Name ______________________ Date ______________________ Class ______________________

Past Paper Questions: Matrices I

1 Let $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$

(a) The transformation in the x-y plane represented by $\mathbf{A}^{-1}$ transforms a triangle of area $30\,\text{cm}^2$ into a triangle of area $d\,\text{cm}^2$.

Find the value of d. [3]

$\det A = 2(1) - 0(1) = 2$.

If A^{-1} transforms a triangle of area 30cm^2 into a triangle of area $d\text{cm}^2$,

$\Rightarrow$ then A transforms a triangle of area $d\text{cm}^2$ into a triangle of area 30cm^2.

Area of image $= |\det A| \times$ Area of object

$30 = 2 \times d$

$d = 15$

(b) Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}.$$ [5]

Let, $P(n)$ be the statement that $A^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}$.

Basic case: $n = 1$.

$L.H.S. = A^n = A^1 = A = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$

$R.H.S. = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix} = \begin{pmatrix} 2^1 & 0 \\ 2^1 - 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$.

Inductive step: Let, $P(n)$ is true for $n = k$. $\Rightarrow A^k = \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix}$

$A^{k+1} = A^k A = \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2^{k+1} + 0 & 0 + 0 \\ 2(2^k - 1) + 1 & 0 + 1 \end{pmatrix}$

$= \begin{pmatrix} 2^{k+1} & 0 \\ 2^{k+1} - 1 & 1 \end{pmatrix}$

$\Rightarrow P(n)$ is true for $n = k+1$.

$\Rightarrow P(n)$ is true for all positive integers n by mathematical induction.

(c) The line $y = 2x$ is invariant under the transformation in the x-y plane represented by $\mathbf{A}^n\mathbf{B}$, where $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 33 & 0 \end{pmatrix}$.

Find the value of n. [5]

$$A^nB = \begin{bmatrix} 2^n & 0 \\ 2^n-1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 33 & 0 \end{bmatrix} = \begin{bmatrix} 2^n+0 & 0+0 \\ 2^n-1+33 & 0+0 \end{bmatrix} = \begin{bmatrix} 2^n & 0 \\ 2^n+32 & 0 \end{bmatrix}$$

Since the line $y = 2x$ is invariant, a point $(t, 2t)$ is mapped to a point $(T, 2T)$.

$$A^nB\begin{bmatrix} t \\ 2t \end{bmatrix} = \begin{bmatrix} T \\ 2T \end{bmatrix}$$

$$\begin{bmatrix} 2^n & 0 \\ 2^n+32 & 0 \end{bmatrix}\begin{bmatrix} t \\ 2t \end{bmatrix} = \begin{bmatrix} T \\ 2T \end{bmatrix}$$

$$\begin{bmatrix} 2^n t + 0 \\ (2^n+32)(t)+0 \end{bmatrix} = \begin{bmatrix} T \\ 2T \end{bmatrix}$$

$$\begin{bmatrix} 2^n t \\ (2^n+32)t \end{bmatrix} = \begin{bmatrix} T \\ 2T \end{bmatrix}$$

$$\Rightarrow \quad 2^n t = T \rightarrow (1)$$

$$\Rightarrow (2^n+32)t = 2T \rightarrow (2)$$

Divide (2) by (1):

$$\frac{(2^n+32)t}{(2^n)t} = \frac{2T}{T}$$

$$2^n + 32 = 2(2^n)$$

$$32 = 2^n$$

$$2^5 = 2^n$$

$$\Rightarrow n = 5.$$

2 The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} k & 0 & 2 \\ 0 & -1 & -1 \\ 1 & 1 & -k \end{pmatrix},$$

where k is a real constant.

(a) Show that $\mathbf{A}$ is non-singular. [3]

$$\det A = k\begin{vmatrix} -1 & -1 \\ 1 & -k \end{vmatrix} - 0\begin{vmatrix} 0 & -1 \\ 1 & -k \end{vmatrix} + 2\begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= k[k+1] - 0 + 2[0+1]$$

$$= k^2 + k + 2$$

For A to be singular, $\det A = 0 \Rightarrow k^2+k+2=0$

$$k = \frac{-1 \pm \sqrt{1^2-4(1)(2)}}{2(1)} = \frac{-1 \pm \sqrt{-7}}{2}$$

No real roots.

Since k is real, A is non-singular (since $\det A \neq 0$).

