5)^2 + 5e^{-0.5} - 10(0.5) + 13$$

$$v = 5.20\ \mathrm{ms^{-1}}.$$

When the string is taut, the acceleration of P is proportional to e^{-x}.

(iii) Find the modulus of elasticity of the string. **[2]**



$$F_{net} = 8x - T - (2e^{-x} + 4)$$

$$ma = 8x - T - 2e^{-x} - 4$$

$$m(ke^{-2x}) = (8x - T - 4) - 2e^{-x}$$

Comparing coefficients gives:

$$8x - T - 4 = 0$$

$$8x - 4 = T$$

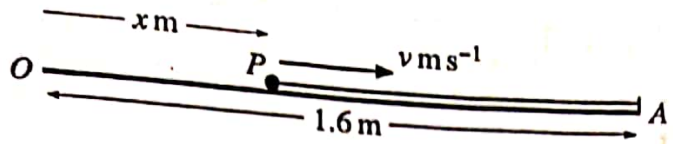
Then $T = \dfrac{\lambda(x - 0.5)}{0.5}$

$$0.5(8x - 4) = \lambda x - 0.5\lambda$$

$$4x - 2 = \lambda x - 0.5\lambda$$

Comparing coefficients of x gives:

$$\underline{\lambda = 4}.$$

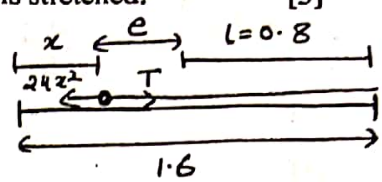


A particle P of mass 0.5 kg is projected along a smooth horizontal surface towards a fixed point A. Initially P is at a point O on the surface, and after projection, P has a displacement from O of x m and velocity v m s^{-1}. The particle P is connected to A by a light elastic string of natural length 0.8 m and modulus of elasticity 16 N. The distance OA is 1.6 m (see diagram). The motion of P is resisted by a force of magnitude $24x^2$ N.

(i) Show that $v\dfrac{dv}{dx} = 32 - 40x - 48x^2$ while P is in motion and the string is stretched. **[3]**

Let, e = extension of string. Then $e = 1.6 - 0.8 - x$

$e = 0.8 - x$



$\therefore T = \dfrac{\lambda e}{L} = \dfrac{16(0.8-x)}{0.8}$

$F = ma$

$T - 24x^2 = 0.5\, v\dfrac{dv}{dx} \Rightarrow \dfrac{16(0.8-x)}{0.8} - 24x^2 = 0.5\, v\dfrac{dv}{dx}$

$\dfrac{16(0.8-x)}{(0.8)(0.5)} - \dfrac{24x^2}{0.5} = v\dfrac{dv}{dx}$

$40(0.8-x) - 48x^2 = v\dfrac{dv}{dx}$

$32 - 40x - 48x^2 = v\dfrac{dv}{dx}$ (Shown).

The maximum value of v is 4.5.

(ii) Find the initial value of v. **[5]**

Maximum velocity occurs when $a = 0 \Rightarrow v\dfrac{dv}{dx} = 0$

$\therefore 32 - 40x - 48x^2 = 0$

$6x^2 + 5x - 4 = 0$

$6x^2 + 8x - 3x - 4 = 0$

$(2x-1)(3x+4) = 0$

$x = \dfrac{1}{2}$ or $x = \dfrac{-4}{3}$ (Ignore)

Then when $x = \dfrac{1}{2}$, $v = 4.5$.

$v\dfrac{dv}{dx} = 32 - 40x - 48x^2$

$\int v\, dv = \int 32 - 40x - 48x^2\, dx$

$\dfrac{v^2}{2} = 32x - \dfrac{40x^2}{2} - \dfrac{48x^3}{3} + c$

Substituting $x = 0.5$ and $v = 4.5$ gives:

$$\frac{4.5^2}{2} = 32(0.5) - \frac{40(0.5)^2}{2} - \frac{48(0.5)^3}{3} + c$$

$$\frac{81}{8} = c + 9$$

$$\Rightarrow \quad c = \frac{9}{8}$$

Then $\quad \dfrac{v^2}{2} = 32x - 20x^2 - 16x^3 + \dfrac{9}{8}.$

For initial velocity, $x = 0$

$$\therefore \quad \frac{v^2}{2} = \frac{9}{8} \Rightarrow v^2 = \frac{9}{4}$$

$$v = 1.5\ ms^{-1}.$$

58 A smooth horizontal surface has two fixed points O and A which are 0.8 m apart. A particle P of mass 0.25 kg is projected with velocity 3 m s^{-1} horizontally from A in the direction away from O. The velocity of P is v m s^{-1} when the displacement of P from O is x m. A force of magnitude kv^2x^{-2} N opposes the motion of P.

