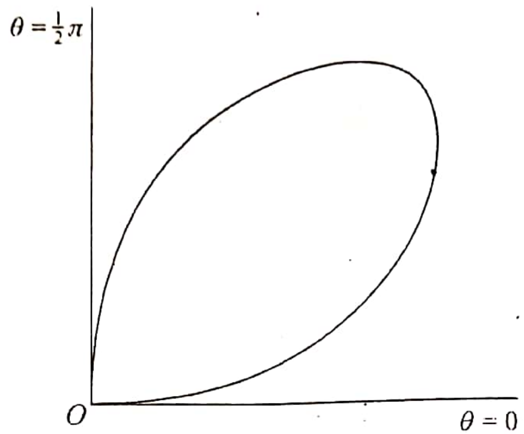
Past Paper Questions: Polar Curves

1



The diagram shows one loop of the curve whose polar equation is $r = a\sin 2\theta$, where a is a positive constant. Find the area of the loop, giving your answer in terms of a and π. [4]

Area of loop

$$= \frac{1}{2}\int_0^{\pi/2} (a\sin 2\theta)^2 \, d\theta = \frac{a^2}{2}\int_0^{\pi/2} \sin^2 2\theta \, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} \frac{1-\cos 4\theta}{2} \, d\theta = \frac{a^2}{4}\int_0^{\pi/2} 1-\cos 4\theta \, d\theta$$

$$= \frac{a^2}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2} = \frac{a^2}{4}\left[\left(\frac{\pi}{2} - 0\right) - (0)\right]$$

$$= \frac{\pi a^2}{8}.$$

2 The curve C has equation

$$(x^2 + y^2)^2 = 4xy.$$

(i) Show that the polar equation of C is $r^2 = 2\sin 2\theta$. [2]

(ii) Draw a sketch of C, indicating any lines of symmetry as well as the form of C at the pole. [5]

(iii) Write down the maximum possible distance of a point of C from the pole. [1]

i)

$$(x^2+y^2)^2 = 4xy$$

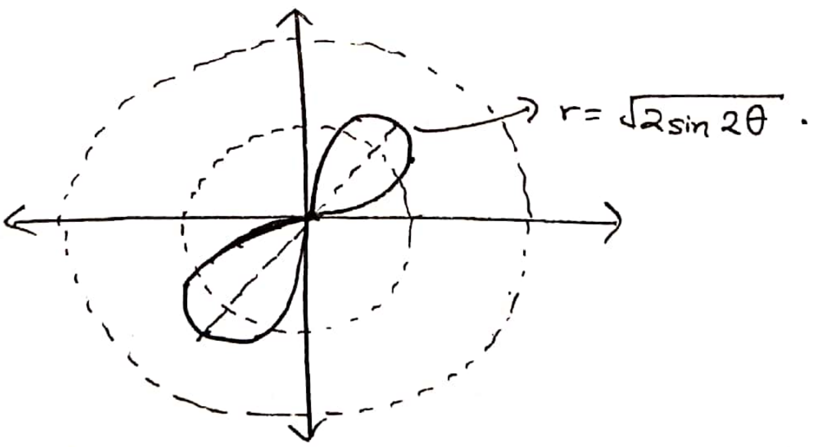
$$(r^2)^2 = 4(r\cos\theta)(r\sin\theta)$$

$$r^4 = 4r^2\sin\theta\cos\theta$$

$$r^2 = 2\sin 2\theta.$$

ii)

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
r	0	1.32	1.41	1.32	0	0	1.32	1.41	1.32	0



Lines of symmetry: $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$.

iii) When $\theta = \frac{\pi}{4}$, $r = \sqrt{2\sin\left(2\times\frac{\pi}{4}\right)} = \sqrt{2\times 1} = \sqrt{2}$.

$\Rightarrow$ Maximum distance from pole $= \sqrt{2}$.

3 The curve C has polar equation

$$r = \frac{\pi - \theta}{\theta},$$

where $\frac{1}{2}\pi \leqslant \theta \leqslant \pi$.

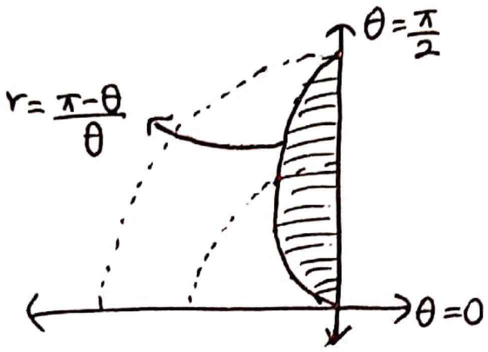
(i) Draw a sketch of C. [3]

(ii) Show that the area of the region bounded by the line $\theta = \frac{1}{2}\pi$ and C is

$$\pi\left(\tfrac{3}{4} - \ln 2\right).$$ [5]

i)

θ	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{4\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π
r	1	0.71	0.5	0.33	0.2	0.09	0



ii) Area $= \frac{1}{2}\int_{\pi/2}^{\pi} \left(\frac{\pi-\theta}{\theta}\right)^2 d\theta = \frac{1}{2}\int_{\pi/2}^{\pi} \frac{\pi^2 - 2\pi\theta + \theta^2}{\theta^2} d\theta$

$= \frac{1}{2}\int_{\pi/2}^{\pi} \pi^2\theta^{-2} - 2\pi\theta^{-1} + 1 \, d\theta$

$= \frac{1}{2}\left[\frac{\pi^2\theta^{-1}}{-1} - 2\pi\ln\theta + \theta\right]_{\pi/2}^{\pi}$

$= \frac{1}{2}\left[\frac{-\pi^2}{\theta} - 2\pi\ln\theta + \theta\right]_{\pi/2}^{\pi}$

$= \frac{1}{2}\left[\left(\frac{-\pi^2}{\pi} - 2\pi\ln\pi + \pi\right) - \left(\frac{-\pi^2}{\left(\frac{\pi}{2}\right)} - 2\pi\ln\left(\frac{\pi}{2}\right) + \frac{\pi}{2}\right)\right]$

$= \frac{1}{2}\left[-\pi - 2\pi\ln\pi + \pi + 2\pi + 2\pi\ln\left(\frac{\pi}{2}\right) - \frac{\pi}{2}\right]$

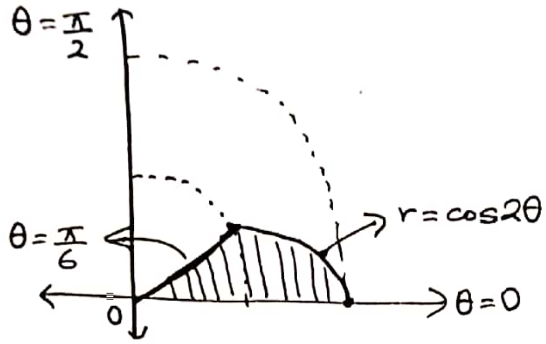
$= \frac{1}{2}\left[\frac{3\pi}{2} + 2\pi\ln\left(\frac{\pi/2}{\pi}\right)\right] = \frac{1}{2}\left[\frac{3\pi}{2} + 2\pi\ln\frac{1}{2}\right]$

$= \frac{1}{2}\left[\frac{3\pi}{2} - 2\pi\ln 2\right] = \pi\left[\frac{3}{4} - \ln 2\right].$

4 Draw a diagram to illustrate the region R which is bounded by the curve whose polar equation is $r = \cos 2\theta$ and the lines $\theta = 0$ and $\theta = \frac{1}{6}\pi$. [2]

Determine the exact area of R. [4]

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$
r	1	0.866	0.5

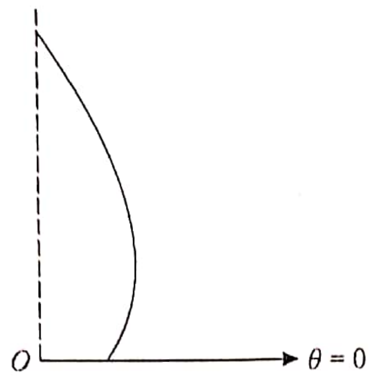


Area of region R

$$= \frac{1}{2}\int_0^{\pi/6} (\cos 2\theta)^2 \, d\theta = \frac{1}{2}\int_0^{\pi/6} \cos^2 2\theta \, d\theta = \frac{1}{2}\int_0^{\pi/6} \frac{1+\cos 4\theta}{2} \, d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{\sin 4\theta}{4}\right]_0^{\pi/6} = \frac{1}{4}\left[\left(\frac{\pi}{6} + \frac{\sqrt{3}}{8}\right) - (0)\right] = \frac{\sqrt{3}}{32} + \frac{\pi}{24}.$$

5



The diagram shows the curve C with polar equation $r = e^{\theta}$, where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. Find the maximum distance of a point of C from the line $\theta = \frac{1}{2}\pi$, giving the answer in exact form. [4]

Find the area of the finite region bounded by C and the lines $\theta = 0$ and $\theta = \frac{1}{2}\pi$, giving the answer in exact form. [3]

$$x = r\cos\theta = e^{\theta}\cos\theta.$$

$$\frac{dx}{d\theta} = e^{\theta}(-\sin\theta) + e^{\theta}(\cos\theta) = e^{\theta}(\cos\theta - \sin\theta).$$

For maximum distance, $\frac{dx}{d\theta} = 0 \Rightarrow e^{\theta}(\cos\theta - \sin\theta) = 0$

$$e^{\theta} \neq 0 \Rightarrow \cos\theta - \sin\theta = 0$$

$$\tan\theta = 1.$$

$$\Rightarrow \theta = \frac{\pi}{4}.$$

When $\theta = \frac{\pi}{4}$, $x = e^{\frac{\pi}{4}}\cos\left(\frac{\pi}{4}\right) = e^{\pi/4}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}e^{\frac{\pi}{4}}.$

$$\text{Area} = \frac{1}{2}\int_0^{\pi/2} (e^{\theta})^2\, d\theta = \frac{1}{2}\int_0^{\pi/2} e^{2\theta}\, d\theta = \frac{1}{2}\left[\frac{e^{2\theta}}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{e^{2\times\frac{\pi}{2}} - e^{2\times 0}}{2}\right] = \frac{e^{\pi} - 1}{4}.$$

6 The curves C_1 and C_2 have polar equations

$$r = \theta + 2 \quad \text{and} \quad r = \theta^2$$

respectively, where $0 \leqslant \theta \leqslant \pi$.

(i) Find the polar coordinates of the point of intersection of C_1 and C_2. [2]

(ii) Sketch C_1 and C_2 on the same diagram. [2]

(iii) Find the area bounded by C_1, C_2 and the line $\theta = 0$. [3]

i) Setting $\theta + 2 = \theta^2$

$$0 = \theta^2 - \theta - 2$$
$$0 = \theta^2 - 2\theta + \theta - 2$$
$$0 = (\theta + 1)(\theta - 2)$$
$$\Rightarrow \theta = -1 \quad \text{or} \quad \theta = 2$$

(Ignore)

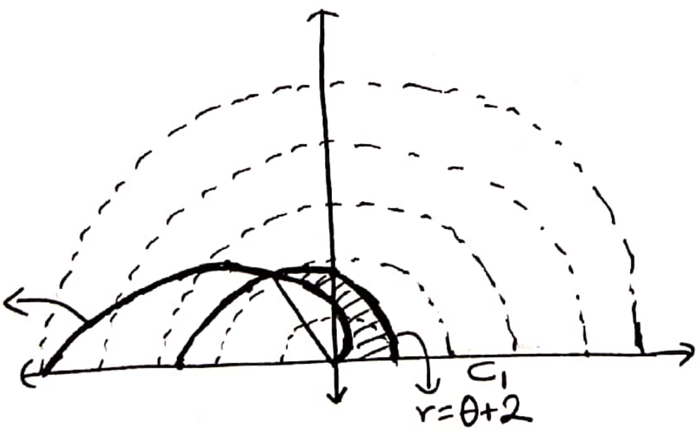
$\theta = 2 \Rightarrow r = 2 + 2 = 4$ ∴ Point of intersection = $(4, 2)$.

ii) $r = \theta + 2$

θ	0	$\frac{\pi}{2}$	π
r	2	3.57	5.14

$r = \theta^2$

θ	0	$\frac{\pi}{2}$	π
r	0	2.47	9.87



iii) $A = \frac{1}{2}\int_0^2 (\theta+2)^2 - (\theta^2)^2 \, d\theta = \frac{1}{2}\int_0^2 \theta^2 + 4\theta + 4 - \theta^4 \, d\theta$

$= \frac{1}{2}\left[\frac{\theta^3}{3} + 2\theta^2 + 4\theta - \frac{\theta^5}{5}\right]_0^2 = \frac{1}{2}\left[\left(\frac{8}{3} + 8 + 8 - \frac{32}{5}\right) - (0)\right]$

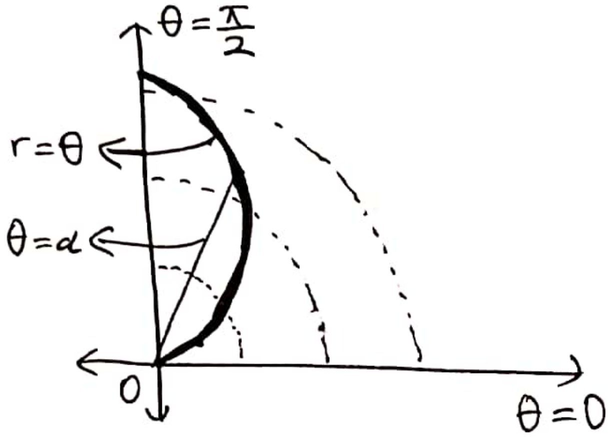
$A = \frac{92}{15}$.

7 Draw a sketch of the curve C whose polar equation is $r = \theta$, for $0 \leqslant \theta \leqslant \frac{\pi}{2}$ [2]

On the same diagram draw the line $\theta = \alpha$, where $0 < \alpha < \frac{1}{2}\pi$. [1]

The region bounded by C and the line $\theta = \frac{1}{2}\pi$ is denoted by R. Find the exact value of α for which the line $\theta = \alpha$ divides R into two regions of equal area. [4]

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{2\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
r	0	0.26	0.52	0.78	1.04	1.30	1.57



θ = α divides R into two equal halves $\Rightarrow$ $\frac{1}{2}\int_0^{\alpha} (\theta)^2 \, d\theta = \frac{1}{2}\int_{\alpha}^{\pi/2} (\theta)^2 \, d\theta$

$$\left[\frac{\theta^3}{3}\right]_0^{\alpha} = \left[\frac{\theta^3}{3}\right]_{\alpha}^{\pi/2}$$

$$\frac{\alpha^3}{3} - 0 = \frac{\pi^3}{24} - \frac{\alpha^3}{3}$$

$$\frac{2\alpha^3}{3} = \frac{\pi^3}{24}$$

$$\alpha^3 = \frac{3\pi^3}{48}$$

$$\alpha^3 = \frac{\pi^3}{16}$$

$$\alpha = \frac{\pi}{\sqrt[3]{16}} = \frac{\pi}{2\sqrt[3]{2}}$$

8 The curve C has polar equation

$$r = a(1 - e^{-\theta}),$$

where a is a positive constant and $0 \leqslant \theta < 2\pi$.

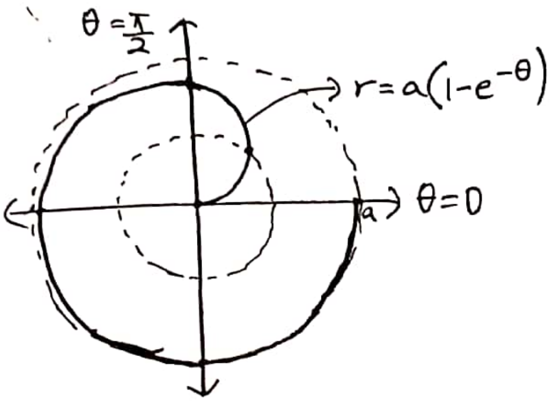
(i) Draw a sketch of C. [3]

(ii) Show that the area of the region bounded by C and the lines $\theta = \ln 2$ and $\theta = \ln 4$ is

$$\tfrac{1}{2}a^2\left(\ln 2 - \tfrac{13}{32}\right).$$ [4]

i)

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	0	$0.5a$	$0.8a$	$0.9a$	$0.96a$	$0.98a$	$0.99a$	$0.99a$	$0.99a$



ii) Area $= \frac{1}{2}\int_{\ln 2}^{\ln 4} \left[a\left(1-e^{-\theta}\right)\right]^2 d\theta = \frac{a^2}{2}\int_{\ln 2}^{\ln 4} 1 - 2e^{-\theta} + e^{-2\theta}\, d\theta$

$= \frac{a^2}{2}\left[\theta - \frac{2e^{-\theta}}{-1} + \frac{e^{-2\theta}}{-2}\right]_{\ln 2}^{\ln 4} = \frac{a^2}{2}\left[\theta + \frac{2}{e^{\theta}} \text{[redacted]} \frac{1}{2e^{2\theta}}\right]_{\ln 2}^{\ln 4}$

$= \frac{a^2}{2}\left[\left(\ln 4 + \frac{2}{e^{\ln 4}} \text{[redacted]} \frac{1}{2e^{2\ln 4}}\right) - \left(\ln 2 + \frac{2}{e^{\ln 2}} \text{[redacted]} \frac{1}{2e^{2\ln 2}}\right)\right]$

$= \frac{a^2}{2}\left[\ln 4 + \frac{2}{4} \text{[redacted]} \frac{1}{32} - \ln 2 - \frac{2}{2} + \frac{1}{8}\right]$

$= \frac{a^2}{2}\left[\ln\left(\frac{4}{2}\right) - \frac{13}{32}\right] = \frac{a^2}{2}\left(\ln 2 - \frac{13}{32}\right)$

9 The curve C has polar equation

$$r = \frac{a}{1+\theta},$$

where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(i) Show that r decreases as θ increases. [2]

(ii) The point P of C is further from the initial line than any other point of C. Show that, at P,

$$\tan\theta = 1 + \theta,$$

and verify that this equation has a root between 1.1 and 1.2. [4]

(iii) Draw a sketch of C. [3]

(iv) Find the area of the region bounded by the initial line, the line $\theta = \frac{1}{2}\pi$ and C, leaving your answer in terms of π and a. [3]

i) $\frac{dr}{d\theta} = -a(1+\theta)^{-2} \cdot 1 = \frac{-a}{(1+\theta)^2}$

Since a is positive, $\frac{-a}{(1+\theta)^2} < 0$ for all θ $\therefore \frac{dr}{d\theta} < 0$

So r decreases as θ increases.

ii) $y = r\sin\theta = \left(\frac{a}{1+\theta}\right)\sin\theta = a\left(\frac{\sin\theta}{1+\theta}\right)$

$\frac{dy}{d\theta} = a\left[\frac{(1+\theta)(\cos\theta) - \sin\theta(1)}{(1+\theta)^2}\right]$

For maximum value of y, $\frac{dy}{d\theta} = 0 \Rightarrow a\left[\frac{(1+\theta)\cos\theta - \sin\theta}{(1+\theta)^2}\right] = 0$

$\Rightarrow (1+\theta)\cos\theta = \sin\theta$

$1+\theta = \tan\theta$ (shown)

~~Let~~, $\tan\theta - \theta - 1 = 0$

Let, $f(\theta) = \tan\theta - \theta - 1$.

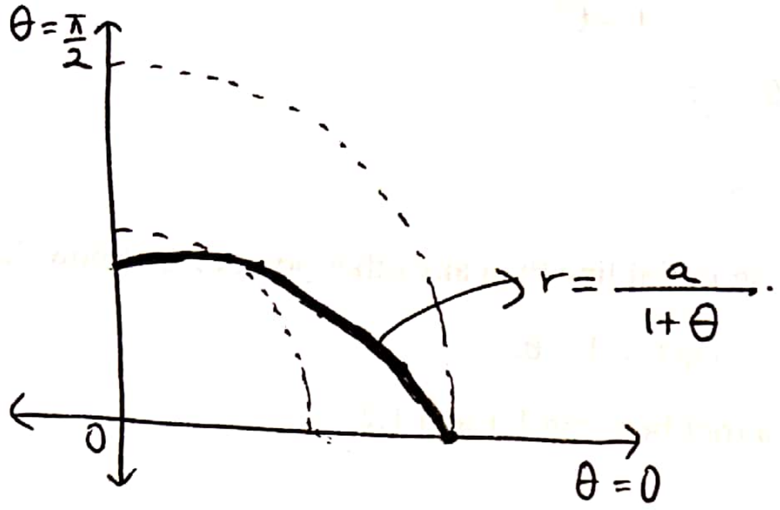
$f(1.1) = \tan(1.1) - 1.1 - 1 = -0.135$

$f(1.2) = \tan(1.2) - 1.2 - 1 = 0.372$

$f(1.1) < 0$ and $f(1.2) > 0 \therefore$ there is a root between 1.1 and 1.2.

iii)

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
r	$1a$	$0.79a$	$0.66a$	$0.56a$	$0.49a$	$0.43a$	$0.39a$



iv) $$\text{Area} = \frac{1}{2}\int_0^{\pi/2}\left(\frac{a}{1+\theta}\right)^2 d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} (1+\theta)^{-2}\, d\theta$$

$$= \frac{a^2}{2}\left[\frac{(1+\theta)^{-1}}{(-1)(1)}\right]_0^{\pi/2}$$

$$= \frac{a^2}{2}\left[\frac{-1}{1+\theta}\right]_0^{\pi/2}$$

$$= \frac{a^2}{2}\left[\frac{-1}{1+\frac{\pi}{2}} - \left(\frac{-1}{1+0}\right)\right]$$

$$= \frac{a^2}{2}\left[1-\frac{1}{1+\frac{\pi}{2}}\right] = \frac{a^2}{2}\left[1-\frac{1}{\frac{2+\pi}{2}}\right]$$

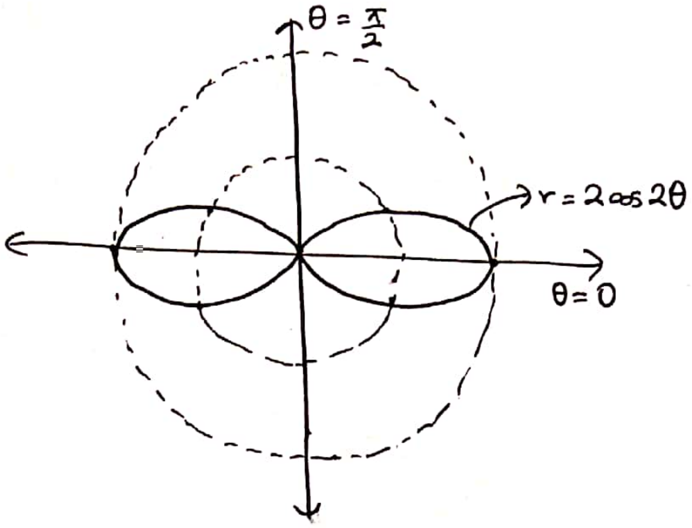
$$= \frac{a^2}{2}\left[1-\frac{2}{2+\pi}\right] = \frac{a^2}{2}\left[\frac{2+\pi-2}{2+\pi}\right]$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2+\pi}\right] = \frac{\pi a^2}{2(\pi+2)}.$$

10 The curve C has polar equation $r = 2\cos 2\theta$. Sketch the curve for $0 \leqslant \theta < 2\pi$. [4]

Find the exact area of one loop of the curve. [4]

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
r	2	1	0	0	1	2	1	0	0	1	2



Area of one loop

$$= 2\left[\frac{1}{2}\int_0^{\pi/4} (2\cos 2\theta)^2 \, d\theta\right] = 4\int_0^{\pi/4} \frac{1+\cos 4\theta}{2} \, d\theta$$

$$= 2\left[\theta + \frac{\sin 4\theta}{4}\right]_0^{\pi/4} = 2\left[\left(\frac{\pi}{4} + \frac{0}{4}\right) - 0\right] = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}.$$

11 The curves C_1 and C_2 have polar equations

$$C_1: \quad r = a,$$
$$C_2: \quad r = 2a\cos 2\theta, \text{ for } 0 \leqslant \theta \leqslant \tfrac{1}{4}\pi,$$

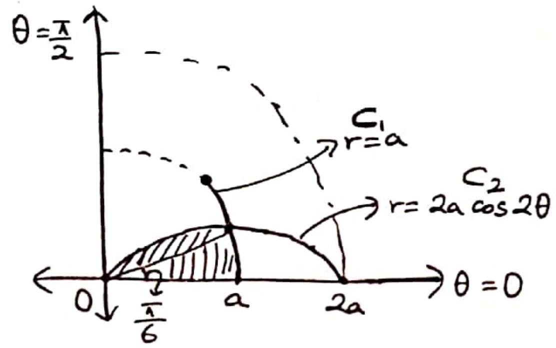
where a is a positive constant. Sketch C_1 and C_2 on the same diagram. [3]

The curves C_1 and C_2 intersect at the point with polar coordinates (a, β). State the value of β. [1]

Show that the area of the region bounded by the initial line, the arc of C_1 from $\theta = 0$ to $\theta = \beta$, and the arc of C_2 from $\theta = \beta$ to $\theta = \frac{1}{4}\pi$ is

$$a^2\left(\tfrac{1}{6}\pi - \tfrac{1}{8}\sqrt{3}\right).$$ [4]

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{8}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$
r	$2a$	$1{\cdot}73a$	$1{\cdot}41a$	a	0



Setting $\quad a = 2a\cos 2\theta \Rightarrow 2\cos 2\theta = 1 \Rightarrow \cos 2\theta = \frac{1}{2}$

$$2\theta = \frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{6}.$$

$$\therefore \beta = \frac{\pi}{6}.$$

Area of region

$$= \frac{a^2\left(\frac{\pi}{6}\right)}{2} + \frac{1}{2}\int_{\pi/6}^{\pi/4} (2a\cos 2\theta)^2\, d\theta$$

$$= \frac{\pi a^2}{12} + 2a^2\int_{\pi/6}^{\pi/4} \cos^2 2\theta\, d\theta = \frac{\pi a^2}{12} + 2a^2\int_{\pi/6}^{\pi/4} \frac{1+\cos 4\theta}{2}\, d\theta$$

$$= \frac{\pi a^2}{12} + a^2\int_{\pi/6}^{\pi/4} 1+\cos 4\theta\, d\theta = \frac{\pi a^2}{12} + a^2\left[\theta + \frac{\sin 4\theta}{4}\right]_{\pi/6}^{\pi/4}$$

$$= \frac{\pi a^2}{12} + a^2\left[\frac{\pi}{4} - \left(\frac{\pi}{6} + \frac{\sin\left(\frac{2\pi}{3}\right)}{4}\right)\right] = \frac{\pi a^2}{12} + a^2\left[\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right]$$

