Past Paper Questions: Projectile Motion

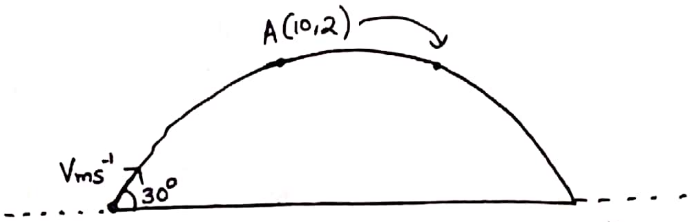
1 A ball is projected from a point O with speed $V\,\text{m s}^{-1}$, at an angle of $30°$ above the horizontal. At time T s after projection, the ball passes through the point A, whose horizontal and vertically upward displacements from O are 10 m and 2 m respectively.

(i) By using the equation of the trajectory, or otherwise, find the value of V. [3]

(ii) Find the value of T. [2]

(iii) Find the angle that the direction of motion of the ball at A makes with the horizontal. [4]

i)



$$y = x\tan\theta - \frac{gx^2}{2u^2}\sec^2\theta .$$

When $x = 10,\ y = 2$

$$\Rightarrow 2 = 10\tan\theta - \frac{10(10)^2}{2V^2}\sec^2\theta$$

Since $\theta = 30° \Rightarrow$ $$2 = 10\tan 30° - \frac{500\sec^2 30°}{V^2}$$

$$2 = \frac{10\sqrt{3}}{3} - \frac{2000}{3V^2}$$

$$\frac{2000}{3V^2} = \frac{10\sqrt{3}}{3} - 2 \Rightarrow V^2 = \frac{2000}{3\left(\frac{10\sqrt{3}}{3} - 2\right)}$$

$$V = 13.29 \approx 13.3\ \text{ms}^{-1}.$$

ii) In ~~vertical~~ horizontal direction,

$$x = v_x \times t$$

$$10 = V\cos 30° \times T$$

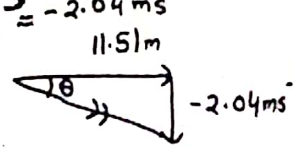
$$10 = 13.29 \times \frac{\sqrt{3}}{2} \times T$$

$$\Rightarrow T = 0.8687\,\text{s} \approx 0.869\,\text{s}.$$

iii) When $T = 0.869\,\text{s}$,

Horizontally, $v_x = V\cos 30° = 13.29 \times \frac{\sqrt{3}}{2} = 11.51\ \text{ms}^{-1}$.

Vertically, $v = u + at \Rightarrow v_y = (V\sin 30°) + (-10)(0.869) = -2.0414\ \text{m/s} \approx -2.04\ \text{ms}^{-1}$



$$\tan\theta = \frac{-2.04}{11.51} = -0.1773 \Rightarrow \theta = -10.05° \approx -10.1°$$

Direction of motion at A = $10.1°$ below horizontal.

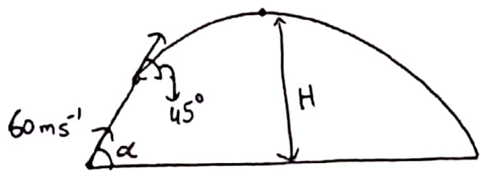
2 A particle is projected with speed $60 \mathrm{~m\,s^{-1}}$ from a point on horizontal ground. The angle of projection is α° above the horizontal. The particle reaches the ground again after 10 s.

(i) Find the value of α. [3]

(ii) Find the greatest height reached by the particle. [2]

(iii) At time T s after the instant of projection the direction of motion of the particle is at an angle of 45° above the horizontal. Find the value of T. [4]

i)



Since particle reaches ground level at 10s, it reaches maximum height at $t=5$.

Vertically, Using $v = u + at$

$$0 = 60\sin\alpha + (-10)(5)$$

$$50 = 60\sin\alpha$$

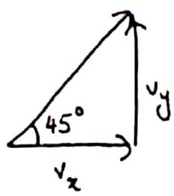
$$\Rightarrow \sin\alpha = \frac{5}{6} \Rightarrow \alpha = \sin^{-1}\left(\frac{5}{6}\right) = 56.44^\circ \approx 56.4^\circ$$

ii) For greatest height, $t=5$.

Using $s = ut + \frac{1}{2}at^2$

$$H = (60\sin\alpha)(5) + \frac{1}{2}(-10)(5)^2 = 60 \times \frac{5}{6} \times 5 - 5 \times 5^2 = 125 \text{m}.$$

iii) When $t = T$,



Horizontally, $v_x = 60\cos\alpha = 33.166 \text{ ms}^{-1}$

Vertically, $v_y = 60\sin\alpha - 10(T) = 60\sin\alpha - 10T = 50 - 10T \text{ ms}^{-1}$

~~I~~ From the vector diagram,

$$\tan 45^\circ = \frac{v_y}{v_x}$$

$$1 = \frac{v_y}{v_x}$$

$$\Rightarrow v_y = v_x$$

$$50 - 10T = 33.166$$

$$16.833 = 10T$$

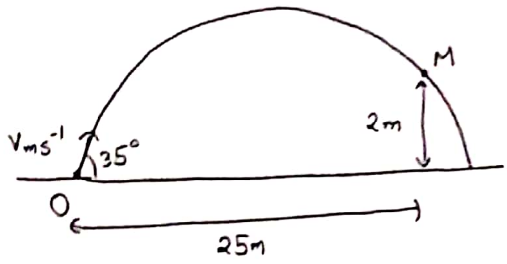
$$\Rightarrow T = 1.683 \approx 1.68\text{s}.$$

3 A particle is projected from a point O on horizontal ground. The velocity of projection has magnitude $V\,\mathrm{m\,s^{-1}}$ and direction upwards at 35° to the horizontal. The particle passes through the point M at time T seconds after the instant of projection. The point M is 2 m above the ground and at a horizontal distance of 25 m from O.

(i) Find the values of V and T. [5]

(ii) Find the speed of the particle as it passes through M and determine whether it is moving upwards or downwards. [4]

i)



Using $y = x\tan\theta - \dfrac{g\sec^2\theta\, x^2}{2u^2}$

when $x = 25, y = 2$ with $\theta = 35°$

$\Rightarrow 2 = 25\tan 35° - \dfrac{10(\sec^2 35°)(25^2)}{2 \times V^2}$

$2 = 25\tan 35° - \dfrac{3125\sec^2 35°}{V^2}$

$V^2 = \dfrac{3125\sec^2 35°}{25\tan 35° - 2} \Rightarrow V = 17.33\,\mathrm{ms^{-1}} \approx 17.3\,\mathrm{ms^{-1}}.$

Horizontally, $v_x \times T = x \Rightarrow 25 = V\cos 35° \times T$

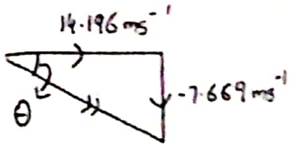
$T = \dfrac{25}{17.33 \times \cos 35°} = 1.760\,s = 1.76\,s.$

ii) At M,

Horizontally, $V_x = V\cos 35° = 17.33 \times \cos 35° = 14.196\,\mathrm{ms^{-1}}.$

Vertically, using $v = u + at$

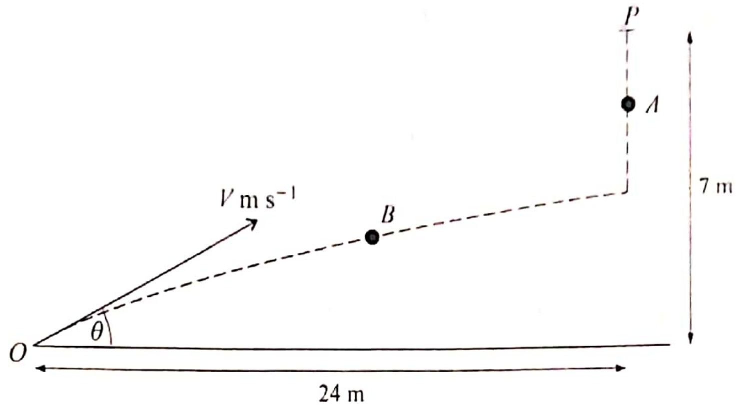
$v_y = 17.33 \times \sin 35° - 10(1.760) = -7.669\,\mathrm{ms^{-1}}.$



$\tan\theta = \dfrac{-7.669}{14.196} \Rightarrow \theta = -28.37° = -28.4°$

At M, direction of motion is 28.4° below horizontal

∴ particle is moving downwards.



A particle A is released from rest at time $t = 0$, at a point P which is 7 m above horizontal ground. At the same instant as A is released, a particle B is projected from a point O on the ground. The horizontal distance of O from P is 24 m. Particle B moves in the vertical plane containing O and P, with initial speed $V\,\mathrm{m\,s^{-1}}$ and initial direction making an angle of θ above the horizontal (see diagram). Write down

(i) an expression for the height of A above the ground at time t s, [1]

(ii) an expression in terms of V, θ and t for

(a) the horizontal distance of B from O, [1]

(b) the height of B above the ground. [1]

At time $t = T$ the particles A and B collide at a point above the ground.

(iii) Show that $\tan\theta = \frac{7}{24}$ and that $VT = 25$. [6]

(iv) Deduce that $7V^2 > 3125$. [3]

i) For particle A, using $s = ut + \frac{1}{2}at^2 = 0(t) + \frac{1}{2}(10)t^2 = 5t^2$.

$\therefore$ height of A above ground $= 7 - 5t^2$.

ii) a) Horizontally, ~~s = ut +~~

$x = v_x \times t$

$x = (V\cos\theta) \times t = Vt\cos\theta$. $\Rightarrow$ Horizontal distance of B from O $= Vt\cos\theta$

b) Vertically, using $s = ut + \frac{1}{2}at^2$

$y = (V\sin\theta)t + \frac{1}{2}(-10)t^2$

$y = Vt\sin\theta - 5t^2 \Rightarrow$ Height of B above ground $= Vt\sin\theta - 5t^2$.

iii) When $t = T$,

height of A $=$ height of B

$7 - 5t^2 = Vt\sin\theta - 5t^2$

$$7 = Vt\sin\theta \rightarrow (1)$$

Horizontally,

$$Vt\cos\theta = 24 \rightarrow (2)$$

Dividing (1) by (2) gives: $\dfrac{Vt\sin\theta}{Vt\cos\theta} = \dfrac{7}{24}$

$$\tan\theta = \frac{7}{24} \text{ (Shown).}$$

Squaring (1) and (2) and adding gives:

$$V^2t^2\sin^2\theta + V^2t^2\cos^2\theta = 7^2 + 24^2$$

$$V^2t^2(\sin^2\theta + \cos^2\theta) = 625$$

$$V^2t^2 = 625$$

$$Vt = 25. \text{ (Shown).}$$

iv) Since collision occurs above ground,

height of A above ground > 0

$$\Rightarrow 7 - 5t^2 > 0$$

Substituting $t = \dfrac{25}{V}$ gives:

$$7 - 5\left(\frac{25}{V}\right)^2 > 0$$

$$7 - \frac{3125}{V^2} > 0$$

$$7V^2 - 3125 > 0$$

$$7V^2 > 3125$$

(Shown).

5 A stone is projected from a point O on horizontal ground with speed $V \text{ m s}^{-1}$ at an angle θ above the horizontal, where $\sin\theta = \frac{3}{5}$. The stone is at its highest point when it has travelled a horizontal distance of 19.2 m.

(i) Find the value of V. [3]

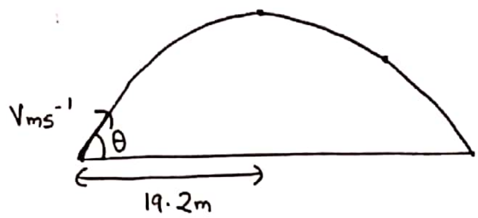
After passing through its highest point the stone strikes a vertical wall at a point 4 m above the ground.

(ii) Find the horizontal distance between O and the wall. [4]

At the instant when the stone hits the wall the horizontal component of the stone's velocity is halved in magnitude and reversed in direction. The vertical component of the stone's velocity does not change as a result of the stone hitting the wall.

(iii) Find the distance from the wall of the point where the stone reaches the ground. [4]

i)



$\sin\theta = \frac{3}{5}$ $\Rightarrow \sin 2\theta = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)$

$\cos\theta = \frac{4}{5}$ $= \frac{24}{25}$

$\tan\theta = \frac{3}{4}$.

Range of stone $= 19.2 \times 2 = 38.4\text{m}$.

$$\text{Range} = \frac{u^2 \sin 2\theta}{g}$$

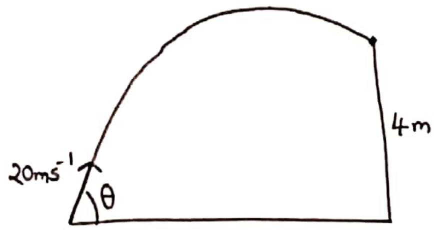
$$38.4 = \frac{V^2 \times \left(\frac{24}{25}\right)}{10}$$

$$V = 20\text{ms}^{-1}.$$

ii) Vertically, using $s = ut + \frac{1}{2}at^2$

$$4 = (20\sin\theta)t + \frac{1}{2}(-10)t^2$$

$$4 = 12t - 5t^2$$



$$5t^2 - 12t + 4 = 0$$

$$5t^2 - 10t - 2t + 4 = 0$$

$$(5t-2)(t-2) = 0$$

$\Rightarrow t = \frac{2}{5}$ or $t = 2$.

(Ignore since particle needs to cross highest point).

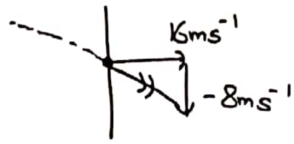
$\therefore t = 2$.

Horizontally, $x = v_x \times t = 20\cos\theta \times 2 = 20\left(\frac{4}{5}\right) \times 2 = 32\text{m}$.

iii) Before collision:

$V_x = 20\cos\theta = 20 \times \frac{4}{5} = 16\,ms^{-1}.$

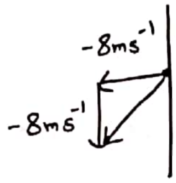
$V_y = 20\sin\theta - 10(2) = -8\,ms^{-1}.$



After collision,

$V_x = -\frac{1}{2}(16ms^{-1}) = -8ms^{-1}$

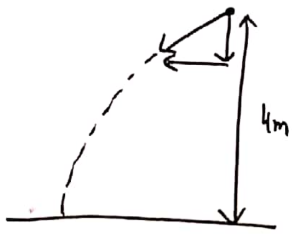
$V_y = -8ms^{-1}$



Vertically, using $s = ut + \frac{1}{2}at^2$

$$-4 = -8t + \frac{1}{2}(-10)t^2$$

$$-4 = -8t - 5t^2$$



$$5t^2 + 8t - 4 = 0$$

$$5t^2 + 10t - 2t - 4 = 0$$

$$(5t - 2)(t + 2) = 0$$

$\Rightarrow \quad t = \frac{2}{5} \quad$ or $\quad t = -2.$
(Ignore).

Horizontally,

$x = V_x \times t = -8 \times \frac{2}{5} = -3.2\ m.$

$\therefore$ Distance from wall when stone hits the ground $= 3.2m.$

6 A particle is projected with speed 65 m s^{-1} from a point on horizontal ground, in a direction making an angle of $\alpha°$ above the horizontal. The particle reaches the ground again after 12 s. Find

(i) the value of α, [3]

(ii) the greatest height reached by the particle, [2]

(iii) the length of time for which the direction of motion of the particle is between 20° above the horizontal and 20° below the horizontal, [5]

(iv) the horizontal distance travelled by the particle in the time found in part (iii). [1]

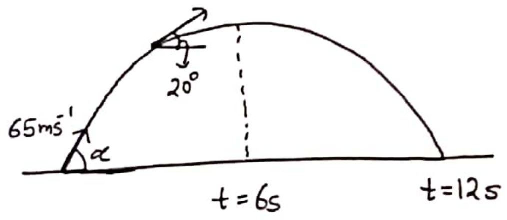
i) Vertically, When $t = 12s$, $y = 0$.

$y = (65 \sin\alpha)t + \frac{1}{2}(-10)t^2$

$0 = (65\sin\alpha)(12) - 5(12)^2$

$60 = 65 \sin\alpha$

$\sin\alpha = \frac{12}{13} \Rightarrow \alpha = \sin^{-1}\left(\frac{12}{13}\right) = 67.38° \approx 67.4°$



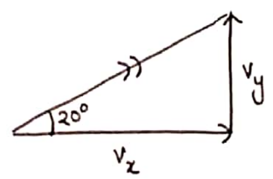
ii) Greatest height is reached when $t = \frac{12}{2} = 6s$.

Vertically, using $s = ut + \frac{1}{2}at^2$

$y = (65\sin\alpha)(6) + \frac{1}{2}(-10)(6)^2 = 180m$

$\therefore$ Greatest height $= 180m$.

iii) Horizontally, $v_x = 65\cos\alpha = 25ms^{-1}$.



Vertically, using $v = u + at$

$v_y = 65\sin\alpha - 10t = 60 - 10t$.

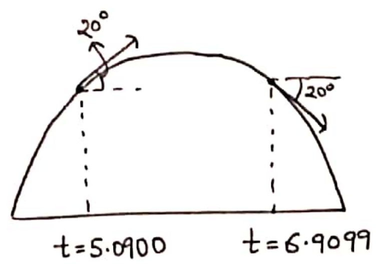
$\tan 20° = \frac{v_y}{v_x} \Rightarrow \tan 20° = \frac{60-10t}{25}$

$25\tan 20° = 60 - 10t$

$10t = 60 - 25\tan 20°$

$t = 5.090$

Using symmetry, direction of particle is 20° below horizontal when $t = 12 - 5.090 = 6.9099$

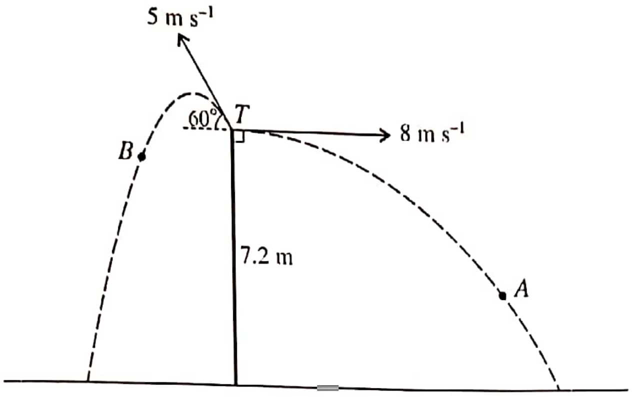


$\therefore$ Length of time for which direction of motion is between 20° above horizontal and 20° below horizontal

$= 6.9099 - 5.0900 = 1.819 \approx 1.82s$.

iv) Horizontal distance $= v_x \times t = 65\cos\alpha \times 1.82 = 45.49m \approx 45.5m$.

1



Particles A and B are projected simultaneously from the top T of a vertical tower, and move in the same vertical plane. T is 7.2 m above horizontal ground. A is projected horizontally with speed $8\,\mathrm{m\,s^{-1}}$ and B is projected at an angle of 60° above the horizontal with speed $5\,\mathrm{m\,s^{-1}}$. A and B move away from each other (see diagram).

(i) Find the time taken for A to reach the ground. [2]

At the instant when A hits the ground,

(ii) show that B is approximately 5.2 m above the ground, [2]

(iii) find the distance AB. [3]

i) For A, vertically, using $s = ut + \frac{1}{2}at^2$

$$-7.2 = 0(t) + \frac{1}{2}(-10)t^2$$

$$t^2 = 1.44 \Rightarrow t = 1.2\,s.$$

ii) For B, vertically, using $s = ut + \frac{1}{2}at^2$

When $t = 1.2s \Rightarrow$ $s = (5\sin 60°)(1.2) + \frac{1}{2}(-10)(1.2)^2 = -2.003\,m$

$\therefore$ Height above ground $= 7.2 + (-2.003) = 5.196 \approx 5.20\,m$ (Shown).

iii) For A, horizontally, $x = v_x \times t$

$$x_A = 8 \times 1.2 = 9.6\,m$$

For B, horizontally, $x = v_x \times t$

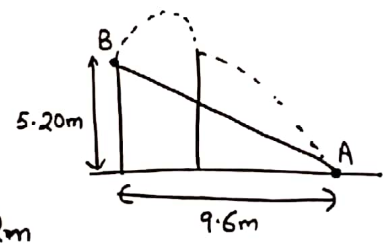
$$x_B = (-5\cos 60°) \times 1.2 = -3\,m.$$

$\therefore$ horizontal Distance between A and B $= 9.6 - (-3) = 12.6\,m.$

Vertical distance between A and B = 5.20 m.

$\Rightarrow$ Distance between A and B $= \sqrt{12.6^2 + 5.20^2} = 13.62\,m$

$\simeq 13.6\,m$



8 A small stone is projected from a point O on horizontal ground with speed $V\,\mathrm{m\,s^{-1}}$ at an angle $\theta°$ above the horizontal. Referred to horizontal and vertically upwards axes through O, the equation of the stone's trajectory is $y = 0.75x - 0.02x^2$, where x and y are in metres. Find

(i) the values of θ and V, [4]

(ii) the distance from O of the point where the stone hits the ground, [2]

(iii) the greatest height reached by the stone. [2]

i) $y = 0.75x - 0.02x^2$

By comparison to $y = x\tan\theta - \dfrac{g\sec^2\theta\, x^2}{2V^2}$

$\tan\theta = 0.75 \Rightarrow \theta = \tan^{-1}(0.75) = 36.86° \approx 36.9°$

and $\dfrac{g\sec^2\theta}{2V^2} = 0.02 \Rightarrow V = \sqrt{\dfrac{g\sec^2\theta}{2(0.02)}} = \sqrt{\dfrac{10\times\sec^2 36.86°}{2\times 0.02}}$

$V = 19.76\,\mathrm{ms^{-1}} \approx 19.8\,\mathrm{ms^{-1}}$

ii) When stone hits ground, $y = 0$

$\Rightarrow 0.75x - 0.02x^2 = 0$

$x(0.75 - 0.02x) = 0$

$x = 0$ or $0.75 - 0.02x = 0$

$x = \dfrac{0.75}{0.02} = 37.5\,\mathrm{m}$

i.e. distance from O when stone hits ground $= 37.5\mathrm{m}$.

iii) For greatest height, $x = \dfrac{37.5}{2} = 18.75\mathrm{m}$

When $x = 18.75$, $y = 0.75(18.75) - 0.02\times 18.75^2 = 7.031\,\mathrm{m} \approx 7.03\mathrm{m}$.

$\therefore$ Greatest height reached $= 7.03\mathrm{m}$.

9 A particle is projected from a point O on horizontal ground. The velocity of projection has magnitude $20\,\text{m s}^{-1}$ and direction upwards at an angle θ to the horizontal. The particle passes through the point which is 7 m above the ground and 16 m horizontally from O, and hits the ground at the point A.

(i) Using the equation of the particle's trajectory and the identity $\sec^2\theta = 1 + \tan^2\theta$, show that the possible values of $\tan\theta$ are $\frac{3}{4}$ and $\frac{17}{4}$. [4]

(ii) Find the distance OA for each of the two possible values of $\tan\theta$. [3]

(iii) Sketch in the same diagram the two possible trajectories. [2]

i) $$y = x\tan\theta - \frac{gx^2\sec^2\theta}{2u^2}$$

When $x = 16\text{m}$, $y = 7\text{m}$ with $u = 20$

$$\Rightarrow \quad 7 = 16\tan\theta - \frac{10(16)^2\sec^2\theta}{2\times 20^2}$$

$$7 = 16\tan\theta - \frac{16\sec^2\theta}{5}$$

$$35 = 80\tan\theta - 16\sec^2\theta$$

$$35 = 80\tan\theta - 16(1+\tan^2\theta)$$

$$35 = 80\tan\theta - 16 - 16\tan^2\theta$$

$$\Rightarrow 16\tan^2\theta - 80\tan\theta + 51 = 0$$

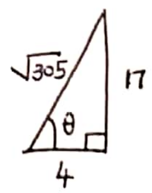
$$16\tan^2\theta - 12\tan\theta - 68\tan\theta + 51 = 0$$

$$(4\tan\theta - 17)(4\tan\theta - 3) = 0$$

Either, $4\tan\theta - 17 = 0$ or $4\tan\theta - 3 = 0$

$\tan\theta = \frac{17}{4}$ or $\tan\theta = \frac{3}{4}$ (Shown)

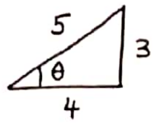
ii) Case 1 : $\tan\theta = \frac{17}{4} \Rightarrow \sin\theta = \frac{17}{\sqrt{305}}$, $\cos\theta = \frac{4}{\sqrt{305}}$



$$\Rightarrow \sin 2\theta = 2\left(\frac{17}{\sqrt{305}}\right)\left(\frac{4}{\sqrt{305}}\right) = \frac{136}{305}.$$

$$\text{Range} = \frac{u^2\sin 2\theta}{g} = \frac{20^2\times\left(\frac{136}{305}\right)}{10} = 17.83 \approx 17.8\text{m}. \Rightarrow OA = 17.8\text{m}$$

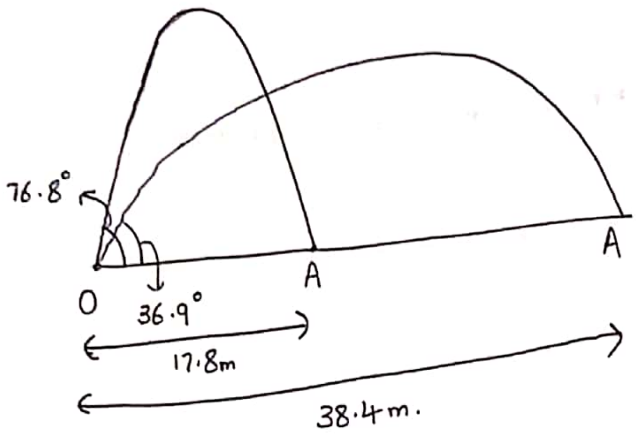
Case 2: $\tan\theta = \frac{3}{4} \Rightarrow \sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$



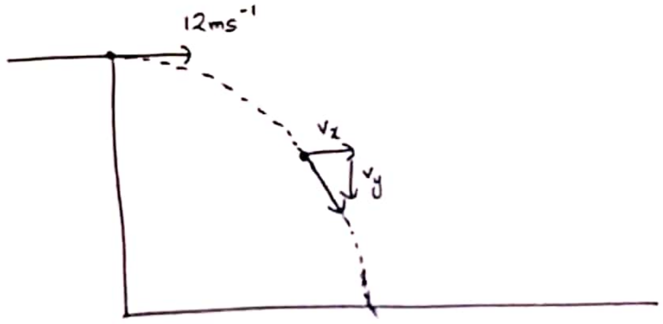
$$\Rightarrow \sin 2\theta = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}.$$

$$\text{Range} = \frac{u^2\sin 2\theta}{g} = \frac{20^2\times\left(\frac{24}{25}\right)}{10} = 38.4\text{m}. \Rightarrow OA = 38.4\text{m}.$$

iii)



10 A particle is projected horizontally with speed $12\,\text{m s}^{-1}$ from the top of a high cliff. Find the direction of motion of the particle after 2 s. [3]

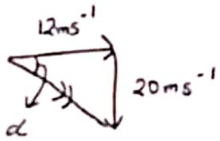


At $t = 2$,

Horizontally, $v_x = 12\ ms^{-1}$.

Vertically, using $v = u + at$

$$v_y = 0 + (10)(2) = 20\ ms^{-1}.$$



$$\tan\alpha = \frac{v_y}{v_x} = \frac{20}{12}$$

$$\alpha = \tan^{-1}\left(\frac{20}{12}\right) = 59.03^\circ \simeq 59.0^\circ$$

$\Rightarrow$ Direction of motion is 59.0° below horizontal.

11 Two particles P and Q are projected simultaneously with speed $40\,\mathrm{m\,s^{-1}}$ from a point O on a horizontal plane. Both particles subsequently pass at different times through the point A which has horizontal and vertically upward displacements from O of 40 m and 15 m respectively.

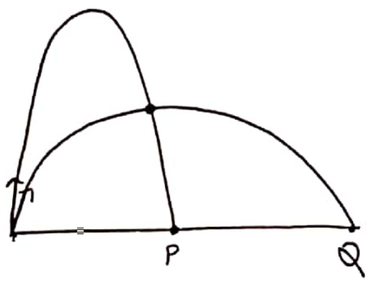
(i) By considering the equation of the trajectory of a projectile, show that each angle of projection satisfies the equation $\tan^2\theta - 8\tan\theta + 4 = 0$. [3]

(ii) Calculate the distance between the points at which P and Q strike the plane. [5]

i) $y = x\tan\theta - \dfrac{gx^2\sec^2\theta}{2u^2}$

When $x = 40, y = 15$ with $u = 40$

$\Rightarrow \quad 15 = 40\tan\theta - \dfrac{10\,(40)^2\sec^2\theta}{2\times 40^2}$



$15 = 40\tan\theta - 5\sec^2\theta$

$15 = 40\tan\theta - 5(1+\tan^2\theta)$

$15 = 40\tan\theta - 5 - 5\tan^2\theta$

$0 = -5\tan^2\theta + 40\tan\theta - 20$

$0 = \tan^2\theta - 8\tan\theta + 4$ (Shown).

ii) $\tan^2\theta - 8\tan\theta + 4 = 0$

$\tan\theta = \dfrac{-(-8)\pm\sqrt{(-8)^2-4(1)(4)}}{2(1)} = \dfrac{8\pm\sqrt{48}}{2} = 4\pm 2\sqrt{3}$

Either, $\tan\theta = 4+2\sqrt{3}$ or $\tan\theta = 4-2\sqrt{3}$

$\theta = 82.36^\circ$ $\qquad$ $\theta = 28.18^\circ$

$\theta \approx 82.4^\circ$ $\qquad$ $\theta \approx 28.2^\circ$

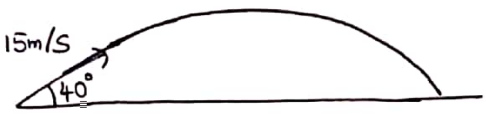
For P, Range $= \dfrac{u^2\sin 2\theta}{g} = \dfrac{40^2\times\sin(2\times 82.4^\circ)}{10} = 42.115\,\mathrm{m}$

For Q, Range $= \dfrac{u^2\sin 2\theta}{g} = \dfrac{40^2\times\sin(2\times 28.2^\circ)}{10} = 133.226\,\mathrm{m}$

Distance between P and Q when they strike the plane

$= 133.226 - 42.115 = 91.11 \approx 91.1\,\mathrm{m}$.