(i) Show that $v\frac{dv}{dx} = -4kv^2x^{-2}$. [1]

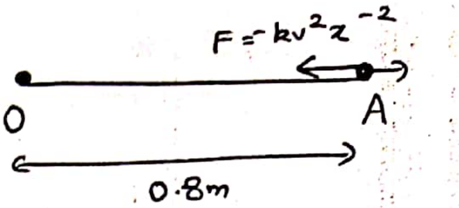
$x = 0.8$m
$V = 3$ms^{-1}

$F = ma$

$-kv^2x^{-2} = 0.25 \times v\frac{dv}{dx}$

$\frac{-kv^2x^{-2}}{0.25} = v\frac{dv}{dx}$

$\therefore v\frac{dv}{dx} = -4kv^2x^{-2}$. (Shown).



(ii) Express v in terms of k and x. [5]

$\int \frac{v}{v^2}\, dv = \int -4kx^{-2}\, dx$

$\int \frac{1}{v}\, dv = -4k\int x^{-2}\, dx$

$\ln|v| = \frac{-4kx^{-1}}{-1} + c$

$\ln|v| = \frac{4k}{x} + c$

when $x = 0.8$, $v = 3$

$\Rightarrow \ln 3 = \frac{4k}{0.8} + c$

$\ln 3 = 5k + c$

$c = \ln 3 - 5k$.

Then $\ln|v| = \frac{4k}{x} + \ln 3 - 5k$.

$\ln v - \ln 3 = \frac{4k}{x} - 5k$

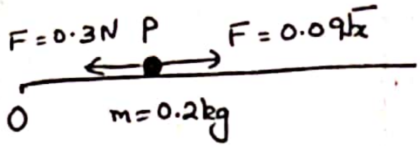
$\ln\left(\frac{v}{3}\right) = \frac{4k}{x} - 5k$

$\frac{v}{3} = e^{\frac{4k}{x} - 5k}$

$\Rightarrow v = 3e^{\frac{4k}{x} - 5k}$.

59 A particle P of mass 0.2 kg is projected horizontally from a fixed point O on a smooth horizontal surface. When the displacement of P from O is x m the velocity of P is $v\,\mathrm{m\,s^{-1}}$. A horizontal force of variable magnitude $0.09\sqrt{x}$ N directed away from O acts on P. An additional force of constant magnitude 0.3 N directed towards O acts on P.

(i) Show that $v\dfrac{dv}{dx} = 0.45\sqrt{x} - 1.5$. **[2]**



$$F = ma$$

$$0.09\sqrt{x} - 0.3 = 0.2\, v\frac{dv}{dx}$$

$$\frac{0.09\sqrt{x} - 0.3}{0.2} = v\frac{dv}{dx}$$

$$v\frac{dv}{dx} = 0.45\sqrt{x} - 1.5$$

(ii) Find the value of x for which the acceleration of P is zero. **[2]**

$$a = v\frac{dv}{dx} = 0.45\sqrt{x} - 1.5 = 0$$

$$\sqrt{x} = \frac{1.5}{0.45} = \frac{10}{3}$$

$$x = \frac{100}{9}\ \text{m.}$$

(iii) Given that the minimum value of v is positive, find the set of possible values for the speed of projection. [5]

$$\int v\,dv = \int 0.45\sqrt{x} - 1.5\,dx$$

$$\frac{v^2}{2} = \frac{0.45x^{\frac{3}{2}}}{\frac{3}{2}} - 1.5x + c$$

$$\frac{v^2}{2} = 0.30x^{\frac{3}{2}} - 1.5x + c.$$

When $x = \frac{100}{9}$, v is minimum. and $v_{min} > 0$

Then $$\frac{v_{min}^2}{2} = 0.30\left(\frac{100}{9}\right)^{\frac{3}{2}} - 1.5\left(\frac{100}{9}\right) + c$$

$$\frac{v_{min}^2}{2} = \frac{-50}{9} + c.$$

Then since $v_{min}^2 > 0$

$$\therefore \frac{v_{min}^2}{2} > 0$$

$$\therefore \frac{-50}{9} + c > 0$$

$$c > \frac{50}{9}$$

Then when $x = 0$,

$$\frac{v^2}{2} = 0.30(0)^2 - 1.5(0) + c$$

$$\frac{v^2}{2} = c$$

Then $\frac{v^2}{2} > \frac{50}{9} \Rightarrow v^2 > \frac{100}{9}.$

$$\therefore v > \frac{10}{3}.$$

60 A particle P of mass m kg moves in a horizontal straight line against a resistive force of magnitude mkv^2 N, where $v\,\mathrm{m\,s^{-1}}$ is the speed of P after it has moved a distance x m and k is a positive constant. The initial speed of P is $u\,\mathrm{m\,s^{-1}}$.

