Past Paper Questions: Rational Functions

1 The curve C has equation $y = \frac{x^2 - 4}{x - 3}$.

(i) Find the equations of the asymptotes of C. [3]

(ii) Draw a sketch of C and its asymptotes. Give the coordinates of the points of intersection of C with the coordinate axes. [4]

[You are not required to find the coordinates of any turning points.]

i) $y = x+3+\frac{5}{x-3}$.

$$\begin{array}{r} x+3 \\ x-3\,\overline{)\,x^2+0x-4} \\ x^2-3x \\ - \quad + \\ \hline 3x-4 \\ 3x-9 \\ - \quad + \\ \hline 5 \\ \hline \end{array}$$

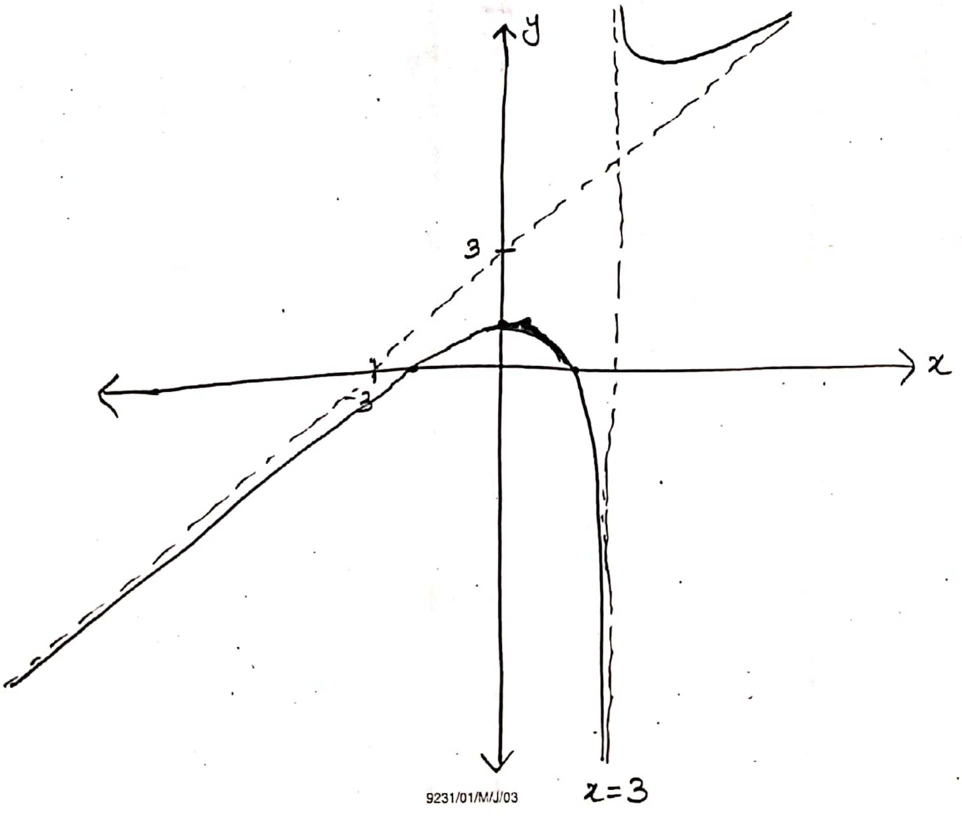
Asymptotes: $y=x+3$

$x = 3$.

ii) x-intercept $\Rightarrow y=0 \Rightarrow \frac{x^2-4}{x-3}=0$

$x = \pm 2 \Rightarrow (2,0)$ and $(-2,0)$.

y-intercept $\Rightarrow x=0 \Rightarrow y=\frac{0^2-4}{0-3}=\frac{+4}{3}$ $\therefore$ $\left(0,+\frac{4}{3}\right)$



2 The curve C has equation

$$y = \frac{x - ax^2}{x - 1},$$

where a is a constant and $a > 1$.

(i) Find the equations of the asymptotes of C. [3]

(ii) Show that the x-coordinates of both the turning points of C are positive. [4]

i) $y = \frac{x - ax^2}{x-1} = -ax + 1 - a + \frac{1-a}{x-1}.$

$$\begin{array}{r l}
 & -ax + 1 - a \\
x - 1 \;\big) & -ax^2 + x \\
 & -ax^2 + ax \\
 & + \quad - \\
\hline
 & (1-a)x \\
 & (1-a)x - 1 + a \\
 & - \quad + \quad - \\
\hline
 & 1 - a \\
\hline
\end{array}$$

Asymptotes: $y = -ax + 1 - a.$

$x = 1.$

ii) $\frac{dy}{dx} = -a - \frac{1-a}{(x-1)^2}$

At turning point, $\frac{dy}{dx} = 0 \Rightarrow -a - \frac{1-a}{(x-1)^2} = 0$

$$-a = \frac{1-a}{(x-1)^2}$$

$$-a(x-1)^2 = 1 - a$$

$$-a(x^2 - 2x + 1) = 1 - a$$

$$-ax^2 + 2ax - a = 1 - a$$

$$-ax^2 + 2ax - 1 = 0$$

$$ax^2 - 2ax + 1 = 0.$$

$$x = \frac{-(-2a) \pm \sqrt{(-2a)^2 - 4(a)(1)}}{2a}$$

$$= \frac{2a \pm \sqrt{4a^2 - 4a}}{2a} = \frac{2a \pm \sqrt{4a^2\left(1 - \frac{1}{a}\right)}}{2a}$$

$$x = \frac{2a \pm 2a\sqrt{1 - \frac{1}{a}}}{2a} = 1 \pm \sqrt{1 - \frac{1}{a}}$$

$a > 1 \Rightarrow \cancel{1 < \frac{1}{a} < a}\; 0 < \frac{1}{a} < 1 \Rightarrow 0 > \frac{-1}{a} > -1$

$$1 > 1 - \frac{1}{a} > 0$$

$$1 > \sqrt{1 - \frac{1}{a}} > 0$$

Let, $\alpha = 1 + \sqrt{1 - \frac{1}{a}}.$

Then $$2 > 1+\sqrt{1-\frac{1}{a}} > 1$$

$$\therefore \alpha = 1+\sqrt{1-\frac{1}{a}} > 0$$

Let, $\beta = 1-\sqrt{1-\frac{1}{a}}$,

$$-1 < -\sqrt{1-\frac{1}{a}} < 0$$

$$0 < 1-\sqrt{1-\frac{1}{a}} < 1$$

Then $\beta = 1-\sqrt{1-\frac{1}{a}} > 0$

$\therefore$ Both turning points have x-coordinates > 0

3 The curve Γ, which has equation

$$y = \frac{ax^2 + bx + c}{x^2 + px + q},$$

has asymptotes $x = 1$, $x = 4$ and $y = 2$. Find the values of a, p and q. [4]

It is given that Γ has a stationary point at $x = 2$.

(i) Find the value of c. [3]

(ii) Show that if $b \neq -10$ then Γ has exactly 2 stationary points. [2]

(iii) Draw a sketch of Γ for the case where $b = -6$. [4]

Vertical asymptotes: $x = 1, x = 4 \therefore$ Denominator $= (x-1)(x-4)$

$= x^2 - 5x + 4$

$\therefore p = -5, q = 4.$

Horizontal asymptote: $y = 2$. As $|x| \mapsto \infty, y \mapsto \frac{a}{1} = a \quad \therefore a = 2.$

i) $y = \frac{2x^2+bx+c}{x^2-5x+4}.$

$$\frac{dy}{dx} = \frac{(x^2-5x+4)(4x+b)-(2x^2+bx+c)(2x-5)}{(x^2-5x+4)^2}$$

when $x = 2$, $\frac{dy}{dx} = 0 \Rightarrow 0 = (-2)(8+b) - (8+2b+c)(-1)$

$0 = -16 - 2b + 8 + 2b + c$

$0 = c - 8$

$\Rightarrow c = 8$

ii) $y = \frac{2x^2+bx+8}{x^2-5x+4}$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2-5x+4)(4x+b)-(2x^2+bx+8)(2x-5)}{(x^2-5x+4)^2}$$

$0 = (x^2-5x+4)(4x+b) - (2x^2+bx+8)(2x-5)$

$0 = (4x^3 - 20x^2 + 16x + bx^2 - 5bx + 4b) - (4x^3 - 10x^2 + 2bx^2 - 5bx + 16x - 40)$

$0 = (b-20)x^2 + (16-5b)x + 4b - ((2b-10)x^2 + (16-5b)x - 40)$

$0 = (-b-10)x^2 + 4b + 40$

$\Rightarrow x^2 = \frac{4b+40}{b+10}$. If $b \neq -10$, then $x^2 = 4 \Rightarrow x = \pm 2$

$\therefore$ ~~Stationary~~ curve has two stationary points.

III) $b = -6$

$$\Rightarrow y = \frac{2x^2 - 6x + 8}{(x-1)(x-4)} = \frac{2(x^2-3x+4)}{(x-1)(x-4)}.$$

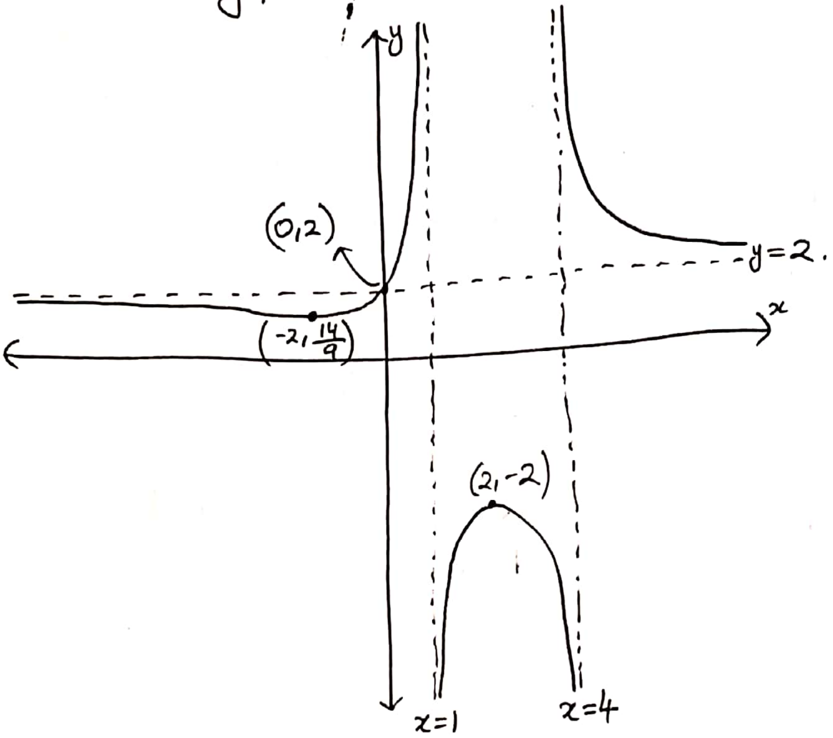
Asymptotes: $x = 1, x = 4, y = 2$.

Intercepts: $x = 0 \Rightarrow y = \frac{2(4)}{(-1)(-4)} = 2$.

$y = 0 \Rightarrow \frac{2(x^2-3x+4)}{(x-1)(x-4)} = 0 \Rightarrow$ No real roots

Then no x-intercepts.

With $b=-6$, two turning points exist.