The matrices $\mathbf{B}$ and $\mathbf{C}$ are given by

$$\mathbf{B} = \begin{pmatrix} 0 & -3 \\ -1 & 3 \\ 0 & 0 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} -3 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

It is given that $\mathbf{CAB} = \begin{pmatrix} -2 & -\frac{3}{2} \\ -1 & -\frac{3}{2} \end{pmatrix}$.

(b) Find the value of k. [3]

$$CAB = \begin{bmatrix} -3 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} k & 0 & 2 \\ 0 & -1 & -1 \\ 1 & 1 & -k \end{bmatrix}\begin{bmatrix} 0 & -3 \\ -1 & 3 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & \frac{-3}{2} \\ -1 & \frac{-3}{2} \end{bmatrix} = \begin{bmatrix} -3k+1 & 2 & -5-k \\ k+2 & 1 & 1-2k \end{bmatrix}\begin{bmatrix} 0 & -3 \\ -1 & 3 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & \frac{-3}{2} \\ -1 & \frac{-3}{2} \end{bmatrix} = \begin{bmatrix} -2 & 9k+3 \\ -1 & -3k-3 \end{bmatrix}$$

By comparison, $9k+3 = \frac{-3}{2} \Rightarrow k = \frac{-\frac{3}{2}-3}{9} = \frac{-1}{2}$.

(c) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plan[e] represented by **CAB**. [5]

$$CAB = \begin{bmatrix} -2 & \frac{-3}{2} \\ -1 & \frac{-3}{2} \end{bmatrix}$$

Let, equation of invariant line : $y = mx$.

$$\Rightarrow \begin{bmatrix} -2 & \frac{-3}{2} \\ -1 & \frac{-3}{2} \end{bmatrix} \begin{bmatrix} t \\ mt \end{bmatrix} = \begin{bmatrix} T \\ mT \end{bmatrix}$$

$$-2t - \frac{3}{2}mt = T \rightarrow (1)$$

$$-t - \frac{3}{2}mt = mT \rightarrow (2)$$

Dividing (2) by (1):

$$\frac{-t - \frac{3}{2}mt}{-2t - \frac{3}{2}mt} = \frac{mT}{T}.$$

$$\frac{-1 - \frac{3}{2}m}{-2 - \frac{3}{2}m} = m$$

$$-1 - \frac{3}{2}m = -2m - \frac{3}{2}m^2$$

$$-2 - 3m = -4m - 3m^2$$

$$3m^2 + m - 2 = 0$$

$$3m^2 + 3m - 2m - 2 = 0$$

$$(3m - 2)(m + 1) = 0$$

$$\Rightarrow m = \frac{2}{3}, -1$$

$\therefore$ Equations of invariant lines : $y = \frac{2}{3}x$ and $y = -x$.

3 The matrix **M** represents the sequence of two transformations in the x-y plane given by a rotation of 60° anticlockwise about the origin followed by a one-way stretch in the x-direction, scale factor d ($d \neq 0$).

(a) Find **M** in terms of d. [4]

$$M = \begin{bmatrix} d & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix}$$

$$= \begin{bmatrix} d & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$M = \begin{bmatrix} \frac{d}{2} + 0 & \frac{-\sqrt{3}}{2}d + 0 \\ 0 + \frac{\sqrt{3}}{2} & -0 + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{d}{2} & -\frac{\sqrt{3}}{2}d \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

(b) The unit square in the x-y plane is transformed by **M** onto a parallelogram of area $\frac{1}{2}d^2$ units^2.

Show that $d = 2$. [2]

Area of unit square $= 1^2 = 1$.

$$\det M = \frac{d}{2}\left(\frac{1}{2}\right) - \left(\frac{-\sqrt{3}d}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{d}{4} + \frac{3d}{4} = d.$$

Area of parallelogram $= |\det M| \times$ Area of unit square

$$\frac{d^2}{2} = |d| \times 1$$

$$\frac{d^2}{2} = d$$

$$d = 2.$$

The matrix **N** is such that $\mathbf{MN} = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

(c) Find **N**. [3]

$$d = 2 \Rightarrow M = \begin{bmatrix} \frac{2}{2} & \frac{-2\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & -\sqrt{3} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$\det M = 1\left(\frac{1}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(-\sqrt{3}\right) = 2.$$

$$M^{-1} = \frac{1}{2}\begin{bmatrix} \frac{1}{2} & \sqrt{3} \\ \frac{-\sqrt{3}}{2} & 1 \end{bmatrix}$$

$$MN = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow N = M^{-1}\begin{bmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{1}{2} & \sqrt{3} \\ -\frac{\sqrt{3}}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$N = \frac{1}{2}\begin{bmatrix} \frac{1+\sqrt{3}}{2} & \frac{1+\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{2} & \frac{1-\sqrt{3}}{2} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1+\sqrt{3} & 1+\sqrt{3} \\ 1-\sqrt{3} & 1-\sqrt{3} \end{bmatrix}$$