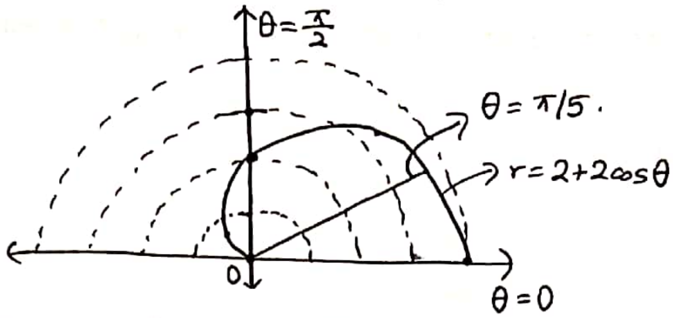
$$= \frac{\pi a^2}{12} + \frac{\pi a^2}{12} - \frac{\sqrt{3}a^2}{8} = \frac{\pi a^2}{6} - \frac{\sqrt{3}a^2}{8} = a^2\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right).$$

12 The curve C has polar equation $r = 2 + 2\cos\theta$, for $0 \leqslant \theta \leqslant \pi$. Sketch the graph of C. [2]

Find the area of the region R enclosed by C and the initial line. [4]

The half-line $\theta = \frac{1}{5}\pi$ divides R into two parts. Find the area of each part, correct to 3 decimal places. [3]

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
r	4	3.41	2	0.59	0



Area of region R $= \frac{1}{2}\int_0^{\pi} (2+2\cos\theta)^2 \, d\theta = 2\int_0^{\pi} 1+2\cos\theta+\cos^2\theta \, d\theta$

$= 2\int_0^{\pi} 1+2\cos\theta+\frac{1+\cos 2\theta}{2} \, d\theta = \int_0^{\pi} 3+4\cos\theta+\cos 2\theta \, d\theta$

$= \left[3\theta+4\sin\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi} = 3\pi-(0) = 3\pi.$

Area of part bound by C, $\theta=0$ and $\theta=\frac{\pi}{5}$

$= \frac{1}{2}\int_0^{\pi/5} (2+2\cos\theta)^2 \, d\theta = \left[3\theta+4\sin\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi/5}$

$= \left[\frac{3\pi}{5}+4\sin\left(\frac{\pi}{5}\right)+\frac{\sin\left(2\times\frac{\pi}{5}\right)}{2}\right] - 0 = 4.7116 \approx 4.712.$

Area of part bound by C, $\theta=\frac{\pi}{5}$ and $\theta=\frac{\pi}{2}$

$= 3\pi - 4.7116 = 4.7131 \approx 4.713$

13 The curve C has cartesian equation

$$(x^2 + y^2)^2 = a^2(x^2 - y^2),$$

where a is a positive constant. Show that C has polar equation

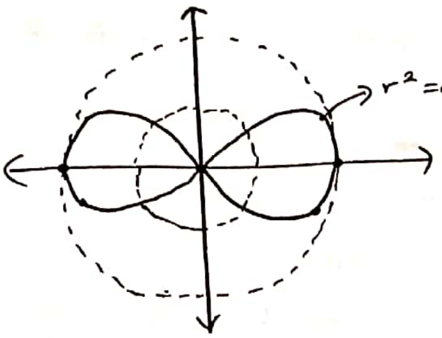
$$r^2 = a^2 \cos 2\theta.$$ [2]

Sketch C for $-\pi < \theta \leqslant \pi$. [2]

Find the area of the sector between $\theta = -\frac{1}{4}\pi$ and $\theta = \frac{1}{4}\pi$. [3]

Find the polar coordinates of all points of C where the tangent is parallel to the initial line. [7]

$$(x^2+y^2)^2 = a^2(x^2-y^2)$$
$$(r^2)^2 = a^2\left[r^2(\cos^2\theta - \sin^2\theta)\right]$$
$$r^4 = r^2a^2(\cos 2\theta)$$
$$r^2 = a^2 \cos 2\theta.$$



Area of sector between $\theta = \frac{-\pi}{4}$ and $\theta = \frac{\pi}{4}$

$$= 2\left[\frac{1}{2}\int_0^{\pi/4} a^2\cos 2\theta \, d\theta\right] = \int_0^{\pi/4} a^2\cos 2\theta \, d\theta = a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$

$$= a^2\left[\frac{\sin\left(2\times\frac{\pi}{4}\right)}{2} - \frac{\sin(2\times 0)}{2}\right] = a^2\left(\frac{1}{2} - 0\right) = \frac{a^2}{2}.$$

$$(x^2+y^2)^2 = a^2(x^2-y^2)$$

Differentiating gives: $2(x^2+y^2)\cdot(2x+2yy') = a^2(2x-2yy')$

To be parallel to initial line, $y' = 0$

$$\Rightarrow 2(x^2+y^2)(2x+2y(0)) = a^2(2x-2y(0))$$
$$2(x^2+y^2)(2x) = 2a^2x$$
$$2\left[2x^3+2xy^2\right] = 2a^2x$$
$$2\left[2x(x^2+y^2)\right] = 2x(a^2)$$
$$2\left[x^2+y^2\right] = a^2$$
$$\Rightarrow 2r^2 = a^2 \Rightarrow r = \frac{a}{\sqrt{2}}.$$

Then $\left(\frac{a}{\sqrt{2}}\right)^2 = a^2\cos 2\theta$

$$\frac{a^2}{2} = a^2 \cos 2\theta$$

$$\frac{1}{2} = \cos 2\theta$$

$$\Rightarrow \quad 2\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{-\pi}{3}, \frac{-5\pi}{3}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{-\pi}{6}, \frac{-5\pi}{6}.$$

$\Rightarrow$ Points where tangent is parallel to initial line:

$$\left(\frac{a}{\sqrt{2}}, \frac{\pi}{6}\right), \left(\frac{a}{\sqrt{2}}, \frac{5\pi}{6}\right), \left(\frac{a}{\sqrt{2}}, \frac{-\pi}{6}\right), \left(\frac{a}{\sqrt{2}}, \frac{-5\pi}{6}\right).$$

$$r = a\sqrt{\cos 2\theta} \Rightarrow y = r\sin\theta = a\sqrt{\cos 2\theta}\,\sin\theta.$$

$$\frac{dy}{d\theta} = a\left[\sqrt{\cos 2\theta}\,(\cos\theta) + \sin\theta\,\frac{1}{\cancel{2}\sqrt{\cos 2\theta}}\cdot -\sin 2\theta\cdot\cancel{2}\right]$$

$$\frac{dy}{d\theta} = a\left[\frac{(\cos 2\theta)(\cos\theta) - \sin\theta\sin 2\theta}{\sqrt{\cos 2\theta}}\right].$$

For tangent to be parallel to initial line,

$$\frac{dy}{d\theta} = 0 \Rightarrow a\left[\frac{\cos 2\theta\cos\theta - \sin\theta\sin 2\theta}{\sqrt{\cos 2\theta}}\right] = 0$$

$$\cos 2\theta\cos\theta - \sin\theta\sin 2\theta = 0$$

$$\cos\theta\left[\cos 2\theta - 2\sin^2\theta\right] = 0$$

$$\cos\theta\left[1 - 2\sin^2\theta - 2\sin^2\theta\right] = 0$$

$$\cos\theta = 0 \quad \text{or} \quad 1 - 4\sin^2\theta = 0$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2} \qquad\qquad \sin\theta = \pm\frac{1}{2}.$$

(Ignore/cusps)

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{-\pi}{6}, \frac{-5\pi}{6}.$$

$$\theta = \frac{\pi}{6} \Rightarrow r = a\sqrt{\cos\left(\frac{2\pi}{6}\right)} = \frac{a}{\sqrt{2}}$$

$$\theta = \frac{5\pi}{6} \Rightarrow r = a\sqrt{\cos\left(\frac{5\pi}{6}\right)} = \frac{a}{\sqrt{2}}.$$

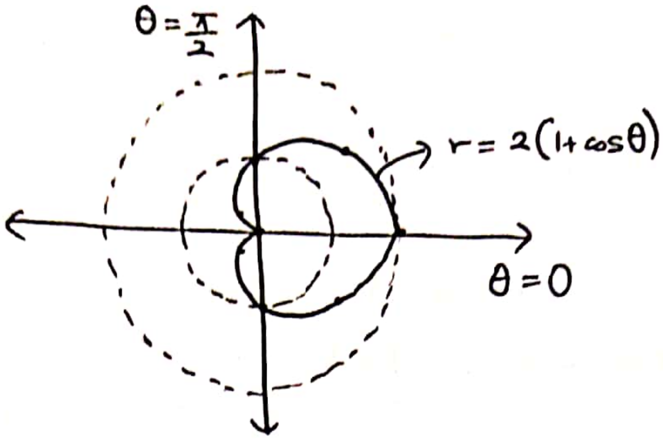
$$\theta = \frac{-\pi}{6} \Rightarrow r = a\sqrt{\cos\left(\frac{-\pi}{6}\right)} = \frac{a}{\sqrt{2}}$$

$$\theta = \frac{-5\pi}{6} \Rightarrow r = a\sqrt{\cos\left(\frac{-5\pi}{6}\right)} = \frac{a}{\sqrt{2}}.$$

$\Rightarrow$ Points where tangent is parallel to initial line:

$$\left(\frac{a}{\sqrt{2}}, \frac{\pi}{6}\right), \left(\frac{a}{\sqrt{2}}, \frac{5\pi}{6}\right), \left(\frac{a}{\sqrt{2}}, \frac{-\pi}{6}\right), \left(\frac{a}{\sqrt{2}}, \frac{-5\pi}{6}\right).$$

14 Find the area of the region enclosed by the curve with polar equation $r = 2(1+\cos\theta)$, for $0 \leqslant \theta < 2\pi$. [4]



$$\text{Area} = 2\left[\frac{1}{2}\int_0^{\pi} \left[2(1+\cos\theta)\right]^2 d\theta\right]$$

$$= 4\int_0^{\pi} 1 + 2\cos\theta + \cos^2\theta \, d\theta$$

$$= 4\int_0^{\pi} 1 + 2\cos\theta + \frac{1+\cos 2\theta}{2} \, d\theta$$

$$= 4\int_0^{\pi} \frac{3 + 2\cos\theta + \cos 2\theta}{2} \, d\theta$$

$$= 2\left[3\theta + 2\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= 2\left[(3\pi + 0 + 0) - (0)\right]$$

$$\text{Area} = 6\pi$$

15 Use the identity $2\sin P\cos Q \equiv \sin(P+Q) + \sin(P-Q)$ to show that

$$2\sin\theta\cos(\theta - \tfrac{1}{4}\pi) \equiv \cos(2\theta - \tfrac{3}{4}\pi) + \frac{1}{\sqrt{2}}.$$ [3]

A curve has polar equation $r = 2\sin\theta\cos(\theta - \frac{1}{4}\pi)$, for $0 \leqslant \theta \leqslant \frac{3}{4}\pi$. Sketch the curve and state the polar equation of its line of symmetry, justifying your answer. [3]

Show that the area of the region enclosed by the curve is $\frac{3}{8}(\pi + 1)$. [6]

$$2\sin\theta\cos\left(\theta - \frac{1}{4}\pi\right) \equiv \sin\left(\theta + \theta - \frac{1}{4}\pi\right) + \sin\left(\theta - \theta + \frac{\pi}{4}\right)$$

$$\equiv \sin\left(2\theta - \frac{\pi}{4}\right) + \sin\left(+\frac{\pi}{4}\right)$$

$$\equiv \cos\left(\frac{\pi}{2} - \left(2\theta - \frac{\pi}{4}\right)\right) + \frac{1}{\sqrt{2}}$$

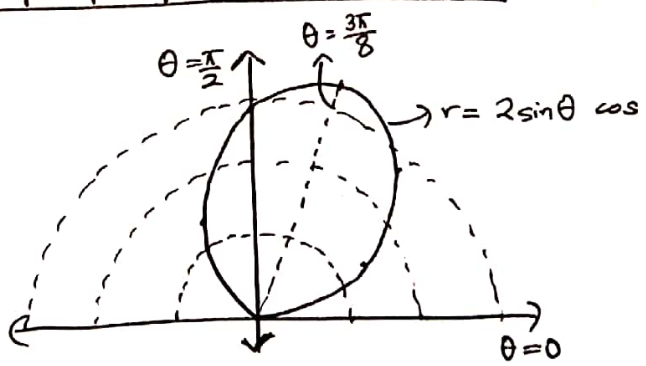
$$\equiv \cos\left(\frac{3\pi}{4} - 2\theta\right) + \frac{1}{\sqrt{2}}$$

$$\equiv \cos\left[-\left(2\theta - \frac{3\pi}{4}\right)\right] + \frac{1}{\sqrt{2}}$$

$$\equiv \cos\left(2\theta - \frac{3\pi}{4}\right) + \frac{1}{\sqrt{2}}.$$

$$r = 2\sin\theta\cos\left(\theta - \frac{1}{4}\pi\right) \equiv \cos\left(2\theta - \frac{3\pi}{4}\right) + \frac{1}{\sqrt{2}}.$$

θ	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$3\pi/4$
r	0	0.71	1.41	1.71	1.41	0.71	0



Line of symmetry : $\theta = \dfrac{3\pi}{8}$.

Justification : At line of symmetry, r is maximum. This happens when $\cos\left(2\theta - \frac{3\pi}{4}\right)$ is maximum $\Rightarrow \cos\left(2\theta - \frac{3\pi}{4}\right) = 1$

$$\Rightarrow 2\theta - \frac{3\pi}{4} = 0$$

$$2\theta = \frac{3\pi}{4}$$

$$\theta = \frac{3\pi}{8}.$$

Area enclosed by curve

$$= \frac{1}{2}\int_0^{3\pi/4} \left[\cos\left(2\theta - \frac{3\pi}{4}\right) + \frac{1}{\sqrt{2}}\right]^2 d\theta$$

$$= \frac{1}{2}\int_0^{3\pi/4} \cos^2\left(2\theta - \frac{3\pi}{4}\right) + 2\cos\left(2\theta - \frac{3\pi}{4}\right) \times \frac{1}{\sqrt{2}} + \left(\frac{1}{\sqrt{2}}\right)^2 d\theta$$

$$= \frac{1}{2}\int_0^{3\pi/4} \frac{1 + \cos\left[2 \times \left(2\theta - \frac{3\pi}{4}\right)\right]}{2} + \frac{2}{\sqrt{2}}\cos\left(2\theta - \frac{3\pi}{4}\right) + \frac{1}{2}\, d\theta$$

$$= \frac{1}{2}\int_0^{3\pi/4} 1 + \frac{\cos\left(4\theta - \frac{3\pi}{2}\right)}{2} + \sqrt{2}\cos\left(2\theta - \frac{3\pi}{4}\right) d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin\left(4\theta - \frac{3\pi}{2}\right)}{8} + \frac{\sqrt{2}\sin\left(2\theta - \frac{3\pi}{4}\right)}{2}\right]_0^{3\pi/4}$$

$$= \frac{1}{2}\left[\left(\frac{3\pi}{4} + \cancel{\frac{\sqrt{2}}{16}} - \frac{1}{8} + \frac{1}{2}\right) - \left(0 + \frac{1}{8} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{3\pi}{4} + \frac{3}{4}\right] = \frac{3\pi}{8} + \frac{3}{8} = \frac{3}{8}(\pi + 1).$$

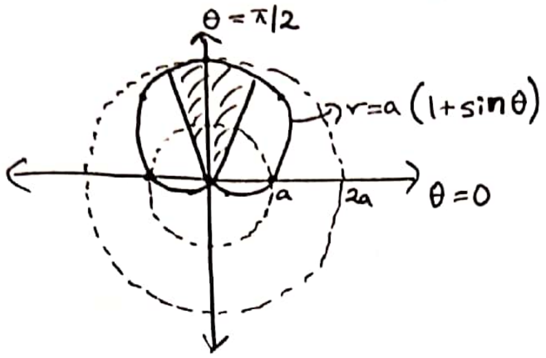
Scanned with CamScanner

16 The curve C has polar equation $r = a(1 + \sin\theta)$, where a is a positive constant and $0 \leqslant \theta < 2\pi$. Draw a sketch of C. [2]

Find the exact value of the area of the region enclosed by C and the half-lines $\theta = \frac{1}{3}\pi$ and $\theta = \frac{2}{3}\pi$. [4]

$r = a(1+\sin\theta)$

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	a	$2a$	a	0	a



$$\text{Area} = 2\left[\frac{1}{2}\int_{\pi/3}^{\pi/2}\left[a(1+\sin\theta)\right]^2 d\theta\right]$$

$$= a^2\int_{\pi/3}^{\pi/2} 1+2\sin\theta+\sin^2\theta\, d\theta = a^2\int_{\pi/3}^{\pi/2} 1+2\sin\theta+\frac{1-\cos 2\theta}{2}\, d\theta$$

$$= a^2\int_{\pi/3}^{\pi/2}\frac{3+4\sin\theta-\cos 2\theta}{2}\, d\theta = \frac{a^2}{2}\left[3\theta-4\cos\theta-\frac{\sin 2\theta}{2}\right]_{\pi/3}^{\pi/2}$$

$$= \frac{a^2}{2}\left[\left(\frac{3\pi}{2}-0\right)-\left(\pi-2-\frac{\sqrt{3}}{4}\right)\right] = \frac{a^2}{2}\left[\frac{\pi}{2}+2+\frac{\sqrt{3}}{4}\right]$$

$$\text{Area} = a^2\left[\frac{\pi}{4}+1+\frac{\sqrt{3}}{8}\right]$$

17 The curve C has cartesian equation $(x^2+y^2)^2 = 2a^2xy$, where a is a positive constant. Show that the polar equation of C is $r^2 = a^2 \sin 2\theta$. [3]

Sketch C for $-\pi < \theta \leqslant \pi$. [2]

Find the area enclosed by one loop of C. [2]

$$(x^2+y^2)^2 = 2a^2xy$$

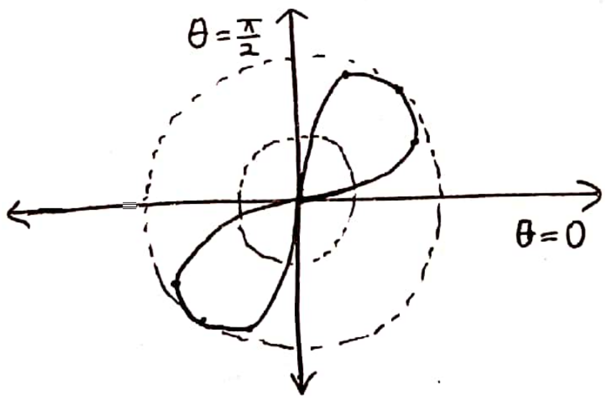
$$(r^2)^2 = 2a^2(r\cos\theta)(r\sin\theta)$$

$$r^4 = 2a^2r^2\sin\theta\cos\theta$$

$$r^2 = a^2\sin 2\theta. \text{ (Shown)}.$$

$$r = a\sqrt{\sin 2\theta}.$$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$-\pi$	$-\frac{5\pi}{6}$	$-\frac{3\pi}{4}$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$
r	0	$0.93a$	a	$0.93a$	0	0	$0.93a$	a	$0.93a$	0



Area of one loop

$$= \frac{1}{2}\int_0^{\pi/2} a^2\sin 2\theta \, d\theta = \frac{a^2}{2}\int_0^{\pi/2}\sin 2\theta \, d\theta = \frac{a^2}{2}\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{-a^2}{4}\left[\cos\left(2\times\frac{\pi}{2}\right) - \cos(2\times 0)\right] = \frac{-a^2}{4}\left[-1-1\right] = \frac{2a^2}{4} = \frac{a^2}{2}.$$

18 The curves C_1 and C_2 have polar equations

$$C_1: \quad r = \frac{1}{\sqrt{2}}, \quad \text{for } 0 \leqslant \theta < 2\pi,$$
$$C_2: \quad r = \sqrt{(\sin \tfrac{1}{2}\theta)}, \quad \text{for } 0 \leqslant \theta \leqslant \pi.$$

Find the polar coordinates of the point of intersection of C_1 and C_2. [2]

Sketch C_1 and C_2 on the same diagram. [3]

Find the exact value of the area of the region enclosed by C_1, C_2 and the half-line $\theta = 0$. [4]

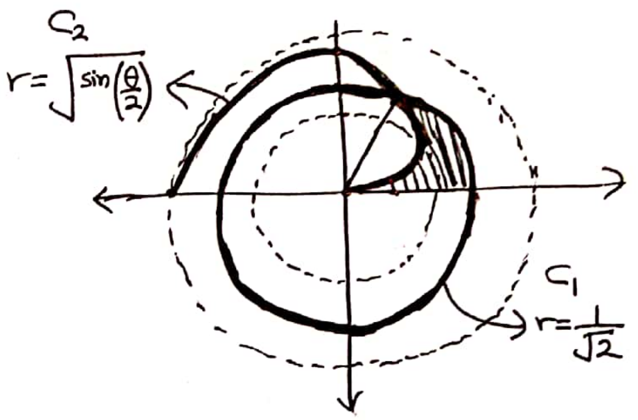
Setting $\frac{1}{\sqrt{2}} = \sqrt{\sin\left(\frac{1}{2}\theta\right)}$

$$\frac{1}{2} = \sin\left(\frac{\theta}{2}\right)$$
$$\frac{\theta}{2} = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$
$$\theta = \frac{\pi}{3}.$$

When $\theta = \frac{\pi}{3}$, $r = \frac{1}{\sqrt{2}}$ $\Rightarrow$ Point of intersection: $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{3}\right)$.

$r = \sqrt{\sin(\theta/2)}$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	0	0.51	0.71	0.84	0.93	0.98	1



Area of region

$$= \frac{\left(\frac{1}{\sqrt{2}}\right)^2 \times \frac{\pi}{3}}{2} - \frac{1}{2}\int_0^{\pi/3} \left[\sqrt{\sin\left(\frac{1}{2}\theta\right)}\right]^2 d\theta = \frac{\pi}{12} - \frac{1}{2}\int_0^{\pi/3} \sin\left(\frac{\theta}{2}\right) d\theta$$

$$= \frac{\pi}{12} - \frac{1}{2}\left[\frac{-\cos\left(\frac{\theta}{2}\right)}{\frac{1}{2}}\right]_0^{\pi/3} = \frac{\pi}{12} + \left[\cos\left(\frac{\theta}{2}\right)\right]_0^{\pi/3} = \frac{\pi}{12} + \left[\cos\left(\frac{\pi}{6}\right) - \cos(0)\right]$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1.$$

19 The curve C has polar equation $r = e^{4\theta}$ for $0 \leqslant \theta \leqslant \alpha$, where α is measured in radians. The length of C is 2015. Find the value of α. [6]

$$\text{Arc length} = \int_0^{\alpha} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$2015 = \int_0^{\alpha} \sqrt{e^{8\theta} + (4e^{4\theta})^2}\, d\theta$$

$$2015 = \int_0^{\alpha} \sqrt{e^{8\theta} + 16e^{8\theta}}\, d\theta$$

$$2015 = \int_0^{\alpha} \sqrt{17e^{8\theta}}\, d\theta$$

$$2015 = \sqrt{17} \int_0^{\alpha} e^{4\theta}\, d\theta$$

$$\frac{2015}{\sqrt{17}} = \left[\frac{e^{4\theta}}{4}\right]_0^{\alpha}$$

$$\frac{2015 \times 4}{\sqrt{17}} = e^{4\alpha} - 1$$

$$1955.84 = e^{4\alpha}$$

$$\Rightarrow 4\alpha = \ln(1955.84)$$

$$\alpha = 1.894 \approx 1.89$$

Scanned with CamScanner

20 A curve C has polar equation $r^2 = 8\operatorname{cosec} 2\theta$ for $0 < \theta < \frac{\pi}{2}$. Find a cartesian equation of C. [3]

Sketch C. [2]

Determine the exact area of the sector bounded by the arc of C between $\theta = \frac{1}{6}\pi$ and $\theta = \frac{1}{3}\pi$, the half-line $\theta = \frac{1}{6}\pi$ and the half-line $\theta = \frac{1}{3}\pi$. [3]

[It is given that $\int \operatorname{cosec} x \, dx = \ln\left|\tan \frac{1}{2}x\right| + c$.]