12 A particle is projected with speed $15\,\mathrm{m\,s^{-1}}$ at an angle of $40°$ above the horizontal from a point on horizontal ground. Calculate the time taken for the particle to hit the ground. [2]



~~Re~~ Total time of flight $= \dfrac{2u\sin\theta}{g} = \dfrac{2\times 15\times \sin 40^\circ}{10}$

$= 1.928\,s = 1.93\,s.$

Alternatively, vertically, using $s = ut + \frac{1}{2}at^2$

$0 = (15\sin 40^\circ)t + \frac{1}{2}(-10)t^2$

$0 = (15\sin 40^\circ - 5t)t$

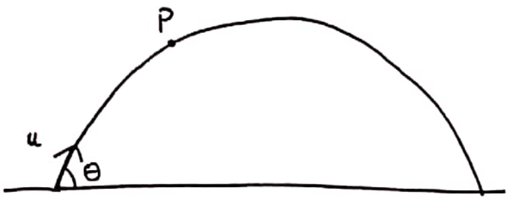
$\Rightarrow t = 0$ or $t = \dfrac{15\sin 40^\circ}{5} = 1.93\,s.$

13 A particle P is projected from a point O on horizontal ground. 0.4 s after the instant of projection, P is 5 m above the ground and a horizontal distance of 12 m from O.

(i) Calculate the initial speed and the angle of projection of P. [6]

(ii) Find the direction of motion of the particle 0.4 s after the instant of projection. [3]

i)



$$y = x\tan\theta - \frac{g\sec^2\theta\, x^2}{2u^2}$$

When $x = 12, y = 5$

$$\Rightarrow 5 = 12\tan\theta - \frac{10\sec^2\theta \times 12^2}{2u^2} \Rightarrow 5 = 12\tan\theta - \frac{720\sec^2\theta}{u^2} \rightarrow (1).$$

In horizontal direction, $x = v_x \times t$

when $t = 0.4\,s$, $x = 12 \Rightarrow 12 = u\cos\theta \times 0.4$

$$30 = u\cos\theta \rightarrow (2)$$

Substituting (2) in (1): $5 = 12\tan\theta - \frac{720}{(u\cos\theta)^2}$

$$5 = 12\tan\theta - \frac{720}{30^2}$$

$$\tan\theta = \frac{1}{12}\left[5 + \frac{720}{30^2}\right] \Rightarrow \tan\theta = \frac{29}{60}$$

$$\Rightarrow \theta = 25.79^\circ \approx 25.8^\circ$$

$$\Rightarrow 30 = u\cos(25.79^\circ) \Rightarrow u = \frac{30}{\cos 25.79^\circ} = 33.32\,ms^{-1} \approx 33.3\,ms^{-1}$$

$\therefore$ Initial speed $= 33.3\,ms^{-1}$, Angle of projection $= 25.8^\circ$

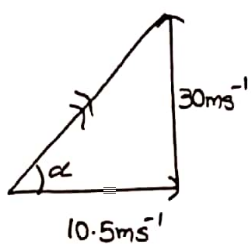
When $t = 0.4\,s$,

Horizontally, $v_x = u\cos\theta = 33.3\cos(25.8^\circ) = 30\,ms^{-1}$.

Vertically, using $v = u + at$

$$v_y = (u\sin\theta) - gt = 33.3\sin(25.8^\circ) - 10 \times 0.4$$

$$= 10.5\,ms^{-1}.$$



$$\tan\alpha = \frac{v_y}{v_x} = \frac{10.5}{30} \Rightarrow \alpha = \tan^{-1}\left(\frac{10.5}{30}\right) = 19.29^\circ \approx 19.3^\circ$$

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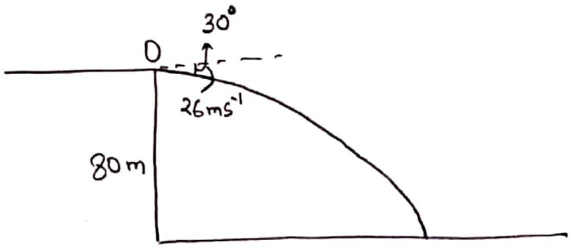
Direction of motion $= 19.3^\circ$ above horizontal.

14 A particle P is projected with speed $26\,\text{m s}^{-1}$ at an angle of $30°$ below the horizontal, from a point O which is 80 m above horizontal ground.

(i) Calculate the distance from O of the particle 2.3 s after projection. [4]

(ii) Find the horizontal distance travelled by P before it reaches the ground. [3]

(iii) Calculate the speed and direction of motion of P immediately before it reaches the ground. [4]



i) When $t = 2.3\text{s}$,

Horizontally, $x = v_x \times t = 26\cos 30° \times 2.3 = 51.788\text{m}$.

Vertically, $y = (26\sin 30°)(2.3) + \frac{1}{2}(10)(2.3)^2 = 56.35\text{m}$.

Distance $= \sqrt{x^2 + y^2} = \sqrt{51.788^2 + 56.35^2} = 76.53\text{m} \approx 76.5\text{m}$.

ii) Vertically using $s = ut + \frac{1}{2}at^2$

$$80 = (26\sin 30°)t + \frac{1}{2}(10)t^2$$

$$80 = 13t + 5t^2$$

$$0 = 5t^2 + 13t - 80$$

$$t = \frac{-13 \pm \sqrt{13^2 - 4(5)(-80)}}{2(5)}$$

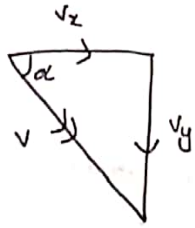
$$t = \frac{-13 \pm \sqrt{1769}}{10} = -5.505,\ 2.9059 \approx 2.91\text{s}.$$

(Ignore).

When $t = 2.91\text{s}$,

$x = v_x \times t = 26\cos 30° \times 2.91 = 65.43\text{m} \approx 65.4\text{m}$.

III) When $t = 2.91$ s,



$v_x = 26\cos 30^\circ = 22.516\ ms^{-1}$.

$v_y = 26\sin 30^\circ + 10(2.91) = 42.059\ ms^{-1}$.

$v = \sqrt{v_x^2 + v_y^2} = \sqrt{22.516^2 + 42.059^2} = 47.70\ ms^{-1} \approx 47.7\ ms^{-1}$

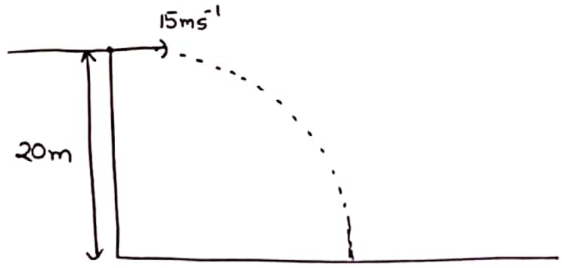
$\tan\alpha = \frac{v_y}{v_x} \Rightarrow \alpha = \tan^{-1}\left(\frac{42.059}{22.516}\right) = 61.83^\circ \approx 61.8^\circ$

$\therefore$ Speed of P $= 47.7\ ms^{-1}$, Direction of motion $= 61.8^\circ$ below horizontal.

15 A stone is thrown with speed 15 m s^{-1} horizontally from the top of a vertical cliff 20 m above the sea. Calculate

(i) the distance from the foot of the cliff to the point where the stone enters the sea, [3]

(ii) the speed of the stone when it enters the sea. [3]



i) Vertically, using $s = ut + \frac{1}{2}at^2$

$20 = 0(t) + \frac{1}{2}(10)t^2$

$20 = 5t^2$

$t^2 = 4$

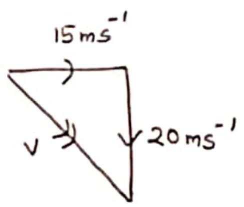
$t = 2$ (Ignore $t = -2$).

Horizontally, using $x = v_x \times t$

$x = 15 \times 2 = 30$ m

ii) When $t = 2$,

$v_x = 15$ ms^{-1}.



Vertically, using $v = u + at$

$v = 0 + 10(2) = 20$ ms^{-1}.

$v = \sqrt{v_x^2 + v_y^2} = \sqrt{15^2 + 20^2} = 25$ ms^{-1}.

16 A small ball B is projected with speed $15\ \mathrm{m\,s^{-1}}$ at an angle of $41°$ above the horizontal from a point O which is 1.6 m above horizontal ground. At time t s after projection the horizontal and vertically upward displacements of B from O are x m and y m respectively.

(i) Express x and y in terms of t and hence show that the equation of the trajectory of B is

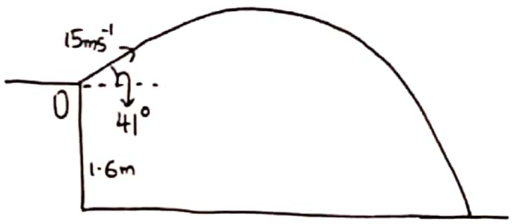
$$y = 0.869x - 0.0390x^2,$$

where the coefficients are correct to 3 significant figures. [4]

A vertical fence is 1.5 m from O and perpendicular to the plane in which B moves. B just passes over the fence and subsequently strikes the ground at the point A.

(ii) Calculate the height of the fence, and the distance from the fence to A. [5]

i)



Horizontally, $x = v_x \times t = 15\cos 41° \times t = (15\cos 41°)t \rightarrow (1)$

Vertically, using $s = ut + \frac{1}{2}at^2$

$y = (15\sin 41°)t + \frac{1}{2}(-10)t^2 = (15\sin 41°)t - 5t^2 \rightarrow (2)$

From (1), $t = \dfrac{x}{15\cos 41°} \rightarrow (3)$

Substituting (3) in (2):

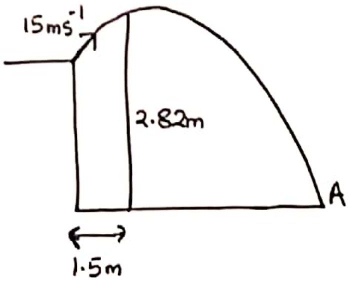
$y = (15\sin 41°)\left(\dfrac{x}{15\cos 41°}\right) - 5\left(\dfrac{x}{15\cos 41°}\right)^2$

$y = 0.8692x - 0.03901x^2$

$y = 0.869x - 0.0390x^2$ (3 s.f.).

ii) When $x = 1.5$, $y = 0.869(1.5) - 0.0390(1.5)^2$

$= 1.216$



$\therefore$ Height of fence $= 1.6 + 1.216 = 2.816\text{m} \approx 2.82\text{m}$.

When ball strikes ground, $y = -1.6$

$\Rightarrow 0.869x - 0.0390x^2 = -1.6$

$0.0390x^2 - 0.869x - 1.6 = 0$

iii)

$$x = \frac{-(-0.869) \pm \sqrt{(-0.869)^2 - 4(0.0390)(-1.6)}}{2(0.0390)}$$

$x = 23.99, -1.709$ (Ignore)

$\therefore$ Distance of A from base of cliff $= 23.99m$

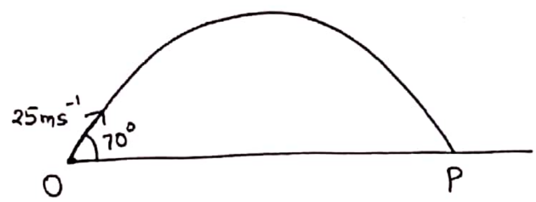
$\Rightarrow$ Distance of A from fence $= 23.99 - 1.5 = 22.49m \approx 22.5m$.

17 A ball is projected with velocity $25\,\mathrm{m\,s^{-1}}$ at an angle of 70° above the horizontal from a point O on horizontal ground. The ball subsequently bounces once on the ground at a point P before landing at a point Q where it remains at rest. The distance PQ is 17.1 m.

(i) Calculate the time taken by the ball to travel from O to P and the distance OP. [3]

(ii) Given that the horizontal component of the velocity of the ball does not change at P, calculate the speed of the ball when it leaves P. [4]

i)



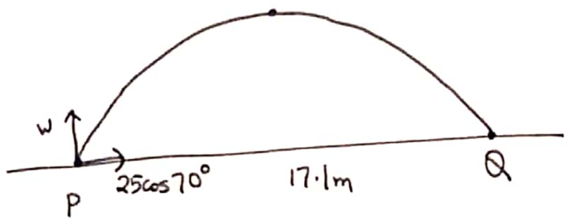
$$\text{Range} = OP = \frac{u^2 \sin 2\theta}{g} = \frac{25^2 \sin(2\times 70^\circ)}{10} = 40.174\,\text{m} \approx 40.2\,\text{m}.$$

Horizontally, $x = v_x \times t$

$40.174 = 25\cos 70^\circ \times t$

$t = 4.698\,\text{s} \approx 4.70\,\text{s}.$

ii) Before impact at P — After impact at P



For distance PQ,

Horizontally, $x = v_x \times t$

$17.1 = 25\cos 70^\circ \times t$

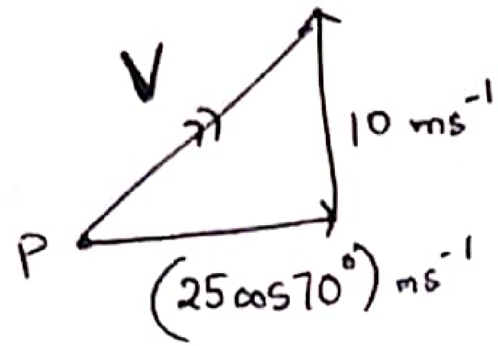
$t = 1.999\,\text{s} \approx 2.00\,\text{s}.$

Time taken to reach maximum height $= \frac{2.00}{2} = 1.00\,\text{s}.$

Vertically, using $v = u + at$

$0 = w + (-10)(1)$

$w = 10\,\text{ms}^{-1}.$

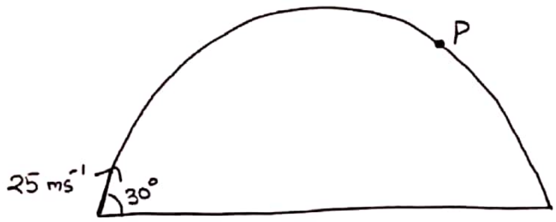


Then after impact at P,

$$V^2 = 10^2 + (25\cos 70^\circ)^2$$

$$V = 13.157\ ms^{-1} \simeq 13.2\ ms^{-1}.$$

18 A particle P is projected with speed $25\,\mathrm{m\,s^{-1}}$ at an angle of $30°$ above the horizontal from a point O on horizontal ground. Calculate the distance OP at the instant 2 s after projection. [4]



Horizontally, $x = v_x \times t = 25\cos 30° \times 2 = 25\sqrt{3}$ m.

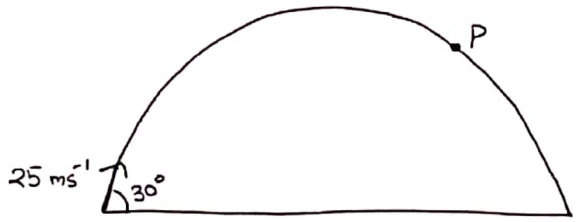
Vertically, using $s = ut + \frac{1}{2}at^2$

$$y = (25\sin 30°)(2) + \frac{1}{2}(-10)(2)^2$$

$$y = 5$$

$\Rightarrow$ $OP = \sqrt{(25\sqrt{3})^2 + 5^2} = 43.58 \approx 43.6\,\mathrm{m}$

18 A particle P is projected with speed $25\,\mathrm{m\,s^{-1}}$ at an angle of $30°$ above the horizontal from a point O on horizontal ground. Calculate the distance OP at the instant 2 s after projection. [4]



Horizontally, $x = v_x \times t = 25\cos 30° \times 2 = 25\sqrt{3}$ m.

Vertically, using $s = ut + \frac{1}{2}at^2$

$$y = (25\sin 30°)(2) + \frac{1}{2}(-10)(2)^2$$

$$y = 5$$

$$\Rightarrow OP = \sqrt{(25\sqrt{3})^2 + 5^2} = 43.58 \approx 43.6\text{m}$$

19 The equation of the trajectory of a projectile is $y = 0.6x - 0.017x^2$, referred to horizontal and vertically upward axes through the point of projection.

(i) Find the angle of projection of the projectile, and show that the initial speed is $20\,\mathrm{m\,s^{-1}}$. [3]

(ii) Find the speed and direction of motion of the projectile when it is at a height of 5.2 m above the level of the point of projection for the second time. [7]

i)
$$y = 0.6x - 0.017x^2$$

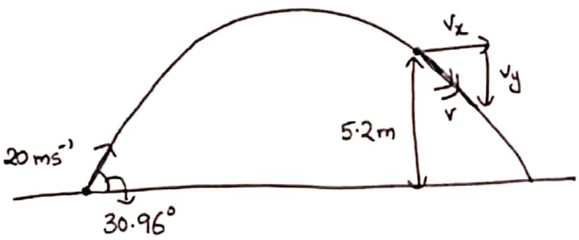
By comparison to $y = x\tan\theta - \dfrac{g\sec^2\theta\, x^2}{2u^2}$,

$$\tan\theta = 0.6 \Rightarrow \theta = \tan^{-1}(0.6) = 30.96^\circ \simeq 31.0^\circ.$$

and $\dfrac{g\sec^2\theta}{2u^2} = 0.017 \Rightarrow \dfrac{10\times\sec^2 31.0^\circ}{2u^2} = 0.017$

$$u = \sqrt{\frac{10\times\sec^2 31.0^\circ}{2\times0.017}} = 20\,\mathrm{ms^{-1}}.\ \text{(Shown)}$$

ii)



Vertically, using $v^2 = u^2 + 2as$

$$v_y^2 = (20\sin 30.96^\circ)^2 + 2(-10)(5.2)$$
$$v_y^2 = 1.859$$
$$v_y = -1.363\,\mathrm{ms^{-1}}$$

Horizontally, $v_x = 20\cos 30.96^\circ = 17.15\,\mathrm{ms^{-1}}$

$$\Rightarrow v = \sqrt{v_x^2 + v_y^2} = \sqrt{17.15^2 + 1.363^2} = 17.20\,\mathrm{ms^{-1}}$$
$$v \simeq 17.2\,\mathrm{ms^{-1}}.$$

$$\tan\alpha = \frac{v_y}{v_x} = \frac{-1.363}{17.15} \Rightarrow \alpha = \tan^{-1}\left(\frac{-1.363}{17.15}\right) = -4.544^\circ$$

$\therefore$ Direction of motion $= 4.54^\circ$ below the horizontal.

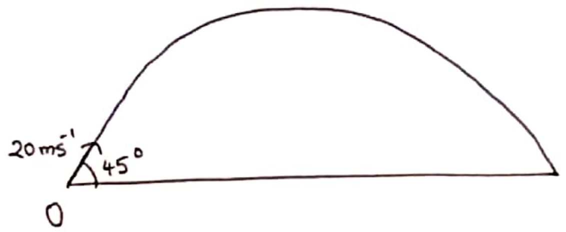
20 A small ball is projected with speed 20 m s^{-1} at an angle of 45° above the horizontal from a point O on horizontal ground. At time t s after projection, the horizontal and vertically upwards displacements of the ball from O are x m and y m respectively.

(i) Express x and y in terms of t. [2]

(ii) Show that the equation of the trajectory of the ball is $y = x - \frac{1}{40}x^2$. [2]

(iii) State the distance from O of the point at which the ball first strikes the ground. [1]

i)



Horizontally, $x = v_x \times t = 20\cos 45° \times t = 10\sqrt{2}\,t \rightarrow (1)$

Vertically, $y = (20\sin 45°)t + \frac{1}{2}(-10)t^2 = 10\sqrt{2}\,t - 5t^2 \rightarrow (2)$

ii) From (1), $t = \frac{x}{10\sqrt{2}} \rightarrow (3)$

Substitute (3) in (2).

$$y = 10\sqrt{2}\left(\frac{x}{10\sqrt{2}}\right) - 5\left(\frac{x}{10\sqrt{2}}\right)^2$$

$$y = x - \frac{x^2}{40} \quad \text{(Shown)}.$$

iii) When ball strikes ground, $y = 0$

$$x - \frac{x^2}{40} = 0$$

$$x\left(1 - \frac{x}{40}\right) = 0$$

$x = 0$ or $1 - \frac{x}{40} = 0$

$x = 40$

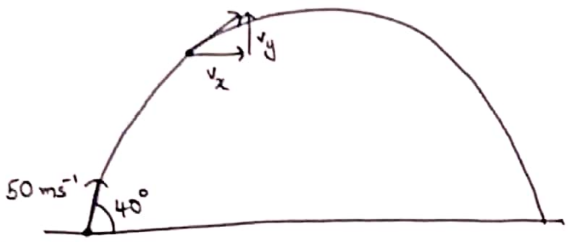
∴ Distance of ball from O when it strikes ground $= 40$ m.

21 A particle P is projected with speed $50\,\mathrm{m\,s^{-1}}$ at an angle of $40°$ above the horizontal from a point O. For the instant 2.5 s after projection, calculate

(i) the speed of P, [3]

(ii) the angle between OP and the horizontal. [4]

i)



When $t = 2.5$ s,

Horizontally, $v_x = 50\cos 40° = 38.30$ m/s.

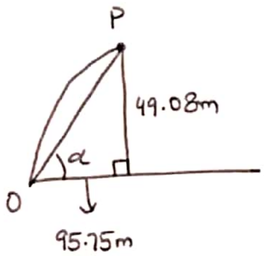
Vertically, $v_y = (50\sin 40°) - 10(2.5) = 7.139$ m/s.

$\therefore\ v = \sqrt{v_x^2 + v_y^2} = \sqrt{38.30^2 + 7.139^2} = 38.96\,\mathrm{m/s} \approx 39.0\,\mathrm{ms^{-1}}$.

ii) $x = v_x \times t = 38.30 \times 2.5 = 95.75$ m.

$y = (50\sin 40°)(2.5) + \frac{1}{2}(-10)(2.5)^2 = 49.098$ m.

$\tan\alpha = \dfrac{49.08}{95.75} \Rightarrow \alpha = 27.14° \simeq 27.1°$



$\therefore$ Angle between OP and horizontal $= 27.1°$.

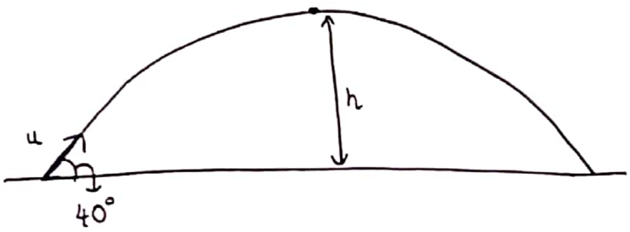
22 A ball B is projected from a point O on horizontal ground at an angle of 40° above the horizontal. B hits the ground 1.8 s after the instant of projection. Calculate

(i) the speed of projection of B, [2]

(ii) the greatest height of B, [2]

(iii) the distance from O of the point at which B hits the ground. [2]

i)



Total time of flight $= \dfrac{2u\sin\theta}{g}$

$1.8 = \dfrac{2 \times u \times \sin 40^\circ}{10}$

$u = 14.00\ ms^{-1} \approx 14.0ms^{-1}$.

ii) For greatest height,

vertically : $v^2 = u^2 + 2as$

$0^2 = (14\sin 40^\circ)^2 + 2(-10)h$

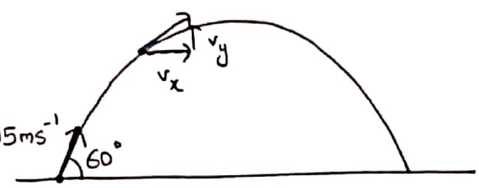
$h = 4.049 \approx 4.05m$.

iii) Horizontally, $x = v_x \times t$

$x = 14\cos 40^\circ \times 1.8$

$x = 19.30 \approx 19.3m$.

23 A particle P is projected with speed $15\ \mathrm{m\,s^{-1}}$ at an angle of $60°$ above the horizontal. Find the direction of motion of P at the instant 0.9 s after projection. [4]



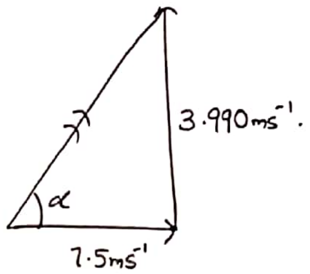
When $t = 0.9\,\mathrm{s}$

Horizontally, $v_x = 15\cos 60° = 7.5\ \mathrm{ms^{-1}}$.

Vertically, $v_y = 15\sin 60° + (-10)(0.9) = 3.990\ \mathrm{ms^{-1}}$.

$$\tan\alpha = \frac{3.990}{7.5}$$

$$\alpha = 28.01° \approx 28.0°$$

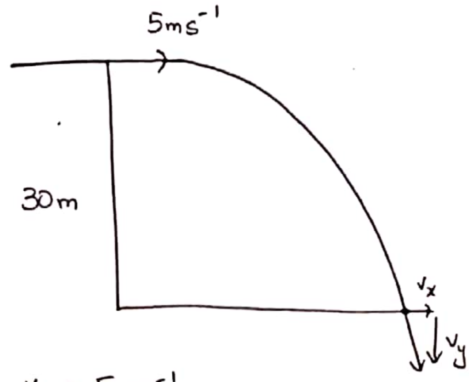


24 A ball is projected horizontally with speed 5 m s^{-1} from the top of a tower which is 30 m high. The tower stands on horizontal ground.

(i) Find the speed and direction of motion of the ball when it reaches the ground. [3]

(ii) Calculate the distance from the foot of the tower to the point where the ball reaches the ground. [3]

i)

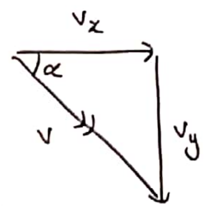


Horizontally, $v_x = 5\ ms^{-1}$.

Vertically, using $v^2 = u^2 + 2as$

$v_y^2 = 0^2 + 2(10)(30)$

$v_y^2 = 600 \Rightarrow v_y = 10\sqrt{6}\ ms^{-1}$.



$v = \sqrt{v_x^2 + v_y^2} = \sqrt{5^2 + (10\sqrt{6})^2} = 25\ ms^{-1}$.

$\tan\alpha = \dfrac{v_y}{v_x} \Rightarrow \tan\alpha = \dfrac{10\sqrt{6}}{5} \Rightarrow \alpha = 78.46^\circ \simeq 78.5^\circ$

$\therefore$ Speed of ball $= 25\ ms^{-1}$, Direction of motion $= 78.5^\circ$ below horizontal.

ii) Vertically, using $v = u + at$

$10\sqrt{6} = 0 + 10(t)$

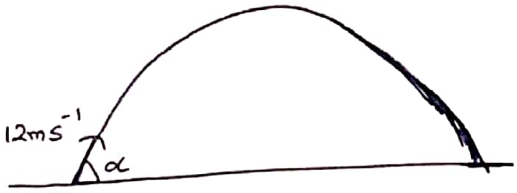
$t = \sqrt{6}\ s$.

Horizontally, $x = v_x \times t$

$x = 5 \times \sqrt{6} = 12.24\ m \approx 12.2m$

i.e. Distance from foot of the tower to point where ball hits ground $= 12.2m$.

25 A particle is projected with speed $12\,\mathrm{m\,s^{-1}}$ from a point on horizontal ground. The particle strikes the ground 1.6 s after the instant of projection. Calculate the angle of projection. [2]



Total time of flight $= \dfrac{2u\sin\theta}{g}$

$$1.6 = \frac{2 \times 12 \times \sin\alpha}{10}$$

$$\sin\alpha = \frac{1.6 \times 10}{2 \times 12}$$

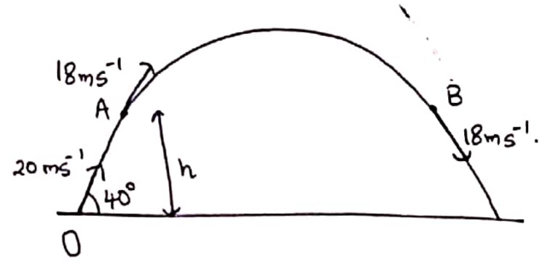
$$\alpha = 41.81^\circ \approx 41.8^\circ$$

26 A particle P is projected with speed $20\,\mathrm{m\,s^{-1}}$ at an angle of $40°$ above the horizontal from a point O on horizontal ground.

(i) Find the height of P above the ground when P has speed $18\,\mathrm{m\,s^{-1}}$. [2]

(ii) Calculate the length of time for which the speed of P is less than $18\,\mathrm{m\,s^{-1}}$, and find the horizontal distance travelled by P during this time. [6]

i)



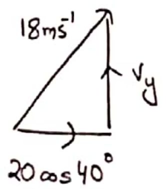
Using conservation of energy:

$$\frac{1}{2} \times m \times 20^2 + mg(0) = \frac{1}{2} \times m \times 18^2 + mgh$$

$$400 = 324 + 20h$$

$$h = 3.8\,\mathrm{m}.$$

ii) At A, $18^2 = (20\cos 40°)^2 + v_y^2$



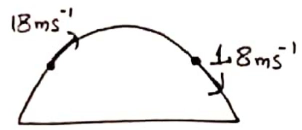
$$v_y = 9.448\,\mathrm{ms^{-1}}.$$

At B, $v_y = -9.448\,\mathrm{ms^{-1}}$.

Vertically, from A to B, using $v = u + at$

$$\Rightarrow -9.448 = 9.448 + (-10)t$$

$$t = 1.889\,\mathrm{s} \approx 1.89\,\mathrm{s}.$$



i.e. time taken to travel from A to B = 1.89s.

Horizontal distance travelled from A to B

$= v_x \times t = 20\cos 40° \times 1.889 = 28.95\,\mathrm{m} \approx 29.0\,\mathrm{m}$

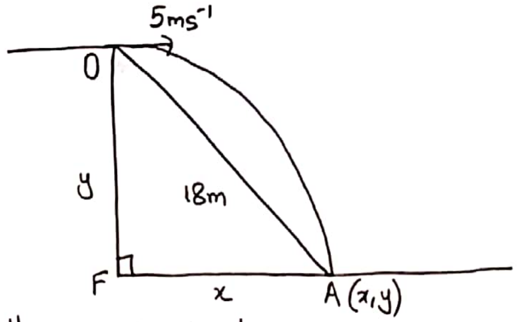
27 A small ball is thrown horizontally with speed $5\ \mathrm{m\,s^{-1}}$ from a point O on the roof of a building. At time t s after projection, the horizontal and vertically downwards displacements of the ball from O are x m and y m respectively.

(i) Express x and y in terms of t, and hence show that the equation of the trajectory of the ball is $y = 0.2x^2$. [3]

The ball strikes the horizontal ground which surrounds the building at a point A.

(ii) Given that $OA = 18$ m, calculate the value of x at A, and the speed of the ball immediately before it strikes the ground at A. [6]

i)



Horizontally, $x = v_x \times t$

$x = 5t \rightarrow (1)$

Vertically, $y = 0(t) + \frac{1}{2}(10)t^2$

$y = 5t^2 \rightarrow (2)$

From (1), $t = \frac{x}{5} \rightarrow (3)$

Substituting (3) in (2)

$y = 5\left(\frac{x}{5}\right)^2 = \frac{x^2}{5}$

$y = 0.2x^2$. (Shown).

ii) let,

OF $= y$ and FA $= x$.