(d) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plan represented by **MN**. [5

Let, equation of invariant line : $y = mx$.

$$\begin{bmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} t \\ mt \end{bmatrix} = \begin{bmatrix} T \\ mT \end{bmatrix}$$

$$\Rightarrow t + mt = T \rightarrow (1)$$

$$\Rightarrow \frac{1}{2}t + \frac{1}{2}mt = mT \rightarrow (2)$$

Divide (2) by (1):

$$\frac{\frac{1}{2}t + \frac{1}{2}mt}{t + mt} = \frac{mT}{T}$$

$$\frac{\frac{1}{2} + \frac{1}{2}m}{1+m} = m$$

$$\frac{1}{2} + \frac{1}{2}m = m + m^2$$

$$1 + m = 2m + 2m^2$$

$$\Rightarrow 2m^2 + m - 1 = 0$$

$$2m^2 + 2m - m - 1 = 0$$

$$(2m-1)(m+1) = 0 \Rightarrow m = \frac{1}{2}, -1.$$

$\therefore$ Invariant lines : $y = \frac{1}{2}x$, $y = -x$.

4 The matrices **A**, **B** and **C** are given by

$$\mathbf{A} = \begin{pmatrix} 2 & k & k \\ 5 & -1 & 3 \\ 1 & 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \quad \mathbf{C} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix},$$

where k is a real constant.

(a) Find **CAB**. [3]

CAB

$$= \begin{bmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & k & k \\ 5 & -1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -1 & 4 \\ 8 & -k-2 & -k+6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -1 \\ 14-k & -k-2 \end{bmatrix}$$

(b) Given that **A** is singular, find the value of k. [3]

$$\det A = 2\begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix} - k\begin{vmatrix} 5 & 3 \\ 1 & 1 \end{vmatrix} + k\begin{vmatrix} 5 & -1 \\ 1 & 0 \end{vmatrix}$$

$$= 2[-1-0] - k[5-3] + k[0+1]$$

$$\det A = -2-2k+k = -2-k$$

Since A is singular, $\det A = 0$

$$\Rightarrow -2-k = 0$$

$$k = -2.$$

(c) Using the value of k from part **(b)**, find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **CAB**. [5]

$$k = -2 \Rightarrow CAB = \begin{bmatrix} 10 & -1 \\ -(-2)+14 & -(-2)-2 \end{bmatrix} = \begin{bmatrix} 10 & -1 \\ 16 & 0 \end{bmatrix}$$

Let, equation of invariant line : $y = mx$.

$$\Rightarrow CAB \begin{bmatrix} t \\ mt \end{bmatrix} = \begin{bmatrix} T \\ mT \end{bmatrix}$$

$$\begin{bmatrix} 10 & -1 \\ 16 & 0 \end{bmatrix} \begin{bmatrix} t \\ mt \end{bmatrix} = \begin{bmatrix} T \\ mT \end{bmatrix}$$

$$\begin{bmatrix} 10t - mt \\ 16t \end{bmatrix} = \begin{bmatrix} T \\ mT \end{bmatrix}$$

$$(10-m)t = T \rightarrow (1)$$

$$16t = mT \rightarrow (2)$$

Dividing (2) by (1):

$$\frac{16t}{(10-m)t} = \frac{mT}{T}$$

$$\frac{16}{10-m} = m$$

$$16 = 10m - m^2$$

$$m^2 - 10m + 16 = 0$$

$$m^2 - 2m - 8m + 16 = 0$$

$$(m-2)(m-8) = 0$$

$$m = 2, 8.$$

$\Rightarrow$ Equations of invariant lines: $y = 2x$, $y = 8x$.