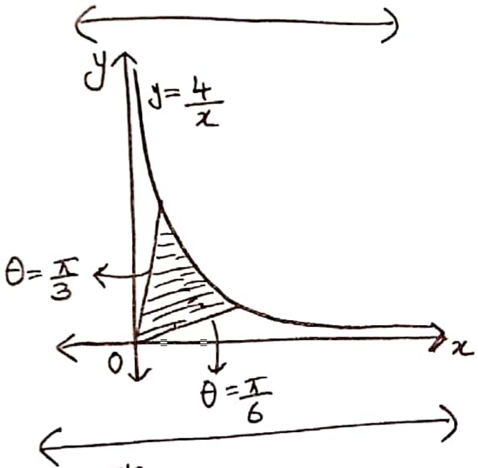
$$r^2 = 8\operatorname{cosec} 2\theta$$

$$r^2 = \frac{8}{\sin 2\theta} = \frac{8}{2\sin\theta\cos\theta}$$

$$\cancel{2}r^2 \sin\theta\cos\theta = 4$$

$$r^2\left(\frac{x}{r}\right)\left(\frac{y}{r}\right) = 4$$

$$xy = 4 \Rightarrow y = \frac{4}{x}.$$



$$\text{Area} = \frac{1}{2}\int_{\pi/6}^{\pi/3} 8\operatorname{cosec} 2\theta \, d\theta$$

$$u = 2\theta$$

$$\frac{du}{d\theta} = 2 \Rightarrow \cdots du = \cdots 2d\theta \Rightarrow \cdots d\theta = \cdots \frac{du}{2}$$

$$\theta = \frac{\pi}{6} \Rightarrow u = 2\theta = \frac{2\pi}{6} = \frac{\pi}{3}.$$

$$\theta = \frac{\pi}{3} \Rightarrow u = 2\theta = \frac{2\pi}{3}$$

$$= \frac{1}{2}\int_{\pi/6}^{\pi/3} 8\operatorname{cosec} 2\theta \, d\theta = 4\int_{\pi/3}^{2\pi/3} \operatorname{cosec} u \frac{du}{2}$$

$$= 2\int_{\pi/3}^{2\pi/3} \operatorname{cosec} u \, du = 2\left[\ln\left|\tan\frac{1}{2}u\right|\right]_{\pi/3}^{2\pi/3}$$

$$= 2\left[\ln\left|\tan\left(\frac{1}{2}\times\frac{2\pi}{3}\right)\right| - \ln\left|\tan\left(\frac{1}{2}\times\frac{\pi}{3}\right)\right|\right] = 2\left[\ln\left|\tan\frac{\pi}{3}\right| - \ln\left|\tan\frac{\pi}{6}\right|\right]$$

$$= 2\ln 3 = \ln 9.$$

21 A curve has polar equation $r = \dfrac{1}{1-\cos\theta}$, for $0 < \theta < 2\pi$. Find, in the form $y^2 = f(x)$, the cartesian equation of the curve. [3]

Hence sketch the curve, and shade the region whose area is given by $\frac{1}{2}\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{1}{(1-\cos\theta)^2}\,d\theta$. [3]

Using the cartesian equation of the curve, find the area of this region. [3]

$$r = \frac{1}{1-\cos\theta}$$

$$r(1-\cos\theta) = 1$$

$$r\left(1-\frac{x}{r}\right) = 1$$

$$r\left(\frac{r-x}{r}\right) = 1$$

$$\frac{r^2 - xr}{r} = 1$$

$$r^2 - xr = r$$

$$r^2 = r + xr$$

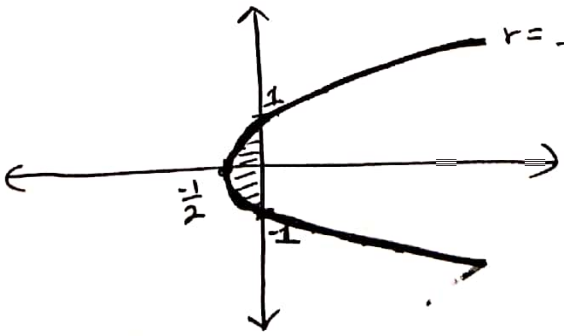
$$r^2 = (1+x)r$$

$$x^2+y^2 = (1+x)\sqrt{x^2+y^2}$$

$$\sqrt{x^2+y^2} = 1+x$$

$$x^2+y^2 = (1+x)^2 \Rightarrow y^2 = 1+2x+x^2-x^2$$

$$y^2 = 1+2x.$$



Shaded region has area $\frac{1}{2}\int_{\pi/2}^{3\pi/2} \frac{1}{(1-\cos\theta)^2}\,d\theta$.

Area of region

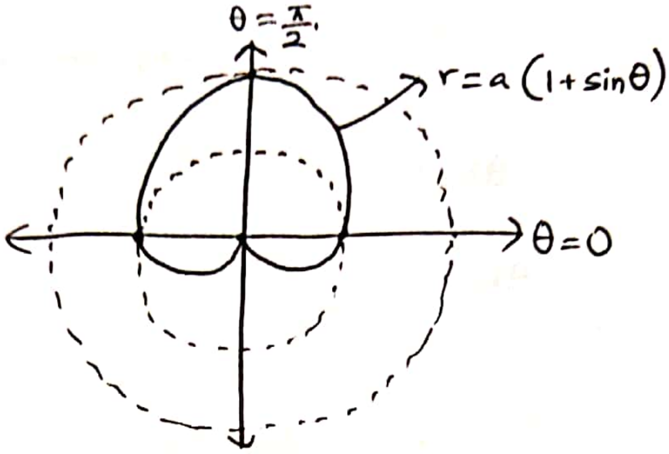
$$= 2\int_{-\frac{1}{2}}^{0} \sqrt{1+2x}\,dx = 2\int_{-1/2}^{0}(1+2x)^{1/2}dx = 2\left[\frac{(1+2x)^{3/2}}{\left(\frac{3}{2}\right)(2)}\right]_{-1/2}^{0}$$

$$= 2\left[\frac{(1+2x)^{3/2}}{3}\right]_{-1/2}^{0} = \frac{2}{3}\left[\left[1+2(0)\right]^{3/2} - \left[1+2\times\frac{-1}{2}\right]^{3/2}\right] = \frac{2}{3}\left[1-0\right] = \frac{2}{3}.$$

22 The curve C has polar equation $r = a(1 + \sin\theta)$ for $-\pi < \theta \leqslant \pi$, where a is a positive constant.

(i) Sketch C. [2]

θ	$-\pi$	$-\pi/2$	0	$\pi/2$	π
r	a	0	a	$2a$	a



(ii) Find the area of the region enclosed by C. [4]

$$\text{Area} = \frac{1}{2}\int_{-\pi}^{\pi} \left[a(1+\sin\theta)\right]^2 d\theta$$

$$= \frac{1}{2}a^2\int_{-\pi}^{\pi} 1+2\sin\theta+\sin^2\theta \, d\theta$$

$$= \frac{a^2}{2}\int_{-\pi}^{\pi} 1+2\sin\theta+\frac{1-\cos 2\theta}{2} \, d\theta$$

$$= \frac{a^2}{2}\int_{-\pi}^{\pi} \frac{3+4\sin\theta-\cos 2\theta}{2} \, d\theta$$

$$= \frac{a^2}{4}\left[3\theta-4\cos\theta-\frac{\sin 2\theta}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{a^2}{4}\left[(3\pi+4)-(-3\pi+4)\right] = \frac{a^2}{4}\left[3\pi+4+3\pi-4\right]$$

$$= \frac{6\pi a^2}{4} = \frac{3\pi a^2}{2}.$$

(iii) Show that the length of the arc of C from the pole to the point furthest from the pole is given by

$$s = (\sqrt{2})a \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sqrt{(1 + \sin\theta)}\, d\theta. \qquad [3]$$

At ~~r = θ,~~ pole, $r = 0 \Rightarrow a(1+\sin\theta) = 0 \Rightarrow \sin\theta = -1 \Rightarrow \theta = \frac{-\pi}{2}$.

At point furthest from pole, $r = 2a \Rightarrow a(1+\sin\theta) = 2a \Rightarrow \sin\theta = 1$

$\Rightarrow \theta = \frac{\pi}{2}$.

$\Rightarrow$ length of arc $= \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$$S = \int_{-\pi/2}^{\pi/2} \sqrt{[a(1+\sin\theta)]^2 + [a(\cos\theta)]^2}\, d\theta$$

$$S = \int_{-\pi/2}^{\pi/2} \sqrt{a^2 + a^2\sin^2\theta + 2a^2\sin\theta + a^2\cos^2\theta}\, d\theta$$

$$S = \int_{-\pi/2}^{\pi/2} \sqrt{a^2 + a^2(\sin^2\theta + \cos^2\theta) + 2a^2\sin\theta}\, d\theta$$

$$S = \int_{-\pi/2}^{\pi/2} \sqrt{2a^2 + 2a^2\sin\theta}\, d\theta = \int_{-\pi/2}^{\pi/2} \sqrt{2a^2(1+\sin\theta)}\, d\theta$$

$$S = \sqrt{2}\, a \int_{-\pi/2}^{\pi/2} \sqrt{1 + \sin\theta}\, d\theta.$$

(iv) Show that the substitution $u = 1 + \sin\theta$ reduces this integral for s to $(\sqrt{2})a \int_0^2 \frac{1}{\sqrt{(2-u)}} \, du$. Hence evaluate s.

[4]

$$S = \sqrt{2}a \int_{-\pi/2}^{\pi/2} \sqrt{1+\sin\theta} \, d\theta .$$

$$u = 1 + \sin\theta \Rightarrow \sin\theta = u - 1 \Rightarrow \cos\theta = \sqrt{1-(u-1)^2} = \sqrt{2u-u^2}$$

$$\frac{du}{d\theta} = \cos\theta \Rightarrow d\theta = \frac{du}{\cos\theta} = \frac{du}{\sqrt{2u-u^2}} .$$

$$\theta = \frac{-\pi}{2} \Rightarrow u = 1 + \sin\left(\frac{-\pi}{2}\right) = 0$$

$$\theta = \frac{\pi}{2} \Rightarrow u = 1 + \sin\left(\frac{\pi}{2}\right) = 2 .$$

$$\Rightarrow S = \sqrt{2}a \int_0^2 \sqrt{u} \frac{du}{\sqrt{2u-u^2}} = \sqrt{2}a \int_0^2 \frac{\sqrt{u}}{\sqrt{u}\sqrt{2-u}} du$$

$$= \sqrt{2}a \int_0^2 \frac{1}{\sqrt{2-u}} du = \sqrt{2}a \int_0^2 (2-u)^{-1/2} du$$

$$= \sqrt{2}a \left[\frac{(2-u)^{\frac{1}{2}}}{(\frac{1}{2})(-1)} \right]_0^2$$

$$= \sqrt{2}a(-2)\left[\sqrt{2-2} - \sqrt{2-0}\right]$$

$$= -2\sqrt{2}a\left(0 - \sqrt{2}\right) = 2(\sqrt{2})^2 a$$

$$S = 4a$$

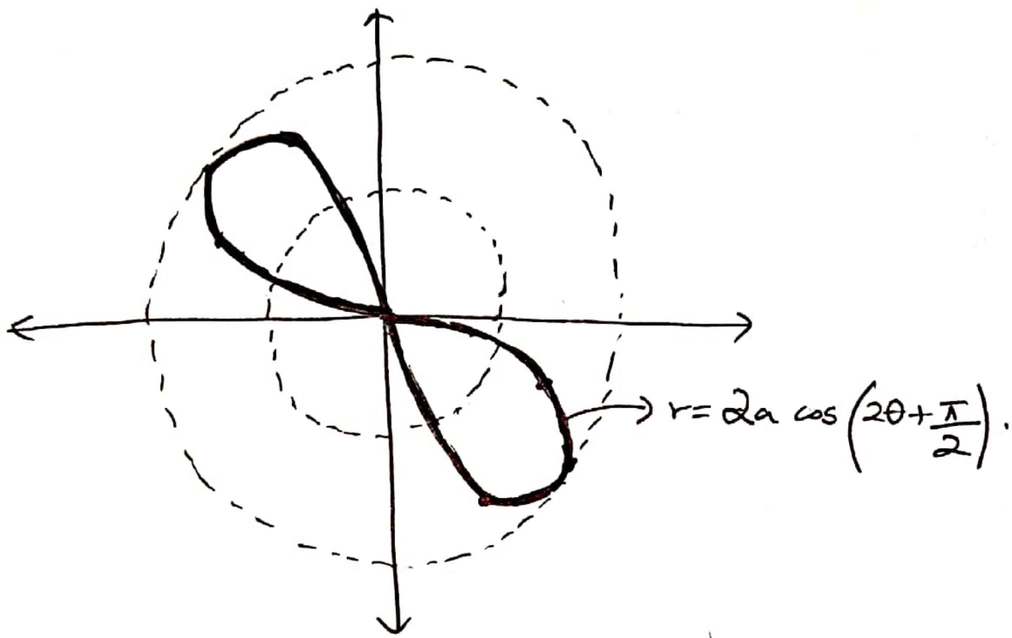
23 A curve C has polar equation $r = 2a\cos(2\theta + \frac{1}{2}\pi)$ for $0 \leqslant \theta < 2\pi$, where a is a positive constant.

(i) Show that $r = -2a\sin 2\theta$ and sketch C. [4]

$$r = 2a\cos\left(2\theta + \frac{\pi}{2}\right) = 2a\left[\cos 2\theta \cos\frac{\pi}{2} - \sin 2\theta \sin\frac{\pi}{2}\right]$$

$$r = 2a\left[\cos 2\theta (0) - \sin 2\theta (1)\right] = -2a\sin 2\theta.$$

θ	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
r	0	1.7a	2a	1.7a	0	0	1.7a	2a	1.7a	0



(ii) Deduce that the cartesian equation of C is

$$(x^2 + y^2)^{\frac{3}{2}} = -4axy.$$ [2]

$$r = -2a\sin 2\theta$$

$$r = -2a(2\sin\theta\cos\theta)$$

$$r = -4a\left(\frac{y}{r}\right)\left(\frac{x}{r}\right)$$

$$r^3 = -4axy$$

$$(x^2+y^2)^{\frac{3}{2}} = -4axy.$$

(iii) Find the area of one loop of C. [5]

Area of one loop

$$= \frac{1}{2}\int_{\pi/2}^{\pi} (-2a\sin 2\theta)^2 \, d\theta = 2a^2 \int_{\pi/2}^{\pi} \sin^2 2\theta \, d\theta$$

$$= 2a^2 \int_{\pi/2}^{\pi} \frac{1-\cos 4\theta}{2} \, d\theta = a^2 \int_{\pi/2}^{\pi} 1 - \cos 4\theta \, d\theta$$

$$= a^2 \left[\theta - \frac{\sin 4\theta}{4}\right]_{\pi/2}^{\pi} = a^2 \left[(\pi - 0) - \left(\frac{\pi}{2} - 0\right)\right]$$

$$= a^2\left(\frac{\pi}{2}\right) = \frac{\pi a^2}{2}.$$

(iv) Show that, at the points (other than the pole) at which a tangent to C is parallel to the initial line,

$$2 \tan \theta = -\tan 2\theta. \qquad [3]$$

$$y = r\sin\theta = -2a\sin 2\theta \sin\theta$$

$$\frac{dy}{d\theta} = -2a\left[\sin 2\theta (\cos\theta) + \sin\theta (\cos 2\theta) \cdot 2\right]$$

$$\frac{dy}{d\theta} = -2a\left[\sin 2\theta \cos\theta + 2\cos 2\theta \sin\theta\right]$$

For tangent to be parallel to initial line,

$$\frac{dy}{d\theta} = 0 \Rightarrow -2a\left[\sin 2\theta \cos\theta + 2\cos 2\theta \sin\theta\right] = 0$$

$$\sin 2\theta \cos\theta + 2\cos 2\theta \sin\theta = 0$$

$$\sin 2\theta \cos\theta = -2\cos 2\theta \sin\theta$$

$$\tan 2\theta = -2\tan\theta$$

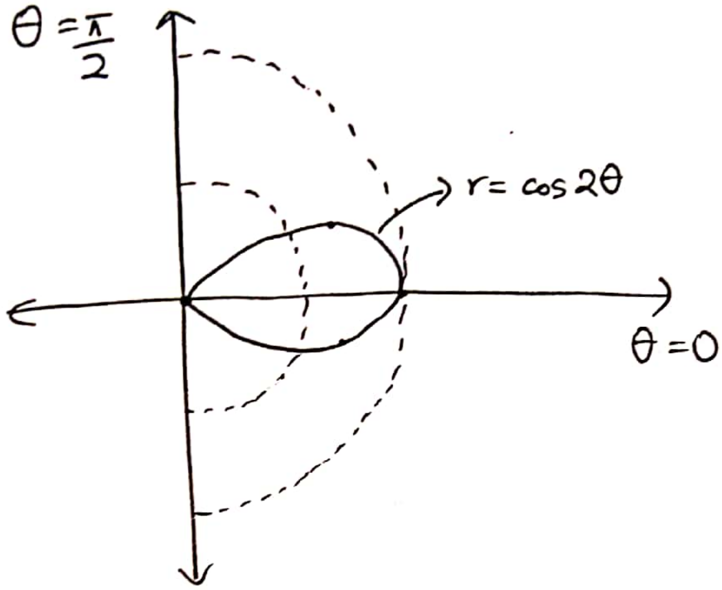
$$-2\tan 2\theta = 2\tan\theta$$

(Shown).

24 The curve C has polar equation $r = \cos 2\theta$, for $-\frac{1}{4}\pi \leqslant \theta \leqslant \frac{1}{4}\pi$

(i) Sketch C. [2]

θ	$-\frac{\pi}{4}$	$-\frac{\pi}{8}$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$
r	0	0.71	1	0.71	0



(ii) Find the area of the region enclosed by C, showing full working. [3]

$$\text{Area} = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta \, d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \frac{1+\cos 4\theta}{2} \, d\theta$$

$$= \frac{1}{4}\int_{-\pi/4}^{\pi/4} 1+\cos 4\theta \, d\theta = \frac{1}{4}\left[\theta + \frac{\sin 4\theta}{4}\right]_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{4}\left[\left(\frac{\pi}{4}+0\right) - \left(\frac{-\pi}{4}+0\right)\right] = \frac{1}{4}\left[\frac{\pi}{2}\right] = \frac{\pi}{8}.$$

(iii) Find a cartesian equation of C. [3]

$$r = \cos 2\theta$$

$$r = \cos^2\theta - \sin^2\theta$$

$$r = \left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2$$

$$r = \frac{x^2 - y^2}{r^2}$$

$$r^3 = x^2 - y^2$$

$$\left(\sqrt{x^2+y^2}\right)^3 = x^2 - y^2$$

$$(x^2+y^2)^{\frac{3}{2}} = x^2 - y^2.$$

25 The curves C_1 and C_2 have polar equations, for $0 \leqslant \theta \leqslant \pi$, as follows:

$$C_1 : r = a,$$
$$C_2 : r = 2a|\cos\theta|,$$

where a is a positive constant. The curves intersect at the points P_1 and P_2.

(i) Find the polar coordinates of P_1 and P_2. [2]

Setting $2a|\cos\theta| = a$

$$|\cos\theta| = \frac{1}{2}$$

$$\cos\theta = \pm\frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{3}, \frac{2\pi}{3}.$$

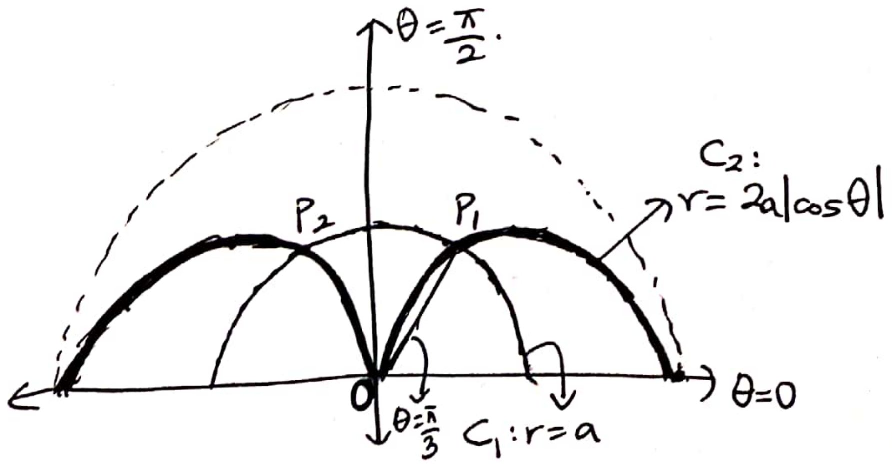
$\theta = \frac{\pi}{3}$, $r = a$, and $\theta = \frac{2\pi}{3}$, $r = a$.

$\therefore P_1\left(a, \frac{\pi}{3}\right)$ and $P_2\left(a, \frac{2\pi}{3}\right)$.

(ii) In a single diagram, sketch C_1, C_2 and their line of symmetry. [3]

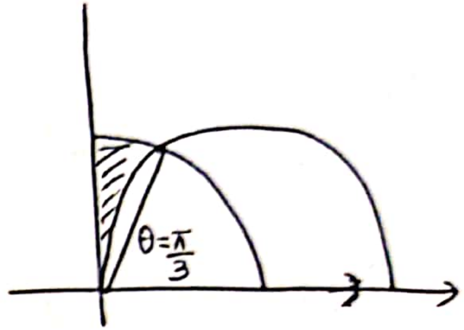
$r = 2a|\cos\theta|.$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	$2a$	$1.73a$	$1.41a$	a	0	a	$1.41a$	$1.73a$	$2a$



(iii) The region R enclosed by C_1 and C_2 is bounded by the arcs OP_1, P_1P_2 and P_2O, where O is the pole. Find the area of R, giving your answer in exact form. [5]

Required area R



$$= 2\left[\frac{a^2 \times \frac{\pi}{6}}{2} - \frac{1}{2}\int_{\pi/3}^{\pi/2} (2a\cos\theta)^2 \, d\theta\right]$$

$$= 2\left[\frac{\pi a^2}{12} - \frac{1}{2}\int_{\pi/3}^{\pi/2} 4a^2\cos^2\theta \, d\theta\right]$$

$$= \frac{\pi a^2}{6} - \int_{\pi/3}^{\pi/2} 4a^2\left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \frac{\pi a^2}{6} - 2a^2\int_{\pi/3}^{\pi/2} 1+\cos 2\theta \, d\theta$$

$$= \frac{\pi a^2}{6} - 2a^2\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/3}^{\pi/2}$$

$$= \frac{\pi a^2}{6} - 2a^2\left[\left(\frac{\pi}{2}+0\right) - \left(\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right)\right]$$

$$= \frac{\pi a^2}{6} - 2a^2\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)$$

$$= \frac{\pi a^2}{6} - \frac{\pi a^2}{3} + \frac{\sqrt{3}a^2}{2} = \frac{\sqrt{3}a^2}{2} - \frac{\pi a^2}{6}.$$

26 The curve C_1 has polar equation $r^2 = 2\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(i) The point on C_1 furthest from the line $\theta = \frac{1}{2}\pi$ is denoted by P. Show that, at P,

$$2\theta \tan\theta = 1$$

and verify that this equation has a root between 0.6 and 0.7. [5]

$$x = r\cos\theta = \sqrt{2\theta}\cos\theta = \sqrt{2}\,\theta^{\frac{1}{2}}\cos\theta.$$

$$\frac{dx}{d\theta} = \sqrt{2}\left[\theta^{\frac{1}{2}}(-\sin\theta) + \frac{\theta^{-\frac{1}{2}}}{2}(\cos\theta)\right] = \sqrt{2}\left[-\sqrt{\theta}\sin\theta + \frac{\cos\theta}{2\sqrt{\theta}}\right]$$

For maximum value of x, $\frac{dx}{d\theta} = 0$

$$\sqrt{2}\left[-\sqrt{\theta}\sin\theta + \frac{\cos\theta}{2\sqrt{\theta}}\right] = 0$$

$$\frac{-2\theta\sin\theta + \cos\theta}{2\sqrt{\theta}} = 0$$

$$-2\theta\sin\theta + \cos\theta = 0$$

$$\cos\theta = 2\theta\sin\theta$$

$$1 = 2\theta\tan\theta \quad \text{(Shown)}.$$

$$2\theta\tan\theta - 1 = 0$$

Let, $f(\theta) = 2\theta\tan\theta - 1$.

$$f(0.6) = 2(0.6)\tan(0.6) - 1 = -0.179$$

$$f(0.7) = 2(0.7)\tan(0.7) - 1 = 0.179.$$

$f(0.6) < 0$ and $f(0.7) > 0 \Rightarrow$ There is a root between 0.6 and 0.7.

The curve C_2 has polar equation $r^2 = \theta \sec^2 \theta$, for $0 \leqslant \theta < \frac{1}{2}\pi$. The curves C_1 and C_2 intersect at the pole, denoted by O, and at another point Q.

(ii) Find the exact value of θ at Q. [2]

Setting $2\theta = \theta \sec^2\theta$

$$2\theta - \theta \sec^2\theta = 0$$

$$(2 - \sec^2\theta)\theta = 0$$

$\theta = 0$ (Ignore) or $2 - \sec^2\theta = 0$

$$\sec^2\theta = 2$$

$$\sec\theta = \sqrt{2}$$

$$\frac{1}{\cos\theta} = \cancel{\frac{1}{\sqrt{2}}}\ \sqrt{2}$$

$$\cos\theta = \frac{1}{\sqrt{2}}$$

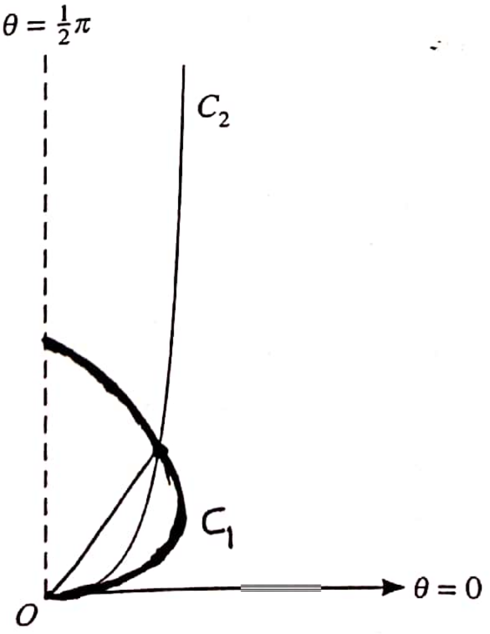
$$\theta = \frac{\pi}{4}.$$

$\therefore$ Value of θ at $Q = \frac{\pi}{4}$.