$\Rightarrow OA^2 = OF^2 + FA^2$

$18^2 = y^2 + x^2$

Since A is a point on the curve, $y = 0.2x^2$

$\Rightarrow 18^2 = (0.2x^2)^2 + x^2$

$0 = 0.04x^4 + x^2 - 324$

$x^2 = \frac{-1 \pm \sqrt{1^2 - 4(0.04)(-324)}}{2(0.04)} = 78.36, -103.36$ (Ignore)

$x^2 = 78.36 \Rightarrow x = 8.852 = 8.85$m.

At A, horizontally, $v_x = 5 ms^{-1}$.

vertically, $y = 0.2 \times 8.85^2 = 15.67 m$.

using $v^2 = u^2 + 2as$

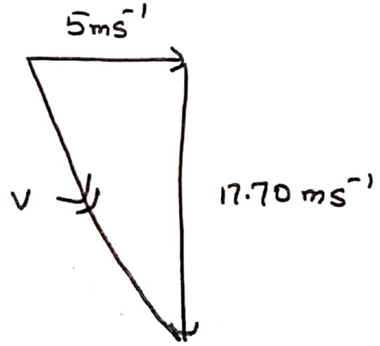
$v_y^2 = 0^2 + 2(10)(15.67)$

$v_y = 17.70 ms^{-1}$.

$v = \sqrt{v_x^2 + v_y^2}$

$= \sqrt{5^2 + 17.70^2}$

$v = 18.39 ms^{-1} \approx 18.4 ms^{-1}$.



28 A stone is projected from a point O on horizontal ground. The equation of the trajectory of the stone is

$$y = 1.2x - 0.15x^2,$$

where x m and y m are respectively the horizontal and vertically upwards displacements of the stone from O. Find

(i) the greatest height of the stone, [2]

(ii) the distance from O of the point where the stone strikes the ground. [2]

i) $y = 1.2x - 0.15x^2$.

For greatest height, $\frac{dy}{dx} = 0$

$1.2 - 0.3x = 0$

$x = \frac{1.2}{0.3} = 4$

When $x = 4$, $y = 1.2(4) - 0.15(4)^2 = 2.4$ m.

i.e. Greatest height of stone $= 2.4$ m.

ii) When stone strikes the ground, $y = 0$

$\Rightarrow 1.2x - 0.15x^2 = 0$

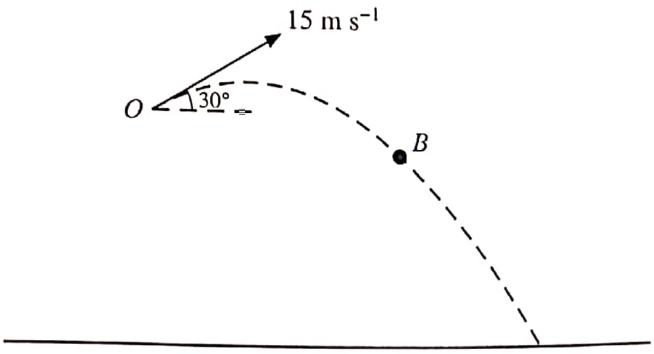
$0.15x(8 - x) = 0$

Either, $0.15x = 0$ or $8 - x = 0$

$x = 0$ $x = 8$.

$\therefore$ Distance from O to point where stone strikes the ground $= 8$m.

29

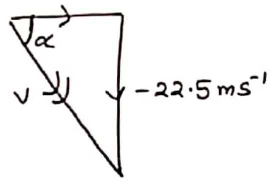


A small ball B is projected from a point O above horizontal ground, with initial speed $15\,\mathrm{m\,s^{-1}}$ at an angle of projection of 30° above the horizontal (see diagram). The ball strikes the ground 3 s after projection.

(i) Calculate the speed and direction of motion of the ball immediately before it strikes the ground. [5]

(ii) Find the height of O above the ground. [2]

i) When $t = 3\,s$,



$v_x = 15\cos 30° = \frac{15\sqrt{3}}{2}\,\mathrm{ms^{-1}} = 12.990\,\mathrm{ms^{-1}}$.

$v_y = 15\sin 30° - 10 \times 3 = -22.5\,\mathrm{ms^{-1}}$.

$v = \sqrt{v_x^2 + v_y^2} = \sqrt{12.990^2 + (-22.5)^2} = 25.98\,\mathrm{ms^{-1}}$

$v \approx 26.0\,\mathrm{ms^{-1}}$.

$\tan\alpha = \frac{-22.5}{12.990} \Rightarrow \alpha = -60°$

$\therefore$ Speed of ball $= 26.0\,\mathrm{ms^{-1}}$, direction of motion $= 60°$ below the horizontal.

ii) Vertically, using $s = ut + \frac{1}{2}at^2$

$y = (15\sin 30°)(3) + \frac{1}{2}(-10)(3)^2$

$y = -22.5\,\mathrm{m}$

$\Rightarrow$ Height of O above the ground $= 22.5\,\mathrm{m}$.

30

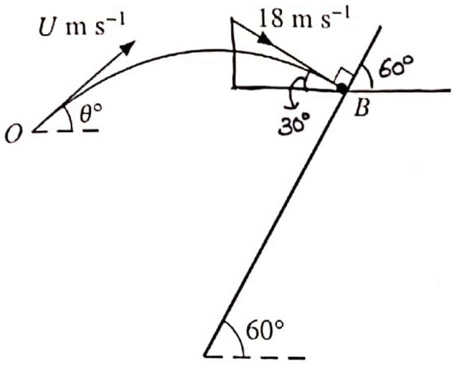


Fig. 1

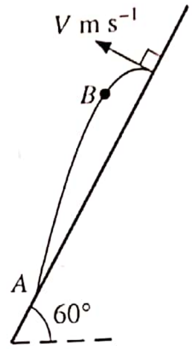


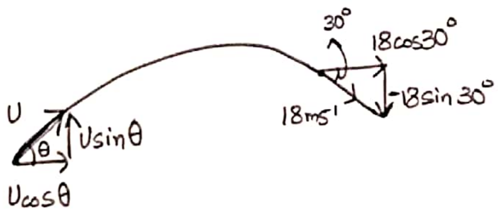
Fig. 2

A small ball B is projected with speed $U\,\mathrm{m\,s^{-1}}$ at an angle of $\theta°$ above the horizontal from a point O. At time 2 s after the instant of projection, B strikes a smooth wall which slopes at 60° to the horizontal. The speed of B is $18\,\mathrm{m\,s^{-1}}$ and its direction of motion is perpendicular to the wall at the instant of impact (see Fig. 1). B bounces off the wall with speed $V\,\mathrm{m\,s^{-1}}$ in a direction perpendicular to the wall. At time 0.8 s after B bounces off the wall, B strikes the wall again at a lower point A (see Fig. 2).

(i) Find U and θ. [5]

(ii) By considering the motion of B after it bounces off the wall, calculate V. [4]

i)



From O to first impact with wall,

horizontally, $U\cos\theta = 18\cos 30° \rightarrow (1)$

vertically, using $v = u + at$

$$-18\sin 30° = U\sin\theta - 10t$$

$$-18\sin 30° = U\sin\theta - 10(2)$$

$$20 - 18\sin 30° = U\sin\theta \rightarrow (2)$$

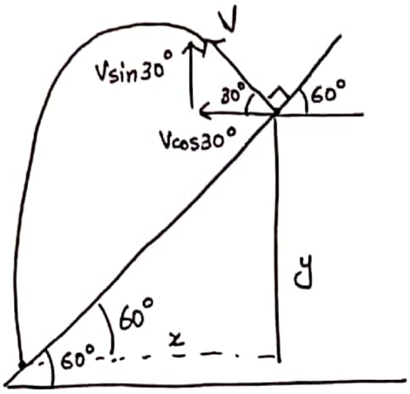
Dividing (2) by (1) gives: $\dfrac{U\sin\theta}{U\cos\theta} = \dfrac{20 - 18\sin 30°}{18\cos 30°}$

$$\tan\theta = \frac{11\sqrt{3}}{27} \Rightarrow \theta = 35.20° \approx 35.2°$$

Squaring (1) and (2) and adding: $U^2\cos^2\theta + U^2\sin^2\theta = (18\cos 30°)^2 + (20 - 18\sin 30°)^2$

$$U^2 = 364 \Rightarrow U = 19.07\,\mathrm{ms^{-1}} \approx 19.1\,\mathrm{ms^{-1}}.$$

ii)



From first impact to second impact with wall,

Horizontally, $x = v_x \times t$

$$x = V\cos 30^\circ \times 0.8 = \frac{2\sqrt{3}}{5} V.$$

Vertically, using $s = ut + \frac{1}{2}at^2$

$$y = (-V\sin 30^\circ)(0.8) + \frac{1}{2}(10)(0.8)^2$$

$$y = -0.4V + 5(0.64)$$

$$y = 3.2 - 0.4V.$$

$$\tan 60^\circ = \frac{y}{x}$$

$$\sqrt{3} = \frac{3.2 - 0.4V}{\frac{2\sqrt{3}}{5} V}$$

$$\frac{6}{5}V = 3.2 - 0.4V$$

$$1.6V = 3.2$$

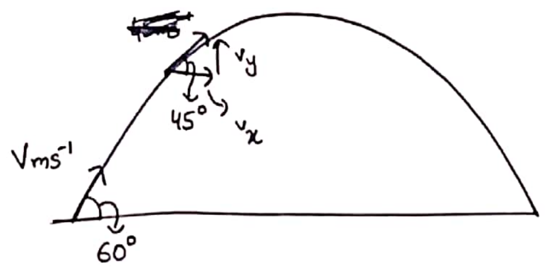
$$V = 2\,ms^{-1}.$$

31 A particle P is projected with speed $V\,\mathrm{m\,s^{-1}}$ at an angle of 60° above the horizontal from a point O on horizontal ground. P is moving at an angle of 45° above the horizontal at the instant 1.5 s after projection.

(i) Find V. [3]

(ii) Hence calculate the horizontal and vertical displacements of P from O at the instant 1.5 s after projection. [2]

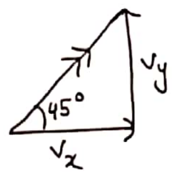
i)



When $t = 1.5s$,

$v_x = V\cos 60° = \frac{V}{2}$

$v_y = V\sin 60° - 10 \times 1.5 = V\sin 60° - 15 = \frac{\sqrt{3}}{2}V - 15.$



$\tan 45° = \frac{v_y}{v_x}$

$1 = \frac{\frac{\sqrt{3}}{2}V - 15}{\frac{V}{2}}$

$\frac{V}{2} = \frac{\sqrt{3}}{2}V - 15$

$V = \frac{-15}{\frac{1}{2} - \frac{\sqrt{3}}{2}} = 40.98\,ms^{-1} \approx 41.0\,ms^{-1}.$

ii) Horizontally, $x = v_x \times t = 41.0\cos 60° \times 1.5 = 30.73\,m \approx 30.7\,m.$

Vertically, $y = (V\sin 60°)t + \frac{1}{2}at^2 = 40.98 \times \sin 60° \times 1.5 + \frac{1}{2}(-10) \times 1.5^2$

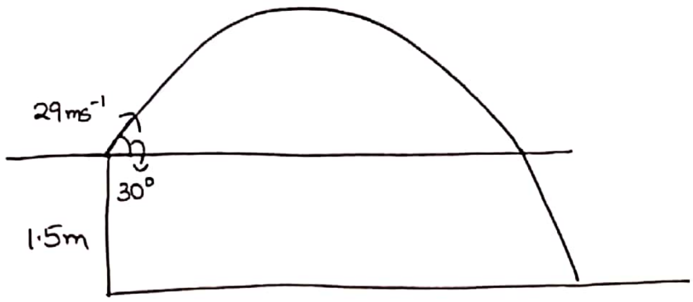
$y = 41.98\,m \approx 42.0\,m.$

32 A small ball B is projected from a point 1.5 m above horizontal ground with initial speed $29\ \mathrm{m\,s^{-1}}$ at an angle of 30° above the horizontal.

(i) Show that B strikes the ground 3 s after projection. [2]

(ii) Find the speed and direction of motion of B immediately before it strikes the ground. [4]

i)



Vertically, using $s = ut + \frac{1}{2}at^2$

$$-1{\cdot}5 = (29\sin 30^\circ)t + \frac{1}{2}(-10)t^2$$

$$-1{\cdot}5 = 14{\cdot}5t - 5t^2$$

$$5t^2 - 14{\cdot}5t - 1{\cdot}5 = 0$$

$$10t^2 - 29t - 3 = 0$$

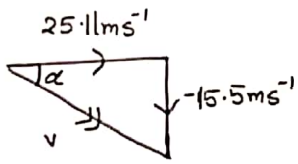
$$10t^2 - 30t + t - 3 = 0$$

$$(10t+1)(t-3) = 0$$

$\Rightarrow t = \frac{-1}{10}$ (Ignore) or $t = 3$.

$\therefore$ B strikes ground after 3 seconds.

ii) When $t = 3$,



Horizontally, $v_x = 29\cos 30^\circ = 25{\cdot}11\ ms^{-1}$.

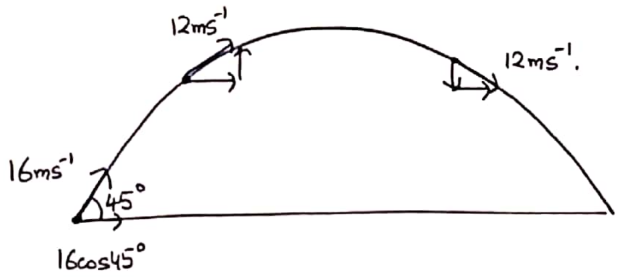
Vertically, $v_y = 29\sin 30^\circ - 10(3) = -15{\cdot}5\ ms^{-1}$.

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{25{\cdot}11^2 + (-15{\cdot}5)^2} = 29{\cdot}51\ ms^{-1}$$

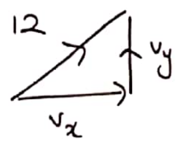
$$\tan\alpha = \frac{-15{\cdot}5}{25{\cdot}11} \Rightarrow \alpha = -31{\cdot}7^\circ$$

$\therefore$ Speed of ball $= 29{\cdot}5\ ms^{-1}$, Direction of ball $= 31{\cdot}7^\circ$ below horizontal.

33 A small ball is projected with speed $16\,\mathrm{m\,s^{-1}}$ at an angle of 45° above the horizontal from a point on horizontal ground. Calculate the period of time, before the ball lands, for which the speed of the ball is less than $12\,\mathrm{m\,s^{-1}}$. [4]



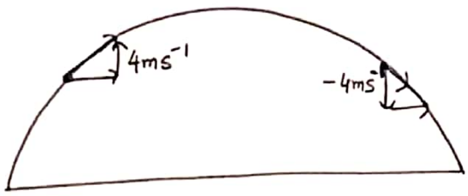
Horizontally, $v_x = 16\cos 45° = 8\sqrt{2}\ \mathrm{ms^{-1}}$.



Vertically, $12^2 = v_x^2 + v_y^2$

$12^2 = (8\sqrt{2})^2 + v_y^2$

$v_y = 4\ \mathrm{ms^{-1}}$.



For time for which speed is less than $12\mathrm{ms^{-1}}$,

Vertically, $v = u + at$

$-4 = 4 + (-10)t$

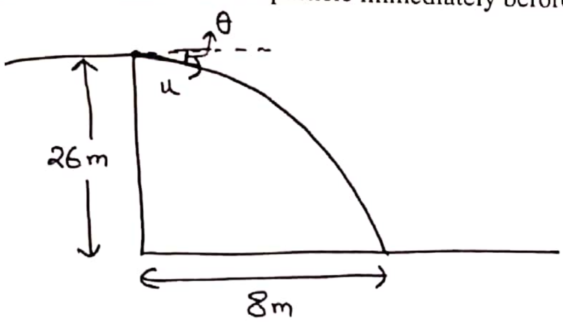
$t = \dfrac{-8}{-10} = 0.8\mathrm{s}$.

$\therefore$ Period of time for which speed of ball is less than $12\mathrm{ms^{-1}} = 0.8\mathrm{s}$

34 A particle is projected at an angle of $\theta°$ below the horizontal from a point at the top of a vertical cliff 26 m high. The particle strikes horizontal ground at a distance 8 m from the foot of the cliff 2 s after the instant of projection. Find

(i) the speed of projection of the particle and the value of θ, [6]

(ii) the direction of motion of the particle immediately before it strikes the ground. [3]



i) Horizontally, $x = v_x \times t$

$$8 = u\cos\theta \times 2$$

$$4 = u\cos\theta \rightarrow (1).$$

Vertically, using $s = ut + \frac{1}{2}at^2$

$$26 = (u\sin\theta)(2) + \frac{1}{2}(10)(2)^2$$

$$6 = 2u\sin\theta$$

$$\Rightarrow u\sin\theta = 3 \rightarrow (2).$$

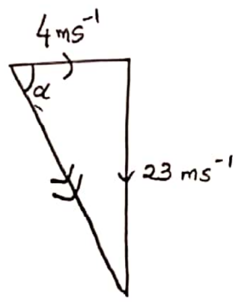
Dividing (2) by (1): $\frac{u\sin\theta}{u\cos\theta} = \frac{3}{4} \Rightarrow \tan\theta = \frac{3}{4} \Rightarrow \theta = 36.86° \approx 36.9°$.

Squaring (1) and (2) and adding:

$$u^2\sin^2\theta + u^2\cos^2\theta = 3^2 + 4^2$$

$$u^2 = 25 \Rightarrow u = 5\ ms^{-1}.$$

ii) When $t = 3s$,



$$v_x = u\cos\theta = 5\cos 36.86° = 4.$$

$$v_y = u\sin\theta + gt = 5\sin 36.86° + 10(2) = 23\ ms^{-1}.$$

$$\tan\alpha = \frac{v_y}{v_x} = \frac{23}{4} \Rightarrow \alpha = \tan^{-1}\left(\frac{23}{4}\right) = 80.13° \approx 80.1°$$

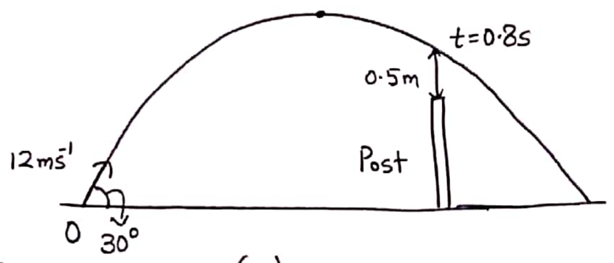
$\Rightarrow$ Direction of motion $= 80.1°$ below horizontal.

5 A small ball B is projected with speed $12\,\mathrm{m\,s^{-1}}$ at an angle of 30° above the horizontal from a point O on horizontal ground. At the instant 0.8 s after projection, B is 0.5 m vertically above the top of a vertical post.

(i) Calculate the height of the top of the post above the ground. [3]

(ii) Show that B is at its greatest height 0.2 s before passing over the post. [2]

i)



When $t = 0{\cdot}8\,s$,

Vertically, $y = (12 \sin 30)(0.8) + \frac{1}{2}(-10)(0.8)^2 = 1{\cdot}6\,m$

$\Rightarrow$ Height of top of post $= 1{\cdot}6\,m - 0{\cdot}5\,m = 1{\cdot}1\,m$.

ii) When ball is at highest point in its trajectory,

$$\text{time} = \frac{u \sin\theta}{g} = \frac{12 \sin 30^\circ}{10} = 0{\cdot}6\,s.$$

$\Rightarrow$ B is at its greatest height 0·2s before passing over the post.

36 The point O is 8 m above a horizontal plane. A particle P is projected from O. After projection, the horizontal and vertically upwards displacements of P from O are x m and y m respectively. The equation of the trajectory of P is

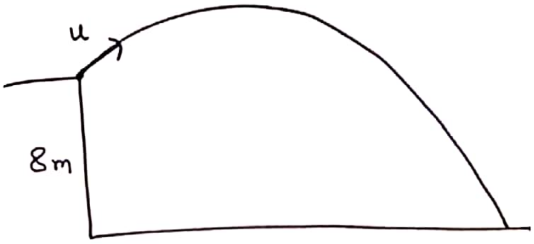
$$y = 2x - x^2.$$

(i) Find the value of x for the point where P strikes the plane. [2]

(ii) Find the angle and speed of projection of P. [3]

(iii) Calculate the speed of P immediately before it strikes the plane. [2]

i)



When particle strikes plane, $y = -8$

$$\Rightarrow 2x - x^2 = -8$$
$$0 = x^2 - 2x - 8$$
$$0 = x^2 - 4x + 2x - 8$$
$$0 = (x-4)(x+2)$$
$$\Rightarrow x = -2 \text{ (Ignore)} \quad \text{or} \quad x = 4$$
$$\therefore x = 4$$

ii)

$$y = 2x - x^2.$$
$$y = x\tan\theta - \frac{g\sec^2\theta\, x^2}{2u^2}.$$

By comparison, ~~to~~ $\tan\theta = 2 \Rightarrow \theta = \tan^{-1}(2) = 63.43°$

and $\frac{g\sec^2\theta}{2u^2} = 1 \Rightarrow \frac{10\sec^2 63.43°}{2u^2} = 1$

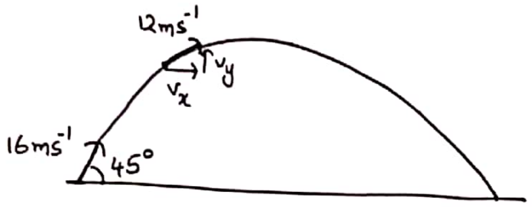
$$\Rightarrow u = \sqrt{\frac{10\sec^2 63.43°}{2}} = 5\,ms^{-1}.$$

$\therefore$ Angle of projection $= 63.4°$, Speed of projection $= 5\,ms^{-1}$.

iii) Using conservation of energy,

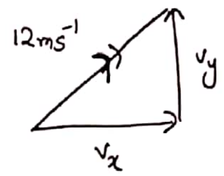
$$\frac{1}{2}mu^2 + mgh = \frac{1}{2}mv^2 + mg(0)$$
$$\frac{1}{2}m \times 5^2 + mg(8) = \frac{1}{2}mv^2 + 0 \Rightarrow 25 + 160 = v^2$$
$$\Rightarrow v = 13.60\,ms^{-1} = 13.6\,ms^{-1}.$$

37 A small ball is projected with speed $16\,\mathrm{m\,s^{-1}}$ at an angle of 45° above the horizontal from a point on horizontal ground. Calculate the period of time, before the ball lands, for which the speed of the ball is less than $12\,\mathrm{m\,s^{-1}}$. [4]



Horizontally, $16\cos 45^\circ = v_x \Rightarrow v_x = 8\sqrt{2}$.

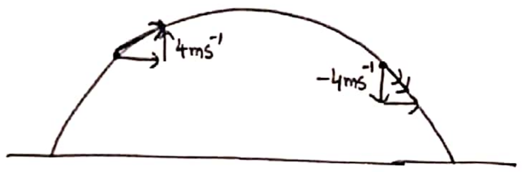
Using vector diagram,



$$12^2 = v_x^2 + v_y^2$$

$$144 = (8\sqrt{2})^2 + v_y^2$$

$$4\,\mathrm{ms^{-1}} = v_y$$



For time when speed of ball is less than $12\,\mathrm{ms^{-1}}$,

Vertically, using $v = u + at$

$$-4 = 4 + (-10)t$$

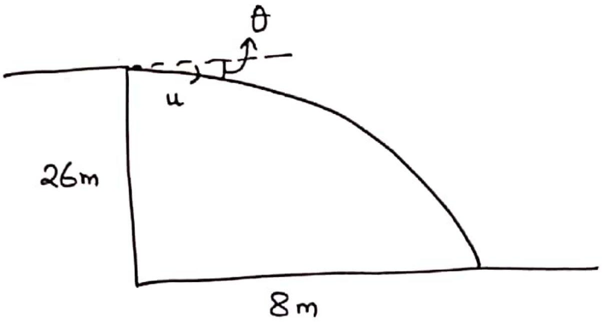
$$t = 0.8\,\mathrm{s}.$$

38 A particle is projected at an angle of $\theta°$ below the horizontal from a point at the top of a vertical cliff 26 m high. The particle strikes horizontal ground at a distance 8 m from the foot of the cliff 2 s after the instant of projection. Find

(i) the speed of projection of the particle and the value of θ, [6]

(ii) the direction of motion of the particle immediately before it strikes the ground. [3]

i)



Horizontally, $x = v_x \times t$

$8 = u\cos\theta \times 2$

$4 = u\cos\theta \rightarrow (1)$

Vertically, $y = (u\sin\theta)t + \frac{1}{2}gt^2$

$26 = (u\sin\theta)(2) + \frac{1}{2}(10)(2)^2$

$6 = 2u\sin\theta$

$\Rightarrow 3 = u\sin\theta \rightarrow (2)$

Dividing (2) by (1):

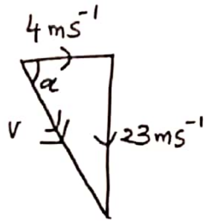
$\frac{u\sin\theta}{u\cos\theta} = \frac{3}{4} \Rightarrow \tan\theta = \frac{3}{4} \Rightarrow \theta = 36.86° \approx 36.9°$

Squaring (1) and (2) and adding:

$u^2\sin^2\theta + u^2\cos^2\theta = 3^2 + 4^2$

$u^2 = 25 \Rightarrow u = 5\,ms^{-1}.$

ii) When $t = 3$,



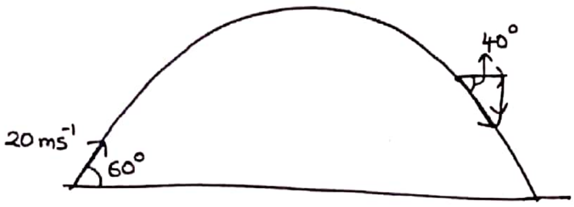
$v_x = u\cos\theta = 5\cos 36.9° = 4\,ms^{-1}.$

$v_y = u\sin\theta = 5\sin 36.9° + 10(2) = 23\,ms^{-1}.$

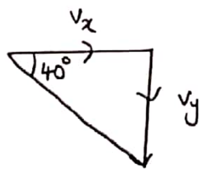
$\tan\alpha = \frac{v_y}{v_x} \Rightarrow \tan\alpha = \frac{23}{4} \Rightarrow \alpha = 80.1°$

$\therefore$ Direction of motion $= 80.1°$ below the horizontal.

39 A particle is projected with speed 20 m s^{-1} at an angle of 60° above the horizontal. Calculate the time after projection when the particle is descending at an angle of 40° below the horizontal. [4]



Horizontally, $v_x = 20\cos 60° = 10\,ms^{-1}$.



Vertically, $v_y = 20\sin 60° - 10t = (10\sqrt{3} - 10t)\,ms^{-1}$.

$$\tan(\pm 40°) = \frac{|v_y|}{|v_x|} \Rightarrow +\tan 40° = \frac{|v_y|}{|v_x|}$$

Since $v_y < 0 \Rightarrow |v_y| = -(v_y) = -(10\sqrt{3} - 10t) = 10t - 10\sqrt{3}$.

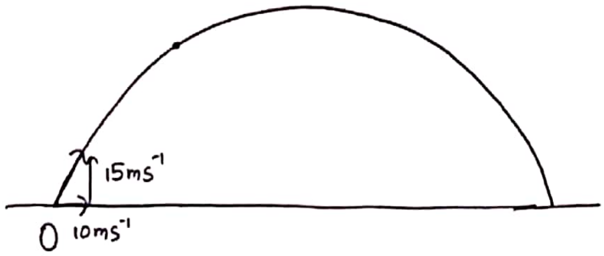
$$\therefore +\tan 40° = \frac{10t - 10\sqrt{3}}{10}$$

$$+8.3909 = 10t - 17.3205$$

$$t = 2.571 \simeq 2.57\,s.$$

40 A particle is projected from a point O on horizontal ground. The initial components of the velocity of the particle are $10\ \text{m s}^{-1}$ horizontally and $15\ \text{m s}^{-1}$ vertically. At time t s after projection, the horizontal and vertically upwards displacements of the particle from O are x m and y m respectively.

(i) Express x and y in terms of t, and hence find the equation of the trajectory of the particle. [4]



Horizontally, $x = v_x \times t = 10t. \Rightarrow x = 10t \rightarrow (1)$

Vertically, $y = 15t + \frac{1}{2}(-10)t^2 \Rightarrow y = 15t - 5t^2 \rightarrow (2).$

From (1), $t = \frac{x}{10} \rightarrow (3)$

Substitute (3) in (2):

$$y = 15\left(\frac{x}{10}\right) - 5\left(\frac{x}{10}\right)^2$$

$$y = 1{\cdot}5x - 0{\cdot}05x^2$$

The horizontal ground is at the top of a vertical cliff. The point O is at a distance d m from the edge of the cliff. The particle is projected towards the edge of the cliff and does not strike the ground before it passes over the edge of the cliff.

(ii) Show that d is less than 30. [2]

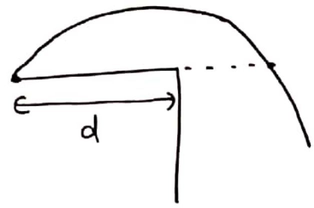
When $y = 0$,

$$1.5x - 0.05x^2 = 0$$

$$x(1.5 - 0.05x) = 0$$

$x = 0$ or $1.5 - 0.05x = 0$

$$x = \frac{1.5}{0.05} = 30$$



Then particle hits horizontal ground at $x = 0$ or $x = 30$ m.

If particle does not strike the ground, then $d < 30$m (shown).

(iii) Find the value of x when the particle is 14 m below the level of O. [2]

When $y = -14$

$$1.5x - 0.05x^2 = -14$$

$$0 = 0.05x^2 - 1.5x - 14$$

$$0 = x^2 - 30x - 280$$

$$x = \frac{-(-30) \pm \sqrt{(-30)^2 - 4(1)(-280)}}{2(1)}$$

$$x = \frac{30 \pm \sqrt{2020}}{2}$$

$x = 37.47, -7.47$ (Ignore)

$\therefore \quad x \approx 37.5$m

41 A particle P is projected from a point O with speed $V\ \mathrm{m\,s^{-1}}$. At time t s after projection the horizontal and vertically upwards displacements of P from O are x m and y m respectively. The equation of the trajectory of P is $y = 2x - \dfrac{25x^2}{V^2}$.

(i) Write down the value of $\tan\theta$, where θ is the angle of projection of P. [1]

By comparison to $y = x\tan\theta - \dfrac{g\sec^2\theta\, x^2}{2u^2}$,

$\tan\theta = 2$.

When $t = 4$, P passes through the point A where $x = y = a$.