5 The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & k & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

(a) Find the set of values of k for which $\mathbf{A}$ is non-singular. [3]

$$\det A = 1\begin{vmatrix} k & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & k \\ 7 & 8 \end{vmatrix} = 9k-48-2(-6)+3(32-7k)$$

$$\det A = 9k-48+12+96-21k = 60-12k.$$

A is non-singular if $\det A \neq 0$

$$\Rightarrow 60-12k \neq 0$$
$$-12k \neq -60$$
$$k \neq 5$$

(b) Given that $\mathbf{A}$ is non-singular, find, in terms of k, the entries in the top row of $\mathbf{A}^{-1}$. [4]

$$a_{11} = \frac{1}{60-12k}\begin{vmatrix} k & 6 \\ 8 & 9 \end{vmatrix} = \frac{9k-48}{60-12k} = \frac{3(3k-16)}{3(20-4k)} = \frac{3k-16}{20-4k}.$$

$$a_{12} = \frac{-1}{60-12k}\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = \frac{-(36-42)}{60-12k} = \frac{6}{6(10-2k)} = \frac{1}{10-2k}.$$

$$a_{13} = \frac{1}{60-12k}\begin{vmatrix} 4 & k \\ 7 & 8 \end{vmatrix} = \frac{32-7k}{60-12k}.$$

$$a_{21} = \frac{-1}{60-12k}\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = \frac{-(18-24)}{60-12k} = \frac{6}{60-12k} = \frac{1}{10-2k}.$$

$$a_{31} = \frac{+1}{60-12k}\begin{vmatrix} 2 & 3 \\ k & 6 \end{vmatrix} = \frac{1}{60-12k}(12-3k) = \frac{12-3k}{60-12k} = \frac{4-k}{20-4k}.$$

$$\begin{bmatrix} \frac{3k-16}{20-4k} & \frac{1}{10-2k} & \frac{32-7k}{60-12k} \\ \frac{1}{10-2k} & \cdot & \\ \frac{4-k}{20-4k} & & \cdot \end{bmatrix}$$

When transpose of matrix is found a_{12} switches position with a_{21} and a_{13} switches position with a_{31}.

$\Rightarrow$ Top row of A^{-1} : $\frac{3k-16}{20-4k}$, $\frac{1}{10-2k}$, $\frac{4-k}{20-4k}$.

(c) Given that $\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, give an example of a matrix $\mathbf{C}$ such that $\mathbf{BAC} = \begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix}$. [4]

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & k & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & k & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & k & 6 \end{bmatrix}$$

Let, $C = \begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix}$.

$$\Rightarrow BAC = \begin{bmatrix} 1 & 2 & 3 \\ 4 & k & 6 \end{bmatrix}\begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix} = \begin{bmatrix} u+2w+3y & v+2x+3z \\ 4u+kw+6y & 4v+kx+6z \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ k & 4 \end{bmatrix}$$

By comparison, $u+2w+3y = 2$ $\quad v+2x+3z = 1$

and $4u+kw+6y = k$ $\quad 4v+kx+6z = 4$

System is satisfied if $u=0, w=1, y=0$

and $v=1, x=0, z=0$

$$\Rightarrow C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(d) Find the set of values of k for which the transformation in the x-y plane represented by $\begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix}$ has two distinct invariant lines through the origin. [6]

Let invariant lines have the equation : $y = mx$.

$$\begin{bmatrix} 2 & 1 \\ k & 4 \end{bmatrix}\begin{bmatrix} t \\ mt \end{bmatrix} = \begin{bmatrix} T \\ mT \end{bmatrix}$$

$$2t + mt = T \rightarrow (1)$$
$$kt + 4mt = mT \rightarrow (2)$$

Dividing (2) by (1) gives:

$$\frac{kt+4mt}{2t+mt} = \frac{mT}{T}$$

$$\frac{k+4m}{2+m} = m$$

$$k + 4m = m^2 + 2m$$

$$0 = m^2 - 2m - k$$

For two distinct invariant lines, we require two distinct values of m

$$\Rightarrow \quad b^2 - 4ac > 0$$
$$(-2)^2 - 4(1)(-k) > 0$$
$$4 + 4k > 0$$
$$k > \frac{-4}{4}$$
$$k > -1.$$

$\det\begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix} = 8 - k$. If $\det\begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix} = 0 \Rightarrow 8 - k = 0 \Rightarrow k = 8.$

Then $\begin{bmatrix} 2 & 1 \\ k & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 8 & 4 \end{bmatrix}$. For invariant line(s):

$$\begin{bmatrix} 2 & 1 \\ 8 & 4 \end{bmatrix}\begin{bmatrix} t \\ mt \end{bmatrix} = \begin{bmatrix} T \\ mT \end{bmatrix}$$

$$2t + mt = T$$
$$8t + 4mt = mT$$

$$\Rightarrow \frac{8t+4mt}{2t+mt} = \frac{mT}{T} \Rightarrow \frac{8+4m}{2+m} = m$$

$$\frac{4(2+m)}{2+m} = m \Rightarrow m = 4$$

$\Rightarrow$ Then there is one invariant line if $m = 4$.