(a) Show that $x = \frac{1}{k}\ln 2$ when $v = \frac{1}{2}u$. [4]

$$F = ma$$

$$-mkv^2 = m \times v\frac{dv}{dx}$$

$$-kv^2 = v\frac{dv}{dx}$$

$$\int -k\,dx = \int \frac{v}{v^2}\,dv$$

$$-kx + c = \ln|v|.$$

When $x = 0$, $v = u$ $\therefore$ $-k(0) + c = \ln|u|$

$$\therefore c = \ln u.$$

$$\therefore \quad -kx + \ln u = \ln|v|.$$

when $v = \frac{1}{2}u$,

$$\Rightarrow -kx + \ln u = \ln\left|\frac{u}{2}\right|$$

$$-kx = \ln\left|\frac{u}{2}\right| - \ln|u|$$

$$-kx = \ln\left|\frac{u}{2} \div u\right|$$

$$-kx = \ln\left|\frac{1}{2}\right|$$

$$\therefore x = \frac{-\ln 2}{-k} = \frac{1}{k}\ln 2$$

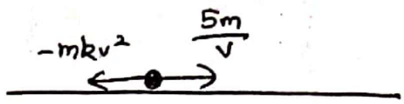
(Shown)

Beginning at the instant when the speed of P is $\frac{1}{2}u$, an additional force acts on P. This force has magnitude $\frac{5m}{v}$ N and acts in the direction of increasing x.

(b) Show that when the speed of P has increased again to $u\,\mathrm{m\,s^{-1}}$, the total distance travelled by P is given by an expression of the form

$$\frac{1}{3k}\ln\left(\frac{A-ku^3}{B-ku^3}\right),$$

stating the values of the constants A and B. **[7]**



$$F = ma$$

$$\frac{5m}{v} - mkv^2 = mv\frac{dv}{dx}$$

$$\left(\frac{5}{v} - kv^2\right) = v\frac{dv}{dx}$$

$$\frac{5-kv^3}{v} = v\frac{dv}{dx}$$

$$\int 1\,dx = \int \frac{v^2}{5-kv^3}\,dv = \frac{-1}{3k}\int \frac{-3kv^2}{5-kv^3}\,dv$$

$$x + c = \frac{-1}{3k}\ln|5-kv^3|$$

When $x = \frac{\ln 2}{k}$, $v = \frac{u}{2}$

$$\therefore \frac{\ln 2}{k} + c = \frac{-1}{3k}\ln\left|5-k\left(\frac{u}{2}\right)^3\right|$$

$$c = \frac{-1}{3k}\ln\left|5-\frac{ku^3}{8}\right| - \frac{\ln 2}{k}$$

$$\therefore x - \frac{1}{3k}\ln\left|5-\frac{ku^3}{8}\right| - \frac{\ln 2}{k} = \frac{-1}{3k}\ln|5-kv^3|.$$

When $v = u$,

$$x - \frac{1}{3k}\ln\left|5-\frac{ku^3}{8}\right| - \frac{\ln 2}{k} = \frac{-1}{3k}\ln|5-ku^3|$$

$$x = \frac{1}{3k}\ln\left|5-\frac{ku^3}{8}\right| - \frac{1}{3k}\ln|5-ku^3| + \frac{\ln 2}{k}$$

$$= \frac{1}{3k}\left[\ln\left|5-\frac{ku^3}{8}\right| - \ln|5-ku^3| + 3\ln 2\right]$$

$$= \frac{1}{3k}\left[\ln\left(\frac{40-ku^3}{8}\times 2^3 \div (5-ku^3)\right)\right]$$

$$x = \frac{1}{3k}\left[\ln\left(\frac{40-ku^3}{8} \times 8 \times \frac{1}{5-ku^3}\right)\right]$$

$$x = \frac{1}{3k}\ln\left(\frac{40-ku^3}{5-ku^3}\right)$$

61 A particle P moving in a straight line has displacement x m from a fixed point O on the line at time t s. The acceleration of P, in $\mathrm{m\,s^{-2}}$, is given by $\frac{200}{x^2} - \frac{100}{x^3}$ for $x > 0$. When $t = 0$, $x = 1$ and P has velocity $10\,\mathrm{m\,s^{-1}}$ directed towards O.