(ii) Calculate V and a. [5]

When $x = a, y = a$

$$\Rightarrow \quad y = 2x - \frac{25x^2}{V^2}$$

$$a = 2a - \frac{25a^2}{V^2}$$

$$\cancel{+}\cancel{a} = \frac{\cancel{+}25a^{\cancel{2}}}{V^2}$$

$$V^2 = 25a \rightarrow (1).$$

Horizontally, $\quad x = V_x \times t$

$$a = V\cos\theta \times 4$$

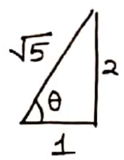
$$a = V \times \frac{1}{\sqrt{5}} \times 4$$

$$a = \frac{4V}{\sqrt{5}} \rightarrow (2)$$

$\tan\theta = 2$

$\Rightarrow \cos\theta = \dfrac{1}{\sqrt{5}}$

$\sin\theta = \dfrac{2}{\sqrt{5}}$.



Substituting (2) in (1):

$$V^2 = \frac{4V}{\sqrt{5}} \times 25$$

$$V = \frac{100}{\sqrt{5}} = 20\sqrt{5} = 44.72\ \mathrm{ms^{-1}}$$

$$\therefore \quad a = \frac{4 \times 20\sqrt{5}}{\sqrt{5}} = 80$$

i.e. $\quad a = 80, \quad V \approx 44.7\ \mathrm{ms^{-1}}$.

[3]

(iii) Find the direction of motion of P when it passes through A.

When $t = 4$,

$v_x = V\cos\theta = 20\sqrt{5} \times \dfrac{1}{\sqrt{5}} = 20\,ms^{-1}$.

$v_x = 20\,ms^{-1}$. $\longrightarrow$

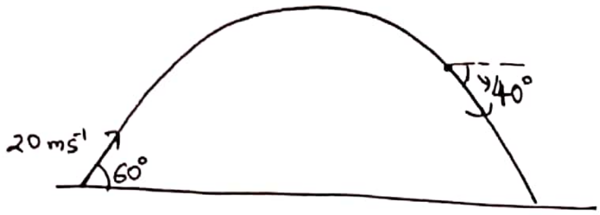
$v_y = v\sin\theta - 10(4) = 20\sqrt{5} \times \dfrac{2}{\sqrt{5}} - 40 = 0\,ms^{-1}$

$\Rightarrow \tan\alpha = \dfrac{v_y}{v_x} \Rightarrow \tan\alpha = \dfrac{0}{20} \Rightarrow \alpha = 0^\circ$

i.e. Direction of motion at A: 0° with horizontal

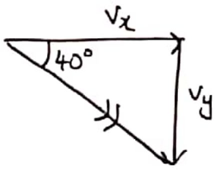
or // to horizontal.

42 A particle is projected with speed 20 m s^{-1} at an angle of 60° above the horizontal. Calculate the time after projection when the particle is descending at an angle of 40° below the horizontal. [4]



Horizontally, $v_x = 20\cos 60° = 10\,ms^{-1}$

In the vector diagram,



$$\tan 40° = \frac{v_y}{v_x}$$

$$\tan 40° = \frac{v_y}{10\,ms^{-1}}$$

$$v_y = 8.3909\,ms^{-1}$$

Vertically, using $v = u + at$

$$v_y = 20\sin 60° + (-10)t$$

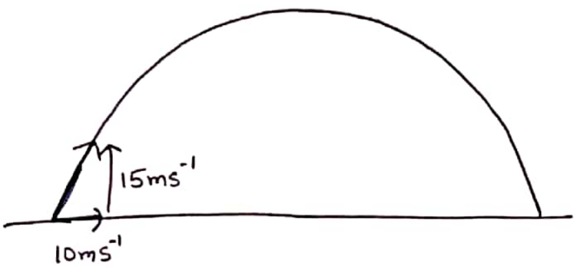
$$-8.3909 = 10\sqrt{3} - 10t$$

$$10t = 25.711$$

$$t = 2.571 \approx 2.57\,s.$$

43 A particle is projected from a point O on horizontal ground. The initial components of the velocity of the particle are $10\,\mathrm{m\,s^{-1}}$ horizontally and $15\,\mathrm{m\,s^{-1}}$ vertically. At time t s after projection, the horizontal and vertically upwards displacements of the particle from O are x m and y m respectively.

(i) Express x and y in terms of t, and hence find the equation of the trajectory of the particle. [4]



Horizontally, $x = v_x \times t$

$x = 10t \rightarrow (1)$

Vertically, using $s = ut + \frac{1}{2}at^2$

$y = 15t + \frac{1}{2}(-10)t^2$

$y = 15t - 5t^2 \rightarrow (2)$

From (1), $t = \frac{x}{10} \rightarrow (3)$

Substituting (3) in (2):

$y = 15\left(\frac{x}{10}\right) - 5\left(\frac{x}{10}\right)^2$

$y = 1.5x - 0.05x^2.$

The horizontal ground is at the top of a vertical cliff. The point O is at a distance d m from the edge of the cliff. The particle is projected towards the edge of the cliff and does not strike the ground before it passes over the edge of the cliff.

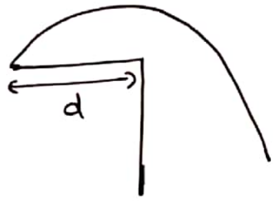
(ii) Show that d is less than 30. [2]

When particle returns to horizontal level, $y = 0$

$$\Rightarrow 1.5x - 0.05x^2 = 0$$

$$0.05x(30 - x) = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad x = 30.$$



Then particle returns to horizontal level when $x = 30$m.

Since particle does not strike the ground before it passes over the edge,

$d < 30$m. (shown).

(iii) Find the value of x when the particle is 14 m below the level of O. [2]

For particle to be 14m below level of O,

$$y = -14$$

$$\Rightarrow 1.5x - 0.05x^2 = -14$$

$$-0.05x^2 + 1.5x + 14 = 0$$

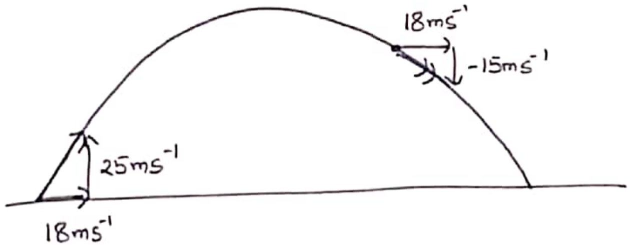
$$x^2 - 30x - 280 = 0$$

$$x = \frac{-(-30) \pm \sqrt{(-30)^2 - 4(1)(-280)}}{2(1)} = \frac{30 \pm \sqrt{2020}}{2}$$

$x = 37.47$ or $x = -7.47$ (Ignore).

$\therefore x \simeq 37.5$m.

4 A small ball B is projected from a point O on horizontal ground. The initial velocity of B has horizontal and vertically upwards components of $18\,\mathrm{m\,s^{-1}}$ and $25\,\mathrm{m\,s^{-1}}$ respectively. For the instant 4 s after projection, find the speed and direction of motion of B. [4]



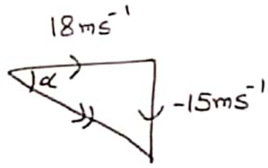
When $t = 4$,

$v_x = 18\,\mathrm{ms^{-1}}$.

$v_y = 25 + (-10)(4) = -15\,\mathrm{ms^{-1}}$

$\Rightarrow v = \sqrt{v_x^2 + v_y^2} = \sqrt{18^2 + (-15)^2} = 23.43\,\mathrm{ms^{-1}}$

$v \simeq 23.4\,\mathrm{ms^{-1}}$



$\alpha = \tan^{-1}\left(\frac{-15}{18}\right) = -39.80^\circ \approx -39.8^\circ$.

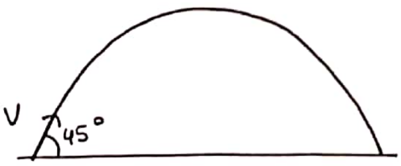
$\therefore$ Speed of B $= 23.4\,\mathrm{ms^{-1}}$, Direction of motion $= 39.8^\circ$ below horizontal.

45 A small object is projected from a point O with speed $V\ \text{m s}^{-1}$ at an angle of 45° above the horizontal. At time t s after projection, the horizontal and vertically upwards displacements of the object from O are x m and y m respectively.

(i) Express x and y in terms of t, and hence find the equation of the path. [4]

Horizontally, $x = v_x \times t$

$$x = V\cos 45° \times t = \frac{\sqrt{2}Vt}{2} \rightarrow (1)$$



Vertically, $y = (V\sin 45°)t + \frac{1}{2}(-10)t^2$

$$y = \frac{\sqrt{2}Vt}{2} - 5t^2 \rightarrow (2)$$

From (1), $t = \frac{2x}{\sqrt{2}V} = \frac{\sqrt{2}x}{V} \rightarrow (3)$

Substitute (3) in (2): $y = \frac{\sqrt{2}V}{2}\left(\frac{\sqrt{2}x}{V}\right) - 5\left(\frac{\sqrt{2}x}{V}\right)^2$

$$y = x - \frac{10x^2}{V^2}$$

The object passes through the point with coordinates (24, 18).

(ii) Find V. [2]

When $x = 24$, $y = 18$ $\Rightarrow$ $18 = 24 - \frac{10(24)^2}{V^2}$

$$-6 = \frac{-5760}{V^2}$$

$$V^2 = 960$$

$$V = 30.98\ \text{ms}^{-1} \approx 31.0\ \text{ms}^{-1}.$$

(iii) The object passes through two points which are 22.5 m above the level of O. Find the values of x for these points. [3]

Substituting $V \approx 30.98$ in $y = x - \frac{10x^2}{V^2} = x - \frac{10x^2}{960}$

$$y = x - \frac{x^2}{96}$$

when $y = 22.5 \Rightarrow$ $22.5 = x - \frac{x^2}{96}$

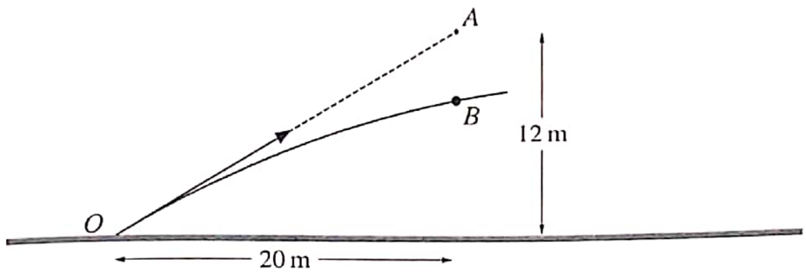
$$\frac{x^2}{96} - x + 22.5 = 0$$

$$x^2 - 96x + 2160 = 0$$

$$x^2 - 36x - 60x + 2160 = 0$$

$$(x-36)(x-60) = 0$$

Either, $x = 36$ or $x = 60$.



A small ball B is projected from a point O on horizontal ground towards a point A 12 m above the ground. 0.9 s after projection B has travelled a horizontal distance of 20 m and is vertically below A (see diagram).

(i) Find the angle and the speed of projection of B. [4]

$\tan\theta = \frac{12}{20} \Rightarrow \theta = \tan^{-1}\left(\frac{3}{5}\right) = 30.96° \approx 31.0°$

Horizontally,
$$x = v_x \times t$$
$$20 = u\cos(30.96°) \times 0.9$$
$$u = \frac{20}{0.9 \times \cos 30.96°} = 25.91 ms^{-1}$$
$$u \simeq 25.9 ms^{-1}.$$

$\therefore$ Angle of projection $= 31.0°$, Speed of projection $= 25.9 ms^{-1}$.

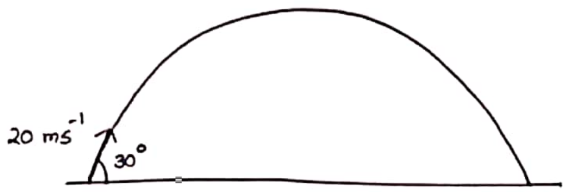
(ii) Calculate the distance AB when B is vertically below A. [2]

Vertically, using $s = ut + \frac{1}{2}at^2$
$$y = (25.9 \sin 31.0°) \times 0.9 + \frac{1}{2}(-10)(0.9)^2$$
$$y = 7.95m$$

$\Rightarrow$ Distance $AB = 12 - 7.95 = 4.05m.$

47 A particle P is projected from a point O on horizontal ground with initial speed $20\,\mathrm{m\,s^{-1}}$ and angle of projection $30°$. At the instant t s after projection, the horizontal and vertically upwards displacements of P from O are x m and y m respectively.

(i) Express x and y in terms of t and hence find the equation of the trajectory of P. [4]



Horizontally, $x = v_x \times t$

$= 20\cos 30° \times t$

$x = 10\sqrt{3}\,t \rightarrow (1)$

Vertically, using $s = ut + \frac{1}{2}at^2$

$y = (20\sin 30°)t + \frac{1}{2}(-10)t^2$

$y = 10t - 5t^2 \rightarrow (2)$

From (1), $t = \dfrac{x}{10\sqrt{3}} \rightarrow (3)$

Substituting (3) in (2):

$$y = 10\left(\frac{x}{10\sqrt{3}}\right) - 5\left(\frac{x}{10\sqrt{3}}\right)^2$$

$$y = \frac{x}{\sqrt{3}} - \frac{x^2}{60}.$$

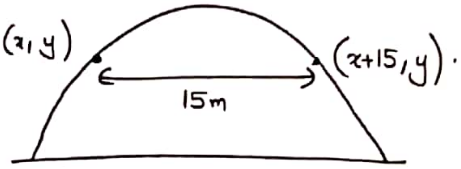
P is at the same height above the ground at two points which are a horizontal distance apart of 15 m.

(ii) Calculate this height. [3]

Let, first point be (x, y)

second point is $(x+15, y)$

Note y-values are equal as both points have the same height above the ground.



At (x,y), $y = \frac{x}{\sqrt{3}} - \frac{x^2}{60} \rightarrow (1)$

At $(x+15, y)$, $y = \frac{x+15}{\sqrt{3}} - \frac{(x+15)^2}{60} \rightarrow (2)$

Substituting (1) in (2):

$$\frac{x}{\sqrt{3}} - \frac{x^2}{60} = \frac{x+15}{\sqrt{3}} - \frac{(x+15)^2}{60}$$

$$\frac{(x+15)^2 - x^2}{60} = \frac{x+15-x}{\sqrt{3}}$$

$$\frac{30x+225}{60} = \frac{15}{\sqrt{3}}$$

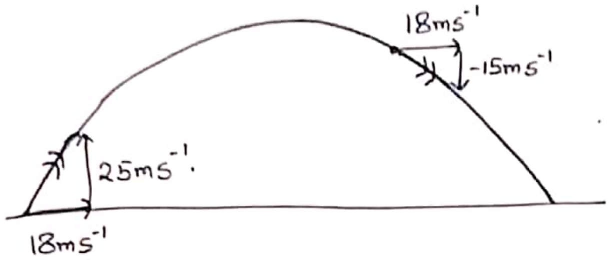
$$30x = 294.615$$

$$x = 9.8205$$

$$\Rightarrow y = \frac{9.8205}{\sqrt{3}} - \frac{9.8205^2}{60} = 4.0625\text{m}$$

$\Rightarrow$ Height of P = 4.06 m.

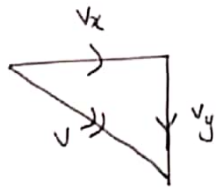
6 A small ball B is projected from a point O on horizontal ground. The initial velocity of B has horizontal and vertically upwards components of $18\ \text{m s}^{-1}$ and $25\ \text{m s}^{-1}$ respectively. For the instant 4 s after projection, find the speed and direction of motion of B. [4]



When $t = 4s$,

$v_x = 18 ms^{-1}$.

$v_y = 25 + (-10)(4) = -15 ms^{-1}$.



$v = \sqrt{v_x^2 + v_y^2} = \sqrt{18^2 + (-15)^2} = 23.43 ms^{-1}$

$\tan\alpha = \frac{v_y}{v_x} \Rightarrow \tan\alpha = \frac{-15}{18} \Rightarrow \alpha = \tan^{-1}\left(\frac{-15}{18}\right) = -39.80°$

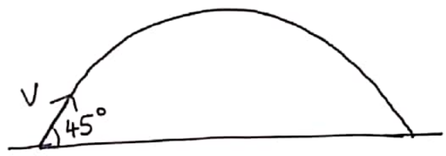
$\Rightarrow$ Speed of motion $= 23.4 ms^{-1}$, Direction of motion $=$ ~~38.~~ $39.8°$ below horizontal.

49 A small object is projected from a point O with speed V m s^{-1} at an angle of 45° above the horizontal. At time t s after projection, the horizontal and vertically upwards displacements of the object from O are x m and y m respectively.

(i) Express x and y in terms of t, and hence find the equation of the path. [4]

Horizontally, $x = v_x \times t$

$$x = V\cos 45° \times t = \frac{\sqrt{2}Vt}{2} \rightarrow (1)$$



Vertically, using $s = ut + \frac{1}{2}at^2$

$$y = (V\sin 45°)t + \frac{1}{2}(-10)t^2$$

$$y = \frac{\sqrt{2}Vt}{2} - 5t^2 \rightarrow (2)$$

From (1), $t = \frac{2x}{\sqrt{2}V} = \frac{\sqrt{2}x}{V} \rightarrow (3)$

Substituting (3) in (2):

$$y = \frac{\sqrt{2}V}{2}\left(\frac{\sqrt{2}x}{V}\right) - 5\left(\frac{\sqrt{2}x}{V}\right)^2$$

$$y = x - \frac{10x^2}{V^2}.$$

The object passes through the point with coordinates (24, 18).

(ii) Find V. [2]

When $x = 24$, $y = 18$

$$\Rightarrow 18 = 24 - \frac{10(24)^2}{V^2}$$

$$-6 = \frac{-5760}{V^2}$$

$$V^2 = 960$$

$$V = \sqrt{960} = 30.98 \text{ ms}^{-1}$$

$$V \approx 31.0 \text{ ms}^{-1}.$$

(iii) The object passes through two points which are 22.5 m above the level of O. Find the values of x for these points. [3]

$$y = x - \frac{10x^2}{V^2}$$

Since $V \approx 31.0 \Rightarrow y = x - \frac{10x^2}{960}$.

When object is 22.5m above level of O,

$$22.5 = x - \frac{10x^2}{960}$$

$$2160 = 96x - x^2$$

$$\Rightarrow x^2 - 96x + 2160 = 0$$

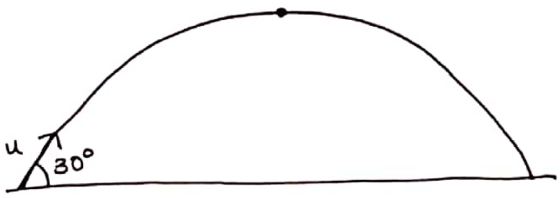
$$x^2 - 36x - 60x + 2160 = 0$$

$$(x-36)(x-60) = 0$$

$$\Rightarrow x = 36 \quad \text{or} \quad x = 60.$$

5 A particle P is projected with speed u at an angle of 30° above the horizontal from a point O on a horizontal plane and moves freely under gravity. The particle reaches its greatest height at time T after projection.

Find, in terms of u, the speed of P at time $\frac{2}{3}T$ after projection. [5]



When particle is at its greatest height,

$$\text{time} = \frac{u\sin\theta}{g}$$

$$T = \frac{u\sin 30^\circ}{10} = \frac{u}{20}$$

$$\Rightarrow \quad u = 20T \Rightarrow (1)$$

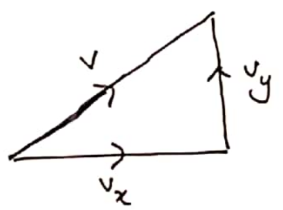
When $t = \frac{2}{3}T$,

$$v_x = u\cos 30^\circ = u\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}u.$$

$$v_y = u\sin 30^\circ + (-10)\left(\frac{2}{3}T\right) = \frac{u}{2} - \frac{20T}{3}$$

Substituting $u = 20T$ gives:

$$v_y = \frac{u}{2} - \frac{u}{3} = \frac{u}{6}.$$

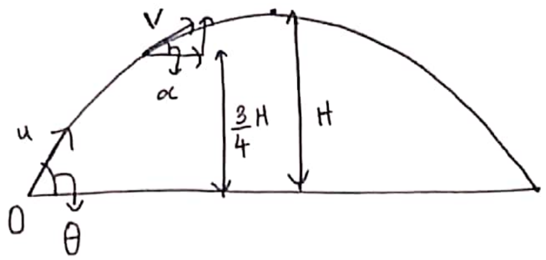


$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{\left(\frac{\sqrt{3}}{2}u\right)^2 + \left(\frac{u}{6}\right)^2}$$

$$v = \sqrt{\frac{7u^2}{9}} = \frac{\sqrt{7}u}{3}.$$

5 A particle P is projected with speed u at an angle θ above the horizontal from a point O on a horizontal plane and moves freely under gravity. The direction of motion of P makes an angle α above the horizontal when P first reaches three-quarters of its greatest height.

(a) Show that $\tan\alpha = \frac{1}{2}\tan\theta$. [6]



For greatest height H, using $v^2 = u^2 + 2as$

$$\cancel{v^2}\quad 0^2 = (u\sin\theta)^2 + 2(-g)H$$

$$H = \frac{(u\sin\theta)^2}{2g}$$

Horizontally, $u\cos\theta = v\cos\alpha \rightarrow (1).$

Vertically, using ~~$v = u + at$~~ $v^2 = u^2 + 2as$

~~$v\sin\alpha = u\sin\theta + (-g)t$~~ $(v\sin\alpha)^2 = (u\sin\theta)^2 + 2(-g)\left(\frac{3}{4}H\right)$

$$v^2\sin^2\alpha = u^2\sin^2\theta - \frac{3gH}{2}$$

Substituting $H = \frac{(u\sin\theta)^2}{2g}$ gives:

$$v^2\sin^2\alpha = u^2\sin^2\theta - \frac{3g}{2}\left(\frac{u^2\sin^2\theta}{2g}\right)$$

$$= u^2\sin^2\theta - \frac{3u^2\sin^2\theta}{4}$$

$$v^2\sin^2\alpha = \frac{u^2\sin^2\theta}{4}$$

$$v\sin\alpha = \frac{u\sin\theta}{2} \rightarrow (2).$$

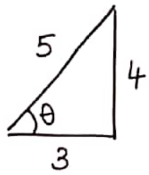
Dividing (2) by (1) gives:

$$\frac{v\sin\alpha}{v\cos\alpha} = \frac{\frac{u\sin\theta}{2}}{u\cos\theta}$$

$$\tan\alpha = \frac{1}{2}\tan\theta. \text{ (Shown).}$$

(b) Given that $\tan\theta = \frac{4}{3}$, find the horizontal distance travelled by P when it first reaches three-quarters of its greatest height. Give your answer in terms of u and g. [4]

$\tan\theta = \frac{4}{3} \Rightarrow \tan\alpha = \frac{1}{2}\left(\frac{4}{3}\right) = \frac{2}{3}$.



$\tan\theta = \frac{4}{3}$

$\sin\theta = \frac{4}{5}$

$\cos\theta = \frac{3}{5}$

When $y = \frac{3}{4}H$,

$v_y = v\sin\alpha = \frac{u\sin\theta}{2} \rightarrow (2)$

Vertically, $v_y = u\sin\theta - gt$

$\frac{u\sin\theta}{2} = u\sin\theta - gt$

$-\frac{u\sin\theta}{2} = -gt$

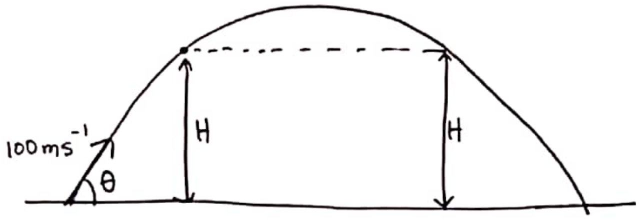
$t = \frac{u\sin\theta}{2g} = \frac{u\left(\frac{4}{5}\right)}{2g} = \frac{2u}{5g}$

Horizontal distance travelled by P when it reaches $\frac{3}{4}$ of its greatest height

$= v_x \times t = u\cos\theta \times \frac{2u}{5g} = u\left(\frac{3}{5}\right)\times\frac{2u}{5g} = \frac{6u^2}{25g}$.

52 A particle P is projected from a point O on a horizontal plane and moves freely under gravity. The initial velocity of P is $100\,\mathrm{m\,s^{-1}}$ at an angle θ above the horizontal, where $\tan\theta = \frac{4}{3}$. The two times at which P's height above the plane is H m differ by 10 s.

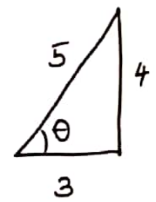
(a) Find the value of H. [5]



$\tan\theta = \frac{4}{3}$

$\sin\theta = \frac{4}{5}$

$\cos\theta = \frac{3}{5}$.



P reaches its greatest height when time $= \dfrac{u\sin\theta}{g}$

$= \dfrac{100\left(\frac{4}{5}\right)}{10} = 8s.$

Then, particle has height H m when $t = 8-5 = 3s$ or $t = 8+5 = 13s$.

Vertically, when $t = 3s$,

$y = (100\sin\theta)(3) + \frac{1}{2}(-10)(3)^2$

$H = 300\left(\frac{4}{5}\right) - 5\times 9 = 195m.$

i.e. $H = 195m$.

(b) Find the magnitude and direction of the velocity of P one second before it strikes the plane. [4]

P strikes plane when time $= 2(8) = 16\text{s}$.

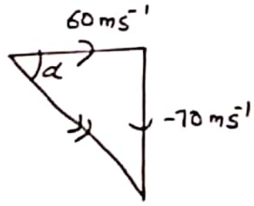
1 second before P strikes plane, $t = 16 - 1 = 15\text{s}$.

Horizontally, $v_x = u\cos\theta = 100 \times \frac{3}{5} = 60\text{ms}^{-1}$.

Vertically, $v_y = u\sin\theta + (-g)t = 100 \times \frac{4}{5} - 10 \times 15 = -70\text{ms}^{-1}$.

$v = \sqrt{60^2 + (-70)^2} = 92.195\text{ ms}^{-1}$

$\Rightarrow \quad v \approx 92.2\text{ ms}^{-1}$.



$\tan\alpha = \frac{-70}{60} \Rightarrow \alpha = \tan^{-1}\left(\frac{-70}{60}\right) = -49.39° \approx -49.4°$

Speed of P $= 92.2\text{ms}^{-1}$, Direction of P $= 49.4°$ below horizontal.

53 A particle P is projected with speed u at an angle θ above the horizontal from a point O on a horizontal plane and moves freely under gravity. The horizontal and vertical displacements of P from O at a subsequent time t are denoted by x and y respectively.

(a) Use the equation of the trajectory given in the List of formulae (MF19), together with the condition $y = 0$, to establish an expression for the range R in terms of u, θ and g. [2]

$$y = x\tan\theta - \frac{g\sec^2\theta\, x^2}{2u^2}$$

When $y=0$,

$$0 = x\tan\theta - \frac{g\sec^2\theta\, x^2}{2u^2}$$

$$0 = \left(\tan\theta - \frac{g\sec^2\theta}{2u^2}x\right)x$$

Either, $x=0$ or $\tan\theta - \frac{g\sec^2\theta\, x}{2u^2} = 0$

$$x = \frac{2u^2\tan\theta}{g\sec^2\theta} = \frac{2u^2\left(\frac{\sin\theta}{\cos\theta}\right)}{g\left(\frac{1}{\cos^2\theta}\right)}$$

$$x = \frac{2u^2\sin\theta\cos\theta}{g} = \frac{u^2\sin 2\theta}{g}$$

$$\therefore\ R = \frac{u^2\sin 2\theta}{g}.$$

(b) Deduce an expression for the maximum height H, in terms of u, θ and g. [2]

For maximum height, $x = \frac{R}{2} = \frac{2u^2\sin\theta\cos\theta}{2g} = \frac{u^2\sin\theta\cos\theta}{g}$.

Substituting: $y = \left(\frac{u^2\sin\theta\cos\theta}{g}\right)\tan\theta - \frac{g\sec^2\theta}{2u^2}\left(\frac{u^2\sin\theta\cos\theta}{g}\right)^2$

$$H = \frac{u^2\sin^2\theta}{g} - \frac{g\sec^2\theta}{2u^2}\times\frac{u^4\sin^2\theta\cos^2\theta}{g^2}$$

$$H = \frac{u^2\sin^2\theta}{g} - \frac{u^2\sin^2\theta}{2g}$$

$$H = \frac{u^2\sin^2\theta}{2g}$$

i.e. $H = \frac{(u\sin\theta)^2}{2g}$.

It is given that $R = \dfrac{4H}{\sqrt{3}}$.

(c) Show that $\theta = 60°$. [1]

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{4H}{\sqrt{3}} = \frac{4}{\sqrt{3}}\left[\frac{u^2 \sin^2\theta}{2g}\right]$$

$$\Rightarrow \frac{\cancel{2}\cancel{u^2}\,\sin\cancel{\theta}\cos\theta}{\cancel{g}} = \frac{\cancel{4}^{2}\cancel{u^2}\,\sin^{\cancel{2}}\theta}{\sqrt{3}(\cancel{2}\cancel{g})}$$

$$\cos\theta = \frac{\sin\theta}{\sqrt{3}} \Rightarrow \tan\theta = \sqrt{3}$$

$$\theta = 60°.$$

It is given also that $u = \sqrt{40}\,\mathrm{m\,s^{-1}}$.

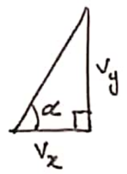
(d) Find, by differentiating the equation of the trajectory or otherwise, the set of values of x for which the direction of motion makes an angle of less than 45° with the horizontal. [4]

$$y = x\tan\theta - \frac{g\sec^2\theta\, x^2}{2u^2}$$

$$\frac{dy}{dx} = \tan\theta - \frac{g\sec^2\theta\, x}{u^2}.$$

Since $u = \sqrt{40} \Rightarrow \dfrac{dy}{dx} = \tan\theta - \dfrac{10\sec^2\theta\, x}{40} = \tan\theta - \dfrac{\sec^2\theta\, x}{4}.$

Since $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{v_y}{v_x} = \tan\alpha$



Since we require $|\alpha| < 45° \Rightarrow |\tan\alpha| < 1$

$$\left|\frac{dy}{dx}\right| < 1 \Rightarrow \left|\tan\theta - \frac{\sec^2\theta}{4}x\right| < 1 \Rightarrow -1 < \tan\theta - \frac{\sec^2\theta}{4}x < 1.$$

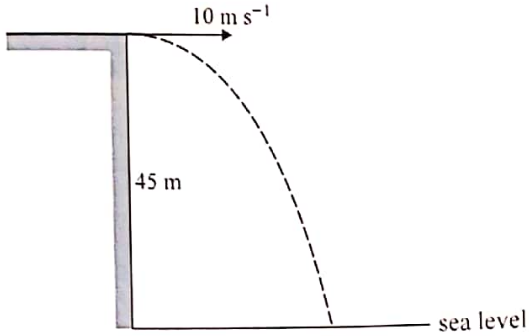
Critical values: $-1 = \tan\theta - \dfrac{\sec^2\theta}{4}x$ and $\tan\theta - \dfrac{\sec^2\theta}{4}x = 1$

$-1 = \tan 60° - \dfrac{\sec^2 60°}{4}x$ $\qquad$ $\tan 60° - \dfrac{\sec^2 60°}{4}x = 1$

$-1 = \sqrt{3} - x$ $\qquad$ $\sqrt{3} - x = 1$

$x = \sqrt{3} + 1.$ $\qquad$ $\sqrt{3} - 1 = x$

Then $\sqrt{3} - 1 < x < \sqrt{3} + 1.$



A stone is projected horizontally, with speed $10\,\mathrm{m\,s^{-1}}$, from the top of a vertical cliff of height 45 m above sea level (see diagram). At time t s after projection the horizontal and vertically upward displacements of the stone from the top of the cliff are x m and y m respectively.