$\Rightarrow$ There are two distinct invariant lines if $k > -1$ and $k \neq +8$.

6 The cubic equation $x^3+5x^2+10x-2=0$ has roots α, β, γ.

(a) Find the value of $\alpha^2+\beta^2+\gamma^2$.

$\Sigma\alpha = \alpha+\beta+\gamma = \frac{-b}{a} = \frac{-5}{1} = -5.$

$\Sigma\alpha\beta = \alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a} = \frac{10}{1} = 10.$

$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$

$= (-5)^2 - 2(10)$

$= 25 - 20$

$= 5.$

(b) Show that the matrix $\begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{pmatrix}$ is singular. [4]

Let, $A = \begin{bmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{bmatrix}$

$\det A = 1\begin{vmatrix} 1 & \gamma \\ \gamma & 1 \end{vmatrix} - \alpha\begin{vmatrix} \alpha & \gamma \\ \beta & 1 \end{vmatrix} + \beta\begin{vmatrix} \alpha & 1 \\ \beta & \gamma \end{vmatrix}$

$= 1[1-\gamma^2] - \alpha[\alpha - \beta\gamma] + \beta[\alpha\gamma - \beta]$

$= 1 - \gamma^2 - \alpha^2 + \alpha\beta\gamma + \alpha\beta\gamma - \beta^2$

$= 1 - (\alpha^2+\beta^2+\gamma^2) + 2\alpha\beta\gamma.$

$\alpha\beta\gamma = \frac{-d}{a} = \frac{-(-2)}{1} = 2.$

$\Rightarrow \det A = 1 - (5) + 2(2)$

$= 1 - 5 + 4$

$= 0$

$\Rightarrow$ Matrix A is singular.

7 Let $\mathbf{A} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, where a is a positive constant.

(a) State the type of the geometrical transformation in the x–y plane represented by $\mathbf{A}$. [1]

Shear in the x-direction with shear factor a.

(b) Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}.$$ [5]

Let, $P(n)$ be the statement that $A^n = \begin{bmatrix} 1 & na \\ 0 & 1 \end{bmatrix}$

Basic case : $n=1$

$L.H.S. = A^n = A^1 = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

$R.H.S. = \begin{bmatrix} 1 & na \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

$\therefore P(1)$ is true.

Inductive step : Let, $P(n)$ is true for $n=k$.

$\Rightarrow A^k = \begin{bmatrix} 1 & ka \\ 0 & 1 \end{bmatrix}$

$A^{k+1} = A^k A = \begin{bmatrix} 1 & ka \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & a+ka \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & (1+k)a \\ 0 & 1 \end{bmatrix}$

$\Rightarrow P(n)$ is true for $n=k+1$.

Then, by mathematical induction, $P(n)$ is true for all positive integers n.

Let $\mathbf{B} = \begin{pmatrix} b & b \\ a^{-1} & a^{-1} \end{pmatrix}$, where b is a positive constant.

(c) Find the equations of the invariant lines, through the origin, of the transformation in the x–y plane represented by $\mathbf{A}^n\mathbf{B}$. [6]

$$A^n B = \begin{bmatrix} 1 & na \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b & b \\ a^{-1} & a^{-1} \end{bmatrix} = \begin{bmatrix} b+n & b+n \\ a^{-1} & a^{-1} \end{bmatrix}$$

Let, equation of invariant line : $y = mx$

$$\begin{bmatrix} b+n & b+n \\ a^{-1} & a^{-1} \end{bmatrix} \begin{bmatrix} t \\ mt \end{bmatrix} = \begin{bmatrix} T \\ mT \end{bmatrix}$$

$$(b+n)t + (b+n)mt = T \rightarrow (1)$$

$$(a^{-1})t + (a^{-1})mt = mT \rightarrow (2)$$

Dividing (2) by (1):

$$\frac{(a^{-1} + a^{-1}m)t}{[(b+n) + (b+n)m]t} = \frac{mT}{T}$$

$$\frac{\frac{1}{a} + \frac{m}{a}}{(b+n) + (b+n)m} = m$$

$$\frac{m+1}{a} = m[(b+n) + (b+n)m]$$

$$(m+1) = am(b+n)(m+1)$$

$$am(b+n)(m+1) - (m+1) = 0$$

$$(m+1)[am(b+n) - 1] = 0$$

Either, $m + 1 = 0$ or $am(b+n) = 1$

$m = -1$ or $m = \frac{1}{a(b+n)}$.

$\Rightarrow$ Equations of invariant lines:

$y = -x$ or $y = \frac{1}{a(b+n)}x$.