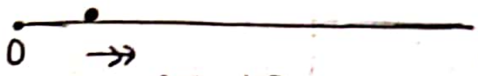
(a) Show that the velocity $v\,\mathrm{m\,s^{-1}}$ of P is given by $v = \frac{10(1-2x)}{x}$. [5]

$a = v\frac{dv}{dx}$.

$v = -10\,\mathrm{m/s}$

$t = 0\,\mathrm{s}$

$x = 1\,\mathrm{m}$



$a = \frac{200}{x^2} - \frac{100}{x^3}$.

$$\frac{200}{x^2} - \frac{100}{x^3} = v\frac{dv}{dx}$$

$$\int \frac{200}{x^2} - \frac{100}{x^3}\,dx = \int v\,dv$$

$$\frac{200x^{-1}}{-1} - \frac{100x^{-2}}{-2} + c = \frac{v^2}{2}$$

$$\frac{-200}{x} + \frac{50}{x^2} + c = \frac{v^2}{2}$$

When $x = 1$, $v = -10$

$$\Rightarrow \frac{-200}{1} + \frac{50}{1^2} + c = \frac{(-10)^2}{2}$$

$$-200 + 50 + c = 50$$

$$c = 200$$

$$\therefore \frac{-200}{x} + \frac{50}{x^2} + 200 = \frac{v^2}{2}$$

$$\frac{-200x + 50 + 200x^2}{x^2} = \frac{v^2}{2}$$

$$\frac{400x^2 - 400x + 100}{x^2} = v^2$$

$$\frac{100\{4x^2 - 4x + 1\}}{x^2} = v^2$$

$$\frac{100(2x-1)^2}{x^2} = v^2$$

$$\therefore v = \pm\frac{10(2x-1)}{x}$$

Since we require $v = -10$ when $x = 1$ $\therefore v = \frac{-10(2x-1)}{x}$

$$\Rightarrow v = \frac{10(1-2x)}{x}.$$

(Shown).

(b) Show that x and t are related by the equation $e^{-40t} = (2x-1)e^{2x-2}$ and deduce what happens to x as t becomes large. [5]

$$V = \frac{dx}{dt} = \frac{10(1-2x)}{x}$$

$$\int \frac{x}{1-2x}\,dx = \int 10\,dt$$

$$1-2x \overline{\big) x}\quad -\tfrac{1}{2}$$
$$x - \tfrac{1}{2}$$
$$- \quad +$$
$$\tfrac{1}{2}$$

$$\therefore \int \frac{-1}{2} + \frac{\frac{1}{2}}{1-2x}\,dx = \int 10\,dt$$

$$\frac{1}{2}\int -1 + \frac{1}{1-2x}\,dx = \int 10\,dt$$

$$\frac{1}{2}\left[-x + \frac{\ln|1-2x|}{-2}\right] = 10t + c.$$

When $t=0$, $x=1$,

$$\frac{1}{2}\left[-1 + \frac{\ln|1-2\times1|}{-2}\right] = 10(0) + c$$

$$\frac{-1}{2} = c$$

$$\therefore \frac{1}{2}\left[-x - \frac{\ln|1-2x|}{2}\right] = 10t - \frac{1}{2}.$$

$$\frac{1}{2}\left[-x - \frac{\ln|1-2x|}{2}\right] = \frac{20t-1}{2}$$

$$\frac{-2x - \ln|1-2x|}{2} = 20t - 1$$

$$-2x - \ln|1-2x| = 40t - 2$$

$$-40t - \ln|1-2x| = 2x - 2$$

$$e^{-40t - \ln|1-2x|} = e^{2x-2}$$

$$\frac{e^{-40t}}{e^{\ln|1-2x|}} = e^{2x-2}$$

$$e^{-40t} = e^{2x-2}\, e^{\ln|1-2x|}$$

$$e^{-40t} = e^{2x-2}\, e^{\ln|-(2x-1)|}$$

$$e^{-40t} = e^{2x-2} \, e^{\ln|2x-1|}$$

$$e^{-40t} = (2x-1)\, e^{2x-2}$$

As $t \mapsto \infty$, $e^{-40t} \mapsto 0$

$$\therefore \quad (2x-1)\left(e^{2x-2}\right) \mapsto 0$$

Since $e^{2x-2} > 0$ for all $x > 0 \therefore$ $2x - 1 \mapsto 0$

$$x \mapsto \frac{1}{2}.$$