(i) Write down expressions for x and y in terms of t, and hence obtain the equation of the stone's trajectory. [3]

(ii) Find the angle the trajectory makes with the horizontal at the point where the stone reaches sea level. [3]

i) Horizontally, $x = v_x \times t$

$x = 10t \rightarrow (1)$

Vertically, $y = 0(t) + \frac{1}{2}(-10)t^2$

$y = -5t^2 \rightarrow (2)$

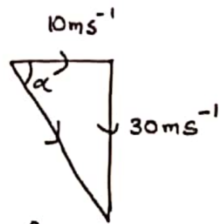
From (1), $t = \frac{x}{10} \rightarrow (3)$

Substituting (3) in (2):

$y = -5\left(\frac{x}{10}\right)^2 = \frac{-x^2}{20}.$

ii) When stone reaches sea level, $y = -45$

$\Rightarrow -45 = -5t^2 \Rightarrow t^2 = 9 \Rightarrow t = 3.$



When $t = 3$,

$v_x = 10\ \mathrm{ms^{-1}}.$

$v_y = 0 + (-10)(3) = -30\ \mathrm{ms^{-1}}$

$\tan\alpha = \frac{v_y}{v_x} \Rightarrow \tan\alpha = \frac{-30}{10} \Rightarrow \alpha = \tan^{-1}(-3) = -71.56° \approx -71.6°$

Direction of stone $= 71.6°$ below the horizontal.

55 A particle is projected from a point O on horizontal ground with speed $50\ \text{m s}^{-1}$ at an angle θ to the horizontal. Given that the speed of the particle when it is at its highest point is $40\ \text{m s}^{-1}$,

(i) show that $\cos\theta = 0.8$, [2]

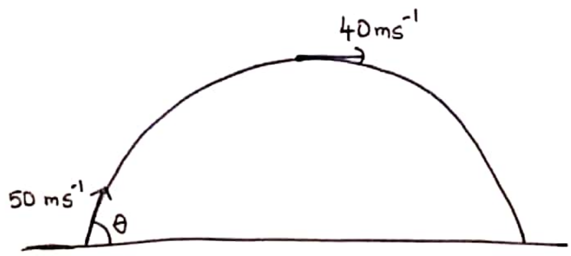
(ii) find, in either order,

(a) the greatest height reached by the particle,

(b) the distance from O at which the particle hits the ground.

[5]

i)



Horizontally,

$$50\cos\theta = 40$$

$$\cos\theta = \frac{4}{5} = 0.8 \quad \text{(Shown)}.$$

ii) a) Vertically, using $v^2 = u^2 + 2as$

$$0^2 = (50\sin\theta)^2 + 2(-10)(H)$$

$$0 = \left(50 \times \frac{3}{5}\right)^2 - 20H$$

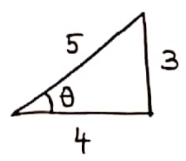
$$20H = 900$$

$$H = 45\text{m}.$$

$\cos\theta = \frac{4}{5}$

$\sin\theta = \frac{3}{5}$

$\tan\theta = \frac{3}{4}$.



b) Total time of flight $= \dfrac{2u\sin\theta}{g} = \dfrac{2 \times 50 \times \frac{3}{5}}{10} = 6\ \cancel{\text{ms}^{-1}}\ \text{s}$

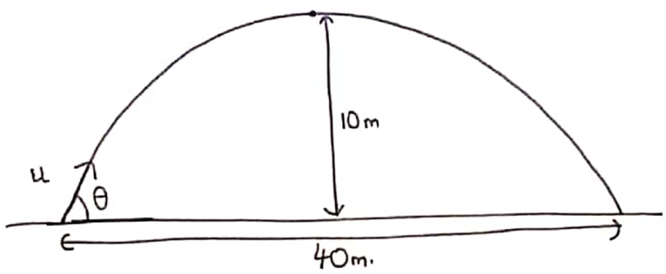
Range $= v_x \times t = 50\cos\theta \times 6 = 50 \times \frac{4}{5} \times 6 = 240\text{m}.$

56 A particle is projected from horizontal ground with speed u m s^{-1} at an angle of θ° above the horizontal. The greatest height reached by the particle is 10 m and the particle hits the ground at a distance of 40 m from the point of projection. In either order,

(i) find the values of u and θ,

(ii) find the equation of the trajectory, in the form $y = ax - bx^2$, where x m and y m are the horizontal and vertical displacements of the particle from the point of projection.

[7]



i) Maximum height $= \dfrac{(u\sin\theta)^2}{2g}$

$10 = \dfrac{(u\sin\theta)^2}{2\times 10} \Rightarrow (u\sin\theta)^2 = 200 \Rightarrow u^2\sin^2\theta = 200 \rightarrow (1)$

Range $= \dfrac{2u^2\sin\theta\cos\theta}{g}$

$40 = \dfrac{2u^2\sin\theta\cos\theta}{10} \Rightarrow u^2\sin\theta\cos\theta = 200 \rightarrow (2).$

Dividing (1) by (2) : $\dfrac{u^2\sin^2\theta}{u^2\sin\theta\cos\theta} = \dfrac{200}{200}$

$\tan\theta = 1$

$\theta = 45°$

$\Rightarrow u^2\sin^2 45° = 200$

$\dfrac{u^2}{2} = 200 \Rightarrow u^2 = 400$

$u = 20\text{ ms}^{-1}.$

ii)

$y = x\tan\theta - \dfrac{g\sec^2\theta\, x^2}{2u^2}$

$y = x\tan 45° - \dfrac{10\sec^2 45°\, x^2}{2(20)^2}$

$y = x - \dfrac{x^2}{40}.$

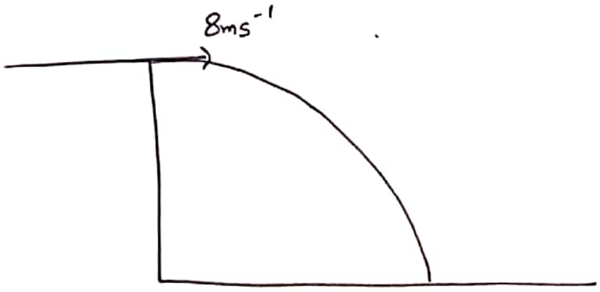
57 A stone is projected horizontally with speed 8 m s^{-1} from a point O at the top of a vertical cliff. The horizontal and vertically upward displacements of the stone from O are x m and y m respectively.

(i) Find the equation of the stone's trajectory. [2]

The stone enters the sea at a horizontal distance of 24 m from the base of the cliff.

(ii) Find the height above sea level of the top of the cliff. [2]

i)



Horizontally, $x = v_x \times t \Rightarrow x = 8t \Rightarrow t = \frac{x}{8} \rightarrow (1)$

Vertically, $y = 0(t) + \frac{1}{2}(-10)t^2 \Rightarrow y = -5t^2 \rightarrow (2)$

Substituting (1) in (2):

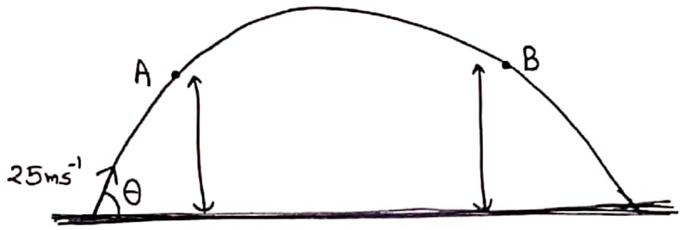
$$y = -5\left(\frac{x}{8}\right)^2 = \frac{-5x^2}{64}$$

ii) When $x = 24$, $y = \frac{-5(24)^2}{64} = -45\text{m}$

$\therefore$ Height above sea level of cliff $= 45\text{ m}$.

58 A stone is projected from a point on horizontal ground with speed 25 m s^{-1} at an angle θ above the horizontal, where $\sin\theta = \frac{4}{5}$. At time 1.2 s after projection the stone passes through the point A.

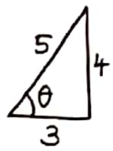
Subsequently the stone passes through the point B, which is at the same height above the ground as A. Find the horizontal distance AB. [5]



$\sin\theta = \frac{4}{5}$

$\cos\theta = \frac{3}{5}$

$\tan\theta = \frac{4}{3}$.



When $t = 1.2$,

$y = (25\sin\theta)t + \frac{1}{2}(-10)t^2 = 25 \times \frac{4}{5} \times 1.2 - 5 \times 1.2^2$

$y = 16.8$ m

When $y = 16.8$,

$y = (25\sin\theta)t - 5t^2$

$16.8 = 20t - 5t^2$

$5t^2 - 20t + 16.8 = 0$

$25t^2 - 100t + 84 = 0$

$25t^2 - 30t - 70t + 84 = 0$

$(5t - 14)(5t - 6) = 0$

$t = \frac{6}{5}$ or $t = \frac{14}{5}$.

When $t = \frac{14}{5}$, $x = v_x \times t = 25 \times \frac{3}{5} \times \frac{14}{5} = 42$m

When $t = \frac{6}{5}$, $x = v_x \times t = 25 \times \frac{3}{5} \times \frac{6}{5} = 18$m.

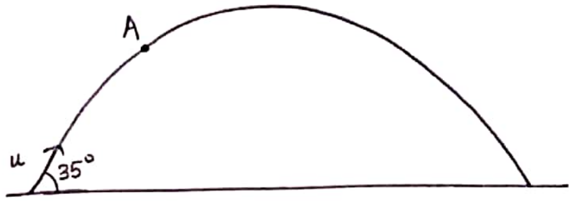
Horizontal distance AB $= 42 - 18 = 24$ m.

59 A particle is projected from a point O at an angle of $35°$ above the horizontal. At time T s later the particle passes through a point A whose horizontal and vertically upward displacements from O are 8 m and 3 m respectively.

(i) By using the equation of the particle's trajectory, or otherwise, find (in either order) the speed of projection of the particle from O and the value of T. [5]

(ii) Find the angle between the direction of motion of the particle at A and the horizontal. [4]

i)



$$y = x\tan\theta - \frac{g\sec^2\theta\, x^2}{2u^2}$$

When $x = 8$, $y = 3$, $\theta = 35°$

$$\Rightarrow\ 3 = 8\tan 35° - \frac{10\sec^2 35°\,(8)^2}{2\times u^2}$$

$$\frac{320\sec^2 35°}{u^2} = 8\tan 35° - 3$$

$$u^2 = \frac{320\sec^2 35°}{8\tan 35° - 3}$$

$$u = 13.53\ \text{ms}^{-1} = 13.5\ \text{ms}^{-1}.$$

Horizontally, $x = v_x \times t$

$$8 = 13.53\times\cos 35°\times T$$

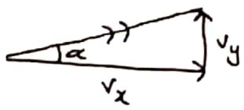
$$T = 0.7213 \simeq 0.721.$$

ii) When $t = 0.7213$,

$$v_x = 13.53\cos 35° = 11.090\ \text{ms}^{-1} \simeq 11.09\ \text{ms}^{-1}.$$

$$v_y = 13.53\sin 35° + (-10)(0.7213) = 0.5526\ \text{ms}^{-1}.$$

$$\tan\alpha = \frac{v_y}{v_x} \Rightarrow \tan\alpha = \frac{0.5526}{11.09}$$



$$\alpha = 2.852° \simeq 2.85°.$$

Direction of motion $= 2.85°$ above the horizontal.

60 A particle P is projected from a point O on horizontal ground with speed V m s^{-1} and direction 60° upwards from the horizontal. At time t s later the horizontal and vertical displacements of P from O are x m and y m respectively.

(i) Write down expressions for x and y in terms of V and t and hence show that the equation of the trajectory of P is

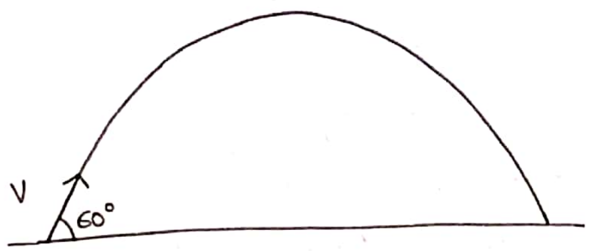
$$y = (\sqrt{3})x - \frac{20x^2}{V^2}.$$ [5]

P passes through the point A at which $x = 70$ and $y = 10$. Find

(ii) the value of V, [2]

(iii) the direction of motion of P at the instant it passes through A. [3]

i)



Horizontally, $x = V_x \times t = V\cos 60° \times t = \frac{Vt}{2} \rightarrow (1)$

Vertically, $y = (V\sin 60°)t + \frac{1}{2}(-g)t^2 = \frac{\sqrt{3}Vt}{2} - 5t^2 \rightarrow (2)$

From (1), $t = \frac{2x}{V} \rightarrow (3)$

Substituting (3) in (2): $y = \frac{\sqrt{3}V}{2}\left(\frac{2x}{V}\right) - 5\left(\frac{2x}{V}\right)^2$

$y = \sqrt{3}x - \frac{20x^2}{V^2}$. (Shown).

ii) When $x = 70$, $y = 10$

$\Rightarrow 10 = \sqrt{3}(70) - \frac{20(70)^2}{V^2}$

$10 = 70\sqrt{3} - \frac{98\,000}{V^2}$

$V^2 = \frac{98000}{70\sqrt{3} - 10}$

$V = 29.68 \text{ ms}^{-1} \approx 29.7 \text{ ms}^{-1}$!

iii)

$$y = \sqrt{3}\,x - \frac{20x^2}{V^2}$$

Substituting $V = 29.68$ gives:

$$y = \sqrt{3}\,x - \frac{20x^2}{880.94}$$

$$\frac{dy}{dx} = \sqrt{3} - \frac{40}{880.94}x$$

When $x = 70$,

$$\frac{dy}{dx} = \sqrt{3} - \frac{40(70)}{880.94} = -1.446$$

Since $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{v_y}{v_x} = \tan\alpha$

$\therefore \tan\alpha = -1.446$

$\alpha = \tan^{-1}(-1.446)$

$\alpha = -55.33^\circ \approx -55.3^\circ$

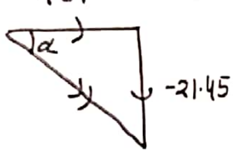
Direction of motion $= 55.3^\circ$ below the horizontal.

Alternatively, $x = 70 \Rightarrow v_x \times t = 70$

$29.68 \cos 60^\circ \times t = 70$

$t = 4.716\,s.$

When $t = 4.716\,s$,



$v_x = 29.68 \cos 60^\circ = 14.84\ ms^{-1}$

$v_y = 29.68 \sin 60^\circ + (-10)(4.716) = -21.45\ ms^{-1}.$

$\tan\alpha = \frac{v_y}{v_x} = \frac{-21.45}{14.84} \Rightarrow \alpha = -55.33^\circ \approx -55.3^\circ$

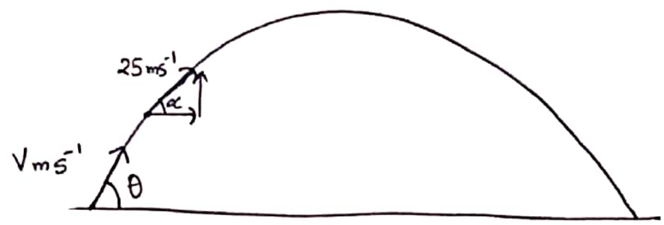
Direction of motion $= 55.3^\circ$ below the horizontal.

61 A particle is projected from a point O with speed $V\,\mathrm{m\,s^{-1}}$ at an angle θ above the horizontal. After 0.3 s the particle is moving with speed $25\,\mathrm{m\,s^{-1}}$ at an angle $\tan^{-1}\left(\frac{7}{24}\right)$ above the horizontal.

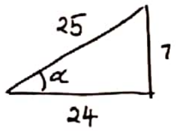
(i) Show that $V\cos\theta = 24$. [2]

(ii) Find the value of $V\sin\theta$, and hence find V and θ. [5]

i)



$\tan\alpha = \frac{7}{24}$



$\sin\alpha = \frac{7}{25}$

$\cos\alpha = \frac{24}{25}$.

Horizontally, $V\cos\theta = 25\cos\alpha$

$V\cos\theta = 25 \times \frac{24}{25}$

$V\cos\theta = 24$ (Shown). $\rightarrow (1)$

ii) Vertically, using $v = u + at$

$25\sin\alpha = V\sin\theta + (-10)(0.3)$

$25\left(\frac{7}{25}\right) = V\sin\theta - 3$

$V\sin\theta = 10 \rightarrow (2)$

Dividing (2) by (1) gives:

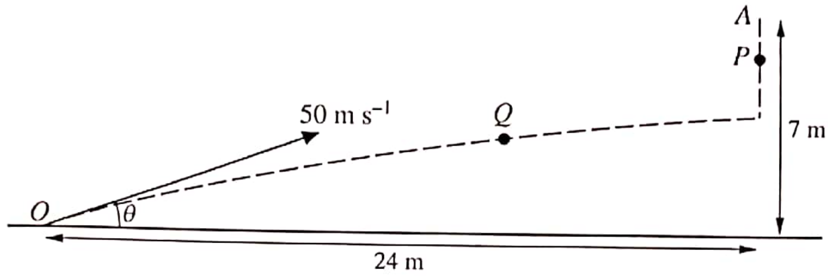
$\frac{V\sin\theta}{V\cos\theta} = \frac{10}{24} \Rightarrow \tan\theta = \frac{5}{12}$

$\theta = \tan^{-1}\left(\frac{5}{12}\right) = 22.61^\circ$

$V\cos 22.61^\circ = 24$

$V = \frac{24}{\cos 22.61^\circ} = 26\,\mathrm{ms^{-1}}$.

$V = 26\,\mathrm{ms^{-1}}$, $\theta = 22.6^\circ$.

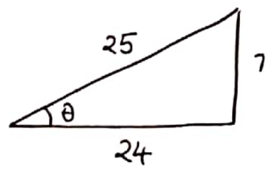


A particle P is released from rest at a point A which is 7 m above horizontal ground. At the same instant that P is released a particle Q is projected from a point O on the ground. The horizontal distance of O from A is 24 m. Particle Q moves in the vertical plane containing O and A, with initial speed 50 m s^{-1} and initial direction making an angle θ above the horizontal, where $\tan\theta = \frac{7}{24}$ (see diagram). Show that the particles collide. [6]

For particle P, $s = ut + \frac{1}{2}at^2$

$$s = 0(t) + \frac{1}{2}(10)t^2 = 5t^2.$$

$\therefore$ Height of particle P $= 7 - 5t^2$.
at time t



$\tan\theta = \frac{7}{24}$

$\sin\theta = \frac{7}{25}$

$\cos\theta = \frac{24}{25}$.

For particle Q,

Horizontally, $x = v_x \times t$

$x = 50\cos\theta \times t$

$x = 48t$

Vertically, $y = (50\sin\theta)t + \frac{1}{2}(-g)t^2$

$y = 14t - 5t^2$

When $x = 24$, $48t = 24 \Rightarrow t = 0.5$

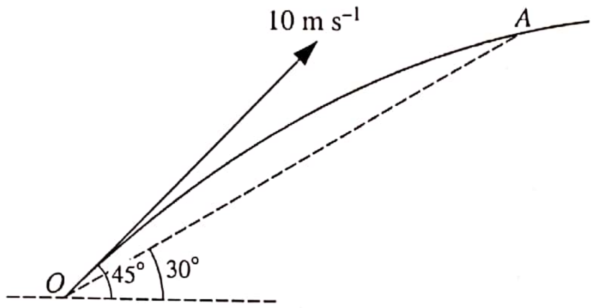
$\Rightarrow y = 14(0.5) - 5(0.5)^2 = 5.75$m.

i.e. When $t = 0.5$m, height of Q above horizontal ground $= 5.75$m.

When $t = 0.5$, Height of P $= 7 - 5(0.5)^2 = 5.75$m.

$\therefore$ Both particles have same horizontal and vertical displacements from O when $t = 0.5$s $\therefore$ ~~the~~ particles collide.

63



A particle P is projected from a point O with initial speed $10\,\mathrm{m\,s^{-1}}$ at an angle of 45° above the horizontal. P subsequently passes through the point A which is at an angle of elevation of 30° from O (see diagram). At time t s after projection the horizontal and vertically upward displacements of P from O are x m and y m respectively.

(i) Write down expressions for x and y in terms of t, and hence obtain the equation of the trajectory of P. [3]

(ii) Calculate the value of x when P is at A. [3]

(iii) Find the angle the trajectory makes with the horizontal when P is at A. [4]

i) Horizontally, $x = v_x \times t = 10\cos 45^\circ \times t = 5\sqrt{2}\,t \rightarrow (1)$

Vertically, $y = (10\sin 45^\circ)t + \frac{1}{2}(-10)t^2 = 5\sqrt{2}t - 5t^2 \rightarrow (2)$

From (1), $t = \frac{x}{5\sqrt{2}} \rightarrow (3)$

Substitute (3) in (2):

$$y = 5\sqrt{2}\left(\frac{x}{5\sqrt{2}}\right) - 5\left(\frac{x}{5\sqrt{2}}\right)^2$$

$$y = x - \frac{x^2}{10}$$

ii) At A, $\tan 30^\circ = \frac{y}{x}$

$$y = \tan 30^\circ x$$

$$y = \frac{1}{\sqrt{3}} x$$

$$\sqrt{3}\,y = x$$

$$\sqrt{3}\left(x - \frac{x^2}{10}\right) = x$$

$$\sqrt{3}x - \frac{\sqrt{3}x^2}{10} - x = 0$$

$$x\left(\frac{-\sqrt{3}}{10}x + \sqrt{3} - 1\right) = 0 \Rightarrow x = 0 \text{ or } x = \frac{-\sqrt{3}+1}{\frac{-\sqrt{3}}{10}} = \cancel{15.77 \approx 15.8\,m}$$

$x = 4.226 \approx 4.23\,m.$

iii)

$$y = x - \frac{x^2}{10}$$

$$\frac{dy}{dx} = 1 - \frac{2x}{10}.$$

When $x = 4.226$, $\frac{dy}{dx} = 1 - \frac{2(4.226)}{10} = 0.1547$

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{v_y}{v_x} = \tan\alpha$$

$$\Rightarrow \tan\alpha = 0.1547$$

$$\alpha = 8.793^\circ \approx 8.79^\circ$$

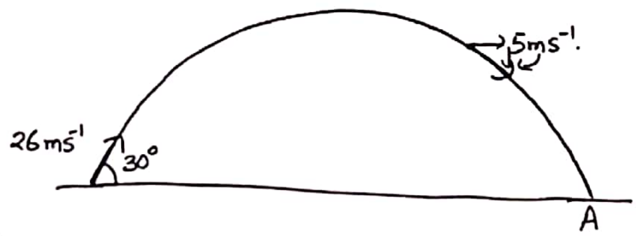
Angle with horizontal at A $\approx 8.79^\circ$.

64 A particle P is projected with speed $26\,\mathrm{m\,s^{-1}}$ at an angle of $30°$ above the horizontal from a point O on a horizontal plane.

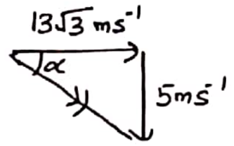
(i) For the instant when the vertical component of the velocity of P is $5\,\mathrm{m\,s^{-1}}$ downwards, find the direction of motion of P and the height of P above the plane. [4]

(ii) P strikes the plane at the point A. Calculate the time taken by P to travel from O to A and the distance OA. [3]

i)



Horizontally, $v_x = 26\cos 30° = 13\sqrt{3}\ \mathrm{m\,s^{-1}}$.



In vector diagram, $\tan\alpha = \dfrac{v_y}{v_x} = \dfrac{-5}{13\sqrt{3}}$

$\alpha = \tan^{-1}\left(\dfrac{-5}{13\sqrt{3}}\right) = -12.51° \approx -12.5°$

Vertically, using $v^2 = u^2 + 2as$

$(-5)^2 = (26\sin 30°)^2 + 2(-10)h$

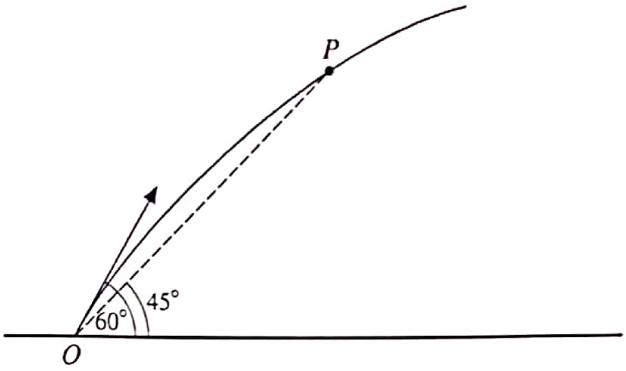
$h = 7.2\,\mathrm{m}$

$\Rightarrow$ Direction of motion $= 12.5°$ below horizontal, height of P above plane $= 7.2\,\mathrm{m}$.

ii) Total time of flight $= \dfrac{2u\sin\theta}{g} = \dfrac{2\times 26\times\sin 30°}{10} = 2.6\,\mathrm{s}$

Distance $OA = v_x \times t = 26\cos 30° \times 2.6 = 58.54\,\mathrm{m} \approx 58.5\,\mathrm{m}$.

65



A particle P is projected from a point O at an angle of 60° above horizontal ground. At an instant 0.6 s after projection, the angle of elevation of P from O is 45° (see diagram).

(i) Show that the speed of projection of P is $8.20\,\text{m s}^{-1}$, correct to 3 significant figures. [4]

(ii) Calculate the time after projection when the direction of motion of P is 45° above the horizontal. [3]

i) Horizontally, $x = v_x \times t$

$$x = V\cos 60° \times t = \frac{Vt}{2}$$

Vertically, $y = (V\sin 60°)t + \frac{1}{2}(-10)t^2 = \frac{\sqrt{3}Vt}{2} - 5t^2.$

At P, $\tan 45° = \frac{y}{x} \Rightarrow y = x\tan 45° \Rightarrow y = x.$ and $t = 0.6$

$$\Rightarrow \frac{V(0.6)}{2} = \frac{\sqrt{3}V(0.6)}{2} - 5(0.6)^2$$

$$0.3V = 0.3\sqrt{3}V - 1.8$$

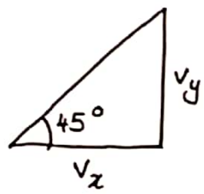
$$(0.3 - 0.3\sqrt{3})V = -1.8$$

$$V = \frac{-1.8}{0.3 - 0.3\sqrt{3}}$$

$$V = 8.196\,\text{ms}^{-1} \approx 8.20\,\text{ms}^{-1}. \text{ (Shown)}.$$

ii) Horizontally, $v_x = V\cos 60° = \frac{V}{2}$

Vertically, $v_y = V\sin 60° + (-10)t = \frac{\sqrt{3}}{2}V - 10t$



$\tan 45° = \frac{v_y}{v_x} \Rightarrow v_x \tan 45° = v_y$

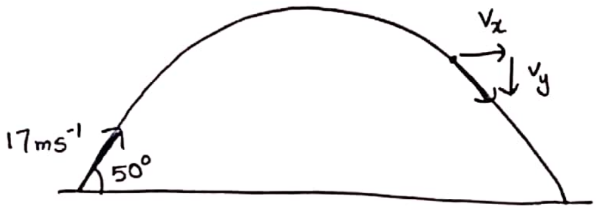
$$v_x = v_y$$

$$\frac{V}{2} = \frac{\sqrt{3}}{2}V - 10t \Rightarrow 10t = \frac{\sqrt{3}}{2}V - \frac{V}{2}$$

$$t = \frac{1}{10}\left[\frac{\sqrt{3}}{2} - \frac{1}{2}\right][8.196]$$

$$t = 0.3\,\text{s}.$$

66 A particle is projected with speed $17\,\mathrm{m\,s^{-1}}$ at an angle of 50° above the horizontal from a point on horizontal ground. Calculate the speed of the particle 2 s after the instant of projection. [3]



When $t = 2s$,

$V_x = 17\cos 50° = 10.927\,ms^{-1}$.

$V_y = 17\sin 50° + (-10)(2) = -6.977$

$V = \sqrt{V_x^2 + V_y^2} = \sqrt{10.927^2 + (-6.977)^2} = 12.96\,ms^{-1} \approx 13.0\,ms^{-1}$.

$\therefore$ Speed of particle $= 13.0\,ms^{-1}$.

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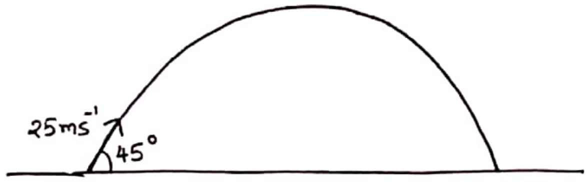
67 A particle P is projected with speed $25\,\mathrm{m\,s^{-1}}$ at an angle of 45° above the horizontal from a point O on horizontal ground. At time t s after projection the horizontal and vertically upward displacements of P from O are x m and y m respectively.