(iii) The diagram below shows the curve C_2. Sketch C_1 on this diagram. [2]

$r = \sqrt{2\theta}$

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
r	0	1.02	1.25	1.45	1.77



Scanned with CamScanner

(iv) Find, in exact form, the area of the region OPQ enclosed by C_1 and C_2. [5]

Area required $= \frac{1}{2}\int_0^{\pi/4} 2\theta - \theta\sec^2\theta \; d\theta$

$= \frac{1}{2}\int_0^{\pi/4} \theta\left(2-\sec^2\theta\right) d\theta.$

$u = \theta \qquad \frac{dv}{d\theta} = 2-\sec^2\theta$

$\frac{du}{d\theta} = 1 \qquad v = 2\theta - \tan\theta$

$= \frac{1}{2}\left\{\left[\theta\left(2\theta-\tan\theta\right)\right]_0^{\pi/4} - \int_0^{\pi/4}\left(2\theta-\tan\theta\right)(1)\,d\theta\right\}$

$= \frac{1}{2}\left\{\left(\frac{\pi}{4}\left(\frac{2\pi}{4}-\tan\frac{\pi}{4}\right)-0\right) - \left[\frac{2\theta^2}{2} - \ln(\sec\theta)\right]_0^{\pi/4}\right\}$

$= \frac{1}{2}\left\{\frac{\pi}{4}\left(\frac{\pi}{2}-1\right) - \left[\theta^2 + \ln(\cos\theta)\right]_0^{\pi/4}\right\}$

$= \frac{1}{2}\left\{\frac{\pi^2}{8} - \frac{\pi}{4} - \left[\left(\frac{\pi}{4}\right)^2 + \ln\left(\cos\frac{\pi}{4}\right)\right] + \left[0 + \ln(\cos 0)\right]\right\}$

$= \frac{1}{2}\left\{\frac{\pi^2}{8} - \frac{\pi}{4} - \frac{\pi^2}{16} - \ln\left(2^{-\frac{1}{2}}\right)\right\}$

$= \frac{1}{2}\left\{\frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\ln 2\right\} = \frac{1}{4}\ln 2 + \frac{\pi^2}{32} - \frac{\pi}{8}$

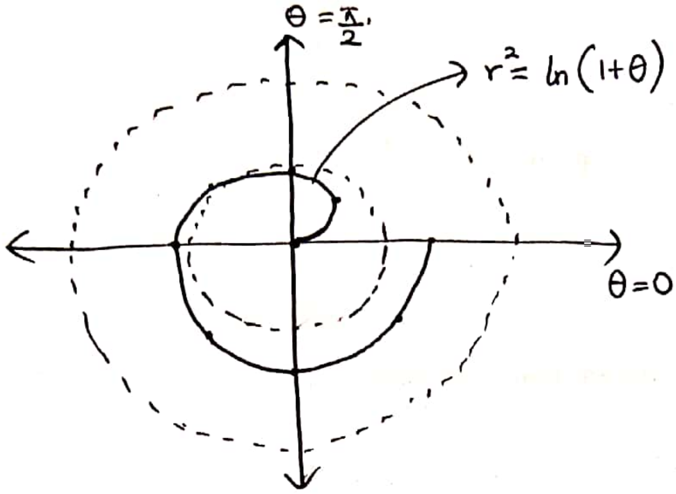
$= \frac{1}{4}\ln 2 + \frac{\pi}{8}\left(\frac{\pi}{4}-1\right).$

27 The curve C has polar equation $r^2 = \ln(1+\theta)$, for $0 \leqslant \theta \leqslant 2\pi$

(i) Sketch C. [2]

$r = \sqrt{\ln(1+\theta)}$

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	0	0.97	1.19	1.32	1.41



(ii) Using the substitution $u = 1 + \theta$, or otherwise, find the area of the region bounded by C and the initial line, leaving your answer in an exact form. [5]

$$\text{Area} = \frac{1}{2}\int_0^{2\pi} r^2 \, d\theta = \frac{1}{2}\int_0^{2\pi} \ln(1+\theta)\, d\theta.$$

$u = 1+\theta$.

$du = d\theta$.

$\theta = 0 \Rightarrow u = 1+0 = 1.$

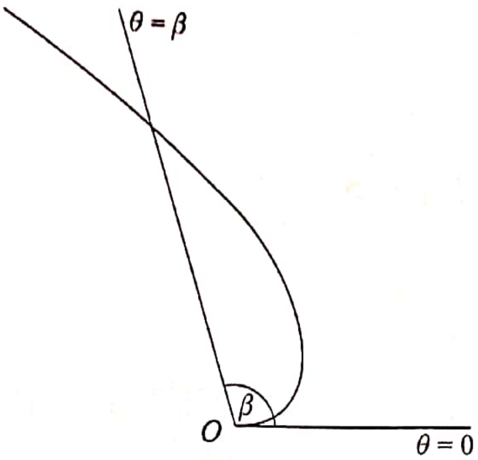
$\theta = 2\pi \Rightarrow u = 1+2\pi$.

$$\text{Area} = \frac{1}{2}\int_0^{2\pi} \ln(1+\theta)\, d\theta = \frac{1}{2}\int_1^{1+2\pi} \ln u \, du = \frac{1}{2}\Big[u\ln u - u\Big]_1^{1+2\pi}$$

$$= \frac{1}{2}\Big[\big[(1+2\pi)\ln(1+2\pi) - (1+2\pi)\big] - \big[1\ln(1) - 1\big]\Big]$$

$$= \frac{(1+2\pi)\ln(1+2\pi) - 1 - 2\pi + 1}{2} = \frac{(1+2\pi)\ln(1+2\pi) - 2\pi}{2}$$

$$= \left(\pi + \frac{1}{2}\right)\ln(1+2\pi) - \pi.$$



The curve C has polar equation

$$r = \theta^{\frac{1}{2}} e^{\theta^2/\pi},$$

where $0 \leqslant \theta \leqslant \pi$. The area of the finite region bounded by C and the line $\theta = \beta$ is π (see diagram). Show that

$$\beta = (\pi \ln 3)^{\frac{1}{2}}.$$ [6]

$$\text{Area} = \frac{1}{2}\int_0^{\beta} \left(\theta^{\frac{1}{2}} e^{\theta^2/\pi}\right)^2 d\theta = \frac{1}{2}\int_0^{\beta} \theta\, e^{2\theta^2/\pi}\, d\theta$$

$$\int \theta\, e^{2\theta^2/\pi}\, d\theta$$

$\cancel{u = \theta}$ $\qquad$ $\cancel{\frac{dv}{d\theta} = e^{\theta^2}}$ $\qquad$ Let, $u = \frac{2\theta^2}{\pi}$.

$\cancel{\frac{du}{d\theta} = 1}$ $\qquad$ $\frac{du}{d\theta} = \frac{4\theta}{\pi} \Rightarrow \theta\, d\theta = \frac{\pi}{4}\, du.$

$$\Rightarrow \int \theta\, e^{2\theta^2/\pi}\, d\theta = \int e^{2\theta^2/\pi}\, \theta\, d\theta = \int e^{u}\left(\frac{\pi}{4}\, du\right) = \frac{\pi}{4}\int e^{u}\, du$$

$$= \frac{\pi}{4}\left(e^{u}\right) + c. = \frac{\pi}{4} e^{2\theta^2/\pi} + c$$

$$\Rightarrow \text{Area} = \frac{1}{2}\int_0^{\beta} \theta\, e^{\theta^2/\pi}\, d\theta$$

$$\pi = \frac{1}{2}\left[\frac{\pi}{4} e^{2\theta^2/\pi}\right]_0^{\beta}$$

$$\pi = \frac{\pi}{8}\left(e^{2\beta^2/\pi} - e^{2(0)^2/\pi}\right)$$

$$8 = e^{2\beta^2/\pi} - 1 \Rightarrow e^{2\beta^2/\pi} = 9$$

$$\frac{2\beta^2}{\pi} = \ln 9$$

$$\frac{2\beta^2}{\pi} = 2\ln 3$$

$$\beta^2 = \pi \ln 3$$

$$\beta = \sqrt{\pi \ln 3} = (\pi \ln 3)^{\frac{1}{3}}.$$

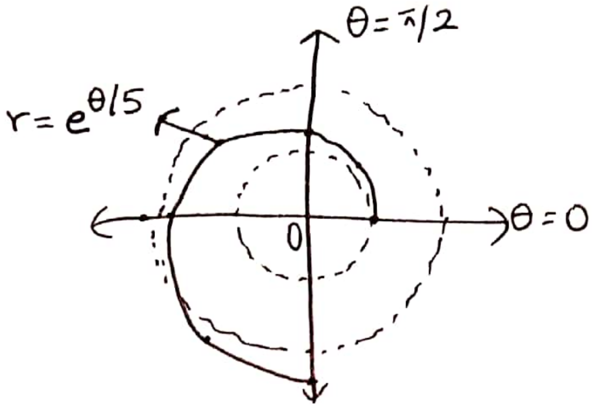
29 The curve C has polar equation

$$r = e^{\frac{1}{5}\theta}, \qquad 0 \leqslant \theta \leqslant \tfrac{3}{2}\pi.$$

(i) Draw a sketch of C. [2]

(ii) Find the length of C, correct to 3 significant figures. [4]

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
r	1	1.17	1.37	1.60	1.87	2.19	2.56



ii) Arc length $= \displaystyle\int_0^{3\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{3\pi/2} \sqrt{\left(e^{\theta/5}\right)^2 + \left(\frac{e^{\theta/5}}{5}\right)^2}\, d\theta$

$= \displaystyle\int_0^{3\pi/2} \sqrt{\frac{e^{2\theta/5} + 25e^{2\theta/5}}{25}}\, d\theta = \int_0^{3\pi/2} \frac{\sqrt{26e^{2\theta/5}}}{5}\, d\theta.$

$= \displaystyle\frac{\sqrt{26}}{5}\int_0^{3\pi/2} e^{\theta/5}\, d\theta = \frac{\sqrt{26}}{5}\left[\frac{e^{\theta/5}}{1/5}\right]_0^{3\pi/2}$

$= \displaystyle\frac{5\sqrt{26}}{5}\left[e^{\theta/5}\right]_0^{3\pi/2}$

$= \sqrt{26}\left[e^{3\pi/10} - e^0\right] = \sqrt{26}\left(e^{\frac{3\pi}{10}} - 1\right) = 7.986$

$\approx 7.99.$

30 The curves C_1 and C_2 have polar equations

$$r = 4\cos\theta \qquad \text{and} \qquad r = 1 + \cos\theta$$

respectively, where $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$.

(i) Show that C_1 and C_2 meet at the points $A\left(\frac{4}{3}, \alpha\right)$ and $B\left(\frac{4}{3}, -\alpha\right)$, where α is the acute angle such that $\cos\alpha = \frac{1}{3}$. [2]

(ii) In a single diagram, draw sketch graphs of C_1 and C_2. [3]

(iii) Show that the area of the region bounded by the arcs OA and OB of C_1, and the arc AB of C_2, is

$$4\pi - \tfrac{1}{3}\sqrt{2} - \tfrac{13}{2}\alpha.$$ [7]

i) Setting $4\cos\theta = 1 + \cos\theta \Rightarrow 3\cos\theta = 1$

$\cos\theta = \frac{1}{3}$

$\cos\alpha = \frac{1}{3} \Rightarrow \alpha = \cos^{-1}\left(\frac{1}{3}\right)$.

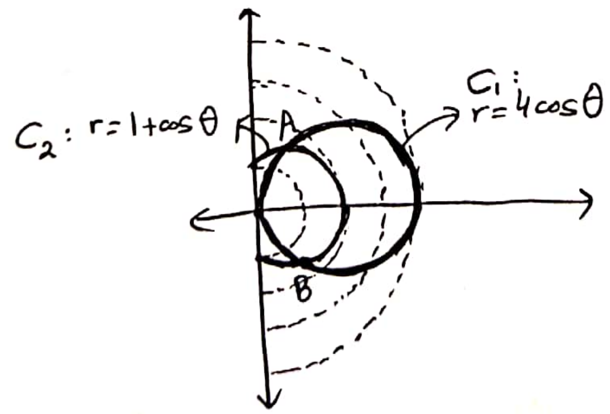
$\theta = \alpha, -\alpha$

When $\theta = \alpha$, $r = 4\cos\alpha = 4\left(\frac{1}{3}\right) = \frac{4}{3}$.

When $\theta = -\alpha$, $r = 4\cos(-\alpha) = 4\left(\frac{1}{3}\right) = \frac{4}{3}$.

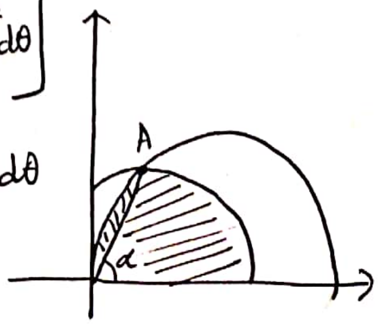
$\therefore$ Points of intersection: $\left(\frac{4}{3}, \alpha\right), \left(\frac{4}{3}, -\alpha\right)$.

ii)



iii) $\text{Area} = 2\left[\frac{1}{2}\int_0^{\alpha}(1+\cos\theta)^2\,d\theta + \frac{1}{2}\int_{\alpha}^{\frac{\pi}{2}}(4\cos\theta)^2\,d\theta\right]$

$= \int_0^{\alpha} 1 + 2\cos\theta + \cos^2\theta\,d\theta + \cancel{8}\int_{\alpha}^{\pi/2} 16\cos^2\theta\,d\theta$



$$= \int_0^{\alpha} 1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\, d\theta + 16\int_{\alpha}^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \int_0^{\alpha} \frac{3 + 4\cos\theta + \cos 2\theta}{2}\, d\theta + 8\int_{\alpha}^{\pi/2} 1 + \cos 2\theta \, d\theta$$

$$= \frac{1}{2}\left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{\alpha} + 8\left[\theta + \frac{\sin 2\theta}{2}\right]_{\alpha}^{\pi/2}$$

$$= \frac{1}{2}\left[3\alpha + 4\sin\alpha + \frac{\sin 2\alpha}{2}\right] + 8\left[\left(\frac{\pi}{2} + 0\right) - \left(\alpha + \frac{\sin 2\alpha}{2}\right)\right]$$

$$= \frac{3\alpha}{2} + 2\sin\alpha + \frac{\sin 2\alpha}{4} + 4\pi - 8\alpha - 4\sin 2\alpha$$

$$= 4\pi - \frac{13\alpha}{2} + 2\sin\alpha - \frac{15\sin 2\alpha}{4}$$

$$\cos\alpha = \frac{1}{3} \Rightarrow \sin\alpha = \frac{2\sqrt{2}}{3}.$$

$$= 4\pi - \frac{13\alpha}{2} + 2\left(\frac{2\sqrt{2}}{3}\right) - \frac{15\left(2 \times \frac{1}{3} \times \frac{2\sqrt{2}}{3}\right)}{4}$$

$$= 4\pi - \frac{13\alpha}{2} - \frac{\sqrt{2}}{3}.$$

31 The curve C has equation

$$r = 10\ln(1+\theta),$$

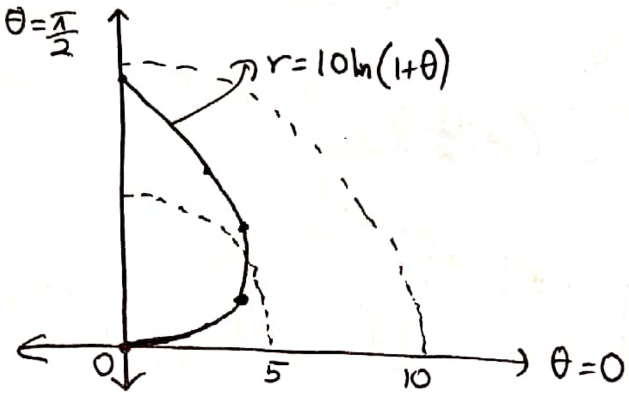
where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. Draw a sketch of C. [2]

Use the substitution $w = \ln(1+\theta)$ to show that the area of the sector bounded by the line $\theta = \frac{1}{2}\pi$ and the arc of C joining the origin to the point where $\theta = \frac{1}{2}\pi$ is

$$50(b^2 - 2b + 2)e^b - 100,$$

where $b = \ln(1 + \frac{1}{2}\pi)$. [6]

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	0	4.21	5.80	7.16	9.44



$$\text{Area} = \frac{1}{2}\int_0^{\pi/2} \left[10\ln(1+\theta)\right]^2 d\theta = 50\int_0^{\pi/2}\left[\ln(1+\theta)\right]^2 d\theta.$$

$$w = \ln(1+\theta) \Rightarrow 1+\theta = e^w$$

$$\frac{dw}{d\theta} = \frac{1}{1+\theta} \Rightarrow d\theta = (1+\theta)\,dw \Rightarrow d\theta = e^w\,dw.$$

$$\theta = 0 \Rightarrow w = \ln(1+\theta) = \ln(1+0) = 0.$$

$$\theta = \frac{\pi}{2} \Rightarrow w = \ln(1+\theta) = \ln\left(1+\frac{\pi}{2}\right).$$

$$\text{Area} = 50\int_0^{\pi/2}\left[\ln(1+\theta)\right]^2 d\theta = \int_0^{\ln(1+\frac{\pi}{2})} w^2 e^w\,dw.$$

$$\int w^2 e^w\,dw$$

$u = w^2$ $\qquad$ $\frac{dv}{dw} = e^w$

$\frac{du}{dw} = 2w$ $\qquad$ $v = e^w$

$$\int w^2 e^w \, dw = w^2 e^w - \int 2w e^w \, dw = w^2 e^w - 2\int w e^w \, dw.$$

$u = w \qquad \frac{dv}{dw} = e^w$

$\frac{du}{dw} = 1 \qquad v = e^w.$

$$\int w e^w \, dw = w e^w - \int e^w (1) \, dw = w e^w - \int e^w \, dw = w e^w - e^w + c.$$

$$\Rightarrow \int w^2 e^w \, dw = w^2 e^w - 2\left(w e^w - e^w\right) = w^2 e^w - 2we^w + 2e^w + c$$

$$= e^w \left(w^2 - 2w + 2\right) + c.$$

$$\therefore \text{Area} = 50 \int_0^{\ln\left(1+\frac{\pi}{2}\right)} w^2 e^w \, dw$$

$$= 50 \left[e^w \left(w^2 - 2w + 2\right) \right]_0^{\ln\left(1+\frac{\pi}{2}\right)}$$

$$= 50 \left[e^{\ln\left(1+\frac{\pi}{2}\right)} \left(\left[\ln\left(1+\frac{\pi}{2}\right)\right]^2 - 2\ln\left(1+\frac{\pi}{2}\right) + 2 \right) - e^0(2) \right]$$

Let; $b = \ln\left(1+\frac{\pi}{2}\right)$.

$$\therefore \text{Area} = 50 \left[e^{\ln\left(1+\frac{\pi}{2}\right)} \left(\left[\ln\left(1+\frac{\pi}{2}\right)\right]^2 - 2\ln\left(1+\frac{\pi}{2}\right) + 2 \right) - 2 \right]$$

$$= 50 \left[e^b \left(b^2 - 2b + 2\right) - 2 \right]$$

$$= 50 e^b \left(b^2 - 2b + 2\right) - 100.$$

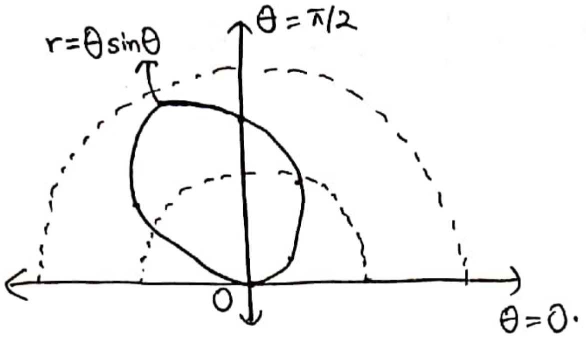
32 The curve C has polar equation

$$r = \theta \sin \theta,$$

where $0 \leqslant \theta \leqslant \pi$. Draw a sketch of C. [2]

Find the area of the region enclosed by C, leaving your answer in terms of π. [7]

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	0	0.26	0.91	1.57	1.81	1.31	0



$$\text{Area} = \frac{1}{2}\int_0^{\pi} (\theta \sin\theta)^2 \, d\theta = \frac{1}{2}\int_0^{\pi} \theta^2 \sin^2\theta \, d\theta = \frac{1}{2}\int_0^{\pi} \theta^2\left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \frac{\theta^2 - \theta^2\cos 2\theta}{2} \, d\theta = \frac{1}{4}\int_0^{\pi} \theta^2 - \theta^2 \cos 2\theta \, d\theta.$$

$$\int \theta^2 \cos 2\theta \, d\theta$$

$u = \theta^2$ $\qquad$ $\frac{dv}{d\theta} = \cos 2\theta.$

$\frac{du}{d\theta} = 2\theta$ $\qquad$ $v = \frac{\sin 2\theta}{2}$

$$\int \theta^2 \cos 2\theta \, d\theta = \frac{\theta^2 \sin 2\theta}{2} - \int 2\theta\left(\frac{\sin 2\theta}{2}\right) d\theta$$

$$= \frac{\theta^2 \sin 2\theta}{2} - \int \theta \sin 2\theta \, d\theta$$

$u = \theta$ $\qquad$ $\frac{dv}{d\theta} = \sin 2\theta$

$\frac{du}{d\theta} = 1$ $\qquad$ $v = \frac{-\cos 2\theta}{2}.$

$$\int \theta \sin 2\theta \, d\theta = \theta\left(\frac{-\cos 2\theta}{2}\right) - \int \frac{-\cos 2\theta}{2}(1)\, d\theta$$

$$= \frac{-\theta\cos 2\theta}{2} + \frac{1}{2}\int \cos 2\theta \, d\theta = \frac{-\theta \cos 2\theta}{2} + \frac{1}{2}\left(\frac{\sin 2\theta}{2}\right)$$

$$= \frac{-\theta \cos 2\theta}{2} + \frac{\sin 2\theta}{4}$$

$$\Rightarrow \int \theta^2 \cos 2\theta \, d\theta = \frac{\theta^2 \sin 2\theta}{2} - \left(\frac{-\theta \cos 2\theta}{2} + \frac{\sin 2\theta}{4} \right) + c$$

$$= \frac{\theta^2 \sin 2\theta}{2} + \frac{\theta \cos 2\theta}{2} - \frac{\sin 2\theta}{4} + c$$

$$\Rightarrow \quad \text{Area} = \frac{1}{4} \int_0^{\pi} \theta^2 - \theta^2 \cos 2\theta \, d\theta$$

$$= \frac{1}{4} \left[\frac{\theta^3}{3} - \left(\frac{\theta^2 \sin 2\theta}{2} + \frac{\theta \cos 2\theta}{2} - \frac{\sin 2\theta}{4} \right) \right]_0^{\pi}$$

$$= \frac{1}{4} \left[\frac{\pi^3}{3} - \frac{\pi}{2} \right] = \frac{\pi^3}{12} - \frac{\pi}{8}.$$

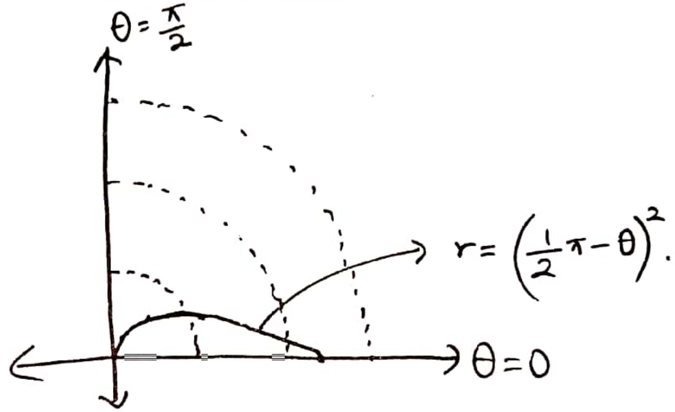
33 The curve C has polar equation

$$r = \left(\tfrac{1}{2}\pi - \theta\right)^2,$$

where $0 \leqslant \theta \leqslant \tfrac{1}{2}\pi$. Draw a sketch of C. [3]

Find the area of the region bounded by C and the initial line, leaving your answer in terms of π. [3]

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	2.46	1.10	0.62	0.27	0



$$A = \frac{1}{2}\int_0^{\pi/2} \left[\left(\frac{1}{2}\pi - \theta\right)^2\right]^2 d\theta = \frac{1}{2}\int_0^{\pi/2} \left(\frac{\pi}{2} - \theta\right)^4 d\theta$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - \theta\right)^5 \Big/ 5(-1)\right]_0^{\pi/2} = \frac{-1}{10}\left[\left(\frac{\pi}{2} - \theta\right)^5\right]_0^{\pi/2}$$

$$= \frac{-1}{10}\left(0^5 - \left(\frac{\pi}{2}\right)^5\right) = \frac{-1}{10}\left(\frac{-\pi^5}{32}\right) = \frac{\pi^5}{320}.$$

34 The curve C has polar equation

$$r = a\sin 3\theta,$$

where $0 \leqslant \theta \leqslant \frac{1}{3}\pi$.

(i) Show that the area of the region enclosed by C is $\frac{1}{12}\pi a^2$. [3]

(ii) Show that, at the point of C at maximum distance from the initial line,

$$\tan 3\theta + 3\tan\theta = 0.$$ [3]

(iii) Use the formula

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

to find this maximum distance. [4]

(iv) Draw a sketch of C. [2]

i) $A = \frac{1}{2}\int_0^{\pi/3} (a\sin 3\theta)^2\, d\theta = \frac{a^2}{2}\int_0^{\pi/3} \sin^2 3\theta \, d\theta = \frac{a^2}{2}\int_0^{\pi/3} \frac{1-\cos 6\theta}{2}\, d\theta$

$= \frac{a^2}{4}\int_0^{\pi/3} 1-\cos 6\theta \, d\theta = \frac{a^2}{4}\left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\pi/3} = \frac{a^2}{4}\left[\left(\frac{\pi}{3} - \frac{\sin(2\pi)}{6}\right) - (0-0)\right]$

$= \frac{a^2}{4}\left(\frac{\pi}{3} - 0\right) = \frac{\pi a^2}{12}$ (Shown).

ii) $y = r\sin\theta = a\sin 3\theta \sin\theta$.