Turning points:

$$\frac{dy}{dx} = \frac{(2x^2-6x+8)(2x-5)-(x^2-5x+4)(4x-6)}{(x^2-5x+4)^2} = 0$$

$$-4x^2 + 16 = 0$$

$$x^2 = 4 \Rightarrow x = \pm 2.$$

$x = 2 \Rightarrow y = -2$

$x = -2 \Rightarrow y = \frac{14}{9}$

$\therefore (2,-2)$ and $\left(-2, \frac{14}{9}\right)$

4 The curve C has equation

$$y = 2x + \frac{3(x-1)}{x+1}.$$

(i) Write down the equations of the asymptotes of C. [2]

(ii) Find the set of values of x for which C is above its oblique asymptote and the set of values of x for which C is below its oblique asymptote. [3]

(iii) Draw a sketch of C, stating the coordinates of the points of intersection of C with the coordinate axes. [4]

i) $y = 2x + 3 - \frac{6}{x+1}$.

$$x+1 \overline{)\,3x-3\,} \quad 3$$
$$3x+3$$
$$-\ -$$
$$-6$$

Asymptotes: $y = 2x+3$

$x = -1$.

ii) To be above oblique asymptote:

$$2x+3 - \frac{6}{x+1} > 2x+3$$

$$\frac{-6}{x+1} > 0$$

$$\frac{6}{x+1} < 0$$

$$\Rightarrow x+1 < 0 \Rightarrow x < -1$$

Then to be below oblique asymptote:

$x > -1$.

iii) Intercepts: x-intercept $\Rightarrow y = 0$ $\because$ $2x+3 - \frac{6}{x+1} = 0$

$$2x+3 = \frac{6}{x+1}$$

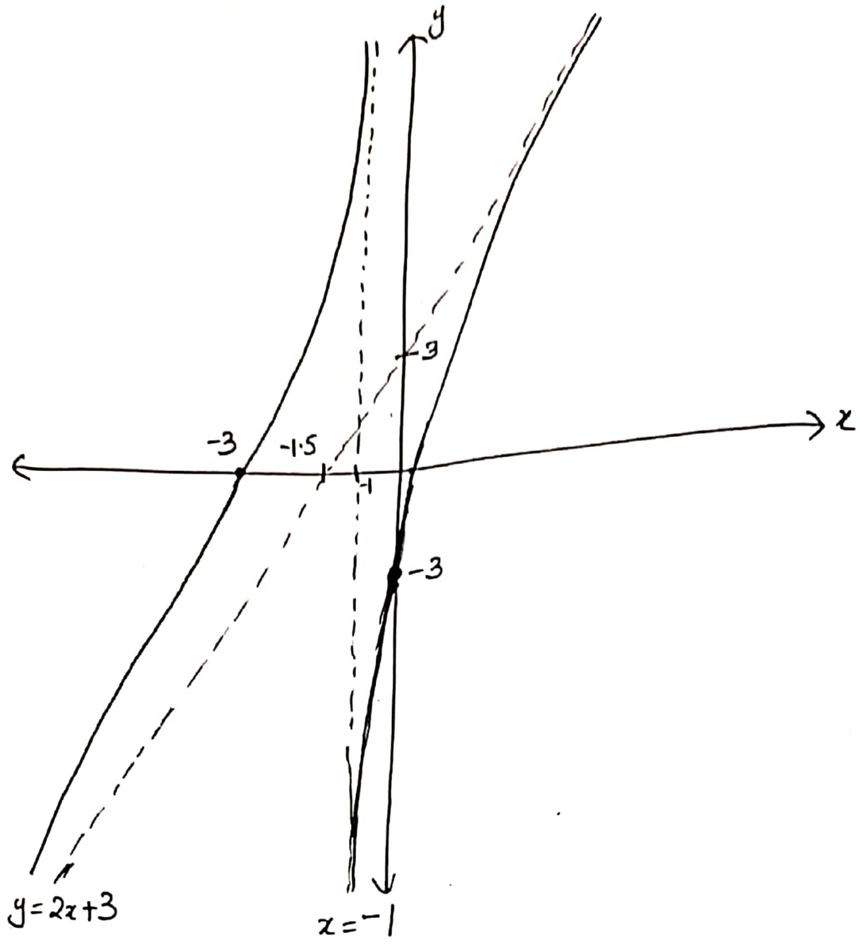
$$2x^2 + 5x - 3 = 0$$

$$2x^2 + 6x - x - 3 = 0$$

$$(2x-1)(x+3) = 0$$

$\Rightarrow x = \frac{1}{2}, -3 \therefore (-3, 0)$ and $(\frac{1}{2}, 0)$.

y-intercept $\Rightarrow x = 0 \Rightarrow y = 2(0)+3 - \frac{6}{0+1} = -3 \therefore (0, -3)$.


y
x
3
-3
-1.5
-1
-3
y=2x+3
x=-1

5 The curve C has equation

$$y = \lambda x + \frac{x}{x+2},$$

where λ is a non-zero constant.

(i) Find the asymptotes of C. [3]

(ii) Show that if $\lambda > 0$ then $\frac{dy}{dx} > 0$ at all points of C. [2]

(iii) Show that, for $\lambda < -\frac{1}{2}$, C has two distinct stationary points, both to the left of the y-axis. [3]

(iv) In separate diagrams draw sketches of C for each of the cases $\lambda > 0$ and $\lambda < -\frac{1}{2}$. [6]

i). $y = \lambda x + 1 - \frac{2}{x+2}$.

$$\begin{array}{r|l} & 1 \\ \hline x+2 & x \\ & x+2 \\ & - \quad - \\ \hline & -2 \\ \hline \end{array}$$

Asymptotes: $y = \lambda x + 1$

$x = -2$.

ii) $\frac{dy}{dx} = \lambda + \frac{2}{(x+2)^2}$.

if $\lambda > 0$, $\frac{2}{(x+2)^2} > 0$

$\lambda + \frac{2}{(x+2)^2} > \lambda > 0 \therefore \frac{dy}{dx} > 0$ if $\lambda > 0$.

iii) Set $\frac{dy}{dx} = 0 \Rightarrow \lambda + \frac{2}{(x+2)^2} = 0 \Rightarrow (x+2)^2 = \frac{-2}{\lambda}$

$x+2 = \pm\sqrt{\frac{-2}{\lambda}}$

$x = -2 \pm \sqrt{\frac{-2}{\lambda}}$.

$-\infty < \lambda < \frac{-1}{2}$

$0 > \frac{1}{\lambda} > -2$

$\Rightarrow 0 > \frac{2}{\lambda} > -4$

$0 < \frac{-2}{\lambda} < 4$

$0 < \sqrt{\frac{-2}{\lambda}} < 2$

Since $\sqrt{\frac{-2}{\lambda}} > 0$, then the equation $\frac{dy}{dx} = 0$ has two roots.

Let, $\alpha = -2 + \sqrt{\frac{-2}{\lambda}}$ and $\beta = -2 - \sqrt{\frac{-2}{\lambda}}$.

$-2 < -2 + \sqrt{\frac{-2}{\lambda}} < 0$ and $-2 > -2 - \sqrt{\frac{-2}{\lambda}} > -4$

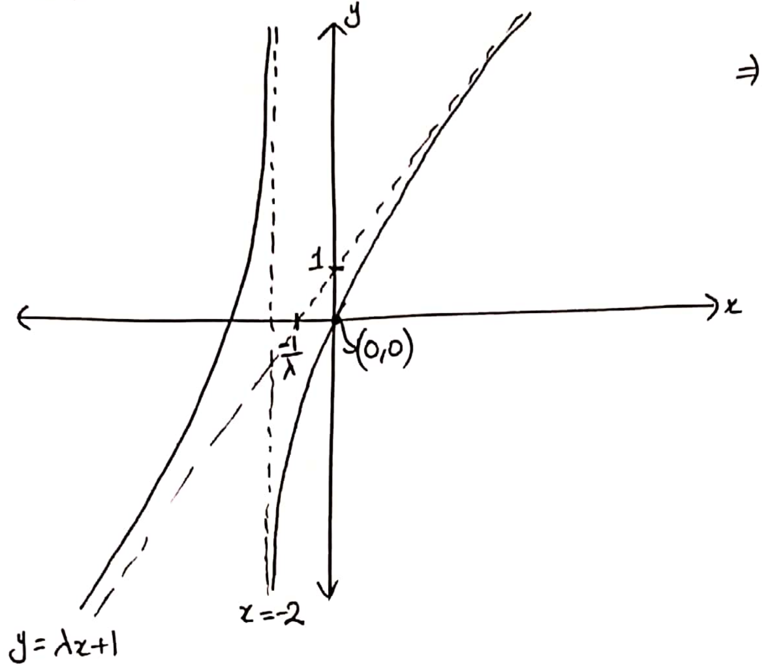
$\therefore$ $-2 < \alpha < 0$ and $-2 > \beta > -4$

i.e. both stationary points have x-coordinates that ~~have~~ are negative.

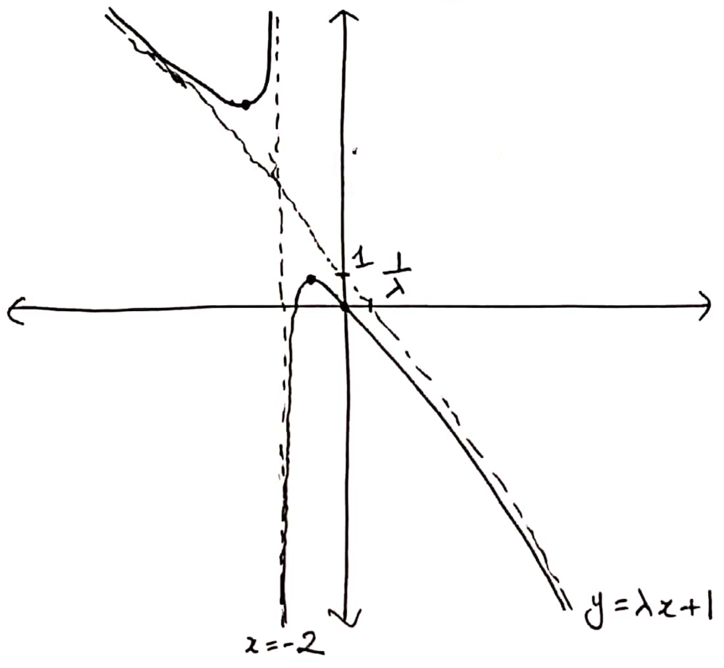
iv) $\lambda > 0$ then $\sqrt{\frac{-2}{\lambda}}$ is not real so there are no stationary points.

$x = 0$

$\Rightarrow y = \lambda(0) + \frac{0}{0+2} = 0$



$\lambda < \frac{-1}{2}$ then the curve has two stationary points.



6 The curve C has equation

$$y = \frac{x^2 - 2x + \lambda}{x+1},$$

where λ is a constant. Show that the equations of the asymptotes of C are independent of λ. [3]

Find the value of λ for which the x-axis is a tangent to C, and sketch C in this case. [4]

Sketch C in the case $\lambda = -4$, giving the exact coordinates of the points of intersection of C with the x-axis. [3]

$$y = x - 3 + \frac{\lambda+3}{x+1}$$

$$\begin{array}{r l} & x - 3 \\ x+1 & \overline{)\,x^2 - 2x + \lambda} \\ & x^2 + x \\ & - \quad - \\ \hline & -3x + \lambda \\ & -3x - 3 \\ & + \quad + \\ \hline & \lambda + 3 \\ \hline \end{array}$$

Oblique asymptote : $y = x - 3$

Vertical asymptote : $x = -1$.

Set $y = 0 \Rightarrow \dfrac{x^2 - 2x + \lambda}{x+1} = 0$

$x^2 - 2x + \lambda = 0$

For x-axis to be a tangent,

$(-2)^2 - 4(1)(\lambda) = 0$

$4 - 4\lambda = 0$

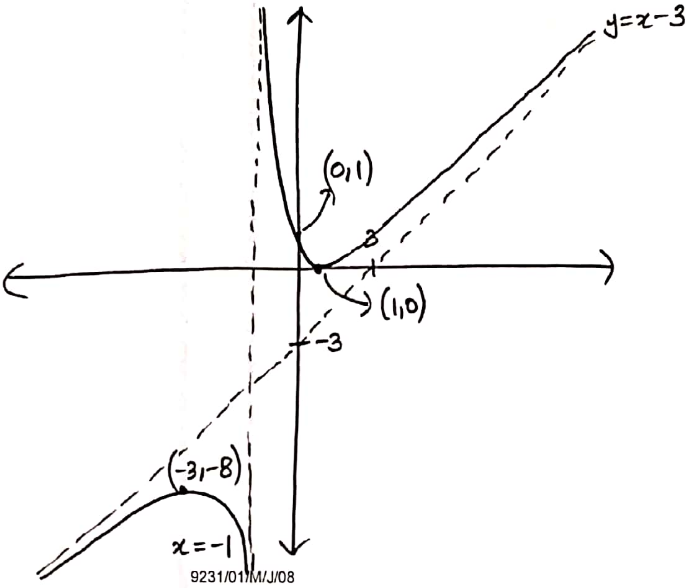
$\lambda = 1.$

$\lambda = 1 \Rightarrow y = \dfrac{x^2 - 2x + 1}{x+1} = x - 3 + \dfrac{4}{x+1} \Rightarrow y' = 1 - \dfrac{4}{(x+1)^2} \Rightarrow x = 1, -3 \;\; \therefore (1,0), (-3,-8)$

$y = 0, -8$

x-intercept $\Rightarrow y = 0 \Rightarrow (1,0)$.

y-intercept $\Rightarrow x = 0 \Rightarrow y = 1 \;\therefore (0,1)$.



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$\lambda = -4 \Rightarrow y = \frac{x^2-2x-4}{x+1} = x-3-\frac{1}{x+1}$.