8 The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, where a and b are positive constants.

(a) The matrix $\mathbf{M}$ represents a sequence of two geometrical transformations.

State the type of each transformation, and make clear the order in which they are applied. [2]

Transformation represented by $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$: Stretch in x-direction with stretch factor a

Transformation represented by $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$: Shear in x-direction with shear factor b.

Order of transformations: Stretch followed by a shear.

The unit square in the x-y plane is transformed by $\mathbf{M}$ onto parallelogram $OPQR$.

(b) Find, in terms of a and b, the matrix which transforms parallelogram $OPQR$ onto the unit square. [2]

$$M = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

$$\det M = a(1) - b(0) = a.$$

$$M^{-1} = \frac{1}{a}\begin{bmatrix} 1 & -b \\ 0 & a \end{bmatrix} = \begin{bmatrix} 1/a & -b/a \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ Matrix that transforms parallelogram OPQR onto unit square

$$= \begin{bmatrix} 1/a & -b/a \\ 0 & 1 \end{bmatrix}.$$

It is given that the area of $OPQR$ is $2\,\text{cm}^2$ and that the line $x+3y=0$ is invariant under the transformation represented by $\mathbf{M}$.

(c) Find the values of a and b. [5]

$M = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \Rightarrow \det M = a(1) - b(0) = a.$

Area of OPQR = $|\det M| \times$ Area of unit square

$2 = |a| \times 1$

$2 = |a| \Rightarrow a = 2.$

Since $x+3y=0 \Rightarrow y = \frac{-x}{3} \Rightarrow$ point $\left(t, \frac{-t}{3}\right)$ is sent to $\left(T, \frac{-T}{3}\right)$.

$\Rightarrow \begin{bmatrix} 2 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ \frac{-t}{3} \end{bmatrix} = \begin{bmatrix} T \\ \frac{-T}{3} \end{bmatrix}$

$2t - \frac{bt}{3} = T \rightarrow (1)$

$\frac{-t}{3} = \frac{-T}{3} \rightarrow (2)$

Divide (2) by (1): $\dfrac{\frac{-t}{3}}{2t - \frac{bt}{3}} = \dfrac{\frac{-T}{3}}{T}$

$\dfrac{\frac{-1}{3}}{2 - \frac{b}{3}} = \frac{-1}{3}$

$1 = 2 - \frac{b}{3}$

$\frac{b}{3} = 1$

$\Rightarrow b = 3.$

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9 The cubic equation $x^3 + cx + 1 = 0$, where c is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^3, \beta^3, \gamma^3$.

Let, $y = x^3 \Rightarrow x = y^{\frac{1}{3}}$.

$$\Rightarrow \left(y^{\frac{1}{3}}\right)^3 + cy^{\frac{1}{3}} + 1 = 0$$

$$y + cy^{\frac{1}{3}} + 1 = 0$$

$$cy^{\frac{1}{3}} = -y - 1$$

$$\left(cy^{\frac{1}{3}}\right)^3 = (-y-1)^3$$

$$c^3 y = -(y+1)^3$$

$$c^3 y = -(y^3 + 3y^2 + 3y + 1)$$

$y^3 + 3y^2 + (c^3+3)y + 1 = 0$ has roots $\alpha^3, \beta^3, \gamma^3$.

(b) Show that $\alpha^6 + \beta^6 + \gamma^6 = 3 - 2c^3$.

$$y^3 + 3y^2 + (c^3+3)y + 1 = 0.$$

$$\Sigma\alpha^3 = \alpha^3 + \beta^3 + \gamma^3 = \frac{-b}{a} = \frac{-3}{1} = -3.$$

$$\Sigma\alpha^3\beta^3 = \alpha^3\beta^3 + \alpha^3\gamma^3 + \beta^3\gamma^3 = \frac{C}{A} = \frac{c^3+3}{1} = c^3 + 3.$$

$$\Rightarrow \Sigma\alpha^6 = (\Sigma\alpha^3)^2 - 2(\Sigma\alpha^3\beta^3)$$

$$= (-3)^2 - 2(c^3+3)$$

$$= 9 - 2c^3 - 6$$

$$= 3 - 2c^3. \text{ (Shown)}.$$

(c) Find the real value of c for which the matrix $\begin{pmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{pmatrix}$ is singular. [5]

Let, $A = \begin{bmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{bmatrix}$.