$\frac{dy}{d\theta} = a\left[\sin 3\theta(\cos\theta) + \sin\theta(3\cos 3\theta)\right] = a(\sin 3\theta\cos\theta + 3\cos 3\theta \sin\theta)$

For maximum value of y, $\frac{dy}{d\theta} = 0 \Rightarrow a(\sin 3\theta\cos\theta + 3\cos 3\theta\sin\theta) = 0$

$\sin 3\theta\cos\theta + 3\cos 3\theta \sin\theta = 0$

$\sin 3\theta\cos\theta = -3\cos 3\theta \sin\theta$

$\frac{\sin 3\theta}{\cos 3\theta} = \frac{-3\sin\theta}{\cos\theta}$

$\tan 3\theta = -3\tan\theta$

$\tan 3\theta + 3\tan\theta = 0$ (Shown).

iii) $\tan 3\theta + 3\tan\theta = 0$

$\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta} + 3\tan\theta = 0$

$3\tan\theta - \tan^3\theta + 3\tan\theta - 9\tan^3\theta = 0$

$6\tan\theta - 10\tan^3\theta = 0$

$$2\tan\theta\left(3-5\tan^2\theta\right)=0$$

$2\tan\theta=0$ or $3-5\tan^2\theta=0$

$\tan\theta=0$ $\qquad$ $\tan^2\theta=\frac{3}{5}$

$\theta=0$ or $\theta=\tan^{-1}\sqrt{\frac{3}{5}}$

When $\theta=0$, $y=a\sin(3\times0)\sin(0)=0$

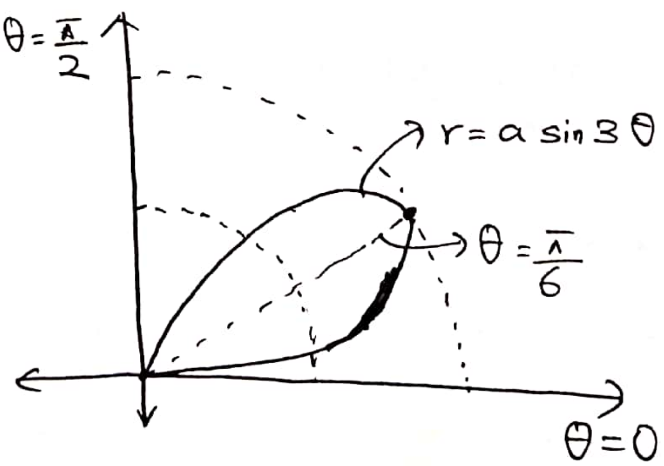
when $\theta=\tan^{-1}\left(\sqrt{\frac{3}{5}}\right)$, $y=a\sin\left(3\tan^{-1}\sqrt{\frac{3}{5}}\right)\sin\left(\tan^{-1}\left(\sqrt{\frac{3}{5}}\right)\right)=\frac{9a}{16}$. (Maximum)

$\Rightarrow$ Maximum distance from initial line $=\frac{9a}{16}$.

iv)

$$r=a\sin3\theta$$

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
r	0	$0.7a$	a	$0.7a$	0



35 The curves C_1 and C_2 have polar equations given by

$$C_1: \quad r = 3\sin\theta, \qquad 0 \leqslant \theta < \pi,$$
$$C_2: \quad r = 1 + \sin\theta, \quad -\pi < \theta \leqslant \pi.$$

(i) Find the polar coordinates of the points, other than the pole, where C_1 and C_2 meet. [2]

(ii) In a single diagram, draw sketch graphs of C_1 and C_2. [3]

(iii) Show that the area of the region which is inside C_1 but outside C_2 is π. [5]

i) Set $3\sin\theta = 1 + \sin\theta$

$2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}.$

When $\theta = \frac{\pi}{6}$, $r = 3\sin\left(\frac{\pi}{6}\right) = \frac{3}{2}.$

When $\theta = \frac{5\pi}{6}$, $r = 3\sin\left(\frac{5\pi}{6}\right) = \frac{3}{2}.$

$\therefore$ Points of intersection: $\left(\frac{3}{2}, \frac{\pi}{6}\right)$ and $\left(\frac{3}{2}, \frac{5\pi}{6}\right)$.

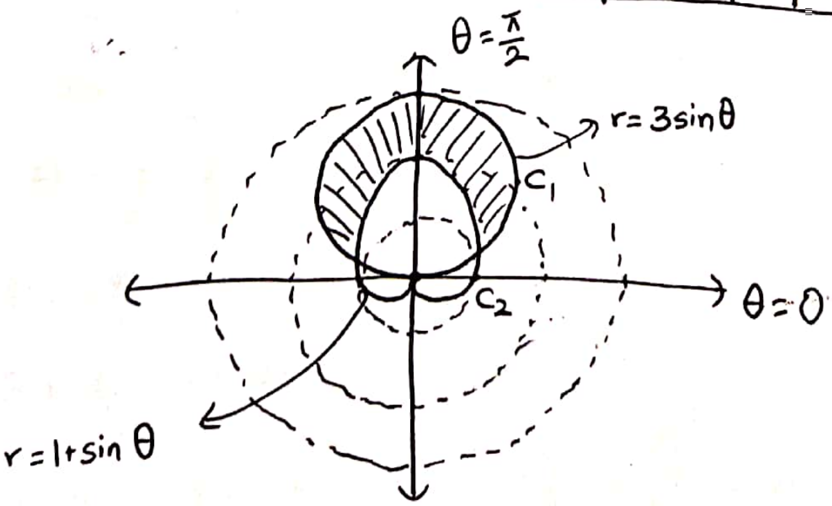
ii)

$r = 3\sin\theta$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
r	0	2.12	3	2.12	0

$r = 1 + \sin\theta$

θ	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π
r	1	0	1	2	1



Scanned with CamScanner

III) Required area

$$= 2\left[\frac{1}{2}\int_{\pi/6}^{\pi/2} (3\sin\theta)^2 - (1+\sin\theta)^2 \, d\theta\right]$$

$$= \int_{\pi/6}^{\pi/2} 9\sin^2\theta - 1 - 2\sin\theta - \sin^2\theta \, d\theta$$

$$= \int_{\pi/6}^{\pi/2} 8\sin^2\theta - 2\sin\theta - 1 \, d\theta$$

$$= \int_{\pi/6}^{\pi/2} 8\left[\frac{1-\cos 2\theta}{2}\right] - 2\sin\theta - 1 \, d\theta$$

$$= \int_{\pi/6}^{\pi/2} 3 - 4\cos 2\theta - 2\sin\theta \, d\theta$$

$$= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\pi/6}^{\pi/2}$$

$$= \left(\frac{3\pi}{2} - 0 + 0\right) - \left(\frac{3\pi}{6} - \sqrt{3} + \sqrt{3}\right)$$

$$= \frac{3\pi}{2} - \frac{\pi}{2} = \pi.$$

36 The curve C has polar equation $r = 3 + 2\cos\theta$, for $-\pi < \theta \leqslant \pi$. The straight line l has polar equation $r\cos\theta = 2$. Sketch both C and l on a single diagram. [3]

Find the polar coordinates of the points of intersection of C and l. [4]

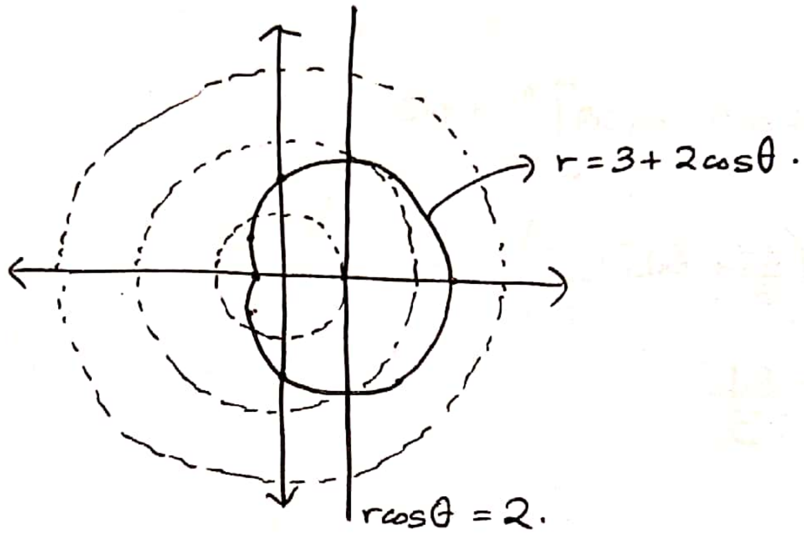
The region R is enclosed by C and l, and contains the pole. Find the area of R. [6]

$r = 3 + 2\cos\theta$

$r\cos\theta = 2$

$\Rightarrow x = 2.$

θ	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π
r	1	3	5	3	1



when curves intersect,

$$3 + 2\cos\theta = \frac{2}{\cos\theta}$$

$$2\cos^2\theta + 3\cos\theta - 2 = 0$$

$$2\cos^2\theta + 4\cos\theta - \cos\theta - 2 = 0$$

$$(2\cos\theta - 1)(\cos\theta + 2) = 0$$

$$\cos\theta = \frac{1}{2} \quad \text{or} \quad \cos\theta = -2.$$

(No solution)

$$\theta = \frac{-\pi}{3}, \frac{\pi}{3}.$$

when $\theta = \frac{-\pi}{3}$, $r = 3 + 2\cos\left(\frac{-\pi}{3}\right) = 4$

when $\theta = \frac{\pi}{3}$, $r = 3 + 2\cos\left(\frac{\pi}{3}\right) = 4$

$\Rightarrow$ Points of intersections: $\left(4, \frac{\pi}{3}\right)$, $\left(4, \frac{-\pi}{3}\right)$.

Required area

$$= 2\left[\frac{1}{2}\int_{\pi/3}^{\pi} \left(3+2\cos\theta\right)^2 d\theta + \frac{1}{2}\times\left(4\cos\frac{\pi}{3}\right)\left(4\sin\frac{\pi}{3}\right)\right]$$

$$= \int_{\pi/3}^{\pi} 9+12\cos\theta+4\cos^2\theta \, d\theta + 4\sqrt{3}$$

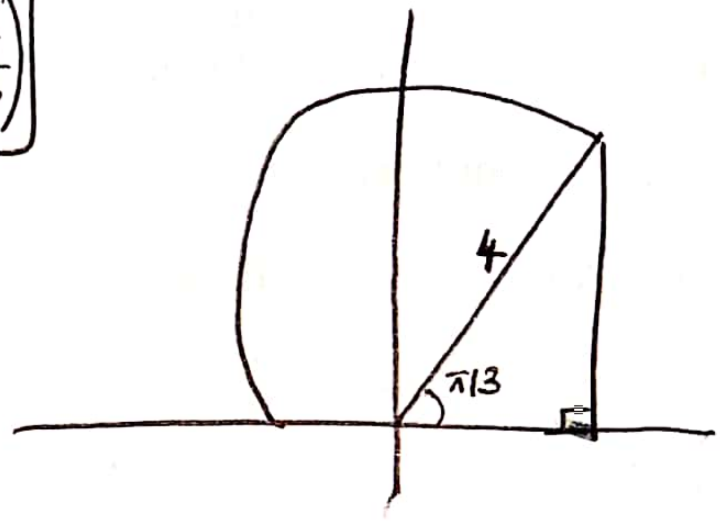
$$= \int_{\pi/3}^{\pi} 9+12\cos\theta + 4\left[\frac{1+\cos 2\theta}{2}\right] d\theta + 4\sqrt{3}$$

$$= \int_{\pi/3}^{\pi} 11+12\cos\theta+2\cos 2\theta \, d\theta + 4\sqrt{3}$$

$$= \Big[11\theta + 12\sin\theta + \sin 2\theta\Big]_{\pi/3}^{\pi} + 4\sqrt{3}$$

$$= 11\pi - \left(\frac{11\pi}{3} + 6\sqrt{3} + \frac{\sqrt{3}}{2}\right) + 4\sqrt{3}$$

$$= \frac{22\pi}{3} - \frac{5\sqrt{3}}{2}.$$

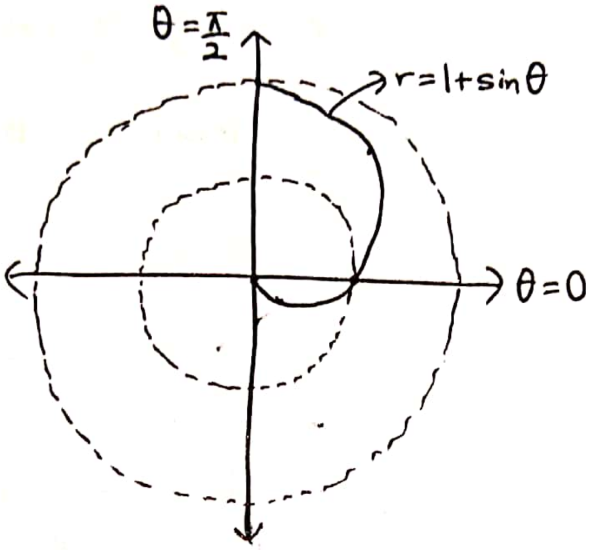


Scanned with CamScanner

37 The curve C has polar equation $r = 1 + \sin\theta$ for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$. Draw a sketch of C. [2]

The area of the region enclosed by the initial line, the half-line $\theta = \frac{1}{2}\pi$, and the part of C for which θ is positive, is denoted by A_1. The area of the region enclosed by the initial line, and the part of C for which θ is negative, is denoted by A_2. Find the ratio $A_1 : A_2$, giving your answer correct to 1 decimal place. [8]

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
r	0	0.29	1	1.70	2



$$A_1 = \frac{1}{2}\int_0^{\pi/2} (1+\sin\theta)^2\,d\theta = \frac{1}{2}\int_0^{\pi/2} 1+2\sin\theta+\frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{3+4\sin\theta-\cos 2\theta}{2}\,d\theta = \frac{1}{4}\int_0^{\pi/2} 3+4\sin\theta-\cos 2\theta\,d\theta$$

$$A_1 = \frac{1}{4}\left[3\theta+4(-\cos\theta)-\frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{1}{4}\left[\frac{3\pi}{2}-(-4)\right] = \frac{3\pi}{8}+1.$$

$$A_2 = \frac{1}{2}\int_{-\pi/2}^{0} (1+\sin\theta)^2\,d\theta = \frac{1}{4}\left[3\theta-4\cos\theta-\frac{\sin 2\theta}{2}\right]_{-\pi/2}^{0}$$

$$= \frac{1}{4}\left[-4-\left(\frac{-3\pi}{2}\right)\right] = \frac{1}{4}\left[\frac{3\pi}{2}-4\right] = \frac{3\pi}{8}-1.$$

$$\Rightarrow A_1 : A_2 = \frac{\frac{3\pi}{8}+1}{\frac{3\pi}{8}-1} = 12.22 \approx 12.2.$$

38 Find the cartesian equation corresponding to the polar equation $r = (\sqrt{2})\sec\left(\theta - \frac{\pi}{4}\right)$. [3]

Sketch the the graph of $r = (\sqrt{2})\sec(\theta - \frac{1}{4}\pi)$, for $-\frac{1}{4}\pi < \theta < \frac{3}{4}\pi$, indicating clearly the polar coordinates of the intersection with the initial line. [2]

$$r = \sqrt{2}\sec\left(\theta - \frac{\pi}{4}\right)$$

$$r = \frac{\sqrt{2}}{\cos\left(\theta - \frac{\pi}{4}\right)}$$

$$r\cos\left(\theta - \frac{\pi}{4}\right) = \sqrt{2}$$

$$r\left[\cos\theta\cos\frac{\pi}{4} + \sin\theta\sin\frac{\pi}{4}\right] = \sqrt{2}$$

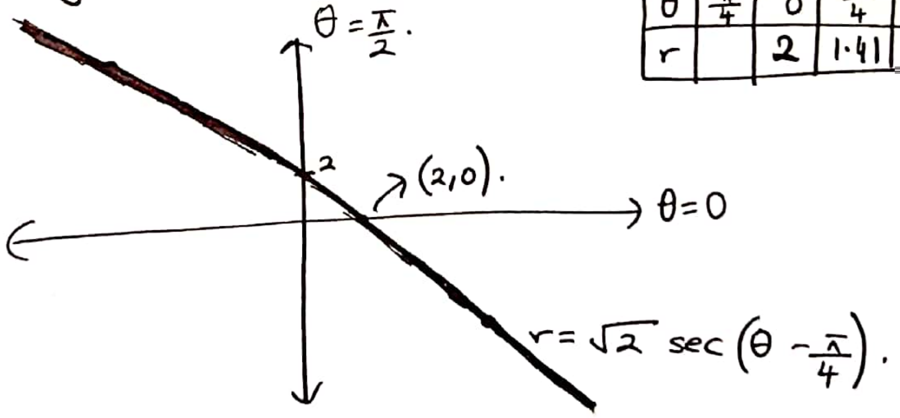
$$r\left[\cos\theta \times \frac{\sqrt{2}}{2} + \sin\theta\,\frac{\sqrt{2}}{2}\right] = \sqrt{2}$$

$$r(\cos\theta + \sin\theta) = 2$$

$$r\cos\theta + r\sin\theta = 2$$

$$x + y = 2$$

θ	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$
r		2	1.41	2	



Point of intersection with initial line: $(2,0)$.

38 Find the cartesian equation corresponding to the polar equation $r = (\sqrt{2})\sec(\theta - \frac{\pi}{4})$. [3]

Sketch the the graph of $r = (\sqrt{2})\sec(\theta - \frac{1}{4}\pi)$, for $-\frac{1}{4}\pi < \theta < \frac{3}{4}\pi$, indicating clearly the polar coordinates of the intersection with the initial line. [2]

$$r = \sqrt{2}\sec\left(\theta - \frac{\pi}{4}\right)$$

$$r = \frac{\sqrt{2}}{\cos\left(\theta - \frac{\pi}{4}\right)}$$

$$r\cos\left(\theta - \frac{\pi}{4}\right) = \sqrt{2}$$

$$r\left[\cos\theta\cos\frac{\pi}{4} + \sin\theta\sin\frac{\pi}{4}\right] = \sqrt{2}$$

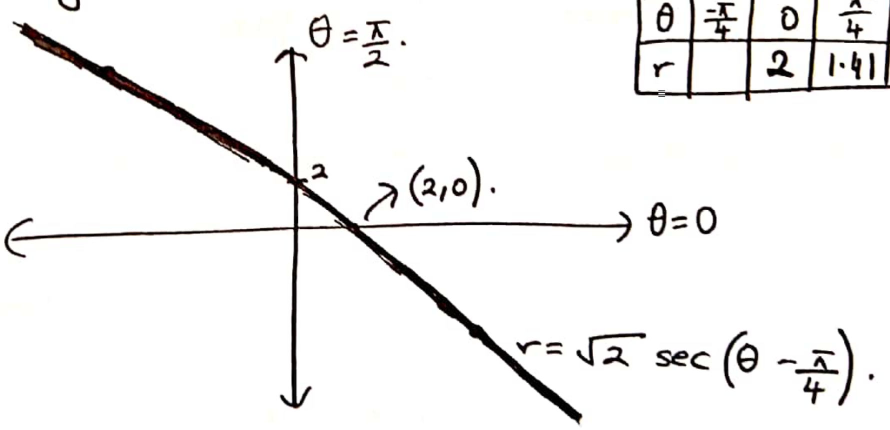
$$r\left[\cos\theta \times \frac{\sqrt{2}}{2} + \sin\theta\frac{\sqrt{2}}{2}\right] = \sqrt{2}$$

$$r(\cos\theta + \sin\theta) = 2$$

$$r\cos\theta + r\sin\theta = 2$$

$$x + y = 2$$

θ	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$
r		2	1.41	2	

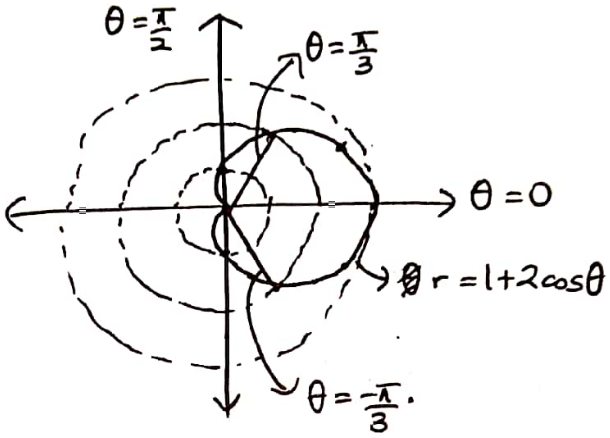


Point of intersection with initial line: $(2,0)$.

39 The curve C has polar equation $r = 1 + 2\cos\theta$. Sketch the curve for $-\frac{2}{3}\pi \leqslant \theta < \frac{2}{3}\pi$. [2]

Find the area bounded by C and the half-lines $\theta = -\frac{1}{3}\pi$, $\theta = \frac{1}{3}\pi$. [4]

θ	$\frac{-2\pi}{3}$	$-\frac{\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{-\pi}{4}$	$\frac{\pi}{4}$
r	0	1	2	2.73	3	2.73	2	1	0	2.41	2.41



$$\text{Area} = 2 \times \frac{1}{2}\int_0^{\pi/3} (1+2\cos\theta)^2\,d\theta = \int_0^{\pi/3} 1 + 4\cos\theta + 4\cos^2\theta\,d\theta$$

$$= \int_0^{\pi/3} 1 + 4\cos\theta + 4\left[\frac{1+\cos 2\theta}{2}\right]d\theta = \int_0^{\pi/3} 3 + 4\cos\theta + 2\cos 2\theta\,d\theta$$

$$= \left[3\theta + 4\sin\theta + \sin 2\theta\right]_0^{\pi/3}$$

$$= \left(\pi + 2\sqrt{3} + \frac{\sqrt{3}}{2}\right) - (0) = \pi + \frac{5\sqrt{3}}{2}.$$

Scanned with CamScanner

40 The curve C has polar equation $r = 2e^{\theta}$, for $\frac{1}{6}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$. Find

(i) the area of the region bounded by the half-lines $\theta = \frac{1}{6}\pi$, $\theta = \frac{1}{2}\pi$ and C, [2]

(ii) the length of C. [3]

i) $A = \frac{1}{2}\int_{\pi/6}^{\pi/2} (2e^{\theta})^2\, d\theta = \frac{4}{2}\int_{\pi/6}^{\pi/2} e^{2\theta}\, d\theta = 2\int_{\pi/6}^{\pi/2} e^{2\theta}\, d\theta$

$= 2\left[\frac{e^{2\theta}}{2}\right]_{\pi/6}^{\pi/2} = \left[e^{2\theta}\right]_{\pi/6}^{\pi/2} = e^{2\times\frac{\pi}{2}} - e^{2\times\frac{\pi}{6}} = e^{\pi} - e^{\pi/3} = 20.29$

$\approx \underline{20.3}$.

ii) Arc length $= \int_{\pi/6}^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{\pi/6}^{\pi/2} \sqrt{(2e^{\theta})^2 + (2e^{\theta})^2}\, d\theta$

$= \int_{\pi/6}^{\pi/2} \sqrt{4e^{2\theta} + 4e^{2\theta}}\, d\theta = \int_{\pi/6}^{\pi/2} \sqrt{8}\, e^{\theta}\, d\theta$

$= \sqrt{8}\int_{\pi/6}^{\pi/2} e^{\theta}\, d\theta = \sqrt{8}\left[e^{\theta}\right]_{\pi/6}^{\pi/2} = \sqrt{8}\left[e^{\pi/2} - e^{\pi/6}\right]$.

$= 8.831 \approx 8.83$.

41 The curve C has polar equation $r = 2\sin\theta(1-\cos\theta)$, for $0 \leqslant \theta \leqslant \pi$. Find $\dfrac{dr}{d\theta}$ and hence find the polar coordinates of the point of C that is furthest from the pole. [5]

Sketch C. [2]

Find the exact area of the sector from $\theta = 0$ to $\theta = \frac{1}{4}\pi$. [6]

$$r = 2\sin\theta(1-\cos\theta) = 2\sin\theta - 2\sin\theta\cos\theta.$$

$$\frac{dr}{d\theta} = 2\cos\theta - 2[\sin\theta(-\sin\theta) + \cos\theta(\cos\theta)] = 2\cos\theta - 2[\cos^2\theta - \sin^2\theta]$$

$$= 2\cos\theta - 2[\cos^2\theta - (1-\cos^2\theta)] = 2\cos\theta - 2[2\cos^2\theta - 1]$$

$$\frac{dr}{d\theta} = 2\cos\theta + 2 - 4\cos^2\theta$$

For maximum distance from pole, $\dfrac{dr}{d\theta} = 0 \Rightarrow 2\cos\theta + 2 - 4\cos^2\theta = 0$

$$2\cos^2\theta - \cos\theta - 1 = 0$$

$$2\cos^2\theta - 2\cos\theta + \cos\theta - 1 = 0$$

$$(2\cos\theta + 1)(\cos\theta - 1) = 0$$

$$\cos\theta = \frac{-1}{2}, 1.$$

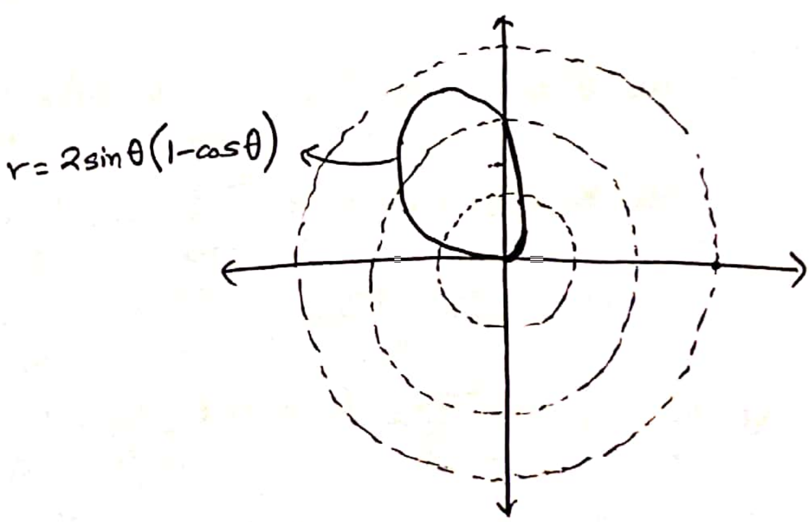
$$\Rightarrow \theta = \frac{2\pi}{3}, 0.$$

When $\theta = \frac{2\pi}{3}$, $r = 2\sin\left(\frac{2\pi}{3}\right)\left[1-\cos\left(\frac{2\pi}{3}\right)\right] = \frac{3\sqrt{3}}{2}$. [Maximum].