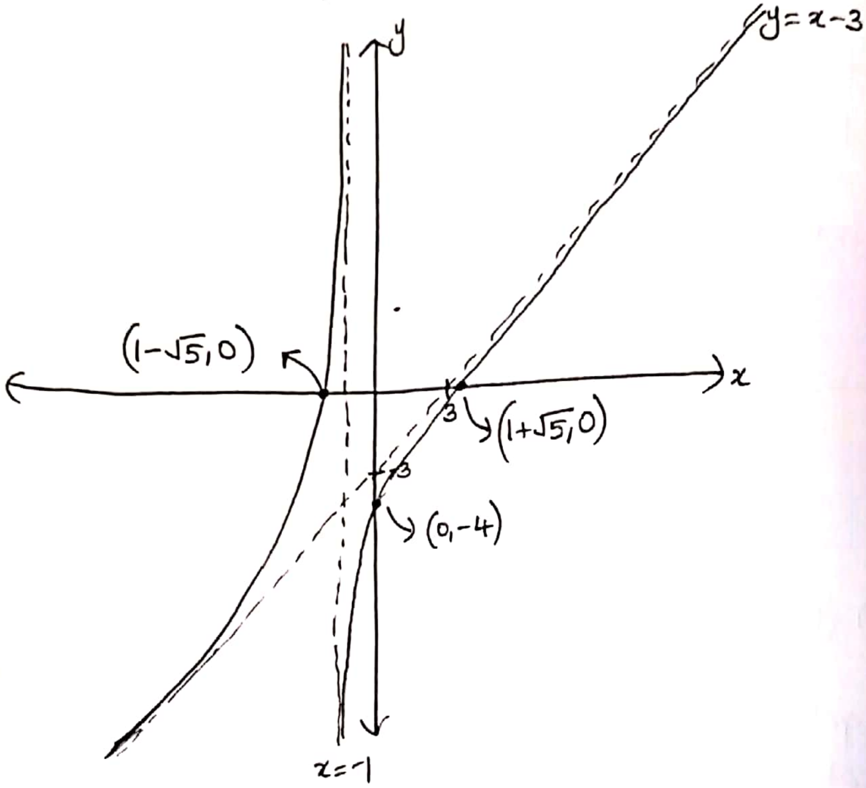
For x-axis, $y=0 \Rightarrow \frac{x^2-2x-4}{x+1} = 0$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2-4(1)(-4)}}{2(1)} = \frac{2 \pm \sqrt{20}}{2}$$

$$x = 1 \pm \sqrt{5}$$

$\therefore$ $(1+\sqrt{5},0)$ and $(1-\sqrt{5},0)$

y-intercept $\Rightarrow x = 0 \therefore y = \frac{-4}{1} = -4 \therefore (0,-4)$.



7 The curve C has equation

$$y = \frac{x^2}{x + \lambda},$$

where λ is a non-zero constant. Obtain the equation of each of the asymptotes of C. [3]

In separate diagrams, sketch C for the cases $\lambda > 0$ and $\lambda < 0$. In both cases the coordinates of the turning points must be indicated. [8]

$$y = x - \lambda + \frac{\lambda^2}{x+\lambda}$$

$$\begin{array}{r|l} & x - \lambda \\ \hline x+\lambda & x^2+0x+0 \\ & x^2+\lambda x \\ & - \quad - \\ \hline & -\lambda x + 0 \\ & -\lambda x - \lambda^2 \\ & + \quad + \\ \hline & \lambda^2 \\ \hline \end{array}$$

Oblique asymptotes: $y = x - \lambda$.

Vertical asymptotes: $x = -\lambda$.

$$\frac{dy}{dx} = 1 - \frac{\lambda^2}{(x+\lambda)^2}$$

For turning points, $\frac{dy}{dx} = 0 \Rightarrow 1 - \frac{\lambda^2}{(x+\lambda)^2} = 0$

$$(x+\lambda) = \pm \lambda$$

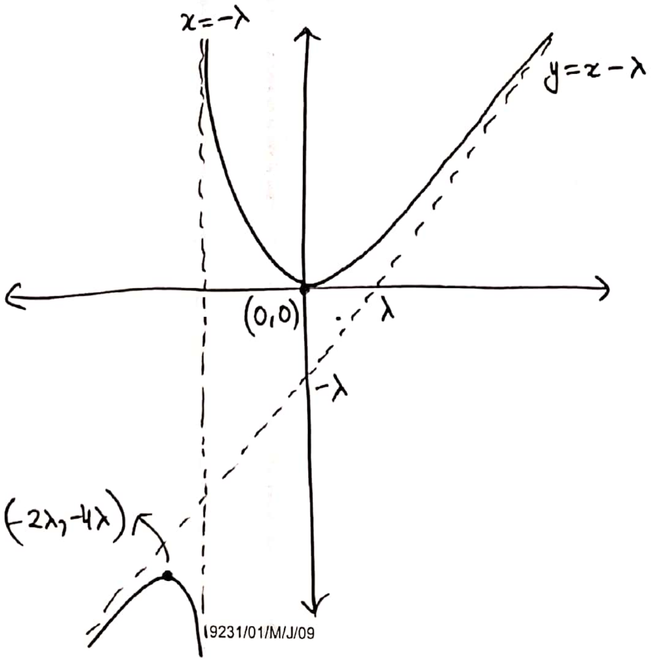
$x + \lambda = \lambda$ or $x + \lambda = -\lambda$

$x = 0$ or $x = -2\lambda$.

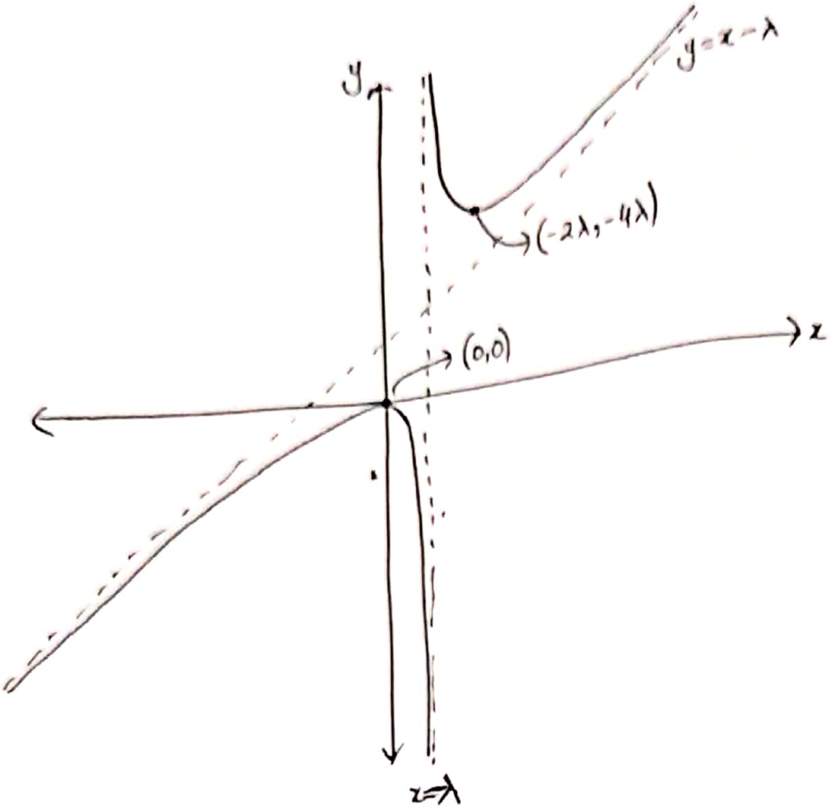
$\Rightarrow y = \frac{0^2}{0+\lambda} = 0$ $\quad y = \frac{(-2\lambda)^2}{-2\lambda+\lambda} = \frac{4\lambda^2}{-\lambda} = -4\lambda$

$\therefore$ Turning points: $\underbrace{(0,0)}_{\text{Also intercept.}}$ and $(-2\lambda, -4\lambda)$.

$\lambda > 0$



$\lambda < 0$



8 The curve C has equation

$$y = \frac{x(x+1)}{(x-1)^2}.$$

(i) Obtain the equations of the asymptotes of C. [3]

(ii) Show that there is exactly one point of intersection of C with the asymptotes and find its coordinates. [2]

(iii) Find $\frac{dy}{dx}$ and hence

(a) find the coordinates of any stationary points of C,

(b) state the set of values of x for which the gradient of C is negative.

[6]

(iv) Draw a sketch of C. [3]

i) $y = \frac{x^2+x}{x^2-2x+1} = 1 + \frac{3x-1}{(x-1)^2}$.

$$\begin{array}{r|l} & 1 \\ \hline x^2-2x+1 & x^2+x \\ & x^2-2x+1 \\ & - \quad + \quad - \\ \hline & 3x-1 \\ \hline \end{array}$$

Asymptotes: $y=1$, $x=1$.

ii) when curve intersects the horizontal asymptote $y=1$

$\Rightarrow 1 = 1 + \frac{3x-1}{(x-1)^2}$

$0 = 3x-1$

$\Rightarrow x = \frac{1}{3}$.

$\therefore \left(\frac{1}{3}, 1\right)$.

iii) $\frac{dy}{dx} = 0 + \frac{(x-1)^2(3) - (3x-1)(2)(x-1)}{(x-1)^4} = \frac{(x-1)\left[3(x-1)-2(3x-1)\right]}{(x-1)^4}$

$\frac{dy}{dx} = \frac{-3x-1}{(x-1)^3}$

a) For stationary point, $\frac{dy}{dx} = 0 \Rightarrow \frac{-3x-1}{(x-1)^3} = 0 \Rightarrow x = \frac{-1}{3}$.

$\Rightarrow y = \frac{\left(\frac{-1}{3}\right)\left(\frac{-1}{3}+1\right)}{\left(\frac{-1}{3}-1\right)^2} = \frac{-1}{8}$ $\therefore$ Stationary point: $\left(\frac{-1}{3}, \frac{-1}{8}\right)$

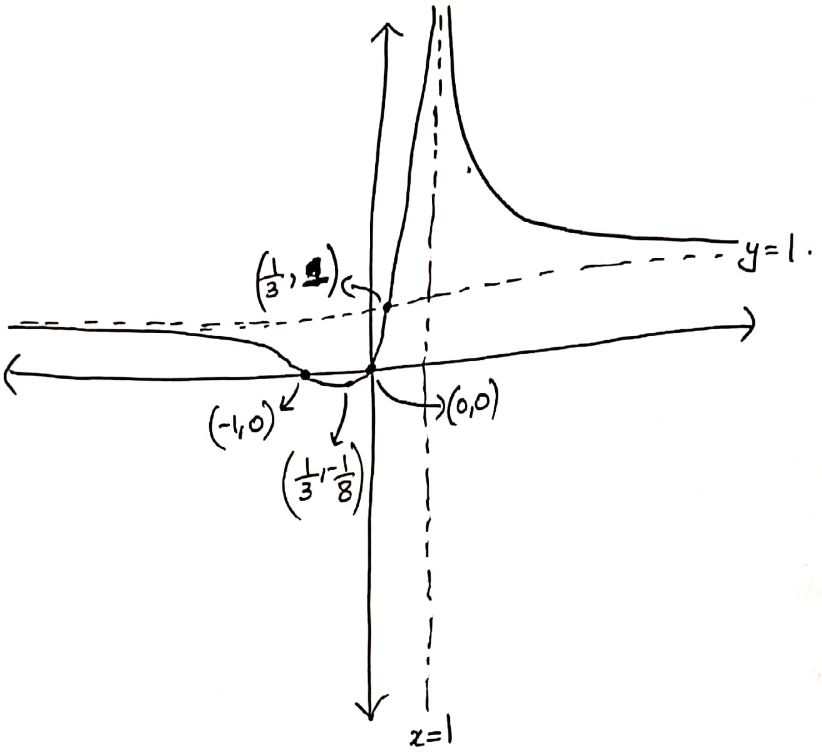
b) $\frac{dy}{dx} < 0 \Rightarrow \frac{-3x-1}{(x-1)^3} < 0 \Rightarrow \frac{-(3x+1)}{(x-1)^3} < 0 \Rightarrow \frac{(3x+1)}{(x-1)^3} > 0$

$\frac{(3x+1)}{(x-1)^3} \times (x-1)^4 > 0 \times (x-1)^4 \Rightarrow (3x+1)(x-1) > 0$

$\Rightarrow x < \frac{-1}{3}$ or $x > 1$.

iv) y-intercept $\Rightarrow$ $x = 0 \therefore y = \frac{0(0+1)}{(0-1)^2} = \frac{0}{1} = 0 \Rightarrow (0,0).$

x-intercept $\Rightarrow$ $y = 0 \therefore 0 = \frac{x(x+1)}{(x-1)^2} \Rightarrow x = 0, -1 \Rightarrow (0,0)$ and $(-1,0)$.