(i) Express x and y in terms of t and hence show that the equation of the path of P is $y = x - 0.016x^2$. [4]

(ii) Calculate the horizontal distance between the two positions at which P is 2.4 m above the ground. [2]

i)



Horizontally, $x = v_x \times t = 25\cos 45^\circ \times t = \dfrac{25\sqrt{2}}{2}t \rightarrow (1)$

Vertically, $y = (25\sin 45^\circ)t + \dfrac{1}{2}(-10)t^2 = \dfrac{25\sqrt{2}}{2}t - 5t^2 \rightarrow (2)$

From (1), $t = \dfrac{2x}{25\sqrt{2}} \rightarrow (3)$

Substituting (3) in (2):

$$y = \frac{25\sqrt{2}}{2}\left[\frac{2x}{25\sqrt{2}}\right] - 5\left[\frac{2x}{25\sqrt{2}}\right]^2 = x - \frac{2x^2}{125} = x - 0.016x^2.$$

$\therefore y = x - 0.016x^2 \rightarrow$ (Shown).

ii) When $y = 2.4$, $\quad 2.4 = x - 0.016x^2$

$0.016x^2 - x + 2.4 = 0$

$$\Rightarrow x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(0.016)(2.4)}}{2(0.016)}$$

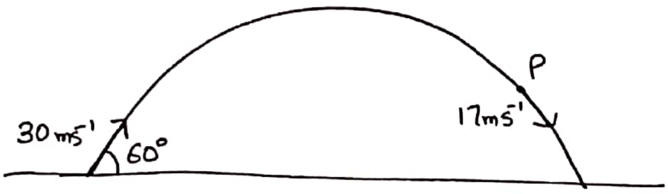
$$x = \frac{1 \pm \sqrt{0.8464}}{0.032} = 2.5, 60$$

$\therefore$ Distance between positions at which P is 2.4 m above the ground $= 60 - 2.5 = 57.5$ m

68 A particle P is projected with speed $30\,\mathrm{m\,s^{-1}}$ at an angle of $60°$ above the horizontal from a point O on horizontal ground. For the instant when the speed of P is $17\,\mathrm{m\,s^{-1}}$ and increasing,

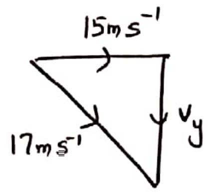
(i) show that the vertical component of the velocity of P is $8\,\mathrm{m\,s^{-1}}$ downwards, [2]

(ii) calculate the distance of P from O. [5]



i) When speed of P is $17\,\mathrm{ms^{-1}}$ and increasing, P has crossed point at which greatest height is obtained.

Horizontally, $v_x = 30\cos 60° = 15\,\mathrm{ms^{-1}}$.



From vector diagram, $17^2 = 15^2 + v_y^{\,2}$

$64 = v_y^{\,2}$

$\Rightarrow v_y = -8\,\mathrm{ms^{-1}}$.

ii) Vertically, using $v = u + at$

$-8 = (30\sin 60°) + (-10)t$

$t = \dfrac{30\sin 60° + 8}{10} = 3.398\,\mathrm{s}$.

Horizontally, $x = v_x \times t = 30\cos 60° \times 3.398 = 50.97\,\mathrm{m}$

Vertically, $y = (30\sin 60°)(3.398) + \frac{1}{2}(-10)(3.398)^2$

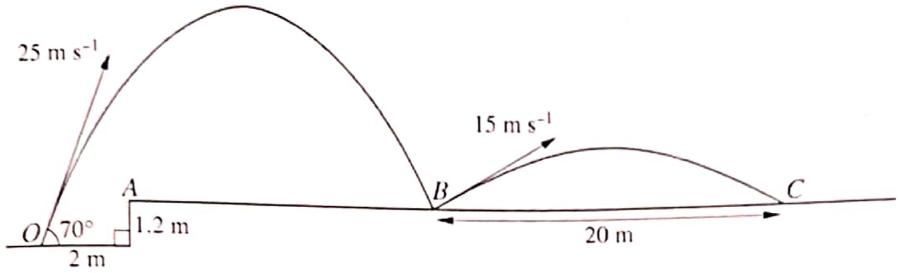
$y = 30.55\,\mathrm{m}$.

$\Rightarrow$ Distance of P from O $= \sqrt{x^2 + y^2}$

$= \sqrt{50.97^2 + 30.55^2}$

$= 59.42\,\mathrm{m} \approx 59.4\,\mathrm{m}$.

69



The point O is 1.2 m below rough horizontal ground ABC. A ball is projected from O with speed $25\,\text{m s}^{-1}$ at an angle of $70°$ to the horizontal. The ball passes over the point A after travelling a horizontal distance of 2 m. The ball subsequently bounces once on the ground at B. The ball leaves B with speed $15\,\text{m s}^{-1}$ and travels a further horizontal distance of 20 m before landing at C (see diagram).

(i) Calculate the height above the level of O of the ball when it is vertically above A. [3]

(ii) Calculate the time after the instant of projection when the ball reaches B. [3]

(iii) Find the angle which the trajectory of the ball makes with the horizontal immediately after it bounces at B. [2]

i) Horizontally, $x = v_x \times t$

$$2 = 25\cos 70° \times t \Rightarrow t = \frac{2}{25\cos 70°} = 0.2339\,\text{s}.$$

Vertically, $y = (25\sin 70°)(0.2339) + \frac{1}{2}(-10)(0.2339)^2$

$$y = 5.221\,\text{m}.$$

∴ Height above level of O when ball is vertically above A $= 5.221 - 1.2 = 4.021 \approx 4.02\,\text{m}$.

ii) Vertically, when $y = 1.2$

$$y = (25\sin 70°)t + \frac{1}{2}(-10)t^2$$

$$1.2 = (25\sin 70°)t - 5t^2$$

$$5t^2 - (25\sin 70°)t + 1.2 = 0$$

$$t = \frac{-(-25\sin 70°) \pm \sqrt{(-25\sin 70°)^2 - 4(5)(1.2)}}{2(5)}$$

$t = 0.05164$ (Ignore) or $t = 4.648 \approx 4.65\,\text{s}$.

Time after instant of projection when ball reaches B $= 4.65\,\text{s}$.

iii)

$$\text{Range} = \frac{u^2 \sin 2\theta}{g}$$

$$20 = \frac{15^2 \times \sin 2\theta}{10}$$

$$\sin 2\theta = \frac{8}{9}$$

$$2\theta = \sin^{-1}\left(\frac{8}{9}\right) = 62.73^\circ$$

$$\theta = 31.36^\circ \approx 31.4^\circ$$

Direction of motion immediately after it bounces at B $= 31.4^\circ$.

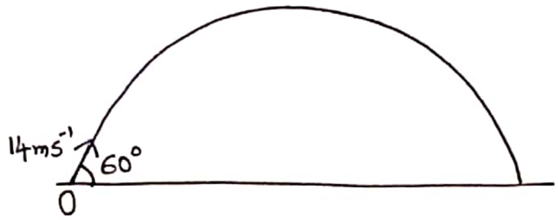
70 A small ball B is projected from a point O with speed 14 m s^{-1} at an angle of 60° above the horizontal.

(i) Calculate the speed and direction of motion of B for the instant 1.8 s after projection. [5]

The point O is 2 m above a horizontal plane.

(ii) Calculate the time after projection when B reaches the plane. [3]

i)

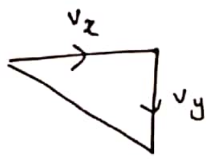


When $t = 1.8s$,

$v_x = 14\cos 60^\circ = 7 ms^{-1}$.

$v_y = 14\sin 60^\circ + (-10)(1.8) = -5.875 \ ms^{-1}$.

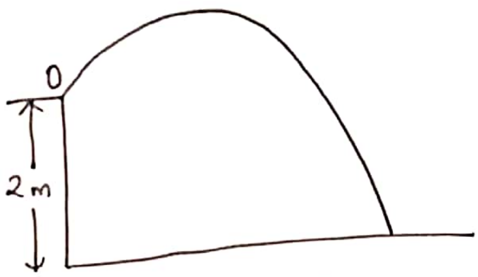
$v = \sqrt{v_x^2 + v_y^2} = \sqrt{7^2 + (-5.875)^2} = 9.139 ms^{-1}$.



$\tan\alpha = \frac{v_y}{v_x} \Rightarrow \alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{-5.875}{7}\right) = -40.00^\circ$

$\therefore$ Speed of motion $= 9.14 ms^{-1}$, Direction of motion $=$ ~~40.0~~ 40.0° below horizontal.

ii)



Vertically, $y = (u\sin\theta)t - \frac{gt^2}{2}$

$-2 = (14\sin 60^\circ)t - 5t^2$

$-2 = 7\sqrt{3}t - 5t^2$

$\Rightarrow 5t^2 - 7\sqrt{3}t - 2 = 0$

$t = \frac{-(-7\sqrt{3}) \pm \sqrt{(-7\sqrt{3})^2 - 4(5)(-2)}}{2(5)} = -0.155, 2.579$

Ignore $t = -0.155$ since $t > 0$.

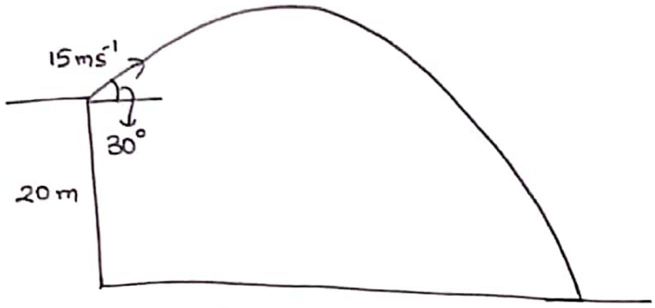
$\Rightarrow t = 2.58s$.

71 The top of a vertical cliff is 20 m above sea level. A particle P is projected with speed 15 m s^{-1} at an angle of 30° above the horizontal from a point O at the top of the cliff. Calculate

(i) the speed and direction of motion of P when it strikes the water, [4]

(ii) the distance OP at the instant P strikes the water. [4]

i)



Vertically, $y = (u\sin\theta)t - \frac{gt^2}{2}$

$-20 = (15\sin 30°)t - 5t^2$

$-20 = 7.5t - 5t^2$

$5t^2 - 7.5t - 20 = 0$

$10t^2 - 15t - 40 = 0$

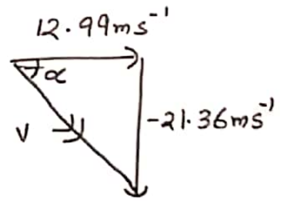
$2t^2 - 3t - 8 = 0 \Rightarrow \cancel{2t^2}\ t = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-8)}}{2(2)}$

$t = 2.886, -1.386$ (Ignore).

$t = 2.886s$

$\Rightarrow v_y = 15\sin 30° + (-10)(2.886) = -21.36 ms^{-1}$.

$v_x = 15\cos 30° = 12.99 ms^{-1}$.



$v = \sqrt{v_x^2 + v_y^2} = \sqrt{12.99^2 + (-21.36)^2}$

$v = 25.0 ms^{-1}$.

$\tan\alpha = \frac{v_y}{v_x} = \frac{-21.36}{12.99} \Rightarrow \alpha = -58.69° \approx -58.7°$.

$\therefore$ Speed of motion $= 25.0 ms^{-1}$, Direction of motion $= 58.7°$ below horizontal.

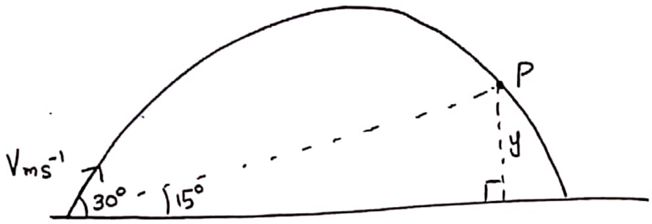
ii) When P strikes water,

$x = v_x \times t = 15\cos 30° \times 2.886 = 37.49 m$.

$y = -20m$

$\Rightarrow OP = \sqrt{37.49^2 + 20^2} = 42.49m \approx 42.5m$

72 A particle P is projected with speed $V\,\mathrm{m\,s^{-1}}$ at an angle of $30°$ above the horizontal from a point O on horizontal ground. At the instant 2 s after projection, OP makes an angle of $15°$ above the horizontal. Calculate V. [4]



when $t=2$,

Horizontally, $x = V\cos 30° \times 2 = 2V\cos 30° = \sqrt{3}\,V$.

Vertically, $y = (V\sin 30°)\times 2 + \frac{1}{2}(-10)(2)^2 = V - 20$.

$$\tan 15° = \frac{y}{x}$$

$$\tan 15° = \frac{V-20}{\sqrt{3}\,V}$$

$$\sqrt{3}\tan 15°\, V = V - 20$$

$$20 = (1-\sqrt{3}\tan 15°)V$$

$$V = \frac{20}{1-\sqrt{3}\tan 15°} = 37.32\,m/s \approx 37.3\,m/s.$$

73 The equation of the trajectory of a small ball B projected from a fixed point O is

$$y = -0.05x^2,$$

where x and y are, respectively, the displacements in metres of B from O in the horizontal and vertically upwards directions.

(i) Show that B is projected horizontally, and find its speed of projection. [3]

(ii) Find the value of y when the direction of motion of B is 60° below the horizontal, and find the corresponding speed of B. [6]

i)

$$y = -0.05x^2$$

By comparison to $y = x\tan\theta - \frac{g\sec^2\theta\, x^2}{2u^2}$,

$$x\tan\theta = 0x \Rightarrow \tan\theta = 0 \Rightarrow \theta = 0°.$$

and $\frac{g\sec^2\theta}{2u^2} = 0.05 \Rightarrow \frac{10 \times \sec^2 0°}{2 \times u^2} = 0.05$

$$\frac{10 \times 1}{2 \times 0.05} = u^2$$

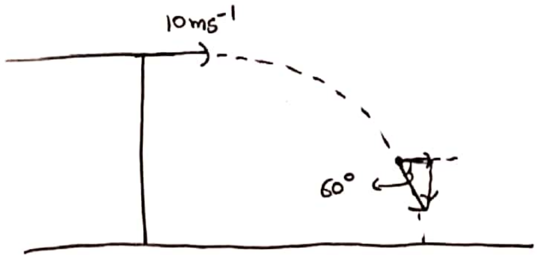
$$100 = u^2$$

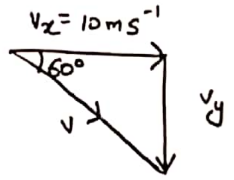
$$\Rightarrow u = 10$$

i.e. Angle of projection = 0° ⇒ B is projected horizontally.

Speed of projection = 10 ms^{-1}.

ii)





In the vector diagram, $\tan 60° = \frac{|v_y|}{|v_x|} = \frac{|v_y|}{10}$

$\Rightarrow |v_y| = 10\tan 60° = 10\sqrt{3} \text{ ms}^{-1}. \Rightarrow v_y = -10\sqrt{3}$

In the vertical direction,

$v_y = 0 + (-10)t \Rightarrow -10\sqrt{3} = -10t \Rightarrow t = \sqrt{3}$ s.

When $t = \sqrt{3}$ s, $x = 10 \times \sqrt{3} = 10\sqrt{3}$ m.

When $x = 10\sqrt{3}$ m, $y = -0.05(10\sqrt{3})^2 = -15$m

i.e. value of y when direction of motion is 60° below horizontal = −15m.

$v = \sqrt{v_x^2 + v_y^2} = \sqrt{10^2 + (-10\sqrt{3})^2} = 20 \text{ ms}^{-1}$. i.e. corresponding speed $= 20 \text{ ms}^{-1}$.

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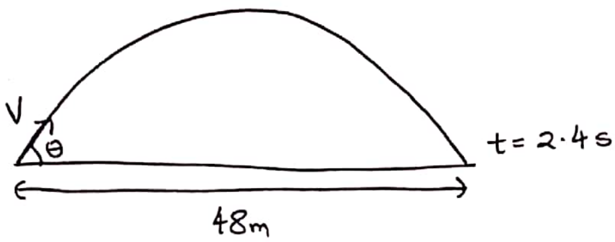
74 A golf ball B is projected from a point O on horizontal ground. B hits the ground for the first time at a point 48 m away from O at time 2.4 s after projection. Calculate the angle of projection. [3]



Horizontally, $x = V\cos\theta \times 2.4$

$\Rightarrow 48 = V\cos\theta \times 2.4$

$\Rightarrow V\cos\theta = 20 \Rightarrow (1)$

Vertically, Ball reaches greatest height when $t = 1.2$ s.

Using $v = u + at$

$0 = V\sin\theta + (-10)(1.2)$

$V\sin\theta = 12 \rightarrow (2)$

Dividing (2) by (1) gives:

$$\frac{V\sin\theta}{V\cos\theta} = \frac{12}{20}$$

$$\tan\theta = \frac{3}{5}$$

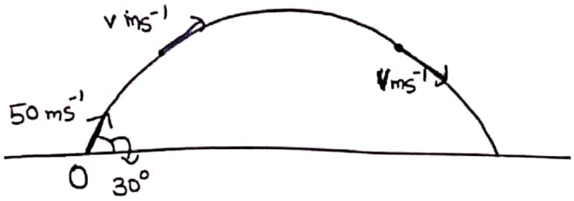
$$\theta = 30.96^\circ \simeq 31.0^\circ.$$

15 A particle P is projected with speed 50 m s^{-1} at an angle of 30° above the horizontal from a point O on a horizontal plane.

(i) Calculate the speed of P when it has been in motion for 4 s, and calculate another time at which P has this speed. [5]

(ii) Find the distance OP when P has been in motion for 4 s. [2]

i)



Horizontally, $v_x = 50\cos 30° = 25\sqrt{3}\ ms^{-1}$.

Vertically, $v_y = 50\sin 30° + (-10)(4) = -15 ms^{-1}$.

$\therefore\ v = \sqrt{v_x^2 + v_y^2} = \sqrt{(25\sqrt{3})^2 + (-15)^2} = 45.83\ ms^{-1} \approx 45.8 ms^{-1}$.

For other time when $v = 45.8 ms^{-1}$, we require $v_y = +15\ ms^{-1}$.

Using $v = u + at$

$15 = (50\sin 30°) + (-10)t$

Other $t = 1$

i.e. Time when P has speed $= 45.8 ms^{-1}$ is $t = 1 s$.

ii) when $t = 4s$, Horizontally, $x = 50\cos 30° \times 4 = 173.2\ m$.

Vertically, $y = 50\sin 30° \times 4 + \frac{1}{2}(-10)(4)^2 = 20 m$.

$\Rightarrow\ OP = \sqrt{x^2 + y^2} = \sqrt{173.2^2 + 20^2} = 174.3 \approx 174\ m$.

76 A particle P is projected with speed V m s^{-1} at an angle of 60° above the horizontal from a point O. At the instant 1 s later a particle Q is projected from O with the same initial speed at an angle of 45° above the horizontal. The two particles collide when Q has been in motion for t s.

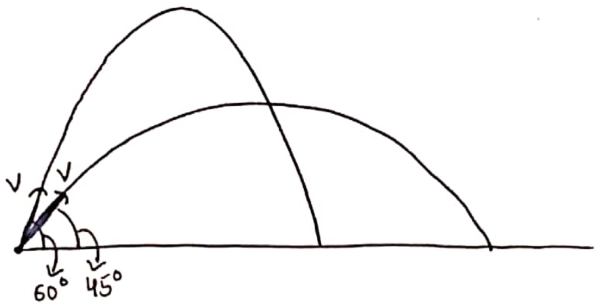
(i) Show that $t = 2.414$, correct to 3 decimal places. [3]

(ii) Find the value of V. [4]

The collision occurs after P has passed through the highest point of its trajectory.

(iii) Calculate the vertical distance of P below its greatest height when P and Q collide. [4]

i)



For P,

Horizontally, $x_P = V\cos 60° \times (t+1)$.

Vertically, $y_P = V\sin 60° \times (t+1) + \frac{1}{2}(-10)(t+1)^2 = V\sin 60°(t+1) - 5(t+1)^2$

For Q,

Horizontally, $x_Q = V\cos 45° \times t$

$y_Q = V\sin 45° \times t + \frac{1}{2}(-10)t^2 = V\sin 45° t - 5t^2$.

When collision occurs, $x_P = x_Q$

$\Rightarrow \cancel{V}\cos 60° \times (t+1) = \cancel{V}\cos 45° \times t$

$\frac{1}{2}(t+1) = \frac{1}{\sqrt{2}}t$

$t+1 = \sqrt{2}t$

$1 = (\sqrt{2}-1)t$

$t = \frac{1}{\sqrt{2}-1} = 1+\sqrt{2} = 2.4142 \approx 2.414$ s.

ii) When collision occurs at $t = 2.414$ s,

$y_P = y_Q$

$V\sin 60° \times (2.414+1) - 5(2.414+1)^2 = V\sin 45° \times 2.414 - 5 \times 2.414^2$

$2.9567V - 58.284 = 1.7071V - 29.142$

$1.2496V = 29.142$

$V = 23.32 = 23.3$ ms^{-1}.

iii) For greatest height, $h = \dfrac{u^2 \sin^2 \theta}{2g}$

$$= \frac{23.32^2 \times \sin^2 60^\circ}{2 \times 10}$$

$$= 20.395\,m.$$

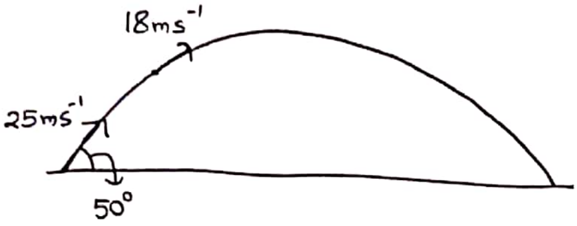
When collision occurs,

$y_P = 23.32 \times \sin 60^\circ \times (3.414) - 5(3.414)^2 = 10.671$

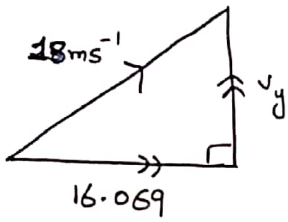
$\therefore$ Vertical distance below greatest height

$= 20.395 - 10.671 = 9.724 \approx 9.72\,m.$

11 A particle is projected with speed 25 m s^{-1} at an angle of 50° above the horizontal. Calculate the time after projection when the particle has speed 18 m s^{-1} and is rising. [4]



Horizontally, $v_x = 25\cos 50° = 16.069\ ms^{-1}$.



From the vector diagram, $v_y = \sqrt{18^2 - 16.069^2} = \cancel{19.151\ ms^{-1}}\ 8.109\ ms^{-1}$.

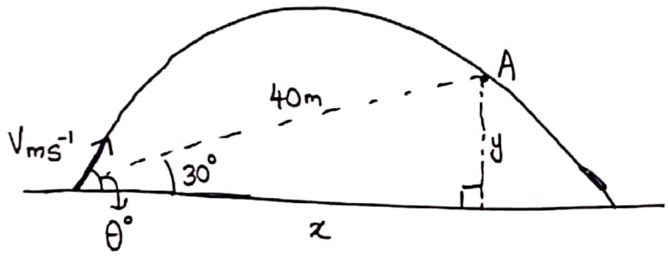
Vertically, using $v = u + at$

$$8.109\ \cancel{19.151} = 25\sin 50° + (-10)t$$

$$t = \frac{25\sin 50° - 8.109}{10}$$

$$t = 1.104\ s \approx 1.10\ s.$$

5 A particle P is projected with speed V m s^{-1} at an angle of $\theta°$ above the horizontal from a point O on horizontal ground. At the instant 4 s after projection the particle passes through the point A, where $OA = 40$ m and the line OA makes an angle of 30° with the horizontal. Calculate V and θ. [5]



When P is at A,

$$x = 40\cos 30° = 20\sqrt{3} \text{ m}$$

$$y = 40\sin 30° = 20 \text{ m}.$$

Horizontally, $x = V\cos\theta \times t$

$$20\sqrt{3} = V\cos\theta \times 4 \Rightarrow V\cos\theta = 5\sqrt{3} \rightarrow (1).$$

Vertically, $y = (V\sin\theta) \times t + \frac{1}{2}(-10)t^2$

$$20 = (V\sin\theta)(4) - 5(4)^2 \Rightarrow 4V\sin\theta = 100 \Rightarrow V\sin\theta = 25 \rightarrow (2).$$

Squaring (1) and (2) and adding gives:

$$V^2\cos^2\theta + V^2\sin^2\theta = (5\sqrt{3})^2 + 25^2$$

$$V^2 = 700$$

$$V = 26.45 \text{ ms}^{-1} \approx 26.5 \text{ ms}^{-1}.$$

Dividing (2) by (1) gives:

$$\frac{V\sin\theta}{V\cos\theta} = \frac{25}{5\sqrt{3}}$$

$$\tan\theta = \frac{5}{\sqrt{3}} \Rightarrow \theta = 70.89° \approx 70.9°.$$

$$\therefore V = 26.5 \text{ ms}^{-1}, \theta = 70.9°$$

79 A particle P is projected with speed 35 m s^{-1} from a point O on a horizontal plane. In the subsequent motion, the horizontal and vertically upwards displacements of P from O are x m and y m respectively. The equation of the trajectory of P is

$$y = kx - \frac{(1+k^2)x^2}{245},$$

where k is a constant. P passes through the points $A\,(14,\ a)$ and $B\,(42,\ 2a)$, where a is a constant.

(i) Calculate the two possible values of k and hence show that the larger of the two possible angles of projection is 63.435°, correct to 3 decimal places. [5]

For the larger angle of projection, calculate

(ii) the time after projection when P passes through A, [2]

(iii) the speed and direction of motion of P when it passes through B. [4]

$$y = kx - \frac{(1+k^2)x^2}{245}$$

Since particle passes through $(14, a)$,

$$\Rightarrow a = 14k - \frac{(1+k^2)\times 196}{245} \rightarrow (1)$$

Since particle passes through $(42, 2a)$,

$$\Rightarrow 2a = 42k - \frac{(1+k^2)\times 1764}{245} \rightarrow (2)$$

Substituting (1) in (2):

$$2\left[14k - \frac{196(1+k^2)}{245}\right] = 42k - \frac{1764(1+k^2)}{245}$$

$$28k - \frac{392(1+k^2)}{245} = 42k - \frac{1964(1+k^2)}{245}$$

$$5\cdot6(1+k^2) - 14k = 0$$

$$5\cdot6k^2 - 14k + 5\cdot6 = 0$$

$$2k^2 - 5k + 2 = 0$$

$$(2k-1)(k-2) = 0 \Rightarrow k = \frac{1}{2}, 2.$$

By comparison to $y = (\tan\theta)x - \dfrac{g\sec^2\theta\, x^2}{2u^2}$, $\tan\theta = k$

$$\therefore \tan\theta = \frac{1}{2} \quad \text{or} \quad \tan\theta = 2$$

$$\theta = 26.565° \quad \text{or} \quad \theta = 63.435°$$

(Shown).

ii) For larger angle of projection, $\theta = 63.435°$

$x = 35\cos 63.435° \times t$

When at A, $x = 14 \Rightarrow 14 = 35\cos 63.435° \times t$

$t = 0.8944 s \approx 0.894 s.$

iii) When at B, $x = 42$

$\Rightarrow 42 = 35\cos 63.435° \times t$

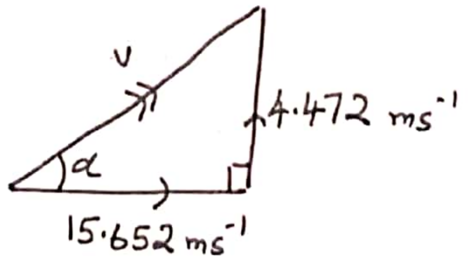
$t = 2.683 s.$

~~$v_y = 35\sin 63.435° \times 2.683 + \frac{1}{2}(-10) \times 2.683^2$~~

$v_y = 35\sin 63.435° + (-10)(2.683) = 4.472 ms^{-1}.$

$v_x = 35\cos 63.435° = 15.652 ms^{-1}.$

$\alpha = \tan^{-1}\left(\frac{4.472}{15.652}\right) = 15.94° \approx 15.9°$



$v = \sqrt{v_x^2 + v_y^2}$

$v = \sqrt{15.652^2 + 4.472^2} = 17.242 ms^{-1}.$

Speed of motion of P = 17.2 ms^{-1}.

Direction of motion of P = 15.9° above horizontal.

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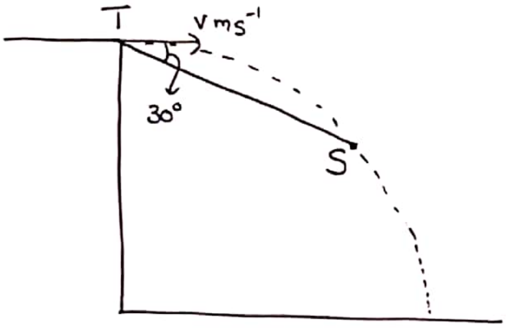
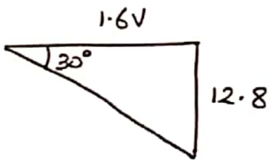
80 A stone S is thrown horizontally from the top T of a high tower. At the instant 1.6 s after S is thrown, the line ST makes an angle of 30° below the horizontal. Find the speed with which S is thrown. [3]



Horizontally, $x = Vt = 1.6V \text{ m}$

Vertically, using $s = ut + \frac{1}{2}at^2$

$$y = 0(1.6) + \frac{1}{2}(10)(1.6)^2 = 12.8 \text{ m}$$



$$\tan 30° = \frac{12.8}{1.6V}$$

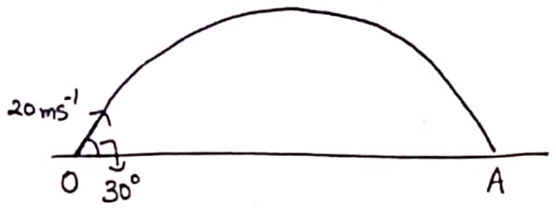
$$V = \frac{12.8}{1.6 \tan 30°} = 13.85 \text{ ms}^{-1} \approx 13.9 \text{ ms}^{-1}.$$

81 A particle P is projected with speed $20 \mathrm{~m~s}^{-1}$ at an angle of $30°$ above the horizontal from a point O on horizontal ground. P subsequently bounces when it first strikes the ground at the point A.

(i) Find the time after projection when P first strikes the ground, and the distance OA. [3]

When P bounces at A the horizontal component of the velocity of P is unchanged. The vertical component of velocity is $8 \mathrm{~m~s}^{-1}$ immediately after bouncing. P strikes the ground for the second time at B where it remains at rest.

(ii) Calculate the first and last times after projection at which the speed of P is $18 \mathrm{~m~s}^{-1}$. [5]



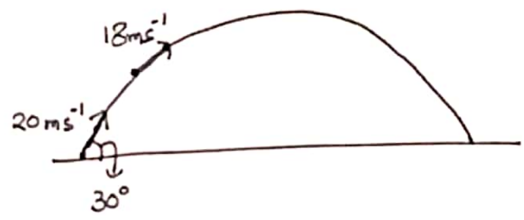
$$\text{Time of flight} = \frac{2u\sin\theta}{g} = \frac{2 \times 20 \times \sin 30°}{10} = 2s.$$

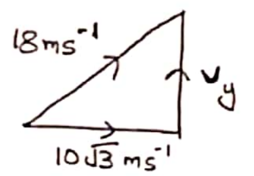
$$\text{Range} = \frac{u^2 \sin 2\theta}{g} = \frac{20^2 \times \sin(60°)}{10} = 34.64\,m \simeq 34.6\,m.$$

i.e. Time after projection when P first strikes ground = 2s.