When $\theta = 0$, $r = 2\sin(0)[1-\cos(0)] = 0$

$\Rightarrow$ Polar coordinates of point furthest from pole: $\left(\frac{3\sqrt{3}}{2}, \frac{2\pi}{3}\right)$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	0	0.13	0.41	0.87	2	2.60	2.41	1.86	0



$$\text{Area} = \frac{1}{2}\int_0^{\pi/4} \left[2\sin\theta\left(1-\cos\theta\right)\right]^2 d\theta$$

$$= \frac{1}{2}\times 4 \int_0^{\pi/4} \sin^2\theta\left(1-2\cos\theta+\cos^2\theta\right) d\theta$$

$$= 2\int_0^{\pi/4} \sin^2\theta - 2\sin^2\theta\cos\theta + \sin^2\theta\cos^2\theta \, d\theta$$

$$= 2\int_0^{\pi/4} \frac{1-\cos 2\theta}{2} - 2\sin^2\theta\cos\theta + \frac{\sin^2 2\theta}{4} \, d\theta$$

$$= 2\int_0^{\pi/4} \frac{1}{2} - \frac{\cos 2\theta}{2} - 2\sin^2\theta\cos\theta + \frac{1-\cos 4\theta}{8} \, d\theta$$

$$= 2\int_0^{\pi/4} \frac{5}{8} - \frac{\cos 2\theta}{2} - 2\sin^2\theta\cos\theta - \frac{\cos 4\theta}{8} \, d\theta$$

$$= 2\left[\frac{5\theta}{8} - \frac{\sin 2\theta}{4} - \frac{2\sin^3\theta}{3} - \frac{\sin 4\theta}{32}\right]_0^{\pi/4}$$

$$= 2\left[\frac{5\pi}{32} - \frac{1}{4} - \frac{\sqrt{2}}{6}\right]$$

$$= \frac{5\pi}{16} - \frac{1}{2} - \frac{\sqrt{2}}{3}.$$

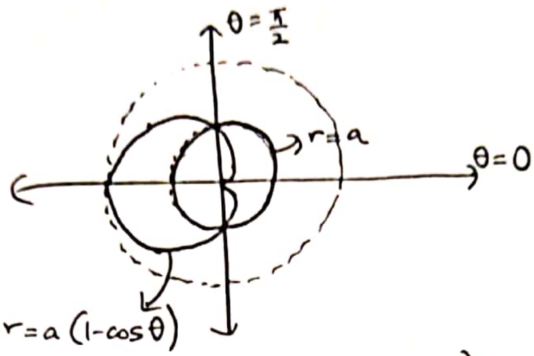
42 A circle has polar equation $r = a$, for $0 \leqslant \theta < 2\pi$, and a cardioid has polar equation $r = a(1 - \cos\theta)$, for $0 \leqslant \theta < 2\pi$, where a is a positive constant. Draw sketches of the circle and the cardioid on the same diagram. [3]

Write down the polar coordinates of the points of intersection of the circle and the cardioid. [2]

Show that the area of the region that is both inside the circle and inside the cardioid is

$$(\tfrac{5}{4}\pi - 2)a^2.$$ [6]

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	0	a	$2a$	a	0



For points of intersection,

$$a = a(1-\cos\theta)$$
$$1 = 1 - \cos\theta$$
$$\cos\theta = 0$$
$$\Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}.$$

$$\theta = \frac{\pi}{2} \Rightarrow r = a, \quad \theta = \frac{3\pi}{2} \Rightarrow r = a.$$

$$\therefore \left(\frac{\pi}{2}, a\right), \left(\frac{3\pi}{2}, a\right).$$

Required area

$$= \text{Area of semicircle} + 2\left[\frac{1}{2}\int_0^{\pi/2} [a(1-\cos\theta)]^2 \, d\theta\right]$$

$$= \frac{1}{2}\pi(a)^2 + \int_0^{\pi/2} a^2(1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= \frac{\pi a^2}{2} + a^2 \int_0^{\pi/2} 1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{\pi a^2}{2} + a^2 \int_0^{\pi/2} \frac{3 - 4\cos\theta + \cos 2\theta}{2}\, d\theta$$

$$= \frac{\pi a^2}{2} + \frac{a^2}{2}\left[3\theta - 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{\pi a^2}{2} + \frac{a^2}{2}\left[\frac{3\pi}{2} - 4\right] = \frac{5\pi}{4}a^2 - 2a^2$$

$$= \left(\frac{5\pi}{4} - 2\right)a^2.$$

43 The curve C has polar equation $r = a(1 - \cos\theta)$ for $0 \leqslant \theta < 2\pi$. Sketch C. [2]

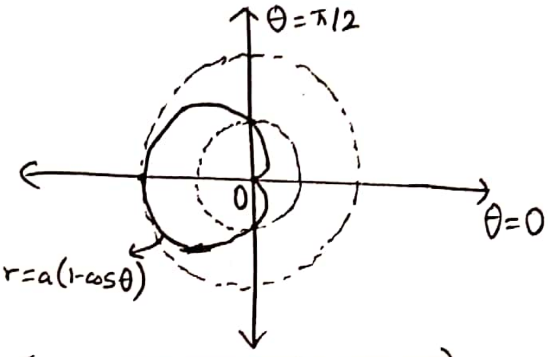
Find the area of the region enclosed by the arc of C for which $\frac{1}{2}\pi \leqslant \theta \leqslant \frac{3}{2}\pi$, the half-line $\theta = \frac{1}{2}\pi$ and the half-line $\theta = \frac{3}{2}\pi$. [5]

Show that

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^2 = 4a^2\sin^2(\tfrac{1}{2}\theta),$$

where s denotes arc length, and find the length of the arc of C for which $\frac{1}{2}\pi \leqslant \theta \leqslant \frac{3}{2}\pi$. [7]

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	0	$0.3a$	a	$1.7a$	$2a$	$1.7a$	a	$0.3a$	0



$$\text{Area} = 2 \times \frac{1}{2}\int_{\pi/2}^{\pi} \left[a(1-\cos\theta)\right]^2 = a^2\int_{\pi/2}^{\pi} 1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= a^2\int_{\pi/2}^{\pi} \frac{3+\cos 2\theta - 4\cos\theta}{2}\, d\theta = \frac{a^2}{2}\left[3\theta + \frac{\sin 2\theta}{2} - 4\sin\theta\right]_{\pi/2}^{\pi}$$

$$= \frac{a^2}{2}\left[\left(3\pi + 0 - 0\right) - \left(\frac{3\pi}{2} + 0 - 4\right)\right] = \frac{a^2}{2}\left[\frac{3\pi}{2} + 4\right]$$

$$= a^2\left(\frac{3\pi}{4} + 2\right).$$

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

$$= \left[a(1-\cos\theta)\right]^2 + \left[a(\sin\theta)\right]^2$$

$$= a^2(1 - 2\cos\theta + \cos^2\theta) + a^2(\sin^2\theta)$$

$$= a^2(2 - 2\cos\theta) = 2a^2(1-\cos\theta) = 2a^2\left(2\sin^2\frac{\theta}{2}\right)$$

$$\left(\frac{ds}{d\theta}\right)^2 = 4a^2\sin^2\left(\frac{\theta}{2}\right) \quad \text{(Shown)}.$$

$$\Rightarrow \frac{ds}{d\theta} = \sqrt{4a^2 \sin^2\left(\frac{\theta}{2}\right)} = 2a \sin\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \text{Arc length} = 2\int_{\pi/2}^{\pi} 2a \sin\left(\frac{\theta}{2}\right) d\theta$$

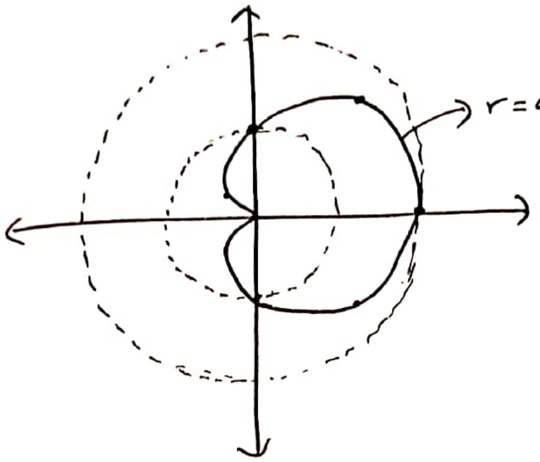
$$= 4a \left[\frac{-\cos\left(\frac{\theta}{2}\right)}{\frac{1}{2}}\right]_{\pi/2}^{\pi} = -8a\left[\cos\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{4}\right)\right]$$

$$\text{Arc length.} = -8a\left[0 - \left(\frac{\sqrt{2}}{2}\right)\right] = 4\sqrt{2}\, a.$$

44 The polar equation of a curve C is $r = a(1 + \cos\theta)$ for $0 \leqslant \theta < 2\pi$, where a is a positive constant.

(i) Sketch C. [2]

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	$2a$	$1.7a$	a	$0.3a$	0	$0.3a$	a	$1.7a$	$2a$



(ii) Show that the cartesian equation of C is

$$x^2 + y^2 = a(x + \sqrt{(x^2 + y^2)}).$$ [2]

$$r = a(1+\cos\theta)$$

$$\sqrt{x^2+y^2} = a\left(1+\frac{x}{r}\right)$$

$$\sqrt{x^2+y^2} = a\left[\frac{\sqrt{x^2+y^2}+x}{\sqrt{x^2+y^2}}\right]$$

$$x^2+y^2 = a\left(x+\sqrt{x^2+y^2}\right).$$

Scanned with CamScanner

(iii) Find the area of the sector of C between $\theta = 0$ and $\theta = \frac{1}{3}\pi$. [4]

$$\text{Area} = \frac{1}{2}\int_0^{\pi/3} \left[a\left(1+\cos\theta\right)\right]^2 d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/3} 1 + 2\cos\theta + \cos^2\theta \, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/3} 1 + 2\cos\theta + \frac{1+\cos 2\theta}{2} \, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/3} \frac{3 + 4\cos\theta + \cos 2\theta}{2} \, d\theta$$

$$= \frac{a^2}{4}\left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/3}$$

$$= \frac{a^2}{4}\left[\pi + 2\sqrt{3} + \frac{\sqrt{3}}{4}\right]$$

$$= \frac{a^2}{4}\left[\frac{4\pi + 9\sqrt{3}}{4}\right]$$

$$= \frac{a^2}{16}\left[4\pi + 9\sqrt{3}\right].$$

(iv) Find the arc length of C between the point where $\theta = 0$ and the point where $\theta = \frac{1}{3}\pi$. [5]

$$\text{Arc length} = \int_0^{\pi/3} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\pi/3} \sqrt{\left[a(1+\cos\theta)\right]^2 + \left[a(-\sin\theta)\right]^2}\, d\theta$$

$$= \int_0^{\pi/3} \sqrt{a^2(1+2\cos\theta+\cos^2\theta)+a^2\sin^2\theta}\, d\theta$$

$$= \int_0^{\pi/3} \sqrt{a^2+2a^2\cos\theta+a^2(\cos^2\theta+\sin^2\theta)}\, d\theta$$

$$= \int_0^{\pi/3} \sqrt{2a^2+2a^2\cos\theta}\, d\theta = \int_0^{\pi/3} \sqrt{2a^2(1+\cos\theta)}\, d\theta$$

$$= \sqrt{2}\,a \int_0^{\pi/3} \sqrt{1+\cos\theta}\, d\theta = \sqrt{2}\,a \int_0^{\pi/3} \sqrt{2\cos^2\frac{\theta}{2}}\, d\theta$$

$$= (\sqrt{2})^2 a \int_0^{\pi/3} \cos\left(\frac{\theta}{2}\right) d\theta = 2a\left[\frac{\sin\left(\frac{\theta}{2}\right)}{\frac{1}{2}}\right]_0^{\pi/3}$$

$$= 4a\left[\sin\left(\frac{\pi}{6}\right) - \sin(0)\right] = 4a\left[\frac{1}{2}-0\right] = 2a.$$

45 The curve C has polar equation

$$r = 5\sqrt{(\cot\theta)},$$

where $0.01 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(i) Find the area of the finite region bounded by C and the line $\theta = 0.01$, showing full working. Give your answer correct to 1 decimal place. [3]

$$\text{Area} = \frac{1}{2}\int_{0.01}^{\pi/2} \left(5\sqrt{\cot\theta}\right)^2 d\theta = \frac{25}{2}\int_{0.01}^{\pi/2} \cot\theta \, d\theta$$

$$= \frac{25}{2}\int_{0.01}^{\pi/2} \frac{\cos\theta}{\sin\theta}\, d\theta = \frac{25}{2}\Big[\ln|\sin\theta|\Big]_{0.01}^{\pi/2}$$

$$= \frac{25}{2}\left[\ln\left|\sin\left(\frac{\pi}{2}\right)\right| - \ln|\sin(0.01)|\right] = \frac{25}{2}\left[-\ln(\sin 0.01)\right]$$

$$\text{Area} = 57.56 \approx 57.6$$

Let P be the point on C where $\theta = 0.01$.

(ii) Find the distance of P from the initial line, giving your answer correct to 1 decimal place. [2]

$$y = r\sin\theta = 5\sqrt{\cot\theta}\,\sin\theta = 5\sqrt{\frac{\cos\theta}{\sin\theta}} \times \sin\theta = 5\sqrt{\cos\theta\,\sin\theta}.$$

when $\theta = 0.01$, $y = 5\sqrt{\cos(0.01)\sin(0.01)} = 0.4999 \approx 0.5$

(iii) Find the maximum distance of C from the initial line. **[3]**

$$y = 5\sqrt{\cos\theta \sin\theta} = 5\sqrt{\frac{\sin 2\theta}{2}} = \frac{5}{\sqrt{2}}\left(\sin 2\theta\right)^{\frac{1}{2}}.$$

$$\frac{dy}{d\theta} = \frac{5}{\sqrt{2}} \times \frac{1}{2}\left(\sin 2\theta\right)^{-\frac{1}{2}} \cdot \cos 2\theta \times 2 = \frac{5\cos 2\theta}{\sqrt{2}\sqrt{\sin 2\theta}} = \frac{5\cos 2\theta}{\sqrt{2\sin 2\theta}}.$$

For maximum distance, $\frac{dy}{d\theta} = 0 \Rightarrow \frac{5\cos 2\theta}{\sqrt{2\sin 2\theta}} = 0$

$$\cos 2\theta = 0$$

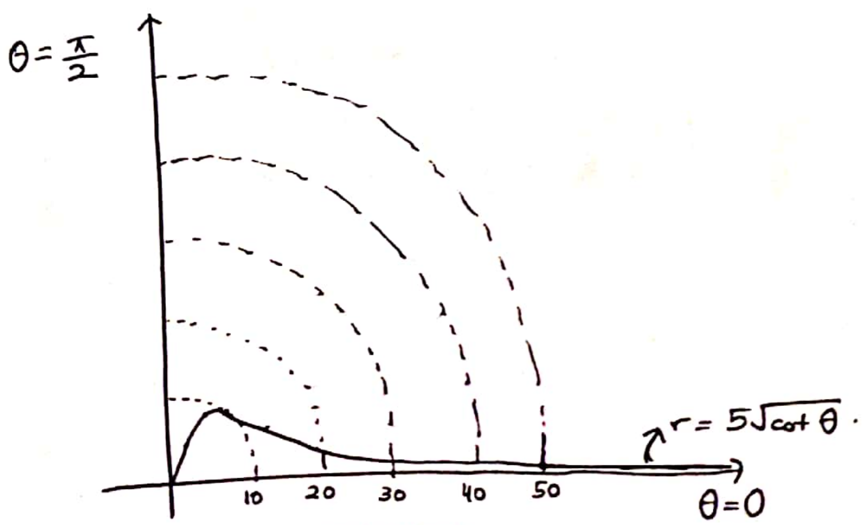
$$\Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}.$$

When $\theta = \frac{\pi}{4}$, $y = \frac{5}{\sqrt{2}}\left[\sin\left(2\times\frac{\pi}{4}\right)\right]^{\frac{1}{2}} = \frac{5}{\sqrt{2}}\left[1\right]^{\frac{1}{2}} = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}.$

$\therefore$ Maximum distance of C from initial line $= \frac{5\sqrt{2}}{2}.$

(iv) Sketch C. **[2]**

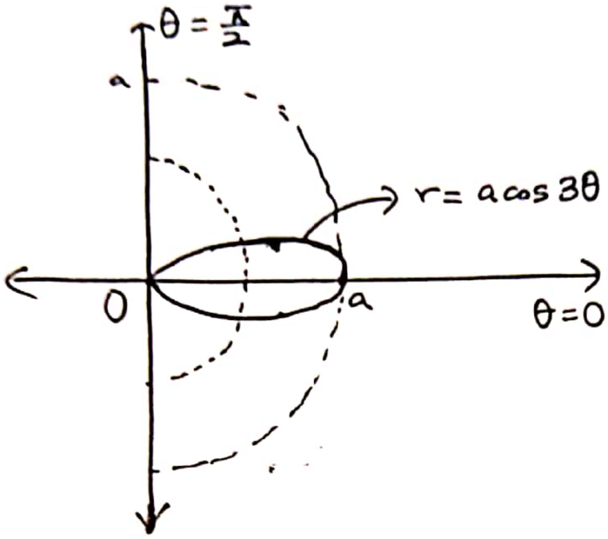
θ	0.01	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
r	50.0	9.7	6.6	5	3.8	2.6	0



46 The curve C has polar equation $r = a\cos 3\theta$, for $-\frac{1}{6}\pi \leqslant \theta \leqslant \frac{1}{6}\pi$, where a is a positive constant.

(i) Sketch C. [2]

θ	$\frac{-\pi}{6}$	$\frac{-\pi}{12}$	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$
r	0	$0.7a$	a	$0.7a$	0



(ii) Find the area of the region enclosed by C, showing full working. [3]

$$A = \frac{1}{2}\int_{-\pi/6}^{\pi/6} (a\cos 3\theta)^2\,d\theta = \frac{a^2}{2}\int_{-\pi/6}^{\pi/6} \cos^2 3\theta\,d\theta$$

$$= \frac{a^2}{2}\int_{-\pi/6}^{\pi/6} \frac{1+\cos 6\theta}{2}\,d\theta = \frac{a^2}{4}\int_{-\pi/6}^{\pi/6} 1+\cos 6\theta\,d\theta$$

$$= \frac{a^2}{4}\left[\theta + \frac{\sin 6\theta}{6}\right]_{-\pi/6}^{\pi/6}$$

$$= \frac{a^2}{4}\left[\left(\frac{\pi}{6}+0\right) - \left(\frac{-\pi}{6}+0\right)\right] = \frac{a^2}{4}\left(\frac{\pi}{3}\right) = \frac{\pi a^2}{12}.$$

(iii) Using the identity $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$, find a cartesian equation of C. [3]

$$r = a\cos 3\theta$$

$$r = a\left[4\cos^3\theta - 3\cos\theta\right]$$

$$r = a\cos\theta\left[4\cos^2\theta - 3\right]$$

$$\sqrt{x^2+y^2} = a\left(\frac{x}{r}\right)\left[4\left(\frac{x}{r}\right)^2 - 3\right]$$

$$\sqrt{x^2+y^2} = \frac{ax}{r}\left[\frac{4x^2-3r^2}{r^2}\right]$$

$$\sqrt{x^2+y^2} = \frac{ax\left[4x^2-3\left(x^2+y^2\right)\right]}{\left(\sqrt{x^2+y^2}\right)^3}$$

$$\left(\sqrt{x^2+y^2}\right)^4 = ax\left(x^2-3y^2\right)$$

$$\left(x^2+y^2\right)^2 = ax\left(x^2-3y^2\right).$$

47 The curves C_1 and C_2 have polar equations, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$, as follows:

$$C_1 : r = 2(e^{\theta} + e^{-\theta}),$$
$$C_2 : r = e^{2\theta} - e^{-2\theta}.$$

The curves intersect at the point P where $\theta = \alpha$.

(i) Show that $e^{2\alpha} - 2e^{\alpha} - 1 = 0$. Hence find the exact value of α and show that the value of r at P is $4\sqrt{2}$. [6]

When $\theta = \alpha$, $2(e^{\alpha}+e^{-\alpha}) = e^{2\alpha} - e^{-2\alpha}$

$$2 = \frac{(e^{\alpha})^2 - (e^{-\alpha})^2}{e^{\alpha}+e^{-\alpha}}$$

$$2 = \frac{(e^{\alpha}-e^{-\alpha})(e^{\alpha}+e^{-\alpha})}{(e^{\alpha}+e^{-\alpha})}$$

$$2 = e^{\alpha} - e^{-\alpha}$$

$$2 = e^{\alpha} - \frac{1}{e^{\alpha}}$$

$$2e^{\alpha} = e^{2\alpha} - 1$$

$$0 = e^{2\alpha} - 2e^{\alpha} - 1$$

$$\Rightarrow e^{\alpha} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)} = \frac{2 \pm \sqrt{4+4}}{2}$$

$$e^{\alpha} = \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$$

Note $e^{\alpha} > 0$ $\therefore$ Ignore $e^{\alpha} = 1 - \sqrt{2}$.

$$\Rightarrow e^{\alpha} = 1 + \sqrt{2}$$

$$\alpha = \ln(1+\sqrt{2}).$$

If $\alpha = \ln(1+\sqrt{2})$, $r = 2\left(e^{\ln(1+\sqrt{2})} + e^{-\ln(1+\sqrt{2})}\right)$

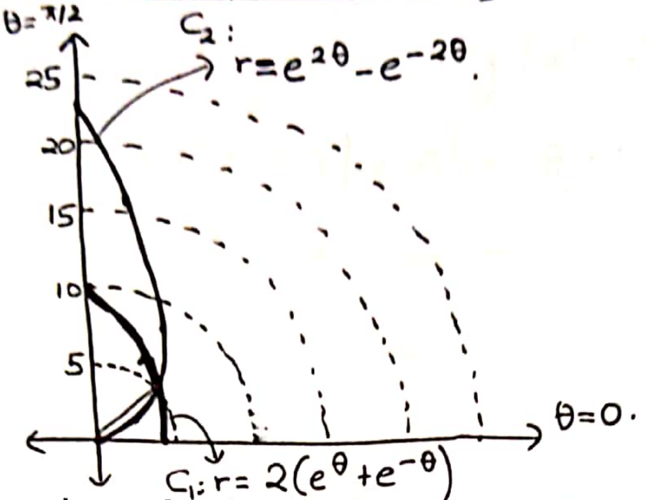
$$= 2\left[1+\sqrt{2} + (1+\sqrt{2})^{-1}\right]$$

$$= 2\left[1+\sqrt{2} + \frac{1}{1+\sqrt{2}}\right] = 2\left[1+\sqrt{2} + \frac{1-\sqrt{2}}{-1}\right]$$

$$= 2\left[1+\sqrt{2} - 1 + \sqrt{2}\right] = 2(2\sqrt{2}) = 4\sqrt{2}.$$

(ii) Sketch C_1 and C_2 on the same diagram. [3]

	θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$r=2(e^{\theta}+e^{-\theta})$	r	4	4.6	6.4	10.0
$r=e^{2\theta}-e^{-2\theta}$	r	0	2.5	8.0	23.1



(iii) Find the area of the region enclosed by C_1, C_2 and the initial line, giving your answer correct to 3 significant figures. [5]

$$\text{Area} = \frac{1}{2}\int_0^{\ln(1+\sqrt{2})} \left[2(e^{\theta}+e^{-\theta})\right]^2 - \left(e^{2\theta}-e^{-2\theta}\right)^2 \, d\theta$$

$$=\frac{1}{2}\int_0^{\ln(1+\sqrt{2})} 4\left(e^{2\theta}+2+e^{-2\theta}\right)-\left(e^{4\theta}-2+e^{-4\theta}\right) d\theta$$

$$=\frac{1}{2}\int_0^{\ln(1+\sqrt{2})} 4e^{2\theta}+8+4e^{-2\theta}-e^{4\theta}+2-e^{-4\theta} \, d\theta$$

$$=\frac{1}{2}\int_0^{\ln(1+\sqrt{2})} 4e^{2\theta}+4e^{-2\theta}-e^{4\theta}-e^{-4\theta}+10 \, d\theta$$

$$=\int_0^{\ln(1+\sqrt{2})} 2e^{2\theta}+2e^{-2\theta}-\frac{e^{4\theta}}{2}-\frac{e^{-4\theta}}{2}+5 \, d\theta$$

$$=\left[e^{2\theta}-e^{-2\theta}-\frac{e^{4\theta}}{8}+\frac{e^{-4\theta}}{8}+5\theta\right]_0^{\ln(1+\sqrt{2})}$$

$$=\left[e^{2\ln(1+\sqrt{2})}-e^{-2\ln(1+\sqrt{2})}-\frac{e^{4\ln(1+\sqrt{2})}}{8}+\frac{e^{-4\ln(1+\sqrt{2})}}{8}+5\ln(1+\sqrt{2})\right]-\left[1-1-\frac{1}{8}+\frac{1}{8}+0\right]$$

$$=5.821 \approx 5.82.$$

48 (a) Show that the curve with Cartesian equation

$$(x^2+y^2)^{\frac{5}{2}} = 4xy(x^2-y^2)$$

has polar equation $r = \sin 4\theta$. [4]

$$\left(x^2+y^2\right)^{\frac{5}{2}} = 4xy\left(x^2-y^2\right)$$

$$\left(r^2\right)^{\frac{5}{2}} = 4\left[r\cos\theta\right]\left[r\sin\theta\right]\left[\left(r\cos\theta\right)^2-\left(r\sin\theta\right)^2\right]$$

$$r^5 = 4r^2\cos\theta\cos\theta\left[r^2\cos^2\theta-r^2\sin^2\theta\right]$$

$$r^5 = 4r^4\cos\theta\sin\theta\left[\cos^2\theta-\sin^2\theta\right]$$

$$r = 4\cos\theta\sin\theta\left[\cos^2\theta-\sin^2\theta\right]$$

$$r = 2\left[2\sin\theta\cos\theta\right]\left[\cos^2\theta-\sin^2\theta\right]$$

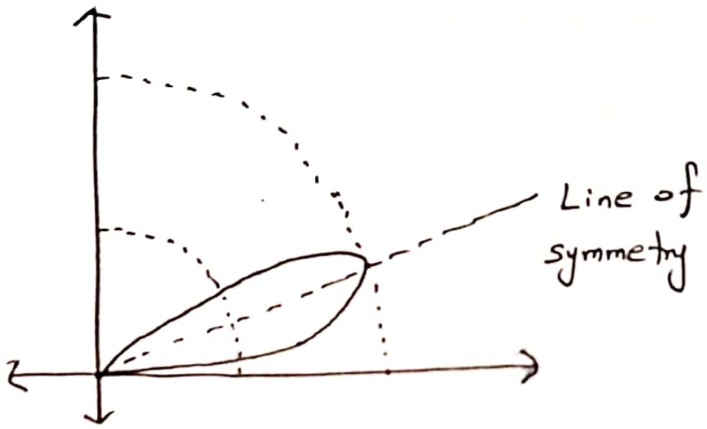
$$r = 2\left(\sin 2\theta\right)\left(\cos 2\theta\right)$$

$$r = \sin 4\theta$$

The curve C has polar equation $r = \sin 4\theta$, for $0 \leqslant \theta \leqslant \frac{1}{4}\pi$.