9 The curve C has equation

$$y = \frac{x^2 - 3x - 7}{x + 1}.$$

(i) Obtain the equations of the asymptotes of C. [3]

(ii) Show that $\frac{dy}{dx} > 1$ at all points of C. [2]

(iii) Draw a sketch of C. [3]

i) $y = x - 4 - \frac{3}{x+1}.$

$$\begin{array}{r} x-4 \\ x+1 \overline{\smash{)}\, x^2-3x-7} \\ x^2+x \\ - \; - \\ \hline -4x-7 \\ -4x-4 \\ + \; + \\ \hline -3 \\ \hline \end{array}$$

Asymptotes: $y = x - 4$

$x = -1.$

ii) $\frac{dy}{dx} = 1 + \frac{3}{(x+1)^2}.$

$\frac{1}{(x+1)^2} > 0$ for all values of x

$\frac{3}{(x+1)^2} > 0$ for all values of x

$\Rightarrow \; 1 + \frac{3}{(x+1)^2} > 1$ for all values of x

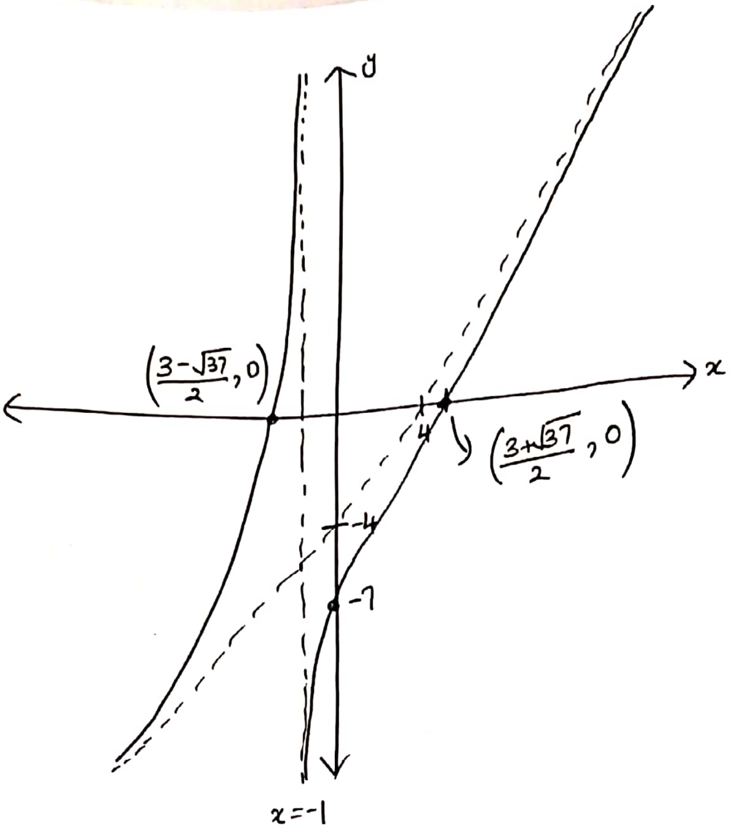
$\Rightarrow \; \frac{dy}{dx} > 1$ at all points of C.

iii) For x-intercept, $y = 0 \Rightarrow \frac{x^2-3x-7}{x+1} = 0 \Rightarrow x = \frac{+3 \pm \sqrt{37}}{2}$

$\left(\frac{3+\sqrt{37}}{2}, 0\right), \left(\frac{3-\sqrt{37}}{2}, 0\right).$

$x = 0 \Rightarrow y = \frac{0^2 - 3(0) - 7}{0+1} = -7 \therefore (0, -7).$

Since $\frac{dy}{dx} > 1$ for all x, $\therefore$ there is no turning point.


y
x
$\left(\frac{3-\sqrt{37}}{2}, 0\right)$
$\left(\frac{3+\sqrt{37}}{2}, 0\right)$
4
-4
-7
x=-1

10 The curve C has equation

$$y = \frac{x^2 + \lambda x - 6\lambda^2}{x + 3},$$

where λ is a constant such that $\lambda \neq 1$ and $\lambda \neq -\frac{3}{2}$.

(i) Find $\frac{dy}{dx}$ and deduce that if C has two stationary points then $-\frac{3}{2} < \lambda < 1$. [5]

(ii) Find the equations of the asymptotes of C. [3]

(iii) Draw a sketch of C for the case $0 < \lambda < 1$. [3]

(iv) Draw a sketch of C for the case $\lambda > 3$. [3]

i) $y = x + \lambda - 3 + \frac{-6\lambda^2 - 3\lambda + 9}{x+3}$.

$$\begin{array}{r} x+\lambda-3 \\ x+3 \enclose{longdiv}{x^2+\lambda x-6\lambda^2} \\ x^2+3x \\ - \quad - \\ \hline (\lambda-3)x-6\lambda^2 \\ (\lambda-3)x+3\lambda-9 \\ - \quad - \quad + \\ \hline -6\lambda^2-3\lambda+9 \\ \hline \end{array}$$

$\frac{dy}{dx} = 1 - \frac{-6\lambda^2-3\lambda+9}{(x+3)^2} = 1 + \frac{6\lambda^2+3\lambda-9}{(x+3)^2}$

Set $\frac{dy}{dx} = 0 \Rightarrow 1 + \frac{6\lambda^2+3\lambda-9}{(x+3)^2} = 0$

$(x+3)^2 = -6\lambda^2 - 3\lambda + 9$

$x^2+6x+9+6\lambda^2+3\lambda-9=0$

$x^2+6x+6\lambda^2+3\lambda=0$

For two stationary points, $6^2-4(1)(6\lambda^2+3\lambda)>0$

$36-24\lambda^2-12\lambda>0$

$2\lambda^2+\lambda-3<0$

$2\lambda^2+3\lambda-\lambda-3<0$

$(2\lambda+3)(\lambda-1)<0$

$\Rightarrow \frac{-3}{2}<\lambda<1$. (Shown).

ii) $y = x+\lambda-3+\frac{-6\lambda^2-3\lambda+9}{x+3}$

Oblique asymptote: $y=x+\lambda-3$, Vertical asymptote: $x=-3$.

iii) $0<\lambda<1 \Rightarrow$ two stationary points exist.

x-intercept $\Rightarrow y=0 \Rightarrow \frac{x^2+\lambda x-6\lambda^2}{x+3}=0 \Rightarrow x=\frac{-\lambda\pm\sqrt{25\lambda^2}}{2}$

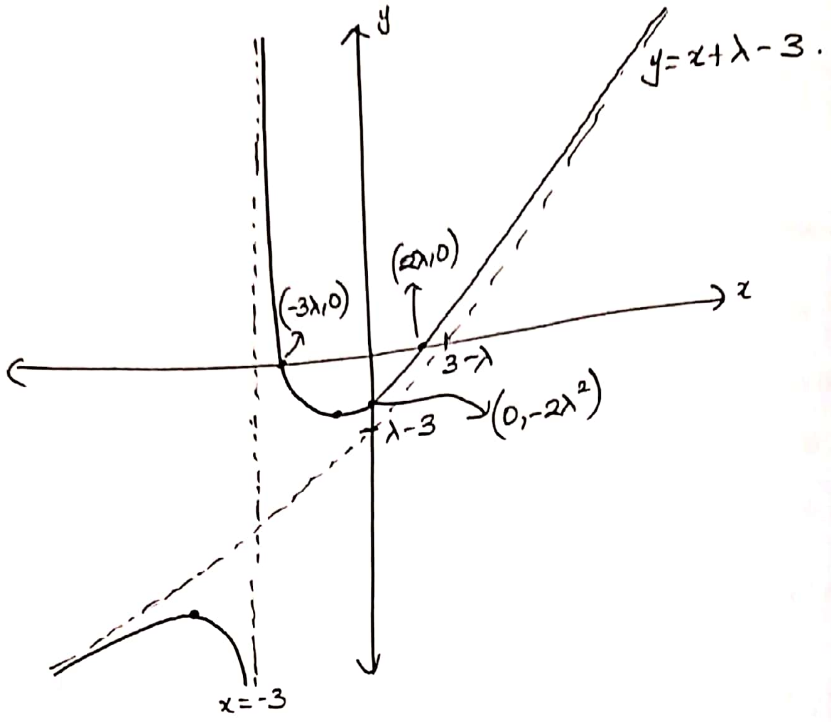
$x=\frac{-\lambda\pm5\lambda}{2}$

$\Rightarrow x=2\lambda, -3\lambda$

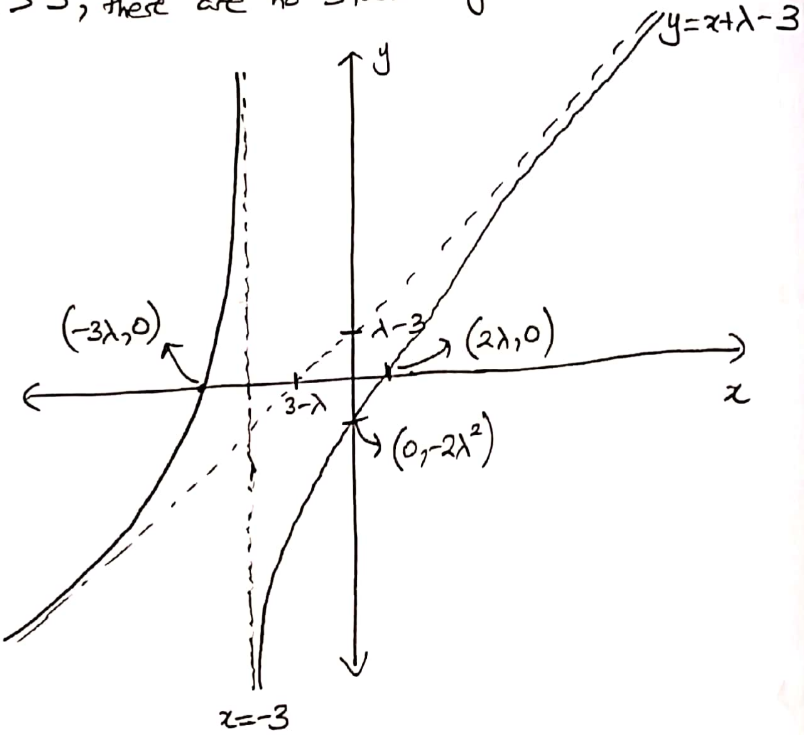
$\therefore (2\lambda, 0), (-3\lambda, 0)$.

y-intercept $\Rightarrow x=0 \Rightarrow y=-2\lambda^2$

$\therefore (0, -2\lambda^2)$



iv) For λ > 3, there are no stationary points.



11 The curve C with equation

$$y = \frac{ax^2 + bx + c}{x - 1},$$

where a, b and c are constants, has two asymptotes. It is given that $y = 2x - 5$ is one of these asymptotes.

(i) State the equation of the other asymptote. [1]

(ii) Find the value of a and show that $b = -7$. [3]

(iii) Given also that C has a turning point when $x = 2$, find the value of c. [3]

(iv) Find the set of values of k for which the line $y = k$ does not intersect C. [4]

i) Vertical asymptote: $x = 1$.

$$\begin{array}{r|l} & ax + b+a \\ \hline x-1 & ax^2+bx+c \\ & ax^2-ax \\ & - \quad + \\ \hline & (b+a)x+c \\ & (b+a)x-b-a \\ & + \quad + \\ \hline & c+b+a \\ \hline \end{array}$$

ii) As $y = ax+b+a + \dfrac{a+b+c}{x-1}$

Oblique asymptote: $y = ax+b+a = 2x-5$

$\therefore a = 2$

and $b+a = -5$

$b+2 = -5$

$b = -7$.

iii) $y = \dfrac{2x^2-7x+c}{x-1} = 2x-5+\dfrac{c-5}{x-1}$.