Distance OA = 34.6m.

ii) For first time when speed of P is $18 ms^{-1}$,





$v_x = 20\cos 30° = 10\sqrt{3}\ ms^{-1}.$

$v_y = \sqrt{18^2 - (10\sqrt{3})^2} = 2\sqrt{6}\ ms^{-1}.$

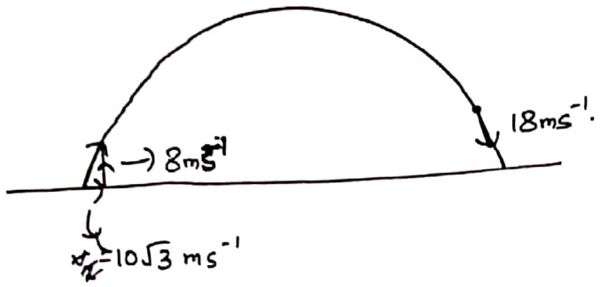
Vertically, $v_y = 20\sin 30° + (-10)t$

$2\sqrt{6} = 10 - 10t$

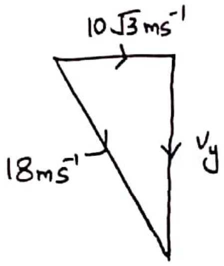
$t = 0.5101 = 0.510s.$

i.e. speed is $18ms^{-1}$ for first time when $t = 0.510s$.

After first bounce,



When speed is $18ms^{-1}$, for last time



$$v_y = -\sqrt{18^2 - (10\sqrt{3})^2} = -2\sqrt{6}\ ms^{-1}$$

Vertically, $v_y = 8 + (-10)t$

$$-2\sqrt{6} = 8 - 10t$$

$$t = 1.2898 \approx 1.29s.$$

$\therefore$ Last time speed is $18ms^{-1} = 2 + 1.29 = 3.29s$.

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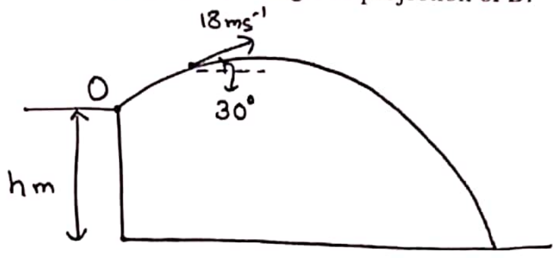
82 A small ball B is projected from a point O which is h m above a horizontal plane. At time 2 s after projection B has speed $18\,\text{m s}^{-1}$ and is moving in the direction $30°$ above the horizontal.

(i) Find the initial speed and the angle of projection of B. [4]



$v_x = 18\cos 30° = 9\sqrt{3}\ \text{ms}^{-1}. \Rightarrow u_x = 9\sqrt{3}\ \text{ms}^{-1}$

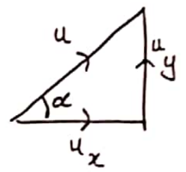
Vertically, $v_y = 18\sin 30° = 9\ \text{ms}^{-1}$

Using $v = u + at \Rightarrow 9 = u_y + (-10)(2)$

$29 = u_y$

$u = \sqrt{u_x^2 + u_y^2} = \sqrt{(9\sqrt{3})^2 + 29^2}$

$u = 32.92\ \text{ms}^{-1} \simeq 32.9\ \text{ms}^{-1}$



$\tan\alpha = \dfrac{u_y}{u_x} \Rightarrow \tan\alpha = \dfrac{29}{9\sqrt{3}}$

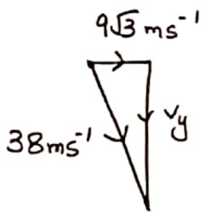
$\alpha = 61.74° \simeq 61.7°$

$\therefore$ Initial speed $= 32.9\ \text{ms}^{-1}$, Angle of projection $= 61.7°$

B has speed $38\,\text{m s}^{-1}$ immediately before it strikes the plane.

(ii) Calculate h. [2]

$v_y = -\sqrt{38^2 - (9\sqrt{3})^2} = -\sqrt{1201}.$



Vertically, using $v^2 = u^2 + 2as$

$(-\sqrt{1201})^2 = (29)^2 + 2(-10)h$

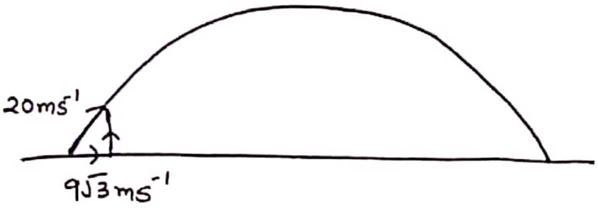
$20h = 360$

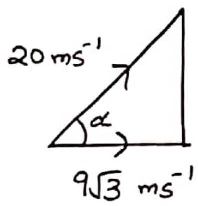
$h = 18$

B bounces when it strikes the plane, and leaves the plane with speed $20\,\mathrm{m\,s^{-1}}$ but with its horizontal component of velocity unchanged.

(iii) Find the total time which elapses between the initial projection of B and the instant when it strikes the plane for the second time. [5]

After bouncing:

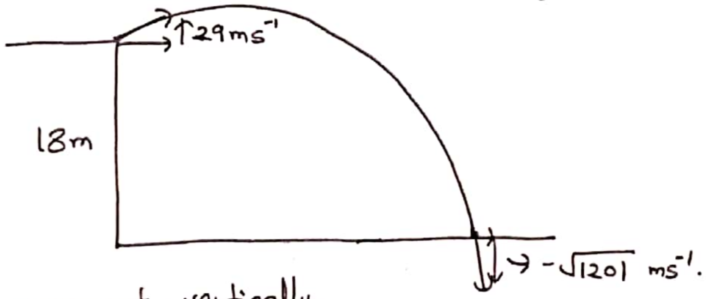




$$\cos\alpha = \frac{9\sqrt{3}}{20} \Rightarrow \alpha = 38.792^\circ$$

$$\text{Time of flight} = \frac{2u\sin\alpha}{g} = \frac{2\times 20\times \sin(38.792^\circ)}{10} = 2.506\ \text{s}.$$

i.e. B hits the ground 2.506s after bouncing for first time.



For initial movement, vertically,

using $\quad v = u + at$

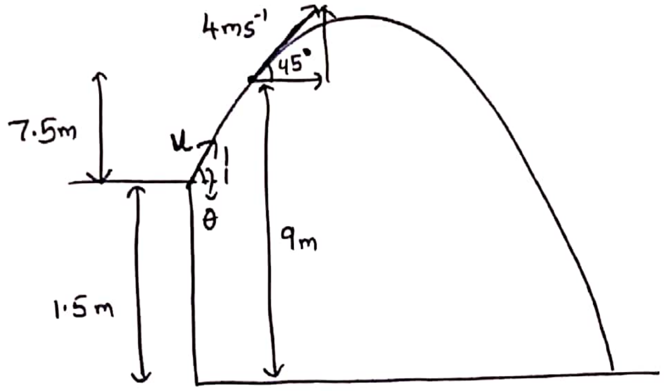
$$-\sqrt{1201} = 29 + (-10)t$$

$$t = 6.3655 \approx 6.366\text{s}$$

$\therefore$ Total time which elapses between initial projection of B and instant it strikes the plane for second time

$$= 2.506 + 6.366 = 8.872\text{s} \simeq 8.87\text{s}.$$

83 A small ball is projected from a point 1.5 m above horizontal ground. At a point 9 m above the ground the ball is travelling at 45° above the horizontal and its velocity is 4 m s^{-1}. Find the angle of projection of the ball. [4]



When $y = 7.5\text{ m}$,

$$V_x = 4\cos 45^\circ = 2\sqrt{2}\ \text{ms}^{-1}.$$

$$V_y = 4\sin 45^\circ = 2\sqrt{2}\ \text{ms}^{-1}.$$

Initially, $u_x = u\cos\theta$, $u_y = u\sin\theta$

Then horizontally, $u\cos\theta = 2\sqrt{2} \rightarrow (1)$.

Vertically, using $v^2 = u^2 + 2as$

$$(2\sqrt{2})^2 = (u\sin\theta)^2 + 2(-10)(7.5)$$

$$8 = u^2\sin^2\theta - 150$$

$$\sqrt{158} = u\sin\theta$$

$$\Rightarrow u\sin\theta = \sqrt{158} \rightarrow (2)$$

Dividing (2) by (1) gives:

$$\frac{u\sin\theta}{u\cos\theta} = \frac{\sqrt{158}}{2\sqrt{2}}$$

$$\tan\theta = \frac{\sqrt{79}}{2}$$

$$\theta = 77.32^\circ \approx 77.3^\circ$$

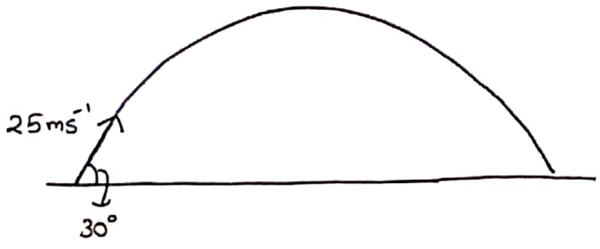
$\Rightarrow$ Angle of projection $= 77.3^\circ$

84 A particle P is projected with speed 25 m s^{-1} at an angle of 30° above the horizontal from a point O on horizontal ground. At time t s after projection the horizontal and vertically upwards displacements of P from O are x m and y m respectively.

(i) Express x and y in terms of t and hence show that the equation of the trajectory of P is

$$y = \frac{x}{\sqrt{3}} - \frac{4x^2}{375}.$$

[4]



Horizontally, $x = 25\cos 30° \times t = \frac{25\sqrt{3}\,t}{2} \Rightarrow x = 12.5\sqrt{3}\,t \rightarrow (1)$

Vertically, $y = 25\sin 30° \times t + \frac{1}{2}(-10)t^2 = 12.5t - 5t^2 \rightarrow (2)$

From (1), $t = \frac{x}{12.5\sqrt{3}} \rightarrow (3)$

Substituting (3) in (2):

$$y = 12.5\left(\frac{x}{12.5\sqrt{3}}\right) - 5\left(\frac{x}{12.5(\sqrt{3})}\right)^2$$

$$y = \frac{x}{\sqrt{3}} - \frac{4x^2}{375} \text{ (Shown)}.$$

(ii) Find the horizontal distance between the two points at which P is 5 m above the ground. [3]

When P is 5m above the ground, $y = 5$

$$\Rightarrow \quad 5 = \frac{x}{\sqrt{3}} - \frac{4x^2}{375}$$

$$5 = \frac{375x - 4\sqrt{3}x^2}{375\sqrt{3}}$$

$$1875\sqrt{3} = 375x - 4\sqrt{3}x^2$$

$$4\sqrt{3}x^2 - 375x + 1875\sqrt{3} = 0$$

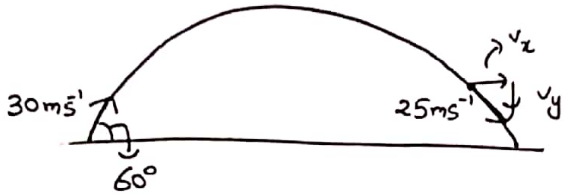
$$x = \frac{-(-375) \pm \sqrt{(-375)^2 - 4(4\sqrt{3})(1875\sqrt{3})}}{2(4\sqrt{3})}$$

$$x = 10.82, 43.30$$

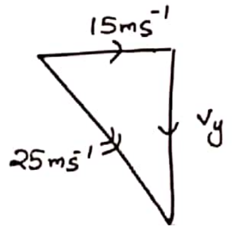
$\therefore$ Horizontal distance between points at which P is 5m above the ground

$= 43.30 - 10.82 = 32.48\text{ m} \simeq 32.5\text{ m}.$

85 A small ball B is projected with speed 30 m s^{-1} at an angle of 60° to the horizontal from a point on horizontal ground. Find the time after projection when the speed of B is 25 m s^{-1} for the second time. [4]



$v_x = 30\cos 60° = 15\text{ms}^{-1}$.



From diagram,

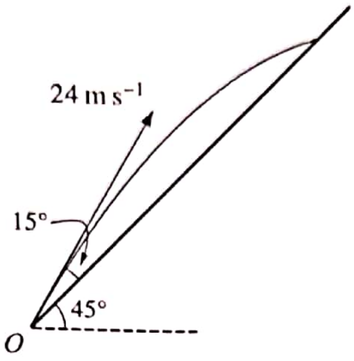
$$v_y = -\sqrt{25^2 - 15^2} = -20\text{ms}^{-1}$$

Then, vertically,

$$v_y = 30\sin 60° + (-10)t$$

$$-20 = 15\sqrt{3} - 10t$$

$$t = \frac{15\sqrt{3} + 20}{10} = 4.598 \approx 4.60\text{s}$$



A small object is projected with speed 24 m s^{-1} from a point O at the foot of a plane inclined at 45° to the horizontal. The angle of projection of the object is 15° above a line of greatest slope of the plane (see diagram). At time t s after projection, the horizontal and vertically upwards displacements of the object from O are x m and y m respectively.

(i) Express x and y in terms of t, and hence find the value of t for the instant when the object strikes the plane. [4]

Horizontally, $x = 24\cos 60^\circ \times t$

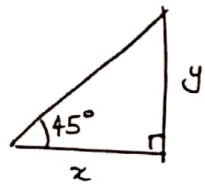
$x = 12t$

Vertically, $y = 24\sin 60^\circ \times t + \frac{1}{2}(-10)t^2$

$y = 12\sqrt{3}\,t - 5t^2$

when object strikes the plane,

$\tan 45^\circ = \frac{y}{x}$



$\Rightarrow \tan 45^\circ(y) = x$

$1(y) = x$

$\Rightarrow 12\sqrt{3}t - 5t^2 = 12t$

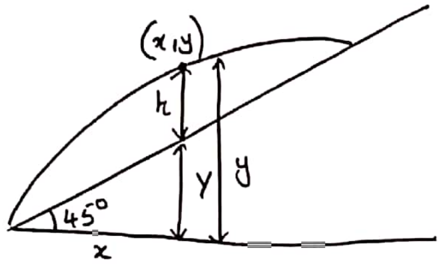
$t\left(12\sqrt{3} - 5t - 12\right) = 0$

$t = 0$ or $t = \frac{12\sqrt{3} - 12}{5}$

$t = 1.756 \simeq 1.76\text{s}$

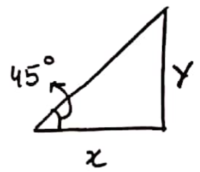
i.e. Value of t when object strikes the plane $= 1.76$s

(ii) Express the vertical height of the object above the plane in terms of t and hence find the greatest vertical height of the object above the plane. [5]



let, vertical height of object above plane $= h$.

In Δ, $\tan 45° = \frac{Y}{x} \Rightarrow Y = x\tan 45° = x(1) = x$.



Then, $h = y - Y$

$= y - x$

$h = 12\sqrt{3}t - 5t^2 - 12t$.

$\Rightarrow \frac{dh}{dt} = 12\sqrt{3} - 10t - 12$.

For greatest vertical height, $\frac{dh}{dt} = 0$

$\Rightarrow 12\sqrt{3} - 10t - 12 = 0$

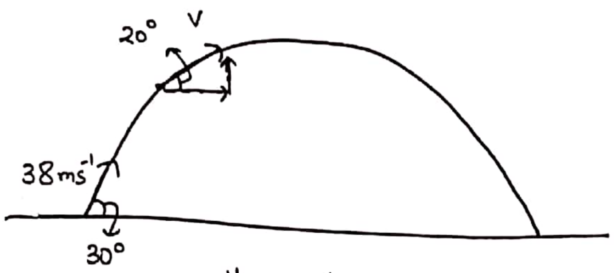
$t = \frac{12 - 12\sqrt{3}}{-10} = 0.8784$

When $t = 0.8784$, $h = 12\sqrt{3}(0.8784) - 5(0.8784)^2 - 12(0.8784)$

$h = 3.858 \simeq 3.86m$

$\therefore$ Greatest vertical height above the plane $= 3.86m$

87 A small ball B is projected with speed 38 m s^{-1} at an angle of 30° to the horizontal from a point on horizontal ground. Find the speed of B when the path of B makes an angle of 20° above the horizontal. [3]



$$u_x = v_x$$

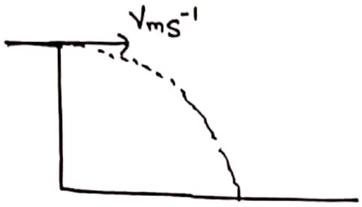
Horizontally, $38\cos 30° = v\cos 20°$

$$v = \frac{38\cos 30°}{\cos 20°}$$

$$v = 35.0 \text{ ms}^{-1}.$$

8 A small object is projected horizontally with speed V m s^{-1} from a point O above horizontal ground. At time t s after projection, the horizontal and vertically upwards displacements of the object from O are x m and y m respectively.

(i) Express x and y in terms of t and hence show that the equation of the path of the object is $y = -\frac{5x^2}{V^2}$. [3]



Horizontally, $x = Vt \rightarrow (1)$

$y = 0(t) + \frac{1}{2}(-10)t^2 = -5t^2 \rightarrow (2)$

From (1), $t = \frac{x}{V} \rightarrow (3)$

Substituting (3) in (2): $y = -5\left(\frac{x}{V}\right)^2 = \frac{-5x^2}{V^2}$

$\therefore \; y = \frac{-5x^2}{V^2}$. (Shown)

The object passes through points with coordinates $(a, -a)$ and $(a^2, -16a)$, where a is a positive constant.

(ii) Find the value of a. [3]

Since object passes through $(a, -a)$:

$\cancel{-}a = \frac{\cancel{-}5a^2}{V^2}$

$V^2 = 5a \rightarrow (1)$

and since object passes through $(a^2, -16a)$:

$\cancel{-}16a = \frac{\cancel{-}5(a^2)^2}{V^2}$

$16V^2 = 5a^3 \rightarrow (2)$

Dividing (1) by (2): $\frac{V^2}{16V^2} = \frac{5a}{5a^3}$

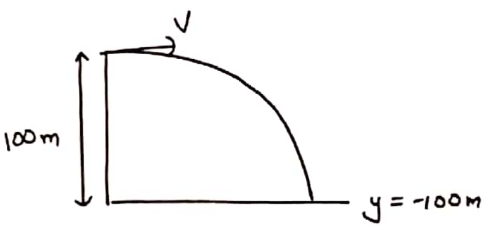
$$\frac{1}{16} = \frac{1}{a^2} \Rightarrow a^2 = 16 \Rightarrow a = 4 .$$

(iii) Given that the object strikes the ground at the point where $x = 5a$, find the height of O above the ground . [2]

When $x = 5a = 5(4) = 20$

and $V^2 = 5a = 5(4) = 20.$

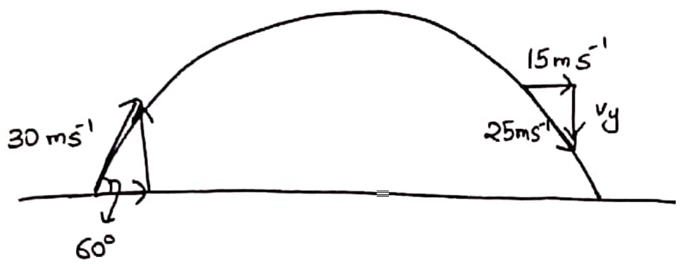
Then $y = \frac{-5x^2}{V^2} = \frac{-5x^2}{20} = \frac{-x^2}{4}.$



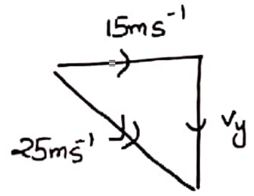
when $x = 20$, $y = \frac{-(20)^2}{4} = -100\text{m}.$

$\therefore$ Height of O above ground $= 100$ m.

89 A small ball B is projected with speed 30 m s^{-1} at an angle of 60° to the horizontal from a point on horizontal ground. Find the time after projection when the speed of B is 25 m s^{-1} for the second time. [4]



$v_x = 30 \cos 60° = 15 m s^{-1}$



$v_y = -\sqrt{25^2 - 15^2} = -20 m/s.$

Vertically, using $v = u + at$

$$-20 = 30\sin 60° + (-10)t$$

$$t = \frac{30\sin 60° + 20}{10}$$

$$t = 4.598 s \approx 4.60 s$$

90 A particle P is projected with speed u at an angle α above the horizontal from a point O on a horizontal plane and moves freely under gravity. The horizontal and vertical displacements of P from O at a subsequent time t are denoted by x and y respectively.

(a) Derive the equation of the trajectory of P in the form

$$y = x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha.$$

[3]

Horizontally, $x = (u\cos\alpha)t \rightarrow (1)$.

Vertically, $y = (u\sin\alpha)t + \frac{1}{2}(-g)t^2 = (u\sin\alpha)t - \frac{g}{2}t^2 \rightarrow (2)$

From (1), $t = \frac{x}{u\cos\alpha} \rightarrow (3)$

Substitute (3) in (2):

$$y = u\sin\alpha\left(\frac{x}{u\cos\alpha}\right) - \frac{g}{2}\left(\frac{x}{u\cos\alpha}\right)^2$$

$$y = x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha \text{ (Shown)}.$$

The point Q is the highest point on the trajectory of P in the case where $\alpha = 45°$.

(b) Show that the x-coordinate of Q is $\frac{u^2}{2g}$.

[3]

With $\alpha = 45°$, $y = x\tan 45° - \frac{gx^2}{2u^2}\sec^2 45°$

$$y = x(1) - \frac{gx^2}{2u^2}(2) = x - \frac{gx^2}{2u^2} \times 2 = x - \frac{gx^2}{u^2}.$$

$$\frac{dy}{dx} = 1 - \frac{2gx}{u^2}.$$

For highest point, $\frac{dy}{dx} = 0 \Rightarrow 1 - \frac{2gx}{u^2} = 0$

$$1 = \frac{2gx}{u^2}$$

$$\frac{u^2}{2g} = x$$

$\Rightarrow$ x-coordinate of $Q = \frac{u^2}{2g}$.

(c) Find the other value of α for which P would pass through the point Q. [4]

When $x = \frac{u^2}{2g}$, $y = \frac{u^2}{2g} - \frac{g}{u^2}\left(\frac{u^2}{2g}\right)^2 = \frac{u^2}{2g} - \frac{g}{u^2} \times \frac{u^4}{4g^2} = \frac{u^2}{2g} - \frac{u^2}{4g}$

$$y = \frac{u^2}{4g}$$

$$\Rightarrow Q = \left(\frac{u^2}{2g}, \frac{u^2}{4g}\right).$$

$$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2u^2}$$

For P to pass through $Q\left(\frac{u^2}{2g}, \frac{u^2}{4g}\right)$,

$$\frac{u^2}{4g} = \left(\frac{u^2}{2g}\right)\tan\alpha - \frac{g}{2u^2}\left(\frac{u^2}{2g}\right)^2\sec^2\alpha$$

$$\frac{u^2}{4g} = \frac{u^2\tan\alpha}{2g} - \frac{gu^4}{8u^2g^2}\sec^2\alpha$$

$$\frac{u^2}{4g} = \frac{u^2}{8g}\left[4\tan\alpha - \sec^2\alpha\right]$$

$$2 = 4\tan\alpha - (1+\tan^2\alpha)$$

$$0 = -\tan^2\alpha + 4\tan\alpha - 3$$

$$0 = \tan^2\alpha - 4\tan\alpha + 3$$

$$0 = (\tan\alpha - 3)(\tan\alpha - 1)$$

$\Rightarrow \tan\alpha = 3$ or $\tan\alpha = 1$

$\alpha = 71.56^\circ$ $\quad$ $\alpha = 45^\circ$

$\alpha \approx 71.6^\circ$

$\therefore$ Other value of $\alpha = 71.6^\circ$

91 A particle P is projected with speed $u\,\mathrm{m\,s^{-1}}$ at an angle of θ above the horizontal from a point O on a horizontal plane and moves freely under gravity. The horizontal and vertical displacements of P from O at a subsequent time t s are denoted by x m and y m respectively.

(a) Starting from the equation of the trajectory given in the List of formulae (MF19), show that

$$y = x\tan\theta - \frac{gx^2}{2u^2}(1+\tan^2\theta). \qquad [1]$$

$y = x\tan\theta - \dfrac{gx^2}{2V^2\cos^2\theta} \Rightarrow y = x\tan\theta - \dfrac{gx^2\sec^2\theta}{2u^2}$ since $V = u$ and $\sec\theta = \dfrac{1}{\cos\theta}$.

$$y = x\tan\theta - \frac{gx^2}{2u^2}(1+\tan^2\theta).$$

(Shown).

When $\theta = \tan^{-1}2$, P passes through the point with coordinates (10, 16).

(b) Show that there is no value of θ for which P can pass through the point with coordinates (18, 30). [6]

$\theta = \tan^{-1}2 \Rightarrow \tan\theta = 2$.

When $x = 10$, $y = 16$

$$\Rightarrow 16 = 10(2) - \frac{10(10)^2}{2u^2}\times(1+2^2)$$

$$16 = 20 - \frac{2500}{u^2}$$

$$\frac{2500}{u^2} = 4 \Rightarrow u^2 = 625 \Rightarrow u = 25\ \mathrm{ms^{-1}}.$$

$$\Rightarrow y = x\tan\theta - \frac{x^2(1+\tan^2\theta)}{125}.$$

When $x = 18, y = 30$

$$30 = 18\tan\theta - \frac{18^2(1+\tan^2\theta)}{125}$$

$$30 = 18\tan\theta - 2.592(1+\tan^2\theta)$$

$$30 = 18\tan\theta - 2.592 - 2.592\tan^2\theta$$

$$0 = -2.592\tan^2\theta + 18\tan\theta - 32.592$$

$$0 = 2.592\tan^2\theta - 18\tan\theta + 32.592$$

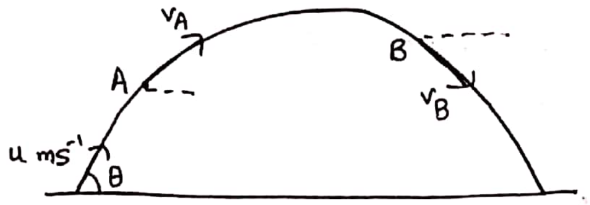
$$\tan\theta = \frac{-(-18) \pm \sqrt{(-18)^2 - 4(2.592)(32.592)}}{2(2.592)}$$

= No real solutions since $b^2 - 4ac < 0$.

Then there is no value of θ for which P passes through (18, 30).

92 A particle P is projected from a point O on a horizontal plane and moves freely under gravity. Its initial speed is $u\,\mathrm{m\,s^{-1}}$ and its angle of projection is $\sin^{-1}(\frac{4}{5})$ above the horizontal. At time 8 s after projection, P is at the point A. At time 32 s after projection, P is at the point B. The direction of motion of P at B is perpendicular to its direction of motion at A.

Find the value of u. [7]



$\theta = \sin^{-1}\left(\frac{4}{5}\right)$

$\sin\theta = \frac{4}{5}$

$\Rightarrow \cos\theta = \frac{3}{5}$.

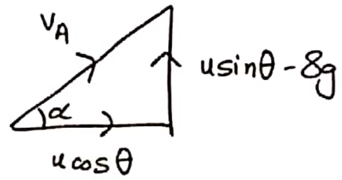
Initially, $u_x = u\cos\theta$, $u_y = u\sin\theta$.

At A,

$v_x = u\cos\theta$

$v_y = u\sin\theta + (-g)(8) = u\sin\theta - 8g$.

$\tan\alpha = \dfrac{u\sin\theta - 8g}{u\cos\theta}$

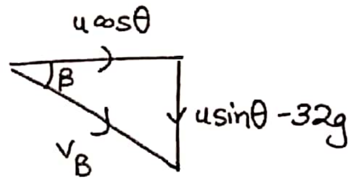


At B,

$v_x = u\cos\theta$

$v_y = u\sin\theta + (-g)(32) = u\sin\theta - 32g$.

$\tan\beta = \dfrac{u\sin\theta - 32g}{u\cos\theta}$.



Velocity at A is perpendicular to velocity at B, i.e. $v_A \perp v_B$.

$\Rightarrow \tan\alpha \times \tan\beta = 1$.

This is because $\alpha - \beta = 90°$ $\Rightarrow \tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \infty$

$\Rightarrow 1 + \tan\alpha\tan\beta = 0$

$\tan\alpha\tan\beta = -1$

$$\Rightarrow \frac{u\sin\theta - 8g}{u\cos\theta} \times \frac{u\sin\theta - 32g}{u\cos\theta} = -1$$

$$\frac{u\left(\frac{4}{5}\right) - 80}{u\left(\frac{3}{5}\right)} \times \frac{u\left(\frac{4}{5}\right) - 320}{u\left(\frac{3}{5}\right)} = -1$$

$$\left(\frac{4u}{5} - 80\right)\left(\frac{4u}{5} - 320\right) = -\frac{9}{25}u^2$$

$$\frac{16u^2}{25} - 64u - 256u + 25600 = \frac{-9}{25}u^2$$

$$u^2 - 320u + 25600 = 0$$

$$(u - 160)^2 = 0$$

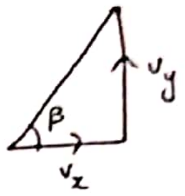
$$\Rightarrow u = 160$$

93 A particle is projected with speed u at an angle α above the horizontal from a point O on a horizontal plane. The particle moves freely under gravity.

(a) Write down the horizontal and vertical components of the velocity of the particle at time T after projection. [2]

Horizontally, $v_x = u\cos\alpha$.

Vertically, $v_y = u\sin\alpha - gT$.



At time T after projection, the direction of motion of the particle is perpendicular to the direction of projection.

(b) Express T in terms of u, g and α. [2]

At time T, $\tan\beta = \frac{v_y}{v_x} = \frac{u\sin\alpha - gT}{u\cos\alpha}$.

Since at time T, direction of motion is perpendicular to direction of projection,

$$\tan\alpha \times \tan\beta = -1$$

$$\left(\frac{\sin\alpha}{\cos\alpha}\right) \times \frac{u\sin\alpha - gT}{u\cos\alpha} = -1$$

$$u\sin^2\alpha - (\sin\alpha)gT = -u\cos^2\alpha$$

$$u\sin^2\alpha + u\cos^2\alpha = gT(\sin\alpha)$$

$$u = gT\sin\alpha$$

$$T = \frac{u}{g\sin\alpha}$$

(c) Deduce that $T > \frac{u}{g}$. [1]

$$0 < \sin\alpha < 1$$

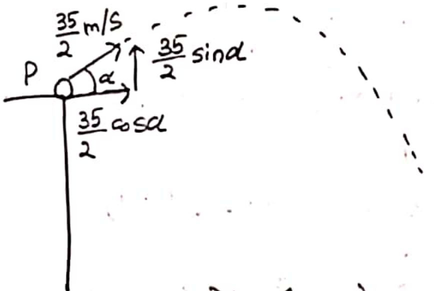
$$\frac{1}{\sin\alpha} > 1$$

$$\frac{u}{g\sin\alpha} > \frac{u}{g}$$

$$\Rightarrow T > \frac{u}{g} \quad \text{(Shown)}.$$

94 Particles P and Q are projected in the same vertical plane from a point O at the top of a cliff. The height of the cliff exceeds 50 m. Both particles move freely under gravity. Particle P is projected with speed $\frac{35}{2}\,\mathrm{m\,s^{-1}}$ at an angle α above the horizontal, where $\tan\alpha = \frac{4}{3}$. Particle Q is projected with speed $u\,\mathrm{m\,s^{-1}}$ at an angle β above the horizontal, where $\tan\beta = \frac{1}{2}$. Particle Q is projected one second after the projection of particle P. The particles collide T s after the projection of particle Q.