$$\det A = 1\begin{vmatrix} 1 & \gamma^3 \\ \gamma^3 & 1 \end{vmatrix} - \alpha^3\begin{vmatrix} \alpha^3 & \gamma^3 \\ \beta^3 & 1 \end{vmatrix} + \beta^3\begin{vmatrix} \alpha^3 & 1 \\ \beta^3 & \gamma^3 \end{vmatrix}$$

$$= 1[1-\gamma^6] - \alpha^3[\alpha^3 - \beta^3\gamma^3] + \beta^3[\alpha^3\gamma^3 - \beta^3]$$

$$= 1 - \gamma^6 - \alpha^6 + \alpha^3\beta^3\gamma^3 + \alpha^3\beta^3\gamma^3 - \beta^6$$

$$\det A = 1 - (\alpha^6 + \beta^6 + \gamma^6) + 2\alpha^3\beta^3\gamma^3$$

From equation $y^3 + 3y^2 + (c^3+3)y + 1 = 0$,

$$\alpha^3\beta^3\gamma^3 = \frac{-d}{a} = \frac{-1}{1} = -1.$$

$$\Rightarrow \det A = 1 - (3 - 2c^3) + 2(-1)$$

$$= 1 - 3 + 2c^3 - 2$$

$$\det A = 2c^3 - 4.$$

For A to be singular, $\det A = 0$

$$\Rightarrow 2c^3 - 4 = 0$$

$$c^3 = 2$$

$$c = \sqrt[3]{2}.$$

10 The matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}.$$

(a) Give full details of the geometrical transformation in the x-y plane represented by $\mathbf{A}$. [1]

Reflection in the line $y = x$.

(b) Give full details of the geometrical transformation in the x-y plane represented by $\mathbf{B}$. [2]

By comparison to $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $\cos\theta = \frac{1}{2}$ $\quad \sin\theta = \frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{3}$ $\quad \theta = \frac{\pi}{3}$.

$\Rightarrow$ Rotation by $\frac{\pi}{3}$ radians in the anticlockwise direction about the origin.

The triangle DEF in the x-y plane is transformed by $\mathbf{AB}$ onto triangle PQR.

(c) Show that the triangles DEF and PQR have the same area. [

$$AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$AB = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-\sqrt{3}}{2} \end{bmatrix}$$

$$\det AB = \frac{\sqrt{3}}{2}\left(-\frac{\sqrt{3}}{2}\right) - \frac{1}{2}\left(\frac{1}{2}\right) = \frac{-3}{4} - \frac{1}{4} = -1.$$

Area of $\Delta PQR = |\det AB| \times$ Area of ΔDEF

$= |-1| \times$ Area of ΔDEF

$\Rightarrow$ Area of $\Delta PQR =$ Area of ΔDEF. (Shown).

(d) Find the matrix which transforms triangle PQR onto triangle DEF. [2]

$$AB = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix}$$

$\det AB = -1.$

$$\Rightarrow (AB)^{-1} = \frac{1}{-1}\begin{bmatrix} -\sqrt{3}/2 & -1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-\sqrt{3}}{2} \end{bmatrix}.$$

$\Rightarrow$ Matrix that transforms ΔPQR onto $\Delta DEF = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix}.$

(e) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **AB**. [5]

Let, equation of invariant line: $y = mx$

$$\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix}\begin{bmatrix} t \\ mt \end{bmatrix} = \begin{bmatrix} T \\ mT \end{bmatrix}$$

$$\begin{bmatrix} \frac{\sqrt{3}}{2}t + \frac{1}{2}mt \\ \frac{1}{2}t - \frac{\sqrt{3}}{2}mt \end{bmatrix} = \begin{bmatrix} T \\ mT \end{bmatrix}$$

$$\frac{\frac{1}{2}t - \frac{\sqrt{3}}{2}mt}{\frac{\sqrt{3}}{2}t + \frac{1}{2}mt} = \frac{mT}{T}$$

$$\Rightarrow \frac{\frac{1}{2} - \frac{\sqrt{3}}{2}m}{\frac{\sqrt{3}}{2} + \frac{m}{2}} = m$$

$$\frac{1}{2} - \frac{\sqrt{3}}{2}m = \frac{\sqrt{3}}{2}m + \frac{m^2}{2}$$

$$1 - \sqrt{3}m = \sqrt{3}m + m^2$$

$$0 = m^2 + 2\sqrt{3}m - 1$$

$$\Rightarrow m = \frac{-2\sqrt{3} \pm \sqrt{(2\sqrt{3})^2 - 4(1)(-1)}}{2(1)} = \frac{-2\sqrt{3} \pm \sqrt{16}}{2}$$

$$m = \frac{-2\sqrt{3} \pm 4}{2} = -\sqrt{3} \pm 2.$$

$\Rightarrow$ Equations of invariant lines: $y = (-\sqrt{3}+2)x$, $y = (-\sqrt{3}-2)x$.