$\dfrac{dy}{dx} = 2 - \dfrac{c-5}{(x-1)^2}$

When $x = 2$, $\dfrac{dy}{dx} = 0 \Rightarrow 0 = 2 - \dfrac{c-5}{(2-1)^2}$

$c-5 = 2$

$c = 7$

iv) $y = k \Rightarrow k = \dfrac{2x^2-7x+7}{x-1}$

$0 = 2x^2-7x+7-kx+k$

$0 = 2x^2+x(-7-k)+k+7$

For line to not intersect curve,

$b^2-4ac < 0$

$\Rightarrow (-7-k)^2-4(2)(k+7) < 0$

$$k^2+14k+49-8k-56<0$$

$$k^2+6k-7<0$$

$$(k+7)(k-1)<0$$

$$\Rightarrow \quad \underline{-7<k<1}.$$

12 The curve C has equation

$$y = \frac{2x^2 + 2x + 3}{x^2 + 2}.$$

Show that, for all x, $1 \leqslant y \leqslant \frac{5}{2}$. [4]

Find the coordinates of the turning points on C. [3]

Find the equation of the asymptote of C. [2]

Sketch the graph of C, stating the coordinates of any intersections with the y-axis and the asymptote. [2]

$$y = \frac{2x^2+2x+3}{x^2+2}$$

$$yx^2+2y = 2x^2+2x+3$$

$$0 = (2-y)x^2+2x+3-2y$$

For x to be real, $b^2-4ac \geqslant 0$

$$(2)^2-4(2-y)(3-2y) \geqslant 0$$

$$4-4[6-7y+2y^2] \geqslant 0$$

$$-20+28y-8y^2 \geqslant 0$$

$$8y^2-28y+20 \leqslant 0$$

$$2y^2-7y+5 \leqslant 0 \Rightarrow 2y^2-2y-5y+5 \leqslant 0$$

$$(2y-5)(y-1) \leqslant 0$$

$$\Rightarrow \frac{5}{2} \leqslant y \leqslant 1. \text{ (Shown)}$$

$$y=1 \Rightarrow 1 = \frac{2x^2+2x+3}{x^2+2}$$

$$0 = x^2+2x+1$$

$$0 = (x+1)^2$$

$$x=-1$$

$$\therefore (-1,1)$$

$$y = \frac{5}{2} \Rightarrow \frac{5}{2} = \frac{2x^2+2x+3}{x^2+2}$$

~~$5x^2+10 = 4x^2+4x+6$~~

$5x^2+10 =$ ~~$8x^2+10x$~~ $4x^2+4x+6$

$$x^2-4x+4 = 0$$

$$(x-2)^2 = 0$$

$$x = 2 \quad \therefore \left(2, \frac{5}{2}\right).$$

i.e. Turning points: $(-1,1)$ and $\left(2,\frac{5}{2}\right)$.

As $|x| \mapsto \infty$, $y \mapsto \frac{2}{1} = 2$ $\therefore$ Horizontal asymptote: $y=2$

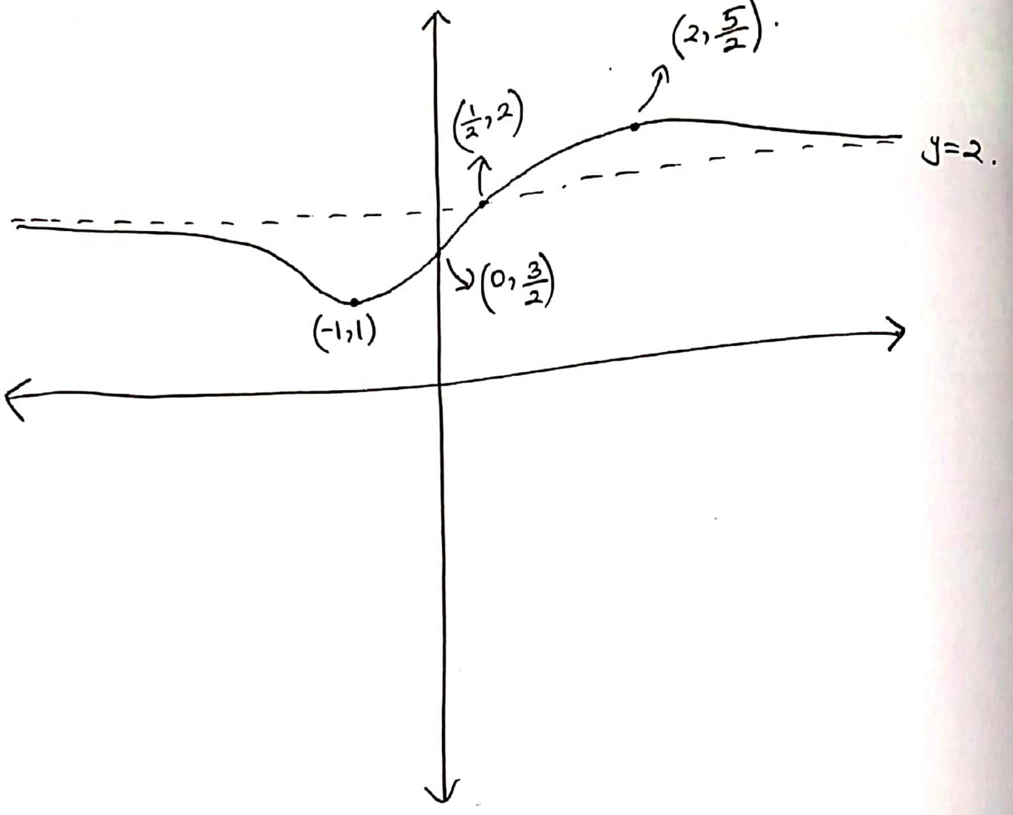
For vertical asymptote: $x^2+2=0 \Rightarrow x = \pm\sqrt{-2}$ (No real roots)

$\Rightarrow$ No vertical asymptotes exist.

x-intercept $\Rightarrow y=0 \therefore \frac{2x^2+2x+3}{x^2+2} = 0$

$$x = \frac{-2 \pm \sqrt{4-24}}{4}$$ = No real roots ∴ No x-intercepts.

y intercept $\Rightarrow x=0 \Rightarrow y = \frac{2x^2+2x+3}{x^2+2} = \frac{3}{2} \therefore \left(0, \frac{3}{2}\right)$.



Set $y=2 \Rightarrow 2 = \frac{2x^2+2x+3}{x^2+2} \Rightarrow 2x^2+4 = 2x^2+2x+3$

$1 = 2x$

$\Rightarrow x = \frac{1}{2}$.

$\therefore \left(\frac{1}{2}, 2\right)$.

13 The curve C has equation $y = \frac{x^2}{x-2}$. Find the equations of the asymptotes of C. [3]

Find the coordinates of the turning points on C. [3]

Draw a sketch of C. [3]

$$y = \frac{x^2}{x-2} = x+2+\frac{4}{x-2}.$$

$$\begin{array}{r|l} & x+2 \\ x-2 & x^2+0x+0 \\ & x^2-2x \\ & -\ \ + \\ \hline & 2x+0 \\ & 2x-4 \\ & -\ \ + \\ \hline & 4 \end{array}$$

Asymptotes: $y=x+2$, $x=2$.

$$\frac{dy}{dx} = 1 - \frac{4}{(x-2)^2}$$

At turning points, $\frac{dy}{dx} = 0$

$$\Rightarrow 1 - \frac{4}{(x-2)^2} = 0$$

$$(x-2)^2 = 4 \Rightarrow x-2 = \pm 2$$

$x-2=2$ or $x-2=-2$

$x=4$ or $x=0$

$\Rightarrow y = \frac{4^2}{4-2} = 8$ or $y = \frac{0^2}{0-2} = 0$

$\therefore$ Turning points: $(4,8)$ and $(0,0)$.

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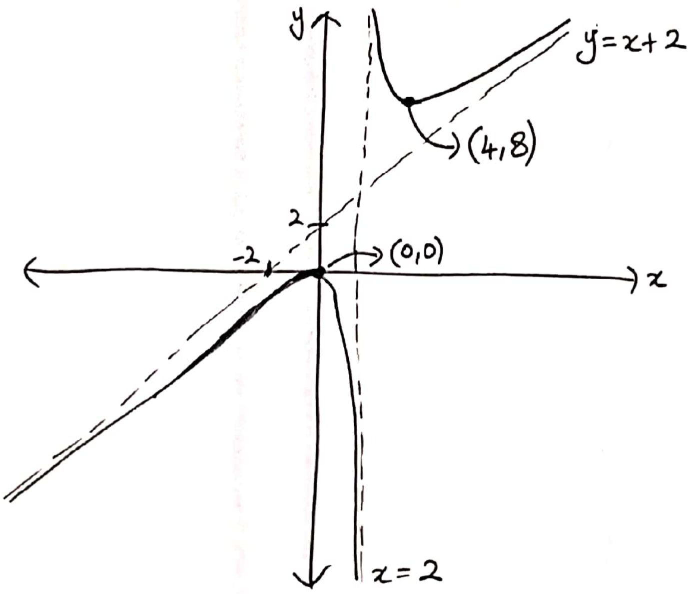
x-intercept $\Rightarrow y=0 \therefore \frac{x^2}{x-2} = 0 \therefore x=0 \Rightarrow (0,0)$.



14 The curve C has equation $y = \frac{2x^2 - 3x - 2}{x^2 - 2x + 1}$. State the equations of the asymptotes of C. [2]

Show that $y \leqslant \frac{25}{12}$ at all points of C. [4]

Find the coordinates of any stationary points of C. [3]

Sketch C, stating the coordinates of any intersections of C with the coordinate axes and the asymptotes. [4]

$$y = 2 + \frac{x-4}{x^2-2x+1} = 2 + \frac{x-4}{(x-1)^2}.$$

$$\begin{array}{r|l} & 2 \\ \hline x^2-2x+1 & 2x^2-3x-2 \\ & 2x^2-4x+2 \\ & - \quad + \quad - \\ \hline & x-4 \\ \hline \end{array}$$

Asymptotes: $y=2$, $(x-1)^2=0 \Rightarrow x=1$

$$\frac{dy}{dx} = \frac{(x^2-2x+1)(1)-(x-4)(2x-2)}{(x-1)^4}$$

$$= \frac{x^2-2x+1-(2x^2-10x+8)}{(x-1)^4} = \frac{-x^2+8x-7}{(x-4)^4}$$

$$\frac{dy}{dx} = \frac{-(x^2-8x+7)}{(x-1)^4} = \frac{-(x-1)(x-7)}{(x-1)^4} = \frac{-(x-7)}{(x-1)^3} = \frac{7-x}{(x-1)^3}.$$

For stationary point, $\frac{dy}{dx}=0 \Rightarrow \frac{-(x-7)}{(x-1)^3}=0 \Rightarrow x=7.$

$$\frac{d^2y}{dx^2} = \frac{(x-1)^3(-1)-(7-x)\left[3(x-1)^2\right]}{(x-1)^6}$$

When $x=7$, $y = 2+\frac{3}{6^2} = \frac{25}{12}$, $\frac{d^2y}{dx^2} = \frac{6^3(-1)-(0)}{6^6} = \frac{-1}{216} < 0 \therefore \left(7, \frac{25}{12}\right)$

is a maximum value. $\therefore$ $y \leq \frac{25}{12}$ at all points on C.

Altenatively:

$$y = \frac{2x^2-3x-2}{x^2-2x+1} \Rightarrow yx^2-2yx+y = 2x^2-3x-2$$

$$(y-2)x^2+(3-2y)x+y+2=0$$

For x to be real, $b^2-4ac \geq 0$

$$(3-2y)^2-4(y-2)(y+2) \geq 0$$

$$9+4y^2-12y-4y^2+16 \geq 0$$

$$-12y \geq -25$$

$$y \leq \frac{25}{12}.$$

When $y=\frac{25}{12} \Rightarrow \frac{25}{12} = \frac{2x^2-3x-2}{x^2-2x+1} \Rightarrow 25x^2-50x+25 = 24x^2-36x-24$

$$x^2-14x+49=0$$

$$x=7 \therefore \left(7, \frac{25}{12}\right)$$

y-intercept $\Rightarrow x=0 \Rightarrow y = \frac{-2}{1} = -2 \therefore (0,-2)$

x-intercept $\Rightarrow y=0 \Rightarrow \frac{2x^2-3x-2}{x^2-2x+1} = 0 \Rightarrow 2x^2-4x+x-2=0$

$(2x+1)(x-2)=0$

$x = \frac{-1}{2}, 2. \quad \therefore \left(\frac{-1}{2}, 0\right)$ and $(2,0)$

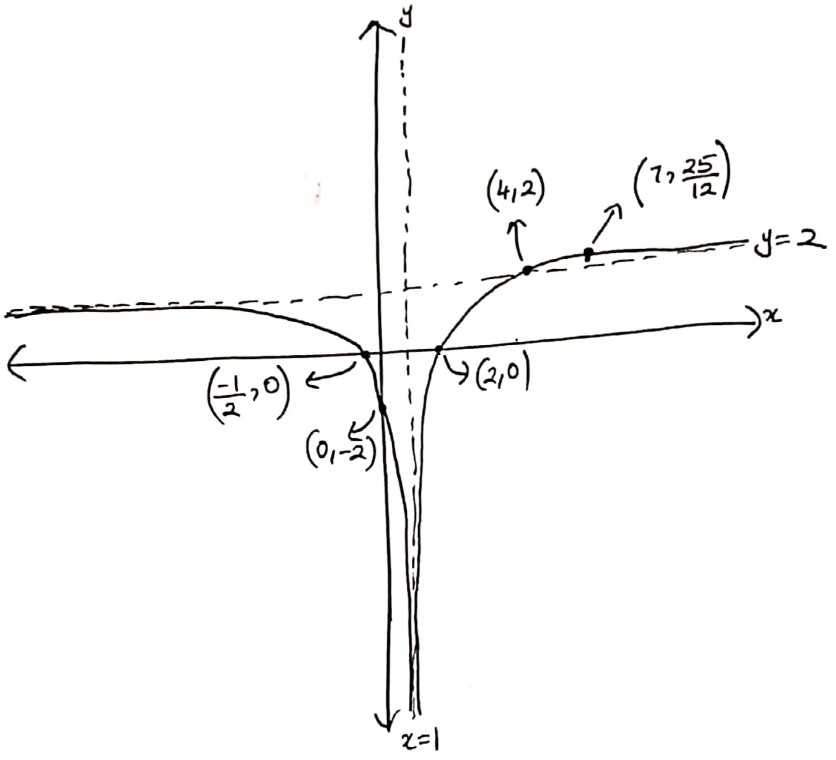
When curve intersects the horizontal asymptote $y=2$

$$\Rightarrow \frac{2x^2-3x-2}{x^2-2x+1} = 2$$

$$2x^2-3x-2 = 2x^2-4x+2$$

$$x = 4$$

$$\therefore (4,2).$$



15 The curve C has equation

$$y = \frac{ax^2 + bx + c}{x + d},$$

where a, b, c and d are constants. The curve cuts the y-axis at $(0, -2)$ and has asymptotes $x = 2$ and $y = x + 1$.