(b) Sketch C and state the equation of the line of symmetry. [3]

θ	0	$\frac{\pi}{16}$	$\frac{\pi}{8}$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$
r	0	0.71	1	0.71	0



Line of symmetry: $\theta = \frac{\pi}{8}$.

(c) Find the exact value of the area of the region enclosed by C. [4]

$$\text{Area} = \frac{1}{2}\int_0^{\pi/4} \sin^2 4\theta \, d\theta = \frac{1}{2}\int_0^{\pi/4} \frac{1-\cos 8\theta}{2} \, d\theta$$

$$= \frac{1}{4}\int_0^{\pi/4} 1-\cos 8\theta \, d\theta = \frac{1}{4}\left[\theta - \frac{\sin 8\theta}{8}\right]_0^{\pi/4}$$

$$= \frac{1}{4}\left[\left(\frac{\pi}{4} - \frac{\sin\left(8\times\frac{\pi}{4}\right)}{8}\right) - 0\right] = \frac{1}{4}\left[\frac{\pi}{4} - 0\right]$$

$$= \frac{\pi}{16}.$$

(d) Using the identity $\sin 4\theta \equiv 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta$, find the maximum distance of C from the line $\theta = \frac{1}{2}\pi$. Give your answer correct to 2 decimal places. [6]

We want maximum value of x:

$$x = r\cos\theta = \sin 4\theta\cos\theta$$
$$= (4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta)\cos\theta$$
$$= 4\sin\theta\cos^4\theta - 4\sin^3\theta\cos^2\theta$$
$$= 4\sin\theta(1-\sin^2\theta)^2 - 4\sin^3\theta(1-\sin^2\theta)$$
$$= 4\sin\theta(1-2\sin^2\theta+\sin^4\theta) - 4\sin^3\theta(1-\sin^2\theta)$$
$$= 4\sin\theta - 8\sin^3\theta + 4\sin^5\theta - 4\sin^3\theta + 4\sin^5\theta$$
$$x = 8\sin^5\theta - 12\sin^3\theta + 4\sin\theta$$

To maximize x, $\frac{dx}{d\theta} = 0$

$$40\sin^4\theta\cos\theta - 36\sin^2\theta\cos\theta + 4\cos\theta = 0$$
$$4\cos\theta(10\sin^4\theta - 9\sin^2\theta + 1) = 0$$

Either $4\cos\theta = 0$

$\cos\theta = 0$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

For these angles, $r = \sin 4\theta = 0$

So x is not maximum here.

or $10\sin^4\theta - 9\sin^2\theta + 1 = 0$

$$\sin^2\theta = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(10)(1)}}{2(10)}$$
$$\sin^2\theta = \frac{9 \pm \sqrt{41}}{20}$$
$$\sin\theta = \pm\sqrt{\frac{9\pm\sqrt{41}}{20}}$$

Either $\sin\theta = \pm\sqrt{\frac{9+\sqrt{41}}{20}}$ or $\sin\theta = \pm\sqrt{\frac{9-\sqrt{41}}{20}}$

$\theta = \pm 1.0708$ $\qquad$ $\theta = \pm 0.3686$

Ignore values of θ which lie outside domain

$\therefore \theta = 1.0708$ or $\theta = 0.3686$

When $\theta = 1.0708$, $\cancel{r} = \sin(4\times 1.0708) \times \cancel{\cos(1.0708)} = -0.909$ (Ignore $\theta = 1.070$)

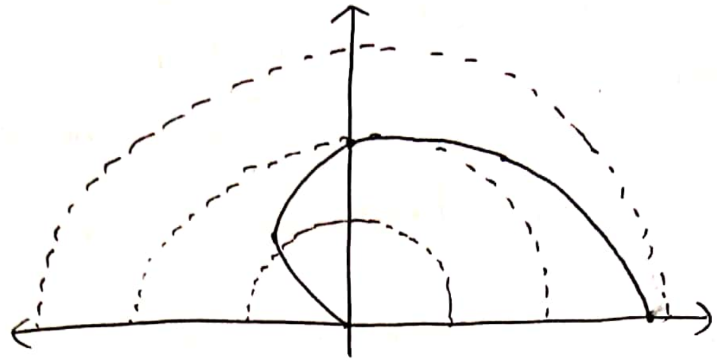
When $\theta = 0.3686$, $x = \sin(4\times 0.3686)\times\cos(0.3686) = 0.9285$

$\therefore$ Maximum distance ≈ 0.93 from $\theta = \frac{\pi}{2}$

49 The curve C has polar equation $r = \ln(1+\pi-\theta)$, for $0 \leqslant \theta \leqslant \pi$.

(a) Sketch C and state the polar coordinates of the point of C furthest from the pole. [3]

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
r	1.4	1.2	0.94	0.58	0



Point furthest from pole occurs when $\theta = 0 \Rightarrow r = \ln(1+\pi-0) = \ln(1+\pi)$

$\Rightarrow (\ln(1+\pi), 0)$.

(b) Using the substitution $u = 1+\pi-\theta$, or otherwise, show that the area of the region enclosed by C and the initial line is

$$\tfrac{1}{2}(1+\pi)\ln(1+\pi)\big(\ln(1+\pi)-2\big)+\pi.$$ [6]

$$A = \frac{1}{2}\int_0^{\pi}\left[\ln(1+\pi-\theta)\right]^2 d\theta$$

Let, $u = 1+\pi-\theta \Rightarrow du = -d\theta \Rightarrow d\theta = -du$.

when $\theta = 0$, $u = 1+\pi-0 = 1+\pi$.

when $\theta = \pi$, $u = 1+\pi-\pi = 1$

$$A = \frac{1}{2}\int_0^{\pi}\left[\ln(1+\pi-\theta)\right]^2 d\theta = \frac{1}{2}\int_{1+\pi}^{1}\left[\ln(u)\right]^2(-du)$$

$$= \frac{-1}{2}\int_{1+\pi}^{1}(\ln u)^2 du.$$

$\int (\ln u)^2 du$

$U = (\ln u)^2 \qquad \frac{dV}{du} = 1$

$\frac{dU}{du} = \frac{1}{u}\times 2\ln u \qquad V = u$

$$\Rightarrow \int(\ln u)^2 du = u(\ln u)^2 - \int u \times \frac{2\ln u}{u}\,du$$

$$= u(\ln u)^2 - 2\int \ln u\, du.$$

$$= u(\ln u)^2 - 2(u\ln u - \cancel{u})$$

$$= u(\ln u)^2 - 2u\ln u + 2u$$

$$\Rightarrow A = \frac{-1}{2}\int_{1+\pi}^{1}(\ln u)^2\,du = \frac{-1}{2}\Big[u(\ln u)^2 - 2u\ln u + 2u\Big]_{1+\pi}^{1}$$

$$= \frac{-1}{2}\Big[\big[1\times(\ln 1)^2 - 2\times(1)\ln(1) + 2\times 1\big] - \big[(1+\pi)[\ln(1+\pi)]^2 - 2\times(1+\pi)\ln(1+\pi) + 2(1+\pi)\big]\Big]$$

$$= \frac{-1}{2}\Big[\cancel{2} - (1+\pi)[\ln(1+\pi)]^2 + 2(1+\pi)\ln(1+\pi) - \cancel{2} - 2\pi\Big]$$

$$= \frac{-1}{2}\Big\{\ln(1+\pi)\big[2(1+\pi) - (1+\pi)\ln(1+\pi)\big] - 2\pi\Big\}$$

$$= \frac{-1}{2}\ln(1+\pi)\cdot(1+\pi)\big[2 - \ln(1+\pi)\big] + \pi$$

$$= \frac{1}{2}(1+\pi)\ln(1+\pi)\big[\ln(1+\pi) - 2\big] + \pi$$

(c) Show that, at the point of C furthest from the initial line,

$$(1+\pi-\theta)\ln(1+\pi-\theta) - \tan\theta = 0$$

and verify that this equation has a root between 1.2 and 1.3. [5]

We want to maximize $y \Rightarrow y = r\sin\theta = \ln(1+\pi-\theta)\sin\theta$

$$\frac{dy}{d\theta} = \ln(1+\pi-\theta)\cos\theta + \sin\theta\left[\frac{1}{1+\pi-\theta}\right]\cdot(-1)$$

$$\frac{dy}{d\theta} = \ln(1+\pi-\theta)\cos\theta - \frac{\sin\theta}{1+\pi-\theta}$$

To maximize y, set $\frac{dy}{d\theta} = 0 \Rightarrow \ln(1+\pi-\theta)\cos\theta - \frac{\sin\theta}{1+\pi-\theta} = 0$

$$\ln(1+\pi-\theta)\cos\theta = \frac{\sin\theta}{1+\pi-\theta}$$

$$(1+\pi-\theta)\ln(1+\pi-\theta) = \tan\theta$$

$$\Rightarrow (1+\pi-\theta)\ln(1+\pi-\theta) - \tan\theta = 0 \quad (\text{shown}).$$

Let, $f(\theta) = (1+\pi-\theta)\ln(1+\pi-\theta) - \tan\theta$.

$f(1.2) = (1+\pi-1.2)\ln(1+\pi-1.2) - \tan(1.2) = 0.602$

$f(1.3) = (1+\pi-1.3)\ln(1+\pi-1.3) - \tan(1.3) = -0.634$

$f(1.2) > 0$ and $f(1.3) < 0$ $\therefore$ there is root between 1.2 and 1.3.

50 The curve C_1 has polar equation $r = \theta\cos\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) The point on C_1 furthest from the line $\theta = \frac{1}{2}\pi$ is denoted by P. Show that, at P,

$$2\theta\tan\theta - 1 = 0$$

and verify that this equation has a root between 0.6 and 0.7. [5]

We want to maximize $x \Rightarrow x = r\cos\theta = (\theta\cos\theta)\cos\theta = \theta\cos^2\theta$.

$\frac{dx}{d\theta} = \theta\left[2\cos\theta(-\sin\theta)\right] + \cos^2\theta\,(1) = -2\theta\cos\theta\sin\theta + \cos^2\theta$.

For maximum value of x, set $\frac{dx}{d\theta} = 0 \Rightarrow -2\theta\cos\theta\sin\theta + \cos^2\theta = 0$

$\cos\theta\,(-2\theta\sin\theta + \cos\theta) = 0$

Note $\cos\theta \neq 0$ since if $\cos\theta = 0$, $\theta = \frac{\pi}{2}$ and $r = 0$ but then P lies on pole so distance from $\theta = \frac{\pi}{2}$ is zero.

$\Rightarrow -2\theta\sin\theta + \cos\theta = 0$

$2\theta\sin\theta = \cos\theta$

$2\theta\tan\theta = 1$

$2\theta\tan\theta - 1 = 0$ (Shown).

Let, $f(\theta) = 2\theta\tan\theta - 1$.

$f(0.6) = 2\times 0.6\times\tan(0.6) - 1 = -0.179$

$f(0.7) = 2\times 0.7\times\tan(0.7) - 1 = 0.179$

Since $f(0.6) < 0$ and $f(0.179) > 0$ ∴ This equation has a root between 0.6 and 0.7.

The curve C_2 has polar equation $r = \theta\sin\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. The curves C_1 and C_2 intersect at the pole, denoted by O, and at another point Q.

(b) Find the polar coordinates of Q, giving your answers in exact form. [2]

Set $\theta\cos\theta = \theta\sin\theta$

$\theta(\cos\theta - \sin\theta) = 0$

$\Rightarrow \theta = 0$ or $\cos\theta - \sin\theta = 0$

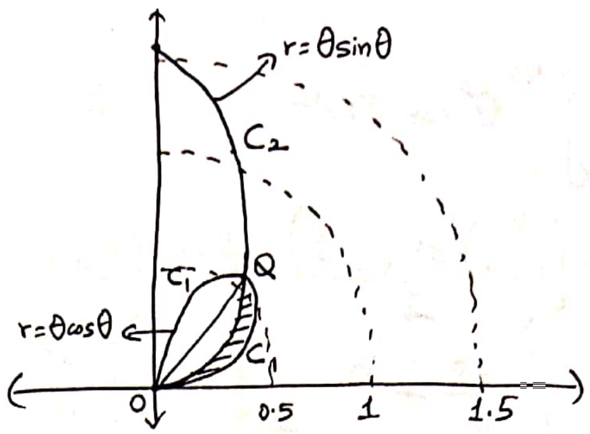
$1 = \tan\theta$

$\Rightarrow \theta = \frac{\pi}{4}$.

When $\theta = \frac{\pi}{4}$, $r = \frac{\pi}{4}\sin\left(\frac{\pi}{4}\right) = \frac{\pi}{4}\times\frac{\sqrt{2}}{2} = \frac{\sqrt{2}\pi}{8}$,

$\Rightarrow Q\left(\frac{\sqrt{2}\pi}{8}, \frac{\pi}{4}\right)$.

(c) Sketch C_1 and C_2 on the same diagram. [3]



(d) Find, in terms of π, the area of the region bounded by the arc OQ of C_1 and the arc OQ of C_2. [7]

Area of region

$$= \frac{1}{2}\int_0^{\pi/4} (\theta\cos\theta)^2 - (\theta\sin\theta)^2\, d\theta = \frac{1}{2}\int_0^{\pi/4} \theta^2\cos^2\theta - \theta^2\sin^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \theta^2(\cos^2\theta - \sin^2\theta)\, d\theta = \frac{1}{2}\int_0^{\pi/4} \theta^2\cos 2\theta\, d\theta$$

$u = \theta^2$ $\qquad \frac{dv}{d\theta} = \cos 2\theta$

$\frac{du}{d\theta} = 2\theta$ $\qquad v = \frac{\sin 2\theta}{2}$

$$\int \theta^2 \cos 2\theta\, d\theta = \frac{\theta^2 \sin 2\theta}{2} - \int 2\theta\, \frac{\sin 2\theta}{2}\, d\theta$$

$$= \frac{\theta^2 \sin 2\theta}{2} - \int \theta \sin 2\theta\, d\theta.$$

$u = \theta$ $\qquad \frac{dv}{d\theta} = \sin 2\theta$

$\frac{du}{d\theta} = 1$ $\qquad v = \frac{-\cos 2\theta}{2}$

$$\Rightarrow \int \theta^2 \cos 2\theta\, d\theta = \frac{\theta^2 \sin 2\theta}{2} - \left[\frac{\theta(-\cos 2\theta)}{2} - \int \frac{-\cos 2\theta}{2}(1)\, d\theta\right]$$

$$= \frac{\theta^2 \sin 2\theta}{2} - \left[\frac{-\theta\cos 2\theta}{2} + \frac{1}{2}\int \cos 2\theta\, d\theta\right]$$

$$= \frac{\theta^2 \sin 2\theta}{2} - \left[-\frac{\theta \cos 2\theta}{2} + \frac{1}{2}\left(\frac{\sin 2\theta}{2}\right)\right]$$

$$= \frac{\theta^2 \sin 2\theta}{2} + \frac{\theta \cos 2\theta}{2} - \frac{\sin 2\theta}{4}$$

$$\therefore \; A = \frac{1}{2}\int_0^{\pi/4} \theta^2 \cos 2\theta \, d\theta = \frac{1}{2}\left[\frac{\theta^2 \sin 2\theta}{2} + \frac{\theta \cos 2\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/4}$$

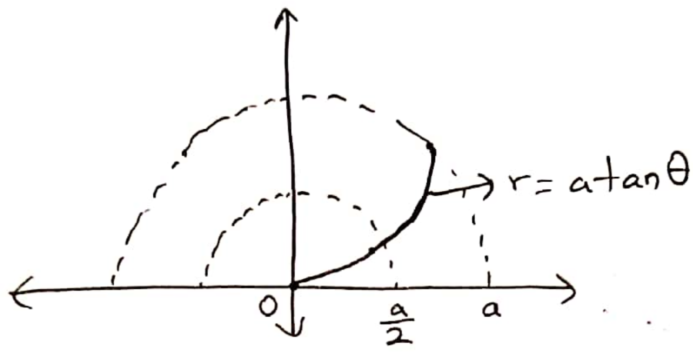
$$= \frac{1}{4}\left[\left(\frac{\left(\frac{\pi}{4}\right)^2 \sin\left(2 \times \frac{\pi}{4}\right)}{1} + \frac{\frac{\pi}{4} \times \cos\left(2 \times \frac{\pi}{4}\right)}{1} - \frac{\sin\left(2 \times \frac{\pi}{4}\right)}{2}\right) - \left(0\right)\right]$$

$$= \frac{1}{4}\left[\frac{\pi^2}{16} + 0 - \frac{1}{2}\right] = \frac{1}{4}\left[\frac{\pi^2}{16} - \frac{1}{2}\right] = \frac{\pi^2}{64} - \frac{1}{8}.$$

51 The curve C has polar equation $r = a\tan\theta$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{4}\pi$.

(a) Sketch C and state the greatest distance of a point on C from the pole. [2]

θ	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$
r	0	$0.41a$	a



Greatest distance from pole $= a$.

(b) Find the exact value of the area of the region bounded by C and the half-line $\theta = \frac{1}{4}\pi$. [4]

$$A = \frac{1}{2}\int_0^{\pi/4} (a\tan\theta)^2 \, d\theta = \frac{a^2}{2}\int_0^{\pi/4} \tan^2\theta \, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/4} \sec^2\theta - 1 \, d\theta = \frac{a^2}{2}\Big[\tan\theta - \theta\Big]_0^{\pi/4}$$

$$= \frac{a^2}{2}\left\{\left[\tan\frac{\pi}{4} - \frac{\pi}{4}\right] - \left[0 - 0\right]\right\} = \frac{a^2}{2}\left(1 - \frac{\pi}{4}\right).$$

(c) Show that C has Cartesian equation $y = \dfrac{x^2}{\sqrt{a^2 - x^2}}$. [3]

$$r = a \tan\theta$$

$$\sqrt{x^2+y^2} = a\left[\frac{y}{x}\right]$$

$$x^2+y^2 = \frac{a^2y^2}{x^2}$$

$$x^4 + x^2y^2 = a^2y^2$$

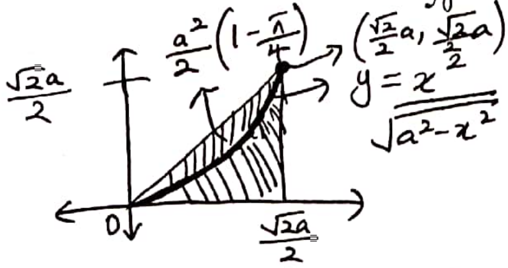
$$x^4 = a^2y^2 - x^2y^2$$

$$x^4 = y^2(a^2-x^2)$$

$$\frac{x^4}{a^2-x^2} = y^2$$

$$\Rightarrow y = \frac{x^2}{\sqrt{a^2-x^2}}.$$

(d) Using your answer to part **(b)**, deduce the exact value of $\displaystyle\int_0^{\frac{1}{2}a\sqrt{2}} \frac{x^2}{\sqrt{a^2-x^2}}\,\mathrm{d}x$. [2]



When $\theta = \frac{\pi}{4}$, $x = r\cos\theta = a\tan\theta\cos\theta = a\tan\frac{\pi}{4}\cos\frac{\pi}{4} = \frac{\sqrt{2}a}{2}$.

$y = r\sin\theta = a\tan\theta\sin\theta = a\tan\frac{\pi}{4}\sin\frac{\pi}{4} = \frac{\sqrt{2}a}{2}$.

$\Rightarrow \displaystyle\int_0^{\frac{1}{2}a\sqrt{2}} \frac{x^2}{\sqrt{a^2-x^2}}\,dx =$ Area of Δ − Area of region between $\theta = \pi/4$ and $r = a\tan\theta$

$$= \frac{1}{2} \times \frac{\sqrt{2}a}{2} \times \frac{\sqrt{2}a}{2} - \frac{a^2}{2}\left(1-\frac{\pi}{4}\right)$$

$$= \frac{a^2}{4} - \frac{a^2}{2} + \frac{\pi a^2}{8} = \frac{\pi a^2}{8} - \frac{a^2}{4}$$

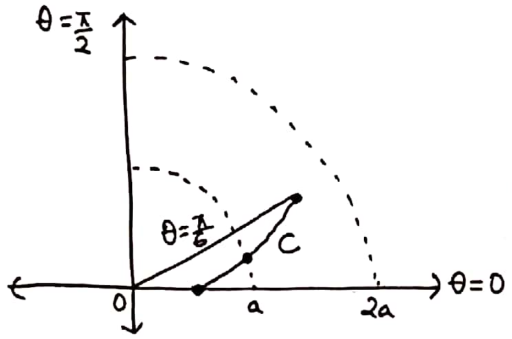
$$= \frac{a^2}{4}\left(\frac{\pi}{2} - 1\right)$$

52 The curve C has polar equation $r = a\cot\left(\frac{1}{3}\pi - \theta\right)$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{6}\pi$.

It is given that the greatest distance of a point on C from the pole is $2\sqrt{3}$.

(a) Sketch C and show that $a = 2$. [3]

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$
r	$\frac{\sqrt{3}}{3}a$	a	$\sqrt{3}a$



From the sketch, greatest distance of a point on C from pole $= \sqrt{3}a$

$$\Rightarrow 2\sqrt{3} = \sqrt{3}a$$
$$\Rightarrow a = 2.$$

(b) Find the exact value of the area of the region bounded by C, the initial line and the half-line $\theta = \frac{1}{6}\pi$. [4]

$$\text{Area} = \int_0^{\pi/6} \frac{1}{2} r^2 \, d\theta = \frac{1}{2}\int_0^{\pi/6} \left[2\cot\left(\frac{\pi}{3} - \theta\right)\right]^2 d\theta$$

$$= \frac{1}{2} \times 4 \int_0^{\pi/6} \cot^2\left(\frac{\pi}{3} - \theta\right) d\theta$$

$$= 2\int_0^{\pi/6} \cot^2\left(\frac{\pi}{3} - \theta\right) d\theta = 2\int_0^{\pi/6} \operatorname{cosec}^2\left(\frac{\pi}{3} - \theta\right) - 1 \, d\theta$$

$$= 2\left[\frac{-\cot\left(\frac{\pi}{3} - \theta\right)}{-1} - \theta\right]_0^{\pi/6} = 2\left[\cot\left(\frac{\pi}{3} - \theta\right) - \theta\right]_0^{\pi/6}$$

$$= 2\left[\left(\sqrt{3} - \frac{\pi}{6}\right) - \left(\frac{\sqrt{3}}{3} - 0\right)\right] = 2\left[\frac{2\sqrt{3}}{3} - \frac{\pi}{6}\right] = \frac{4\sqrt{3}}{3} - \frac{\pi}{3}$$

$$= \frac{4\sqrt{3} - \pi}{3}.$$

(c) Show that C has Cartesian equation $2(x+y\sqrt{3}) = (x\sqrt{3}-y)\sqrt{x^2+y^2}$. [3]

$$r = a\cot\left(\frac{1}{3}\pi - \theta\right)$$

$$r\tan\left(\frac{\pi}{3}-\theta\right) = a$$

$$r\left[\frac{\tan\left(\frac{\pi}{3}\right)-\tan\theta}{1+\tan\left(\frac{\pi}{3}\right)\tan\theta}\right] = a$$

$$r\left[\frac{\sqrt{3}-\tan\theta}{1+\sqrt{3}\tan\theta}\right] = a$$

Since $r = \sqrt{x^2+y^2}$ and $\tan\theta = \frac{y}{x}$

$$\Rightarrow \sqrt{x^2+y^2}\left[\frac{\sqrt{3}-\frac{y}{x}}{1+\sqrt{3}\frac{y}{x}}\right] = a$$

$$\frac{\sqrt{x^2+y^2}\left[\frac{\sqrt{3}x-y}{x}\right]}{\frac{x+\sqrt{3}y}{x}} = a$$

$$\frac{\sqrt{x^2+y^2}\,(\sqrt{3}x-y)}{x} = \frac{a\,(x+\sqrt{3}y)}{x}$$

$$\Rightarrow \sqrt{x^2+y^2}\,(\sqrt{3}x-y) = a\,(x+\sqrt{3}y).$$

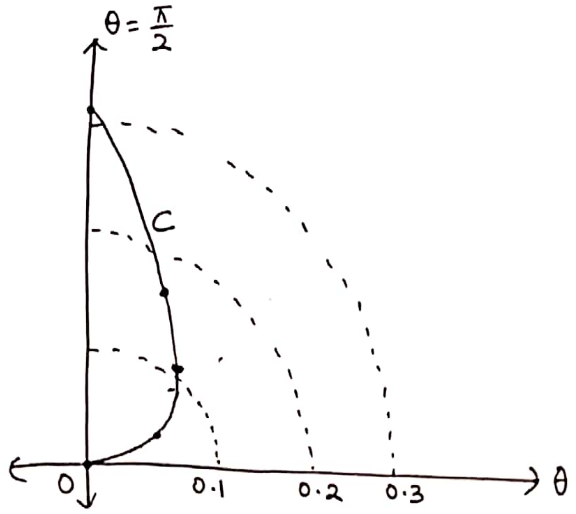
Since $a = 2$

$$\therefore \sqrt{x^2+y^2}\,(\sqrt{3}x-y) = 2\,(x+\sqrt{3}y).$$ (Shown).