$y - (2-\sqrt{3})x = 0$, $y + (2+\sqrt{3})x = 0$.

11 The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) The matrix $\mathbf{M}$ represents a sequence of two geometrical transformations.

State the type of each transformation, and make clear the order in which they are applied. [2

Transformation represented by $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$: Rotation by θ radians in the anticlockwise direction about the

Transformation represented by $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$: Stretch in x-direction with stretch factor 3.

Order of transformations: Stretch followed by a rotation.

(b) Find the values of θ, for $0 \leqslant \theta \leqslant \pi$, for which the transformation represented by $\mathbf{M}$ has exactl one invariant line through the origin, giving your answers in terms of π. [9

$$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3\cos\theta & -\sin\theta \\ 3\sin\theta & \cos\theta \end{bmatrix}.$$

Let, equation of invariant line : $y = mx$

$$\Rightarrow \begin{bmatrix} 3\cos\theta & -\sin\theta \\ 3\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} t \\ mt \end{bmatrix} = \begin{bmatrix} T \\ mT \end{bmatrix}$$

$$3t\cos\theta - mt\sin\theta = T \rightarrow (1)$$
$$3t\sin\theta + mt\cos\theta = mT \rightarrow (2)$$

Divided (2) by (1):

$$\frac{3t\sin\theta + mt\cos\theta}{3t\cos\theta - mt\sin\theta} = \frac{mT}{T}$$

$$\frac{3\sin\theta + m\cos\theta}{3\cos\theta - m\sin\theta} = m.$$

$$3\sin\theta + m\cos\theta = 3m\cos\theta - m^2\sin\theta$$

$$(\sin\theta)m^2 + (m - 3m)\cos\theta + 3\sin\theta = 0$$

$$(\sin\theta)m^2 - 2m\cos\theta + 3\sin\theta = 0.$$

Since there is only one invariant line, m has a repeated value.

$$\Rightarrow b^2 - 4ac = 0$$
$$(-2\cos\theta)^2 - 4(\sin\theta)(3\sin\theta) = 0$$
$$4\cos^2\theta - 12\sin^2\theta = 0$$

$$4\cos^2\theta = 12\sin^2\theta$$

$$\frac{1}{3} = \tan^2\theta$$

$$\Rightarrow \tan\theta = \pm\frac{1}{\sqrt{3}}. \quad \text{for } 0 \leq \theta \leq \pi.$$

Since $\tan\theta$ is positive and negative, θ lies in 1^{st} or 2^{nd} quadrant.

$$\tan\alpha = \frac{1}{\sqrt{3}}$$

$$\alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

$$\Rightarrow \theta = \alpha, \pi - \alpha$$

$$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$

12 **(a)** Give full details of the geometrical transformation in the x-y plane represented by the matrix $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$. [1]

Enlargement with scale factor 6 with center at the origin.

Let $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix}$.

(b) The triangle DEF in the x-y plane is transformed by $\mathbf{A}$ onto triangle PQR.

Given that the area of triangle DEF is $13\,\text{cm}^2$, find the area of triangle PQR. [2]

$\det A = 3(2) - 4(2) = 6 - 8 = -2.$

Area of $\Delta PQR = |\det A| \times$ Area of $\Delta DEF \Rightarrow$ Area of $\Delta PQR = |-2| \times 13$

$= 26\text{ cm}^2.$

(c) Find the matrix $\mathbf{B}$ such that $\mathbf{AB} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$. [2]

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 2 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1.5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \Rightarrow B = A^{-1}\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1.5 \end{bmatrix}\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} -6 & 12 \\ 6 & -9 \end{bmatrix}$$

(d) Show that the origin is the only invariant point of the transformation in the x-y plane represented by $\mathbf{A}$. [4]

Let, (x,y) be invariant point: $\begin{bmatrix} 3 & 4 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

$3x + 4y = x \Rightarrow 2x + 4y = 0 \Rightarrow x = -2y. \rightarrow (1)$

$2x + 2y = y \Rightarrow y + 2x = 0 \Rightarrow y = -2x \rightarrow (2)$

Substitute (1) in (2): $y = -2(-2y)$

$y = 4y$

$0 = 3y$

$y = 0$

$\Rightarrow x = -2(0) = 0$

$\Rightarrow$ Invariant point: $(0,0)$.