(i) Write down the value of d. [1]

(ii) Determine the values of a, b and c. [6]

(iii) Show that, at all points on C, either $y \leqslant 3 - 2\sqrt{6}$ or $y \geqslant 3 + 2\sqrt{6}$. [7]

i) Vertical asymptote at $x+d=0 \Rightarrow x=-d$

By comparing to $x=2$ $\therefore -d=2 \Rightarrow d=-2.$

ii) $y = \dfrac{ax^2+bx+c}{x-2}$

$$\begin{array}{r|l} & ax+b+2a \\ \hline x-2 & ax^2+bx+c \\ & ax^2-2ax \\ & -\qquad + \\ \hline & (b+2a)x+c \\ & (b+2a)x-2b-4a \\ & -\qquad\quad +\quad + \\ \hline & c+2b+4a \\ \hline \end{array}$$

$y = ax+b+2a+\dfrac{c+2b+4a}{x-2}.$

Oblique asymptote: $y = x+1 = ax+b+2a$

$\therefore a=1,\ b+2a=1$

$b+2=1 \Rightarrow b=-1.$

When $x=0$, $y=-2$

$\therefore -2 = \dfrac{a(0)^2+b(0)+c}{0-2}$

$4 = c$

$\therefore y = \dfrac{x^2-x+4}{x-2}.$

iii) $yx-2y = x^2-x+4$

$0 = x^2-x-yx+4+2y$

$0 = x^2+x(-1-y)+4+2y$

For x to be real, $b^2-4ac \geq 0$

$(-1-y)^2-4(1)(4+2y) \geq 0$

$1+2y+y^2-16-8y \geq 0$

$y^2-6y-15 \geq 0$

Critical value: $y^2-6y-15=0 \Rightarrow y = \dfrac{6\pm\sqrt{96}}{2} = \dfrac{6\pm4\sqrt{6}}{2} = 3\pm2\sqrt{6}.$

Then $y \leq 3-2\sqrt{6}$ or $y \geq 3+2\sqrt{6}.$

(Shown).

16 Express $\dfrac{2x^2 - x + 5}{x^2 - 1}$ in the form $2 + \dfrac{A}{x-1} + \dfrac{B}{x+1}$, where A and B are integers to be found. [3]

The curve C has equation $y = \dfrac{2x^2 - x + 5}{x^2 - 1}$. Show that there are two distinct values of x for which $\dfrac{dy}{dx} = 0$. [4]

Sketch C, stating the equations of the asymptotes and giving the coordinates of any points of intersection with the coordinate axes and with the asymptotes. You do not need to find the coordinates of the turning points. [7]

$$\frac{2x^2-x+5}{x^2-1} = 2 + \frac{7-x}{x^2-1}$$

$$\begin{array}{r|l} & \quad 2 \\ \hline x^2-1 & 2x^2-x+5 \\ & 2x^2 \quad -2 \\ & - \qquad + \\ \hline & \quad -x+7 \\ \hline \end{array}$$

$$\frac{7-x}{x^2-1} \equiv \frac{A}{x-1} + \frac{B}{x+1}$$

$$7-x \equiv A(x+1)+B(x-1)$$

$x=1 \Rightarrow \quad 6 = A(2) \Rightarrow A=3.$

$x=-1 \Rightarrow \quad 8 = B(-2) \Rightarrow B=-4$

$$\therefore \quad \frac{2x^2-x+5}{x^2-1} = 2+\frac{3}{x-1} - \frac{4}{x+1}.$$

$$y = \frac{2x^2-x+5}{x^2-1} = 2+\frac{3}{x-1}-\frac{4}{x+1}.$$

$$\frac{dy}{dx} = -\frac{3}{(x-1)^2} + \frac{4}{(x+1)^2} = 0 \Rightarrow \frac{4}{(x+1)^2} = \frac{3}{(x-1)^2}$$

$$4(x-1)^2 = 3(x+1)^2$$

$$4[x^2-2x+1] = 3[x^2+2x+1]$$

$$x^2-14x+1=0$$

$$\Rightarrow x = \frac{14 \pm \sqrt{192}}{2} = \frac{14 \pm 8\sqrt{3}}{2} = 7 \pm 4\sqrt{3}.$$

Then there are two distinct turning points.

When $x = 7+4\sqrt{3}$, $y = 1.96$ $\Rightarrow$ Turning points: $(13.9, 1.96)$, $(0.072, -4.96)$.

When $x = 7-4\sqrt{3}$, $y = -4.96$

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As $|x| \mapsto \infty$, $y \mapsto \frac{2}{1} = 2.$

$x^2-1=0 \Rightarrow x = \pm 1$ $\quad \therefore$ Asymptotes: $y=2$, $x=1$, $x=-1$.

When curve intersects the horizontal asymptote $y=2$:

$$\frac{2x^2-x+5}{x^2-1}=2$$

$$2x^2-x+5=2x^2-2$$

$$7=x$$

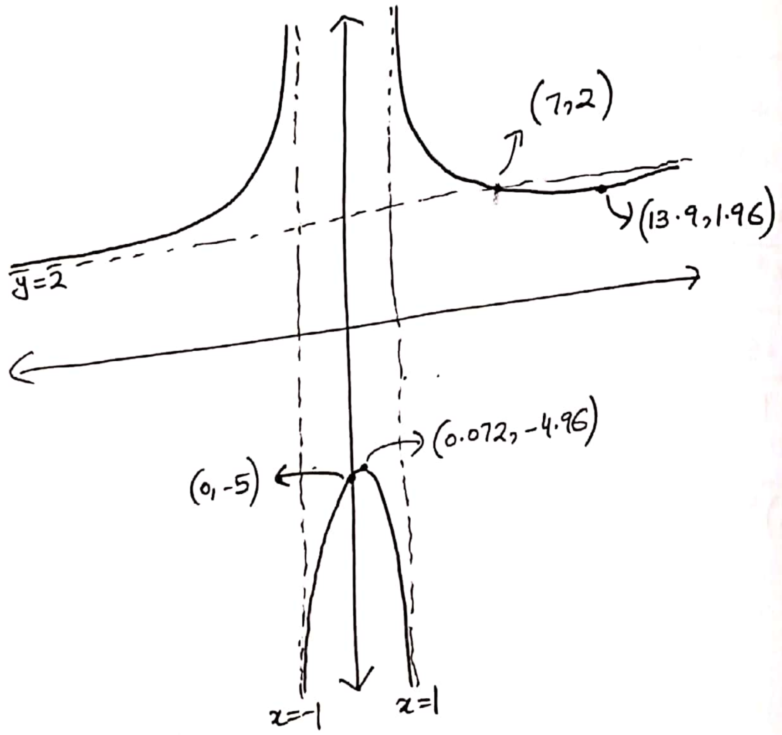
∴ Intersection point with asymptote : (7,2).

x-intercept ⇒ $y=0$ ⇒ $\frac{2x^2-x+5}{x^2-1}=0$

$$\therefore x=\frac{1\pm\sqrt{1-40}}{4}=\frac{1\pm\sqrt{-39}}{4}$$ (No real roots)

∴ No x-intercepts.

y-intercept ⇒ $x=0$ ⇒ $y=\frac{2\times0^2-0+5}{0^2-1}=-5$. ∴ (0,-5).



17 The curve C has equation $y = \frac{4x^2 - 3x}{x^2 + 1}$. Verify that the equation of C may be written in the form $y = -\frac{1}{2} + \frac{(3x-1)^2}{2(x^2+1)}$ and also in the form $y = \frac{9}{2} - \frac{(x+3)^2}{2(x^2+1)}$. [3]

Hence show that $-\frac{1}{2} \leqslant y \leqslant \frac{9}{2}$. [2]

Without differentiating, write down the coordinates of the turning points of C. [2]

State the equation of the asymptote of C. [1]

Sketch the graph of C, stating the coordinates of the intersections with the coordinate axes and the asymptote. [3]

$$y = \frac{(4x^2-3x)\times 2}{(x^2+1)\times 2} = \frac{8x^2-6x}{2(x^2+1)} = \frac{9x^2-6x+1-x^2-1}{2(x^2+1)}$$

$$y = \frac{(3x-1)^2-(x^2+1)}{2(x^2+1)} = \frac{-(x^2+1)}{2(x^2+1)} + \frac{(3x-1)^2}{2(x^2+1)} = \frac{-1}{2} + \frac{(3x-1)^2}{2(x^2+1)}. \text{ (Shown).}$$

$$y = \frac{4x^2-3x}{x^2+1} = \frac{8x^2-6x}{2(x^2+1)} = \frac{9x^2+9-x^2-6x-9}{2(x^2+1)} = \frac{9(x^2+1)-(x+3)^2}{2(x^2+1)}$$

$$y = \frac{9}{2} - \frac{(x+3)^2}{2(x^2+1)} \text{ (Shown).}$$

$$\frac{(3x-1)^2}{2(x^2+1)} \geqslant 0 \Rightarrow \frac{-1}{2} + \frac{(3x-1)^2}{2(x^2+1)} \geqslant \frac{-1}{2} \Rightarrow y \geqslant \frac{-1}{2}.$$

$$\frac{(x+3)^2}{2(x^2+1)} \geqslant 0 \Rightarrow -\frac{(x+3)^2}{2(x^2+1)} \leqslant 0 \Rightarrow \frac{9}{2} - \frac{(x+3)^2}{2(x^2+1)} \leqslant \frac{9}{2} \Rightarrow y \leqslant \frac{9}{2}$$

$$\therefore \frac{-1}{2} \leqslant y \leqslant \frac{9}{2} \text{ (Shown)}$$

When $y = \frac{-1}{2} \Rightarrow \frac{-1}{2} = \frac{-1}{2} + \frac{(3x-1)^2}{2(x^2+1)}$

$0 = (3x-1)^2$

$\Rightarrow x = \frac{1}{3}$

When $y = \frac{9}{2} \Rightarrow \frac{9}{2} = \frac{9}{2} - \frac{(x+3)^2}{2(x^2+1)}$

$0 = \frac{(x+3)^2}{2(x^2+1)}$

$\Rightarrow x = -3.$

$\therefore$ Turning points: $\left(\frac{1}{3}, \frac{-1}{2}\right), \left(-3, \frac{9}{2}\right).$

As $|x| \mapsto \infty$, $y \mapsto \frac{4}{1} = 4$ $\therefore$ Horizontal asymptote: $y = 4$

Vertical asymptote: $x^2+1=0 \Rightarrow x = \pm\sqrt{-1}$ (No real roots) $\therefore$ No vertical asymptotes.

For y-intercept, $x=0 \Rightarrow y = \frac{4\times0^2-3\times0}{0^2+1} = 0$

$\therefore (0,0)$

For x-intercept : $y=0 \Rightarrow \frac{4x^2-3x}{x^2+1} = 0 \Rightarrow x(4x-3)=0$

$x=0$ or $x=\frac{3}{4}$.

$\therefore (0,0)$ and $\left(\frac{3}{4},0\right)$.