53 The curve C has polar equation $r = \dfrac{1}{\pi - \theta} - \dfrac{1}{\pi}$, where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Sketch C. [3]

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	0	0.06	0.11	0.16	0.32



(b) Show that the area of the region bounded by the half-line $\theta = \frac{1}{2}\pi$ and C is $\dfrac{3 - 4\ln 2}{4\pi}$. [6]

$$A = \int_0^{\pi/2} \frac{1}{2} r^2 \, d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}\left[\frac{1}{\pi-\theta} - \frac{1}{\pi}\right]^2 d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{1}{(\pi-\theta)^2} - 2\left(\frac{1}{\pi-\theta}\right)\left(\frac{1}{\pi}\right) + \left(\frac{1}{\pi}\right)^2 d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} (\pi-\theta)^{-2} - \frac{2}{\pi}\left[\frac{1}{\pi-\theta}\right] + \frac{1}{\pi^2} \, d\theta$$

$$= \frac{1}{2}\left[\frac{(\pi-\theta)^{-1}}{(-1)(-1)} - \frac{2}{\pi}\,\frac{\ln(\pi-\theta)}{(-1)} + \frac{\theta}{\pi^2}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{1}{\pi-\theta} + \frac{2}{\pi}\ln(\pi-\theta) + \frac{\theta}{\pi^2}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{1}{\pi-\frac{\pi}{2}} + \frac{2}{\pi}\ln\left(\pi-\frac{\pi}{2}\right) + \frac{\frac{\pi}{2}}{\pi^2}\right) - \left(\frac{1}{\pi-0} + \frac{2}{\pi}\ln(\pi-0) + \frac{0}{\pi^2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{2}{\pi} + \frac{2}{\pi}\ln\left(\frac{\pi}{2}\right) + \frac{1}{2\pi} - \frac{1}{\pi} - \frac{2}{\pi}\ln(\pi)\right]$$

$$= \frac{1}{2}\left[\frac{2}{\pi} + \frac{1}{2\pi} - \frac{1}{\pi} + \frac{2}{\pi}\ln\left(\frac{\frac{\pi}{2}}{\pi}\right)\right] = \frac{1}{2}\left[\frac{4+1-2}{2\pi} + \frac{2}{\pi}\ln\left(\frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{3}{2\pi} + \frac{2}{\pi}\ln(2^{-1})\right] = \frac{1}{2}\left[\frac{3}{2\pi} - \frac{2\ln 2}{\pi}\right]$$

$$= \frac{1}{2}\left[\frac{3 - 4\ln 2}{2\pi}\right]$$

$$= \frac{3-4\ln 2}{4\pi} \text{ (Shown).}$$

54 The curve C has polar equation $r = 2\cos\theta(1+\sin\theta)$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Find the polar coordinates of the point on C that is furthest from the pole. [5]

$r = 2\cos\theta(1+\sin\theta) \quad 0 \le \theta \le \frac{\pi}{2}.$

$$\frac{dr}{d\theta} = 2\cos\theta(\cos\theta) + (1+\sin\theta)(-2\sin\theta)$$

$$= 2\cos^2\theta - 2\sin\theta(1+\sin\theta)$$

$$\frac{dr}{d\theta} = 2\cos^2\theta - 2\sin\theta - 2\sin^2\theta$$

For maximum distance from pole, set $\frac{dr}{d\theta} = 0$

$$\Rightarrow 2\cos^2\theta - 2\sin\theta - 2\sin^2\theta = 0$$

$$\cos^2\theta - \sin\theta - \sin^2\theta = 0$$

$$1 - \sin^2\theta - \sin\theta - \sin^2\theta = 0$$

$$-2\sin^2\theta - \sin\theta + 1 = 0$$

$$2\sin^2\theta + \sin\theta - 1 = 0$$

$$2\sin^2\theta + 2\sin\theta - \sin\theta - 1 = 0$$

$$(2\sin\theta - 1)(\sin\theta + 1) = 0$$

$\sin\theta = \frac{1}{2}$ or $\sin\theta = -1$

Ignore since $\sin\theta > 0$ for $0 \le \theta \le \frac{\pi}{2}$.

$$\theta = \sin^{-1}\left(\frac{1}{2}\right)$$

$$\theta = \frac{\pi}{6}.$$

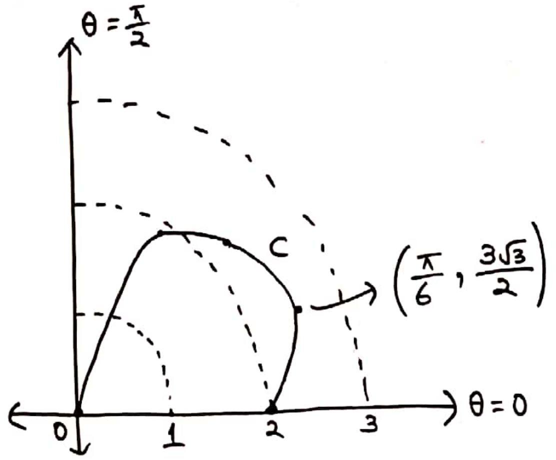
$$\Rightarrow r = 2\cos\left(\frac{\pi}{6}\right)\left(1+\sin\left(\frac{\pi}{6}\right)\right)$$

$$r = \frac{3\sqrt{3}}{2}$$

$\Rightarrow$ Point furthest from pole $= \cancel{\left(\frac{\pi}{6}, \frac{3\sqrt{3}}{2}\right)}$. $\left(\frac{3\sqrt{3}}{2}, \frac{\pi}{6}\right)$

(b) Sketch C. [2]

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	2	2.60	2.41	1.87	0



(c) Find the area of the region bounded by C and the initial line, giving your answer in exact form. [6]

$$\text{Area} = \int_0^{\frac{\pi}{2}} \frac{1}{2} r^2 \, d\theta = \int_0^{\pi/2} \frac{1}{2}\left[2\cos\theta\left(1+\sin\theta\right)\right]^2 d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2} \cdot 4\cos^2\theta\left(1+\sin\theta\right)^2 d\theta$$

$$= 2\int_0^{\pi/2} \cos^2\theta\left[1+2\sin\theta+\sin^2\theta\right] d\theta = 2\int_0^{\pi/2} \cos^2\theta + 2\sin\theta\cos^2\theta + \sin^2\theta\cos^2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \frac{1+\cos 2\theta}{2} + 2\sin\theta\cos^2\theta + \left(\sin\theta\cos\theta\right)^2 d\theta$$

$$= \int_0^{\pi/2} 1+\cos 2\theta \, d\theta + 4\int_0^{\pi/2} \sin\theta\cos^2\theta \, d\theta + 2\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2 d\theta$$

$$= \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} + 4\left[-\int_0^{\pi/2} -\sin\theta\cos^2\theta \, d\theta\right] + \frac{1}{2}\int_0^{\pi/2} \sin^2 2\theta \, d\theta$$

$$= \left(\frac{\pi}{2}+0\right) - \left(0+0\right) - 4\left[\frac{\cos^3\theta}{3}\right]_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2} \frac{1-\cos 4\theta}{2} \, d\theta$$

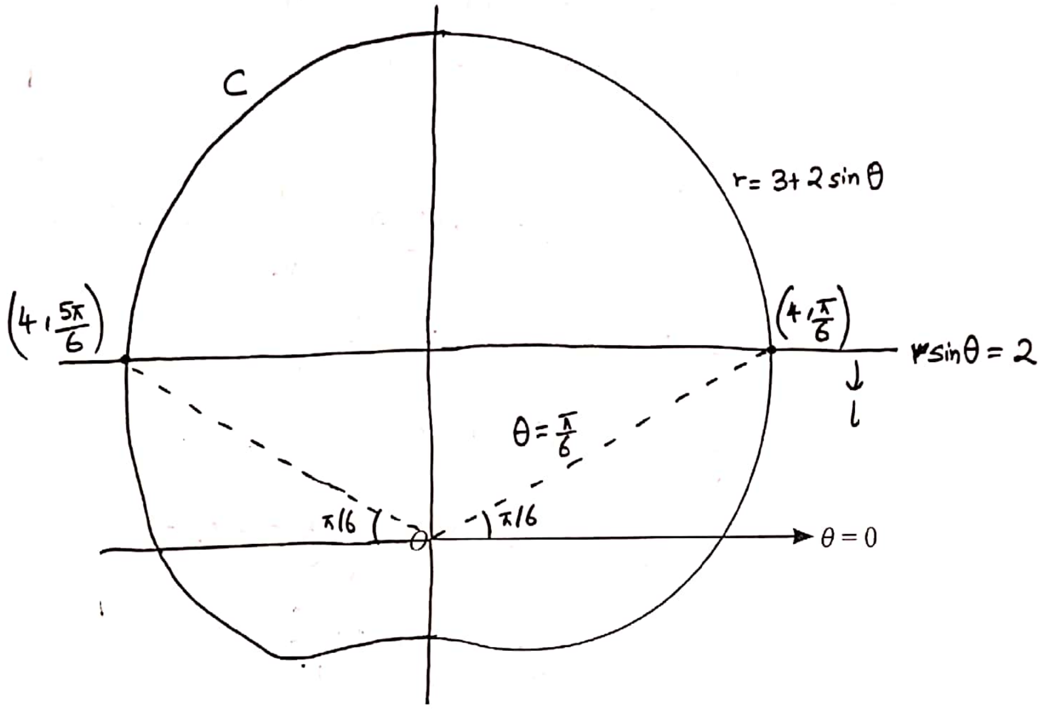
$$= \frac{\pi}{2} - 4\left[\frac{0-1}{3}\right] + \frac{1}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2}$$

$$= \frac{\pi}{2} - 4\left(\frac{-1}{3}\right) + \frac{1}{4}\left[\left(\frac{\pi}{2}-0\right) - \left(0-0\right)\right]$$

$$= \frac{\pi}{2} + \frac{4}{3} + \frac{1}{4}\left[\frac{\pi}{2}\right] = \frac{\pi}{2} + \frac{\pi}{8} + \frac{4}{3} = \frac{5\pi}{8} + \frac{4}{3}.$$

55 The curve C has polar equation $r = 3 + 2\sin\theta$, for $-\pi < \theta \leqslant \pi$.

(a) The diagram shows part of C. Sketch the rest of C on the diagram. [1]



The straight line l has polar equation $r\sin\theta = 2$.

(b) Add l to the diagram in part **(a)** and find the polar coordinates of the points of intersection of C and l. [5]

$L: \; r\sin\theta = 2 \Rightarrow y = 2.$

$r = 3 + 2\sin\theta \rightarrow (1)$

$r\sin\theta = 2 \rightarrow (2)$

Substitute (1) in (2):

$(3 + 2\sin\theta)\sin\theta = 2$

$3\sin\theta + 2\sin^2\theta - 2 = 0$

$2\sin^2\theta + 3\sin\theta - 2 = 0$

$2\sin^2\theta + 4\sin\theta - \sin\theta - 2 = 0$

$(2\sin\theta - 1)(\sin\theta + 2) = 0$

$\Rightarrow \sin\theta = \frac{1}{2}$ or $\sin\theta = -2$ (Ignore since $-1 \leq \sin\theta \leq 1$).

$\sin\theta > 0 \therefore \theta$ lies in 1st or 2nd quadrant.

$\sin\alpha = \frac{1}{2} \Rightarrow \alpha = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

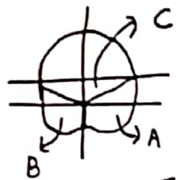
$\Rightarrow \theta = \alpha, \pi - \alpha \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}.$

$\theta = \frac{\pi}{6} \Rightarrow r = 3 + 2\left(\frac{1}{2}\right) = 4$ and $\theta = \frac{5\pi}{6} \Rightarrow r = 3 + 2\left(\frac{1}{2}\right) = 4 \Rightarrow$ Points of intersection: $\left(4, \frac{\pi}{6}\right), \left(4, \frac{5\pi}{6}\right).$

(c) The region R is enclosed by C and l, and contains the pole.

Find the area of R, giving your answer in exact form. [6]

Area of region R = Area of region A + Area of region B + Area of Δ C



By symmetry, Area of region A = Area of region B

$$= \int_{-\pi/2}^{\pi/6} \frac{1}{2}(3+2\sin\theta)^2\, d\theta$$

$$= \int_{-\pi/2}^{\pi/6} \frac{1}{2}\left[9+12\sin\theta+4\sin^2\theta\right] d\theta = \int_{-\pi/2}^{\pi/6} \frac{1}{2}\left[9+12\sin\theta+4\left(\frac{1-\cos 2\theta}{2}\right)\right] d\theta$$

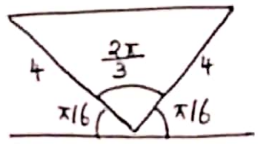
$$= \frac{1}{2}\left[9\theta - 12\cos\theta + 2\left(\theta - \frac{\sin 2\theta}{2}\right)\right]_{-\pi/2}^{\pi/6}$$

$$= \frac{1}{2}\left[11\theta - 12\cos\theta - \sin 2\theta\right]_{-\pi/2}^{\pi/6}$$

$$= \frac{1}{2}\left[\left(\frac{11\pi}{6} - 12\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2}\right) - \left(\frac{-11\pi}{2} - 12(0) - 0\right)\right]$$

$$= \frac{1}{2}\left[\frac{22\pi}{3} - \frac{13\sqrt{3}}{2}\right] = \frac{11\pi}{3} - \frac{13\sqrt{3}}{4}$$

Area of Δ C $= \frac{1}{2} \times 4 \times 4 \times \sin\left(\frac{2\pi}{3}\right) = 4\sqrt{3}$.



$\Rightarrow$ Area of region R

$$= 2\left(\frac{11\pi}{3} - \frac{13\sqrt{3}}{4}\right) + 4\sqrt{3}$$

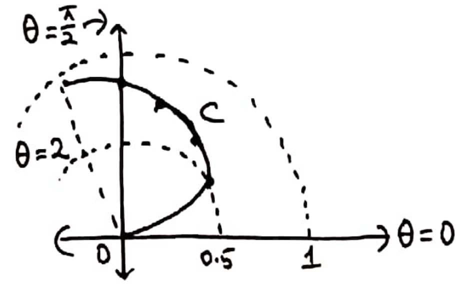
$$= \frac{22\pi}{3} - \frac{13\sqrt{3}}{2} + 4\sqrt{3} = \frac{22\pi}{3} - \frac{5\sqrt{3}}{2}.$$

56 The curve C has polar equation $r^2 = \tan^{-1}\left(\frac{1}{2}\theta\right)$, where $0 \leqslant \theta \leqslant 2$.

(a) Sketch C and state, in exact form, the greatest distance of a point on C from the pole. [3]

$r = \sqrt{\tan^{-1}\left(\frac{\theta}{2}\right)}$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	2
r	0	0.51	0.61	0.69	0.82	0.89



For greatest distance from pole, $\theta = 2 \Rightarrow r^2 = \tan^{-1}\left(\frac{1}{2}\times 2\right) \Rightarrow r^2 = \frac{\pi}{4} \Rightarrow r = \sqrt{\frac{\pi}{4}}$.

$\therefore$ Greatest distance $= \sqrt{\frac{\pi}{4}}$.

(b) Find the exact value of the area of the region bounded by C and the half-line $\theta = 2$. [5]

$$\text{Area} = \int_0^2 \frac{1}{2} r^2 \, d\theta = \int_0^2 \frac{1}{2} \tan^{-1}\left(\frac{1}{2}\theta\right) d\theta$$

$$u = \tan^{-1}\left(\frac{\theta}{2}\right) \qquad \frac{dv}{d\theta} = \frac{1}{2}$$

$$\frac{du}{d\theta} = \frac{1}{1+\left(\frac{\theta}{2}\right)^2}\cdot\frac{1}{2} \qquad v = \frac{\theta}{2}$$

$$= \frac{1}{1+\frac{\theta^2}{4}}\cdot\frac{1}{2} = \frac{2}{4+\theta^2}$$

$$= \cancel{\left[\frac{\theta}{2}\cdot\frac{2}{4+\theta^2}\right]_0^2 - \int_0^2}$$

$$= \left[\frac{\theta}{2}\tan^{-1}\left(\frac{\theta}{2}\right)\right]_0^2 - \int_0^2 \frac{\theta}{2}\cdot\frac{2}{4+\theta^2}\, d\theta$$

$$= \left[\frac{2}{2}\tan^{-1}\left(\frac{2}{2}\right) - 0\right] - \int_0^2 \frac{\theta}{4+\theta^2}\, d\theta$$

$$= \frac{\pi}{4} - \frac{1}{2}\int_0^2 \frac{2\theta}{4+\theta^2}\, d\theta = \frac{\pi}{4} - \frac{1}{2}\left[\ln\left[4+\theta^2\right]\right]_0^2$$

$$= \frac{\pi}{4} - \frac{1}{2}\left[\ln 8 - \ln 4\right] = \frac{\pi}{4} - \frac{1}{2}\ln\left(\frac{8}{4}\right) = \frac{\pi}{4} - \frac{1}{2}\ln 2.$$

Now consider the part of C where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(c) Show that, at the point furthest from the half-line $\theta = \frac{1}{2}\pi$,

$$(\theta^2 + 4)\tan^{-1}\left(\tfrac{1}{2}\theta\right)\sin\theta - \cos\theta = 0$$

and verify that this equation has a root between 0.6 and 0.7. [5]

$x = r\cos\theta \Rightarrow x = \sqrt{\tan^{-1}\left(\frac{\theta}{2}\right)}\cos\theta$

For point furthest from half-line $\frac{\pi}{2}$, $\frac{dx}{d\theta} = 0$

$$\frac{dx}{d\theta} = \sqrt{\tan^{-1}\left(\frac{\theta}{2}\right)}\,(-\sin\theta) + \cos\theta\left[\frac{1}{2}\left[\tan^{-1}\left(\frac{\theta}{2}\right)\right]^{-\frac{1}{2}}\right] \cdot \frac{1}{1+\left(\frac{\theta}{2}\right)^2} \cdot \frac{1}{2}$$

$$= -\sin\theta\sqrt{\tan^{-1}\left(\frac{\theta}{2}\right)} + \frac{\cos\theta}{2\sqrt{\tan^{-1}\left(\frac{\theta}{2}\right)}} \cdot \frac{1}{\frac{4+\theta^2}{4}} \cdot \frac{1}{2}$$

$$= -\sin\theta\sqrt{\tan^{-1}\left(\frac{\theta}{2}\right)} + \frac{\cos\theta}{(4+\theta^2)\sqrt{\tan^{-1}\left(\frac{\theta}{2}\right)}}$$

$$\frac{dx}{d\theta} = 0 \Rightarrow \quad -\sin\theta\sqrt{\tan^{-1}\left(\frac{\theta}{2}\right)} + \frac{\cos\theta}{(4+\theta^2)\sqrt{\tan^{-1}\left(\frac{\theta}{2}\right)}} = 0$$

$$\frac{\cos\theta}{(4+\theta^2)\sqrt{\tan^{-1}\left(\frac{\theta}{2}\right)}} = \sin\theta\sqrt{\tan^{-1}\left(\frac{\theta}{2}\right)}$$

$$\cos\theta = \sin\theta\tan^{-1}\left(\frac{\theta}{2}\right)(4+\theta^2)$$

$$\Rightarrow (\theta^2+4)\tan^{-1}\left(\frac{\theta}{2}\right)\sin\theta - \cos\theta = 0 \text{ (shown).}$$

Let, $f(\theta) = (\theta^2+4)\left(\tan^{-1}\left(\frac{\theta}{2}\right)\right)\sin\theta - \cos\theta$.

$f(0.6) = -0.108 < 0$

$f(0.7) = 0.209 > 0$

Since there is a change of sign, the equation has a root between 0.6 and 0.7.

57 The curve C has Cartesian equation $x^2+xy+y^2=a$, where a is a positive constant.

(a) Show that the polar equation of C is $r^2=\dfrac{2a}{2+\sin 2\theta}$. [3]

$x^2+xy+y^2=a$.

Since $x=r\cos\theta$, $y=r\sin\theta$ and $x^2+y^2=r^2$

$\Rightarrow (x^2+y^2)+xy=a$

$r^2+r^2\sin\theta\cos\theta=a$

$r^2(1+\sin\theta\cos\theta)=a$

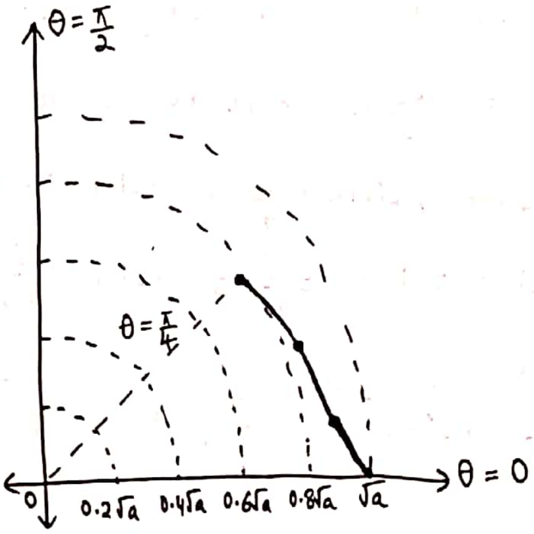
$r^2=\dfrac{a}{1+\sin\theta\cos\theta}=\dfrac{a}{1+\frac{\sin 2\theta}{2}}$

$r^2=\dfrac{a}{\frac{2+\sin 2\theta}{2}}$

$r^2=\dfrac{2a}{2+\sin 2\theta}$ (shown).

(b) Sketch the part of C for $0\leqslant\theta\leqslant\frac{1}{4}\pi$. [2]

θ	0	$\frac{\pi}{8}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$
r	$\sqrt{a}$	$0.86\sqrt{a}$	$0.84\sqrt{a}$	$0.79\sqrt{a}$



The region R is enclosed by this part of C, the initial line and the half-line $\theta = \frac{1}{4}\pi$.

(c) It is given that $\sin 2\theta$ may be expressed as $\frac{2\tan\theta}{1+\tan^2\theta}$. Use this result to show that the area of R is

$$\frac{1}{2}a\int_0^{\frac{1}{4}\pi} \frac{1+\tan^2\theta}{1+\tan\theta+\tan^2\theta}\,d\theta$$

and use the substitution $t = \tan\theta$ to find the exact value of this area. [8]

Area of region $R = \int_0^{\frac{\pi}{4}} \frac{1}{2} r^2 \, d\theta = \int_0^{\pi/4} \frac{1}{2}\left[\frac{2a}{2+\sin 2\theta}\right] d\theta$

$= a\int_0^{\pi/4} \frac{1}{2+\frac{2\tan\theta}{1+\tan^2\theta}}\, d\theta = a\int_0^{\pi/4} \frac{1}{\frac{2+2\tan^2\theta+2\tan\theta}{1+\tan^2\theta}}\, d\theta$

$= a\int_0^{\pi/4} \frac{1+\tan^2\theta}{2(1+\tan\theta+\tan^2\theta)}\, d\theta = \frac{a}{2}\int_0^{\pi/4} \frac{1+\tan^2\theta}{1+\tan\theta+\tan^2\theta}\, d\theta$ (shown).

Let, $t = \tan\theta$

$\Rightarrow \frac{dt}{d\theta} = \sec^2\theta = 1+\tan^2\theta \Rightarrow \frac{dt}{d\theta} = 1+t^2 \Rightarrow d\theta = \frac{dt}{1+t^2}$.

$\theta = 0 \Rightarrow t = \tan(0) = 0$

$\theta = \frac{\pi}{4} \Rightarrow t = \tan\left(\frac{\pi}{4}\right) = 1$.

$\Rightarrow$ Area of region $R = \frac{a}{2}\int_0^1 \frac{1+t^2}{1+t+t^2}\,\frac{dt}{1+t^2} = \frac{a}{2}\int_0^1 \frac{1}{1+t+t^2}\,dt$.

$t^2+t+1 = t^2+t+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1 = \left(t+\frac{1}{2}\right)^2-\frac{1}{4}+1 = \left(t+\frac{1}{2}\right)^2+\frac{3}{4}$.

Let, $t+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan\alpha \Rightarrow dt = \frac{\sqrt{3}}{2}\sec^2\alpha\, d\alpha$

$t=0 \Rightarrow 0+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan\alpha \Rightarrow \tan\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}$.

$t=1 \Rightarrow 1+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan\alpha \Rightarrow \tan\alpha = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$

$\Rightarrow$ Area of region $R = \frac{a}{2}\int_0^1 \frac{1}{1+t+t^2}\,dt = \frac{a}{2}\int_0^1 \frac{1}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}\,dt$

$= \frac{a}{2}\int_{\pi/6}^{\pi/3} \frac{1}{\left(\frac{\sqrt{3}}{2}\tan\alpha\right)^2+\frac{3}{4}}\,\frac{\sqrt{3}}{2}\sec^2\alpha\, d\alpha = \frac{a}{2}\int_{\pi/6}^{\pi/3} \frac{1}{\frac{3}{4}(1+\tan^2\alpha)}\,\frac{\sqrt{3}}{2}\sec^2\alpha\, d\alpha$

$= \frac{a}{2}\int_{\pi/6}^{\pi/3} \frac{2\sqrt{3}}{3}\, d\alpha = \frac{\sqrt{3}a}{3}\Big[\alpha\Big]_{\pi/6}^{\pi/3} = \frac{\sqrt{3}a}{3}\left[\frac{\pi}{3}-\frac{\pi}{6}\right] = \frac{\sqrt{3}a}{3}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}\pi a}{18}$.