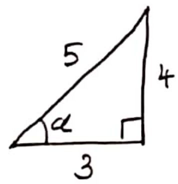
(a) Write down expressions, in terms of T, for the horizontal displacements of P and Q from O when they collide and hence show that $4uT = 21\sqrt{5}(T+1)$. [4]



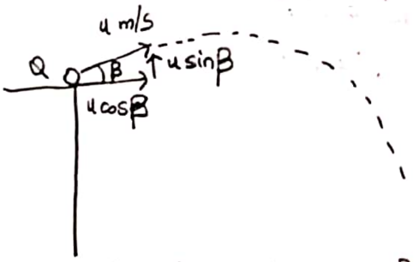
$\tan\alpha = \frac{4}{3}$

$\sin\alpha = \frac{4}{5}$

$\cos\alpha = \frac{3}{5}$



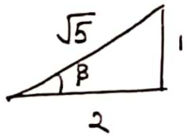
$$x_P = \left(\frac{35}{2}\cos\alpha\right) \times (T+1) = \frac{35}{2} \times \frac{3}{5} \times (T+1) = 10{\cdot}5(T+1) = 10{\cdot}5T + 10{\cdot}5.$$



$\tan\beta = \frac{1}{2}$

$\sin\beta = \frac{1}{\sqrt{5}}$

$\cos\beta = \frac{2}{\sqrt{5}}$.



$$x_Q = (u\cos\beta) \times T = u \times \frac{2}{\sqrt{5}} \times T = \frac{2uT}{\sqrt{5}}.$$

At point of collision, $x_P = x_Q$

$$\Rightarrow 10{\cdot}5T + 10{\cdot}5 = \frac{2uT}{\sqrt{5}}$$

$$\frac{21T}{2} + \frac{21}{2} = \frac{2uT}{\sqrt{5}}$$

$$21(T+1)\sqrt{5} = 4uT$$

$$\Rightarrow 4uT = 21\sqrt{5}(T+1).$$ (Shown). $\rightarrow$ (1)

(b) Find the value of T. [4]

Vertically, $y_P = \left(\frac{35}{2}\sin\alpha\right)(T+1) + \frac{1}{2}(-g)(T+1)^2 = \frac{35}{2} \times \frac{4}{5} \times (T+1) - \frac{10}{2}(T+1)^2$

$y_P = 14(T+1) - 5(T+1)^2 = 14T + 14 - 5(T^2+2T+1) = -5T^2+4T+9$

and $y_Q = (u\sin\beta)T + \frac{1}{2}(-g)T^2 = \left(u \cdot \frac{1}{\sqrt{5}}\right)T - \frac{10}{2}T^2 = \frac{uT}{\sqrt{5}} - 5T^2.$

Then, at point of collision, $y_P = y_Q \Rightarrow -5T^2+4T+9 = \frac{uT}{\sqrt{5}} - 5T^2$

$$4T+9 = \frac{uT}{\sqrt{5}}$$

$$\frac{(4T+9)\sqrt{5}}{T} = u \rightarrow (2)$$

Substitute (2) in (1) $\Rightarrow 4\left[\frac{(4T+9)\sqrt{5}}{T}\right]T = 21\sqrt{5}(T+1)$

$$4(4T+9)\cancel{\sqrt{5}} = 21\cancel{\sqrt{5}}(T+1)$$

$$16T+36 = 21T+21$$

$$15 = 5T$$

$$\Rightarrow T = 3.$$

(c) Find the horizontal and vertical displacements of the particles from O when they collide. [3]

$T = 3 \Rightarrow x_P = 10.5(3+1) = 42$

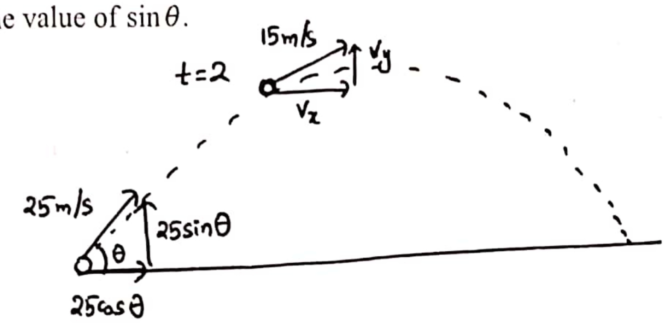
and $y_P = 14(3+1) - 5(3+1)^2 = -24$

i.e. horizontal displacement from O when particles collide = 42 m

vertical displacement from O when particles collide = -24 m.

95 A particle P is projected with speed $25\,\mathrm{m\,s^{-1}}$ at an angle θ above the horizontal from a point O on a horizontal plane and moves freely under gravity. After 2 s the speed of P is $15\,\mathrm{m\,s^{-1}}$.

(a) Find the value of $\sin\theta$. [5]



At $t=2$, $v_x = 25\cos\theta$

$v_y = u_y + at$

$v_y = 25\sin\theta + (-10)(2)$

$v_y = 25\sin\theta - 20$

$v = 15 \Rightarrow \sqrt{v_x^2 + v_y^2} = 15$

$\sqrt{(25\cos\theta)^2 + (25\sin\theta - 20)^2} = 15$

$625\cos^2\theta + 625\sin^2\theta - 1000\sin\theta + 400 = 225$

$625(\sin^2\theta + \cos^2\theta) - 1000\sin\theta = 225 - 400$

$625(1) - 1000\sin\theta = -175$

$-1000\sin\theta = -800$

$\sin\theta = \frac{-800}{-1000}$

$\Rightarrow \sin\theta = \frac{4}{5}$

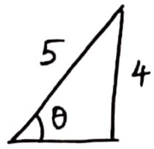
(b) Find the range of the flight. [3]

$$\text{Range} = \frac{u^2 \sin 2\theta}{g}$$

$$= \frac{25^2 \times \frac{24}{25}}{10}$$

$$\text{Range} = 60\text{m}$$

$$\sin\theta = \frac{4}{5}$$



$$\Rightarrow \text{adj} = \sqrt{5^2 - 4^2} = 3$$

$$\Rightarrow \cos\theta = \frac{3}{5}$$

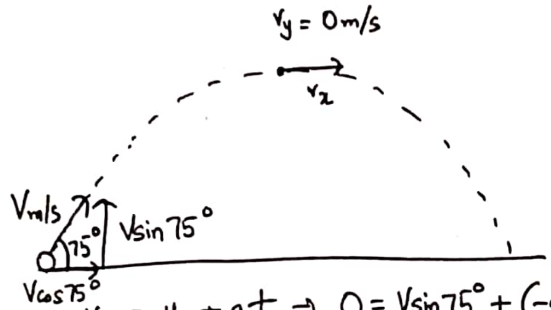
$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$= 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)$$

$$\sin 2\theta = \frac{24}{25}$$

96 A particle P is projected with speed $V\,\mathrm{m\,s^{-1}}$ at an angle $75°$ above the horizontal from a point O on a horizontal plane. It then moves freely under gravity.

(a) Show that the total time of flight, in seconds, is $\dfrac{2V}{g}\sin 75°$. [2]



Vertically, $V_y = u_y + at \Rightarrow 0 = V\sin 75° + (-g)t$

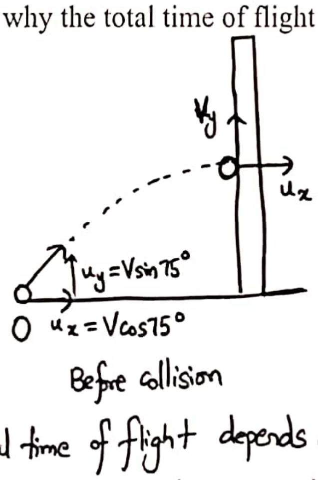
$V\sin 75° = gt \Rightarrow t = \dfrac{V\sin 75°}{g}$

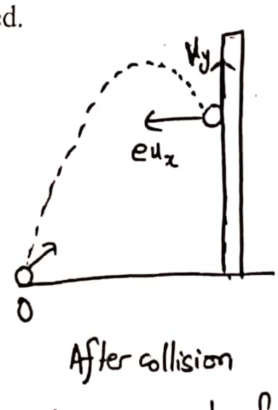
Total time of flight $= 2 \times$ (Time taken to reach maximum height)

$= 2 \times \dfrac{V\sin 75°}{g} = \dfrac{2V\sin 75°}{g}$. (Shown).

A smooth vertical barrier is now inserted with its lower end on the plane at a distance 15 m from O. The particle is projected as before but now strikes the barrier, rebounds and returns to O. The coefficient of restitution between the barrier and the particle is $\frac{3}{5}$.

(b) Explain why the total time of flight is unchanged. [1]





The total time of flight depends on the vertical component of velocity. During the collision, the horizontal component of velocity decreases but the velocity parallel to the barrier i.e. the vertical component of velocity remains unchanged. This ensures that the time taken to reaches the maximum height is unchanged and so the total time of flight remains unchanged.

(c) Find an expression for V in terms of g. [7]

Horizontally, Time taken to reach barrier $= \frac{\text{Distance}}{\text{Speed}} = \frac{15}{V\cos 75^\circ}$.

After collision, horizontal speed $= eu_x = \frac{3}{5} V\cos 75^\circ$

$\Rightarrow$ Time taken to return to O from barrier $= \frac{\text{Distance}}{\text{Speed}} = \frac{15}{\frac{3}{5} V\cos 75^\circ} = \frac{25}{V\cos 75^\circ}$.

Total time of flight $= \frac{15}{V\cos 75^\circ} + \frac{25}{V\cos 75^\circ}$

$$\frac{2V\sin 75^\circ}{g} = \frac{40}{V\cos 75^\circ}$$

$$2V^2 \sin 75^\circ \cos 75^\circ = 40g$$

$$V^2 = \frac{40g}{2\sin 75^\circ \cos 75^\circ}$$

$$V^2 = 80g$$

$$V = \sqrt{80g}$$

$$V = 4\sqrt{5g}.$$

97 A particle P is projected with speed $u\,\mathrm{m\,s^{-1}}$ at an angle of θ above the horizontal from a point O on a horizontal plane and moves freely under gravity. The horizontal and vertical displacements of P from O at a subsequent time t s are denoted by x m and y m respectively.

(a) Show that the equation of the trajectory is given by

$$y = x\tan\theta - \frac{gx^2}{2u^2}(1+\tan^2\theta).$$ [4]

Horizontally, Speed = $\frac{\text{Distance}}{\text{time}} \Rightarrow u\cos\theta = \frac{x}{t} \Rightarrow t = \frac{x}{u\cos\theta} \rightarrow (1)$.

Vertically, $s = ut + \frac{1}{2}at^2 \Rightarrow y = (u\sin\theta)t + \frac{1}{2}(-g)t^2 \Rightarrow y = (u\sin\theta)t - \frac{gt^2}{2} \rightarrow (2)$

Substitute (1) in (2):

$$y = (u\sin\theta)\times\frac{x}{u\cos\theta} - \frac{g}{2}\left(\frac{x}{u\cos\theta}\right)^2$$

$$= x\tan\theta - \frac{gx^2\sec^2\theta}{2u^2}$$

$$y = x\tan\theta - \frac{gx^2}{2u^2}(1+\tan^2\theta) \text{ (Shown)}. \rightarrow (1)$$

In the subsequent motion P passes through the point with coordinates (30, 20).

(b) Given that one possible value of $\tan\theta$ is $\frac{4}{3}$, find the other possible value of $\tan\theta$. [5]

Substitute $x=30, y=20$ in (1):

$$20 = 30\tan\theta - \frac{10 \times 30^2}{2u^2}\left(1+\tan^2\theta\right)$$

$$20 = 30\tan\theta - \frac{4500\left(1+\tan^2\theta\right)}{u^2} \rightarrow (1)$$

Since one possible value of $\tan\theta$ is $\frac{4}{3}$,

$$20 = 30\left(\frac{4}{3}\right) - \frac{4500\left(1+\left(\frac{4}{3}\right)^2\right)}{u^2}$$

$$20 = 40 - \frac{12500}{u^2}$$

$$\frac{12500}{u^2} = 20$$

$$u^2 = 625$$

$$u = 25 \text{ m/s.}$$

Substituting $u = 25$ m/s in (1):

$$20 = 30\tan\theta - \frac{4500\left(1+\tan^2\theta\right)}{25^2}$$

$$20 = 30\tan\theta - 7.2\left(1+\tan^2\theta\right)$$

$$20 = 30\tan\theta - 7.2 - 7.2\tan^2\theta$$

$$7.2\tan^2\theta - 30\tan\theta + 27.2 = 0.$$

Let, $m = \tan\theta \Rightarrow 7.2m^2 - 30m + 27.2 = 0$

$$m = \frac{-(-30) \pm \sqrt{(-30)^2 - 4(7.2)(27.2)}}{2(7.2)}$$

$$m = \frac{30 \pm \sqrt{116.64}}{14.4} = \frac{30 \pm 10.8}{14.4}$$

$$\Rightarrow m = \frac{30+10.8}{14.4} \quad \text{or} \quad m = \frac{30-10.8}{14.4}$$

$$m = \frac{17}{6} \quad \text{or} \quad m = \frac{4}{3}$$

$$\Rightarrow \tan\theta = \frac{17}{6} \quad \text{or} \quad \tan\theta = \frac{4}{3}$$

$\therefore$ Other possible value of $\tan\theta = \frac{17}{6}$.

98 At time t s, a particle P is projected with speed $40\,\mathrm{m\,s^{-1}}$ at an angle θ above the horizontal from a point O on a horizontal plane and moves freely under gravity. The greatest height achieved by P during its flight is H m and the corresponding time is T s.

(a) Obtain expressions for H and T in terms of θ. [2]

Vertically, $v_y = u_y + at \Rightarrow 0 = 40\sin\theta + (-10)T$

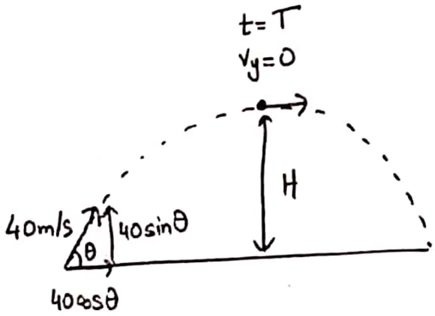
$$10T = 40\sin\theta$$

$$T = 4\sin\theta.$$

and $v_y^2 = u_y^2 + 2as \Rightarrow 0^2 = (40\sin\theta)^2 + 2(-10)(H)$

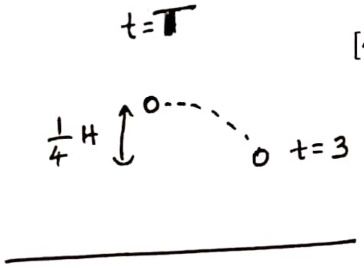
$$20H = 1600\sin^2\theta$$

$$H = 80\sin^2\theta.$$



During the time between $t = T$ and $t = 3$, P descends a distance $\frac{1}{4}H$.

(b) Find the value of θ. [4]



Vertically, $s = ut + \frac{1}{2}at^2$

$$\frac{1}{4}H = 0(3-T) + \frac{1}{2}(g)(3-T)^2$$

$$\frac{H}{4} = \frac{10}{2}(3-T)^2$$

$$H = 20(3-T)^2.$$

Substituting $H = 80\sin^2\theta$ and $T = 4\sin\theta$:

$$80\sin^2\theta = 20(3-4\sin\theta)^2$$

$$80\sin^2\theta = 20[9 - 24\sin\theta + 16\sin^2\theta]$$

$$80\sin^2\theta = 180 - 480\sin\theta + 320\sin^2\theta$$

$$0 = 240\sin^2\theta - 480\sin\theta + 180$$

$$0 = 4\sin^2\theta - 8\sin\theta + 3$$

Let, $m = \sin\theta \Rightarrow 4m^2 - 8m + 3 = 0$

$$4m^2 - 6m - 2m + 3 = 0$$

$$(2m-1)(2m-3) = 0$$

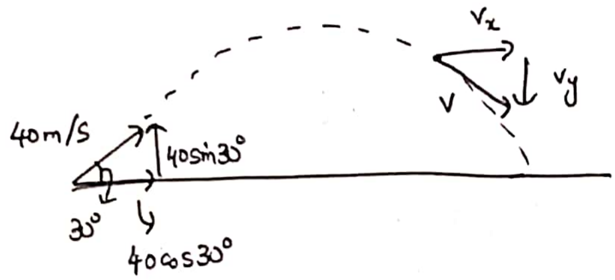
$\Rightarrow m = \frac{1}{2}$ or $m = \frac{3}{2}$.

$\sin\theta = \frac{1}{2}$ $\qquad$ $\sin\theta = \frac{3}{2}$

$\theta = \sin^{-1}\left(\frac{1}{2}\right)$ $\qquad$ (No solutions since $-1 \le \sin\theta \le 1$)

$\theta = 30^\circ.$

(c) Find the speed of P when $t = 3$. [3] .



At $t = 3$, $v_x = 40\cos 30° = 20\sqrt{3}$ m/s.

Vertically, $v_y = u_y + at$

$= 40\sin 30° + (-10)(3)$

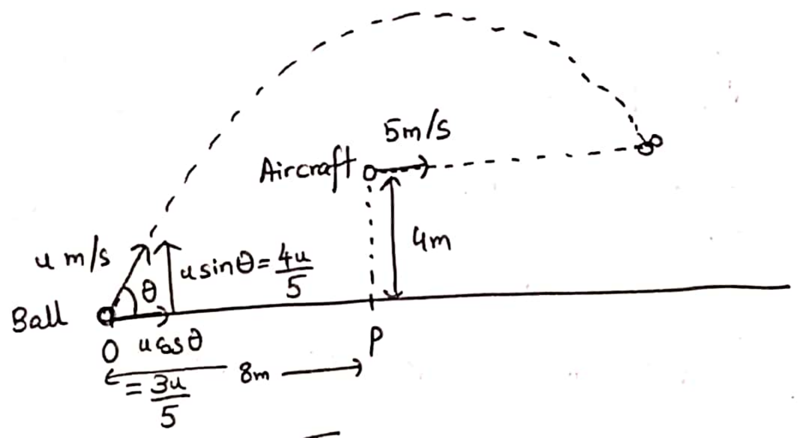
$v_y = -10$ m/s

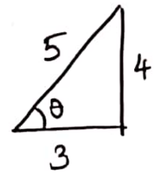
$\Rightarrow v = \sqrt{v_x^2 + v_y^2} = \sqrt{(20\sqrt{3})^2 + (-10)^2} = 10\sqrt{13} \approx 36.1$ m/s.

99 The points O and P are on a horizontal plane, a distance 8 m apart. A ball is thrown from O with speed $u\,\mathrm{m\,s^{-1}}$ at an angle θ above the horizontal, where $\tan\theta = \frac{4}{3}$. At the same instant, a model aircraft is launched with speed $5\,\mathrm{m\,s^{-1}}$ parallel to the horizontal plane from a point 4 m vertically above P. The model aircraft moves in the same vertical plane as the ball and in the same horizontal direction as the ball. The model aircraft moves horizontally with a constant speed of $5\,\mathrm{m\,s^{-1}}$. After T s, the ball and the model aircraft collide.

(a) Find the value of T. [6]

$\tan\theta = \frac{4}{3}$





$\sin\theta = \frac{4}{5}$

$\cos\theta = \frac{3}{5}$.

For ball, $x_B = (u\cos\theta) \times T = \frac{3uT}{5}$.

$y_B = (u\sin\theta)T + \frac{1}{2}(-10)T^2 = \frac{4uT}{5} - 5T^2$

For aircraft, $x_A = 8 + 5T$

$y_A = 4$

At point of collision, $x_B = x_A \Rightarrow \frac{3uT}{5} = 8 + 5T \Rightarrow u = \frac{5(8+5T)}{3T} \rightarrow (1)$

and $y_B = y_A \Rightarrow \frac{4uT}{5} - 5T^2 = 4 \Rightarrow \frac{4uT}{5} = 4 + 5T^2 \Rightarrow u = \frac{5(4+5T^2)}{4T} \rightarrow (2)$

Substitute (1) in (2):

$$\frac{\cancel{5}(8+5T)}{3\cancel{T}} = \frac{\cancel{5}(4+5T^2)}{4\cancel{T}}$$

$$32 + 20T = 12 + 15T^2$$

$$0 = 15T^2 - 20T - 20$$

$$0 = 3T^2 - 4T - 4$$

$$0 = 3T^2 - 6T + 2T - 4$$

$$0 = (3T+2)(T-2)$$

$\Rightarrow T = \frac{-2}{3}$ or $T = 2$

(Ignore).

$\therefore T = 2$.

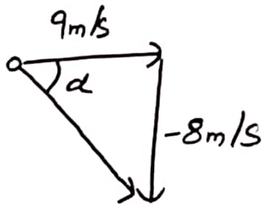
(b) Find the direction in which the ball is moving immediately before the collision. [3]

$T = 2 \Rightarrow v_{Bx} = \frac{3u}{5}.$

$v_{By} = u\sin\theta + (-10)(2) = u\sin\theta - 20$

Substituting $T = 2$ in (1):

$u = \frac{5(8 + 5\times 2)}{3\times 2} = 15.$



$\Rightarrow v_{Bx} = \frac{3\times 15}{5} = 9\ m/s.$

$v_{By} = 15\left(\frac{4}{5}\right) - 20 = -8\ m/s.$

$\tan\alpha = \frac{-8}{9} \Rightarrow \alpha = -41.63^\circ$

$\therefore$ Direction of ball immediately before collision $= 41.6^\circ$.

100 A particle P is projected with speed $u\,\mathrm{m\,s^{-1}}$ at an angle θ above the horizontal from a point O on a horizontal plane and moves freely under gravity. During its flight P passes through the point which is a horizontal distance $3a$ from O and a vertical distance $\frac{3}{8}a$ above the horizontal plane. It is given that $\tan\theta = \frac{1}{3}$.

(a) Show that $u^2 = 8ag$. [2]

$$y = x\tan\theta - \frac{gx^2\sec^2\theta}{2u^2} = x\tan\theta - \frac{gx^2(1+\tan^2\theta)}{2u^2}$$

$$\frac{3a}{8} = 3a\left(\frac{1}{3}\right) - \frac{g(3a)^2\left(1+\left(\frac{1}{3}\right)^2\right)}{2u^2}$$

$$\frac{3a}{8} = a - \frac{10ga^2}{2u^2}$$

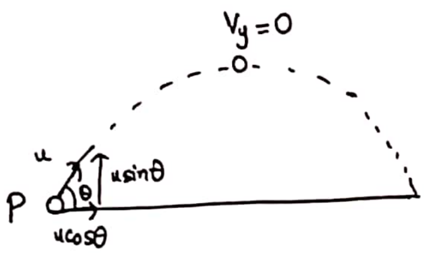
$$\frac{-5a}{8} = \frac{-10ga^2}{2u^2}$$

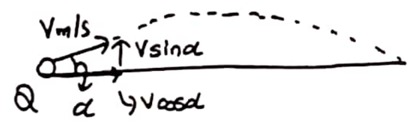
$$-10au^2 = -80ga^2$$

$$u^2 = 8ag \text{ (Shown)}.$$

A particle Q is projected with speed $V\,\mathrm{m\,s^{-1}}$ at an angle α above the horizontal from O at the instant when P is at its highest point. Particles P and Q both land at the same point on the horizontal plane at the same time.

(b) Find V in terms of a and g. [7]





For P, $\quad v_y = u\sin\theta + (-g)t$

$$0 = u\sin\theta - gt$$

$$t = \frac{u\sin\theta}{g}.$$

$\Rightarrow$ Total time of flight, $T = 2t = \dfrac{2u\sin\theta}{g}$.

$\Rightarrow$ Range of $P = u\cos\theta \times T = u\cos\theta \times \dfrac{2u\sin\theta}{g}$

For Q, Total time of flight $= t = \dfrac{u\sin\theta}{g}$.

Range of $Q = V\cos\alpha \times \dfrac{u\sin\theta}{g}$.

$$\text{Range of } P = \text{Range of } Q$$

$$u\cos\theta \times \frac{2u\sin\theta}{g} = V\cos\alpha \times \frac{u\sin\theta}{g}$$

$$2u\cos\theta = V\cos\alpha \rightarrow (1).$$

Vertically for Q, $s = ut + \frac{1}{2}at^2$

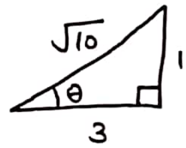
$$0 = (V\sin\alpha)\left(\frac{u\sin\theta}{g}\right) + \frac{1}{2}(-g)\left(\frac{u\sin\theta}{g}\right)^2$$

$$0 = \frac{u\sin\theta}{g}\left[V\sin\alpha - \frac{g}{2}\left(\frac{u\sin\theta}{g}\right)\right]$$

$$0 = V\sin\alpha - \frac{u\sin\theta}{2}$$

$$\frac{u\sin\theta}{2} = V\sin\alpha \rightarrow (2)$$

$\tan\theta = \frac{1}{3}$



$\sin\theta = \frac{1}{\sqrt{10}}$

$\cos\theta = \frac{3}{\sqrt{10}}$

Squaring (1) and (2) and adding:

$$(V\cos\alpha)^2 + (V\sin\alpha)^2 = (2u\cos\theta)^2 + \left(\frac{u\sin\theta}{2}\right)^2$$

$$V^2 = 4u^2\cos^2\theta + \frac{u^2\sin^2\theta}{4}$$

$$\Rightarrow V^2 = 4u^2\left(\frac{3}{\sqrt{10}}\right)^2 + \frac{u^2}{4}\left(\frac{1}{\sqrt{10}}\right)^2$$

$$V^2 = \frac{29u^2}{8}$$

$$V = \sqrt{\frac{29u^2}{8}}$$

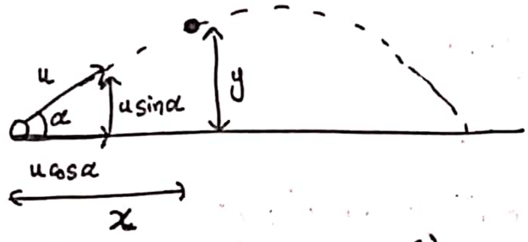
Substitute $u^2 = 8ag \Rightarrow V = \sqrt{\frac{29 \times 8ag}{8}}$

$$\Rightarrow V = \sqrt{29ag}.$$

101 A particle P is projected with speed u at an angle α above the horizontal from a point O on a horizontal plane and moves freely under gravity. The horizontal and vertical displacements of P from O at a subsequent time t are denoted by x and y respectively.

(a) Derive the equation of the trajectory of P in the form

$$y = x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha. \qquad [3]$$



Horizontally, $x = (u\cos\alpha)t \Rightarrow t = \dfrac{x}{u\cos\alpha} \rightarrow (1).$

Vertically, $y = (u\sin\alpha)t + \frac{1}{2}(-g)t^2 = (u\sin\alpha)t - \frac{g}{2}t^2 \rightarrow (2)$

Substitute (1) in (2):

$$y = (u\sin\alpha)\left(\frac{x}{u\cos\alpha}\right) - \frac{g}{2}\left(\frac{x}{u\cos\alpha}\right)^2$$

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$

$$\Rightarrow y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2u^2}$$

During its flight, P must clear an obstacle of height h m that is at a horizontal distance of 32 m from the point of projection. When $u = 40\sqrt{2}\,\mathrm{m\,s^{-1}}$, P just clears the obstacle. When $u = 40\,\mathrm{m\,s^{-1}}$, P only achieves 80% of the height required to clear the obstacle.

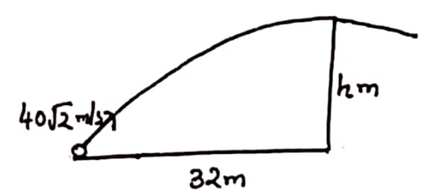
(b) Find the two possible values of h. [6]

$u = 40\sqrt{2} \Rightarrow u^2 = 3200.$

$y = h,\ x = 32$

$$\Rightarrow h = 32\tan\alpha - \frac{10(32)^2\sec^2\alpha}{2\times 3200}$$

$$h = 32\tan\alpha - 1.6\sec^2\alpha \rightarrow (1)$$

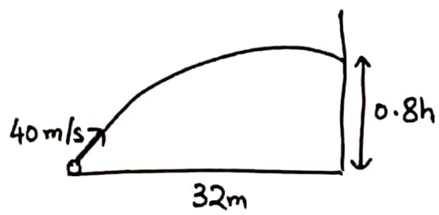


$u = 40\,\mathrm{m/s} \Rightarrow u^2 = 1600$

$y = 0.8h,\ x = 32$

$$\Rightarrow 0.8h = 32\tan\alpha - \frac{10(32)^2\sec^2\alpha}{2\times 1600}$$

$$0.8h = 32\tan\alpha - 3.2\sec^2\alpha \rightarrow (2)$$



Substitute (1) in (2):

$$0.8(32\tan\alpha - 1.6\sec^2\alpha) = 32\tan\alpha - 3.2\sec^2\alpha$$

$$25.6\tan\alpha - 1.28\sec^2\alpha = 32\tan\alpha - 3.2\sec^2\alpha$$

$$1.92\sec^2\alpha = 6.4\tan\alpha$$

$$3\sec^2\alpha = 10\tan\alpha$$
$$3(1+\tan^2\alpha) = 10\tan\alpha$$
$$3\tan^2\alpha - 10\tan\alpha + 3 = 0$$

Let, $m = \tan\alpha$

$$\Rightarrow 3m^2 - 10m + 3 = 0$$
$$3m^2 - 9m - m + 3 = 0$$
$$(3m-1)(m-3) = 0.$$
$$m = \frac{1}{3}, 3$$

$$\Rightarrow \tan\alpha = \frac{1}{3}, 3$$
$$\alpha = \tan^{-1}\left(\frac{1}{3}\right), \tan^{-1}(3).$$

Substitute $\alpha = \tan^{-1}\left(\frac{1}{3}\right)$ in (1)

$$h = 32\tan\left[\tan^{-1}\left(\frac{1}{3}\right)\right] - 1.6\sec^2\left[\tan^{-1}\left(\frac{1}{3}\right)\right] = \frac{80}{9}$$

Substitute $\alpha = \tan^{-1}(3)$ in (1):

$$h = 32\left[\tan\left(\tan^{-1}3\right)\right] - 1.6\sec^2\left(\tan^{-1}(3)\right) = 80.$$

$\therefore$ Possible values of $h = \frac{80}{9}, 80.$