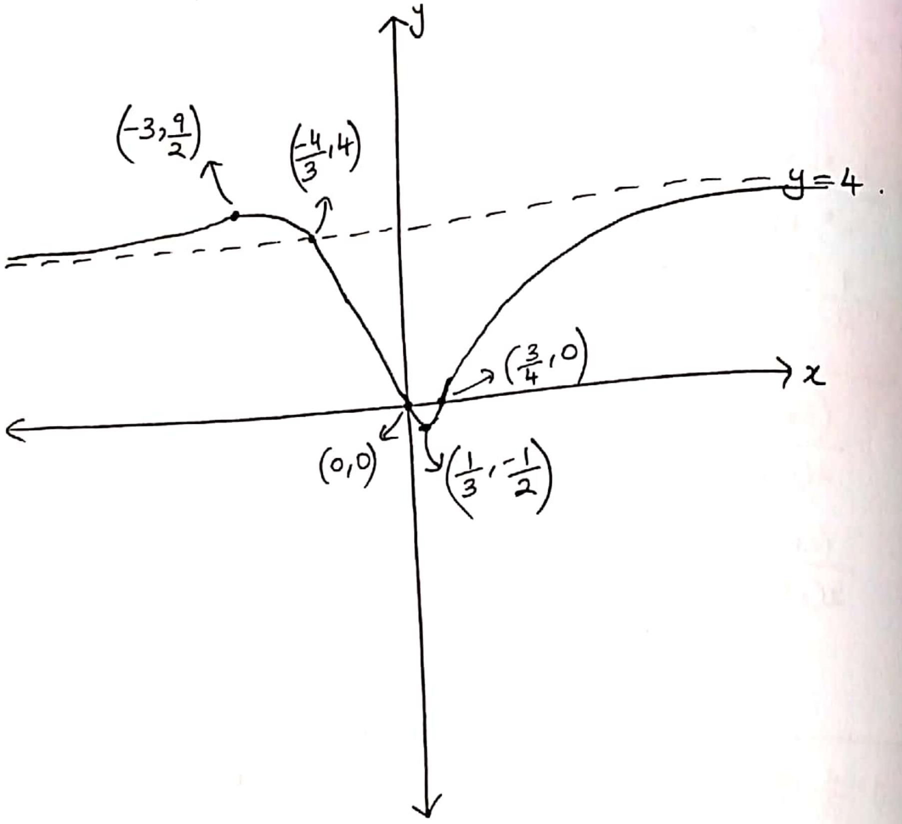
When curve intersects the horizontal line $y=4$,

$$\frac{4x^2-3x}{x^2+1} = 4$$

$$4x^2-3x = 4x^2+4$$

$$x = \frac{-4}{3}.$$

$\therefore$ Intersection point for curve with horizontal asymptote : $\left(\frac{-4}{3},4\right)$



18 A curve C has equation $y = \frac{x^2}{x-2}$. Find the equations of the asymptotes of C. [3]

Show that there are no points on C for which $0 < y < 8$. [4]

Sketch C, giving the coordinates of the turning points. [3]

$$y = x+2+\frac{4}{x-2}$$

$$\begin{array}{r} x+2 \\ x-2 \,\big)\, \overline{x^2} \\ x^2-2x \\ - \quad + \\ \hline 2x \\ 2x-4 \\ - \quad + \\ \hline 4 \\ \hline \end{array}$$

Asymptotes: $y = x+2$, $x = 2$.

$$y = \frac{x^2}{x-2}$$

$$yx - 2y = x^2$$

$$0 = x^2 - yx + 2y$$

For x to not be real, $(-y)^2 - 4(1)(2y) < 0$

$$y^2 - 8y < 0$$

$$y(y-8) < 0$$

$$\Rightarrow \quad 0 < y < 8. \text{ (shown)}$$

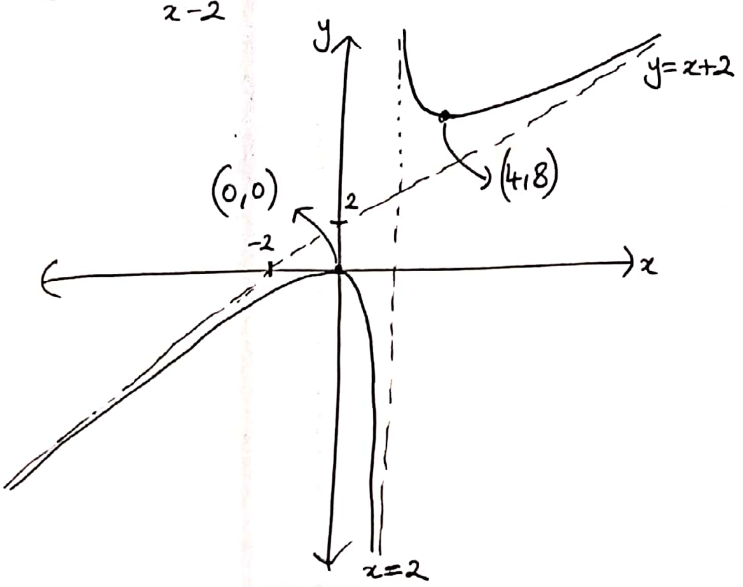
When $y = 0 \Rightarrow \frac{x^2}{x-2} = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0. \therefore (0,0).$

When $y = 8 \Rightarrow \frac{x^2}{x-2} = 8 \Rightarrow x^2 - 8x + 16 = 0 \Rightarrow (x-4)^2 = 0 \Rightarrow x = 4 \therefore (4,8).$

Turning points: $(0,0)$ and $(4,8)$.

x-intercept $\Rightarrow y = 0 \therefore \frac{x^2}{x-2} = 0 \Rightarrow x = 0 \therefore (0,0)$

y-intercept



19 The curve C has equation $y = \frac{x+2}{x^2-9}$. Show that $\frac{dy}{dx} < 0$ at all points on C. [3]

State the equations of the asymptotes of C. [2]

Sketch C, showing the coordinates of any points of intersection with the coordinate axes. [3]

$$\frac{dy}{dx} = \frac{(x^2-9)(1)-(x+2)(2x)}{(x^2-9)^2} = \frac{x^2-9-2x^2-4x}{(x^2-9)^2} = \frac{-x^2-4x-9}{(x^2-9)^2}$$

$$= \frac{-(x^2+4x+9)}{(x^2-9)^2} = \frac{-[x^2+4x+2^2-2^2+9]}{(x^2-9)^2} = \frac{-(x+2)^2-5}{(x^2-9)^2}.$$

$$\frac{dy}{dx} = \frac{-(x+2)^2}{(x^2-9)^2} - \frac{5}{(x^2-9)^2}$$

$$\frac{dy}{dx} = -\left[\frac{(x+2)^2}{(x^2-9)^2} + \frac{5}{(x^2-9)^2}\right] < 0 \quad \text{as} \quad (x+2)^2, (x^2-9)^2 > 0$$

(Shown)

$$y = \frac{x+2}{x^2-9}$$

As $|x| \mapsto \infty$, $y \mapsto 0$ ∴ Horizontal asymptote: $y=0$.

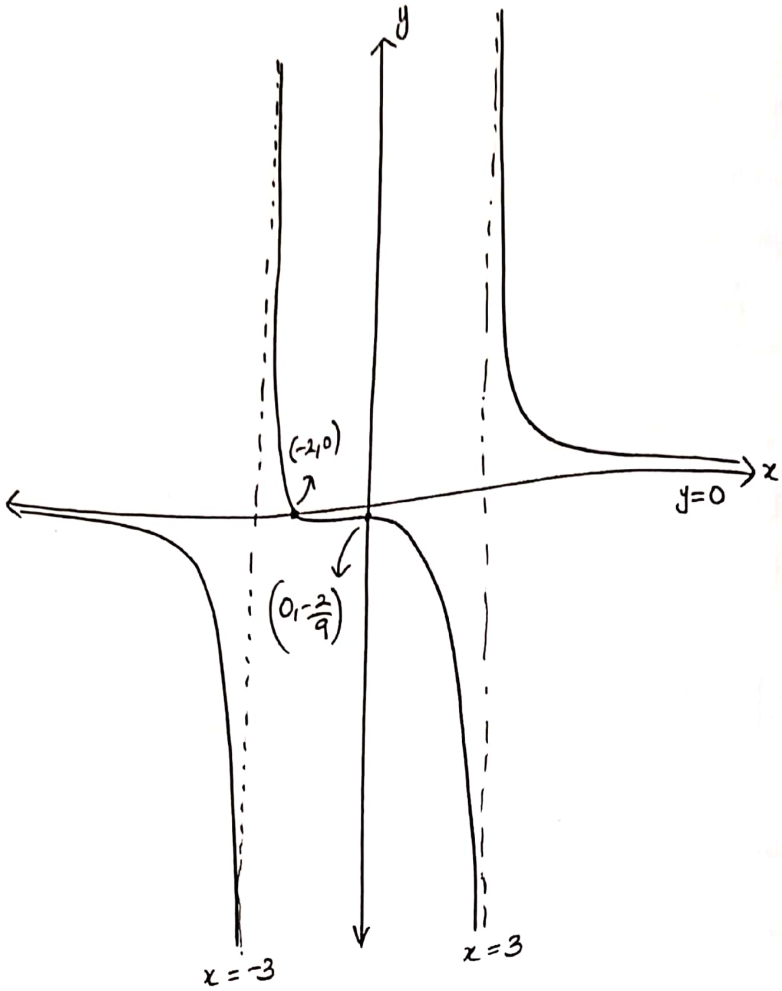
~~As~~ Vertical asymptotes: $x^2-9=0 \Rightarrow x=\pm 3$

$\Rightarrow$ Vertical asymptotes: $x=3, x=-3$.

x-intercept $\Rightarrow y=0 \Rightarrow \frac{x+2}{x^2-9}=0 \Rightarrow x=-2$. ∴ $(-2,0)$

y-intercept $\Rightarrow x=0 \Rightarrow y=\frac{0+2}{0^2-9}=\frac{-2}{9} \Rightarrow \left(0, \frac{-2}{9}\right)$.

Since $\frac{dy}{dx}<0$ ∴ there are no turning points.


y
x
(-2,0)
$\left(0,-\frac{2}{9}\right)$
y=0
x=-3
x=3

20 The curve C has equation $y = \frac{x^2 - 3x + 6}{1 - x}$.

(i) Find the equations of the asymptotes of C. [3]

$$y = -x + 2 + \frac{4}{-x+1} = -x + 2 + \frac{4}{1-x}.$$

$$\begin{array}{r|l} & -x+2 \\ \hline -x+1 & x^2 - 3x + 6 \\ & x^2 - x \\ & - \quad + \\ \hline & -2x + 6 \\ & -2x + 2 \\ & + \quad - \\ \hline & 4 \\ \hline \end{array}$$

Asymptotes: $y = -x + 2$

$x = 1$

(ii) Find the coordinates of the turning points of C. [3]

$$\frac{dy}{dx} = -1 - \frac{4}{(1-x)^2} \cdot -1 = -1 + \frac{4}{(1-x)^2}.$$

At turning point, $\frac{dy}{dx} = 0 \Rightarrow -1 + \frac{4}{(1-x)^2} = 0$

$(1-x)^2 = 4$

$1 - x = \pm 2$

$1 - x = 2$ or $1 - x = -2$

$-x = 1$ $\quad$ $-x = -3$

$x = -1$ $\quad$ $x = 3.$

$$\Rightarrow x = -1 \Rightarrow y = \frac{(-1)^2 - 3(-1) + 6}{1 - (-1)} = 5$$

$$x = 3 \Rightarrow y = \frac{3^2 - 3(3) + 6}{1 - 3} = -3$$

$\therefore$ Turning points: $(-1, 5)$ and $(3, -3)$.

(iii) Find the coordinates of any intersections with the coordinate axes. [2]

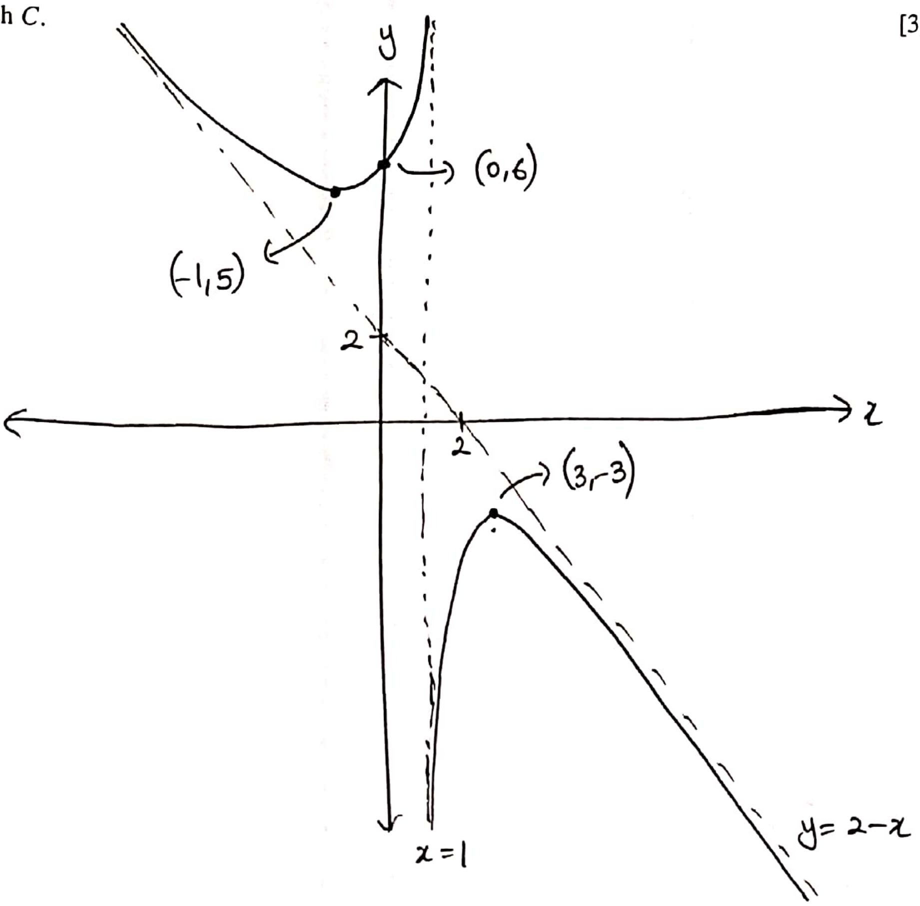
For y-intercept, $x = 0 \Rightarrow y = \frac{6}{1} = 6 \therefore (0,6)$

For x-intercept, $y = 0 \Rightarrow \frac{x^2 - 3x + 6}{1 - x} = 0$

$$x = \frac{3 \pm \sqrt{-15}}{2} \quad \text{(No real roots)}$$

$\Rightarrow$ No x-intercepts exist.

(iv) Sketch C. [3]



21 The curve C has equation

$$y = \frac{x^2 + b}{x + b},$$

where b is a positive constant.

(i) Find the equations of the asymptotes of C. [3]

$$y = x - b + \frac{b + b^2}{x + b}.$$

$$\begin{array}{r} x - b \\ x + b \overline{\big) x^2 + 0x + b} \\ x^2 + bx \\ - \quad - \\ \hline -bx + b \\ -bx - b^2 \\ + \quad + \\ \hline b + b^2 \\ \hline \end{array}$$

$\therefore$ Asymptotes: $y = x - b$

$x = -b$.

(ii) Show that C does not intersect the x-axis. [1]

For x-intercept, $y = 0$

$$\Rightarrow 0 = \frac{x^2 + b}{x + b} \Rightarrow x^2 = -b$$

$x = \pm\sqrt{-b}$

No real roots.

$\therefore$ C does not intersect the x-axis.

(iii) Justifying your answer, find the number of stationary points on C. [2]

$$\frac{dy}{dx} = 1 - \frac{b+b^2}{(x+b)^2}$$

For stationary points, $\frac{dy}{dx} = 0 \Rightarrow 1 - \frac{b+b^2}{(x+b)^2} = 0$

$$1 = \frac{b+b^2}{(x+b)^2}$$

$$x^2+2bx+b^2 = b+b^2$$

$$x^2+2bx-b = 0$$

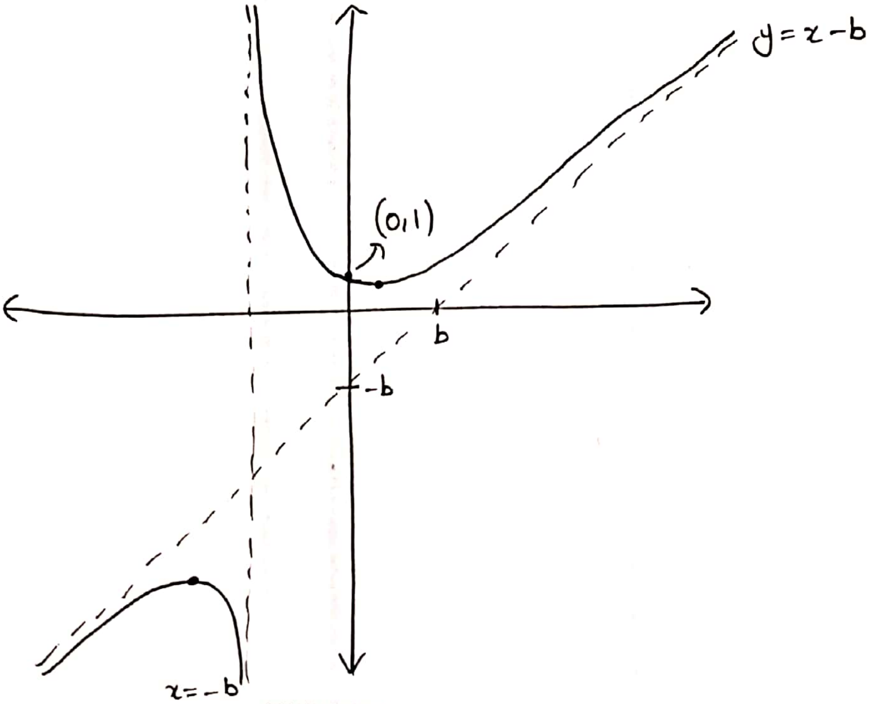
Note Discriminant $= (2b)^2-4(1)(-b) = 4b^2+4b > 0$

$\therefore$ There are two distinct real roots for $\frac{dy}{dx} = 0$

$\Rightarrow$ There are two stationary points on C.

(iv) Sketch C. Your sketch should indicate the coordinates of any points of intersection with the y-axis. You do not need to find the coordinates of any stationary points. [3]

For y-intercept $\Rightarrow x=0 \Rightarrow y = \frac{0^2+b}{0+b} = 1 \therefore (0,1)$.



22 The curve C has equation

$$y = \frac{x^2 + 7x + 6}{x - 2}.$$

(i) Find the coordinates of the points of intersection of C with the axes. [2]

y-intercept $\Rightarrow x = 0 \Rightarrow y = \frac{0^2+7(0)+6}{0-2} = -3 \therefore (0,-3).$

x-intercept $\Rightarrow y = 0 \Rightarrow \frac{x^2+7x+6}{x-2} = 0 \Rightarrow (x+1)(x+6) = 0$

$x = -1, -6$

$\therefore (-1,0), (-6,0).$

(ii) Find the equation of each of the asymptotes of C. [3]

$y = x+9 + \frac{24}{x-2}.$

Asymptotes: $y = x+9$

$x = 2.$

$$\begin{array}{r|l} & x+9 \\ \hline x-2 & x^2+7x+6 \\ & x^2-2x \\ & -\quad + \\ \hline & 9x+6 \\ & 9x-18 \\ & -\quad + \\ \hline & 24 \\ \hline \end{array}$$

(iii) Sketch C. [3]

$$\frac{dy}{dx} = 1 - \frac{24}{(x-2)^2} = 0$$

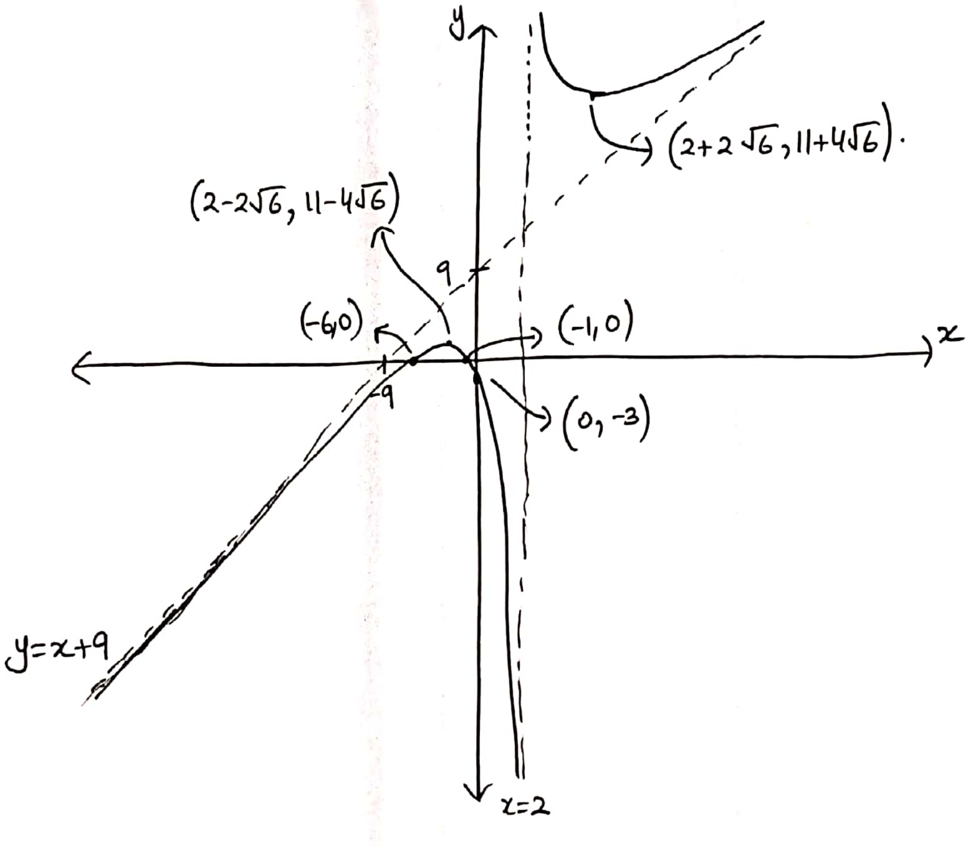
$$(x-2)^2 = 24$$

$$x-2 = \pm\sqrt{24} = \pm 2\sqrt{6}$$

$$x = 2 \pm 2\sqrt{6}$$

$x = 2+2\sqrt{6} \Rightarrow y = 11+4\sqrt{6}$ $\therefore$ Turning points: $(6.89, 20.8)$, $(-2.89, 1.20)$.

$x = 2-2\sqrt{6} \Rightarrow y = 11-4\sqrt{6}$



23 The curves C_1 and C_2 have equations

$$y = \frac{ax}{x+5} \quad \text{and} \quad y = \frac{x^2 + (a+10)x + 5a + 26}{x+5}$$

respectively, where a is a constant and $a > 2$.

(i) Find the equations of the asymptotes of C_1. [2]

$$y = a - \frac{5a}{x+5}.$$

Asymptotes : $y = a, \ x = -5$.

$$\begin{array}{r|l} & a \\ \hline x+5 & ax \\ & ax+5a \\ & - \quad - \\ \hline & -5a \\ \hline \end{array}$$

(ii) Find the equation of the oblique asymptote of C_2. [2]

$$y = x+a+5 + \frac{1}{x+5}.$$

Asymptotes : $y = x + a + 5$.

$$\begin{array}{r|l} & x + a + 5 \\ \hline x+5 & x^2 + (a+10)x + 5a + 26 \\ & x^2 + 5x \\ & - \quad - \\ \hline & (a+5)x + 5a + 26 \\ & (a+5)x + 5a + 25 \\ & - \quad - \quad - \\ \hline & 1 \\ \hline \end{array}$$

(iii) Show that C_1 and C_2 do not intersect. [2]

$$y = \frac{ax}{x+5} \rightarrow (1), \quad y = \frac{x^2 + (a+10)x + 5a + 26}{x+5} \rightarrow (2)$$

Substitute (1) in (2) : $\dfrac{ax}{x+5} = \dfrac{x^2 + (a+10)x + 5a + 26}{x+5}$

$$ax = x^2 + ax + 10x + 5a + 26$$

$$0 = x^2 + 10x + 5a + 26$$

Discriminant $= (10)^2 - 4(1)(5a+26) = 100 - 20a - 104 = -20a - 4$

Since $a > 2$

$$-20a < -40$$

$$-20a - 4 < -44$$

i.e. Discriminant < 0 $\therefore$ C_1 and C_2 do not intersect.

(iv) Find the coordinates of the stationary points of C_2. [3]

$$y = x + a + 5 + \frac{1}{x+5}.$$

$$\frac{dy}{dx} = 1 - \frac{1}{(x+5)^2} = 0$$

$$1 = \frac{1}{(x+5)^2}$$

$$(x+5)^2 = 1$$

$$\Rightarrow x+5 = \pm 1$$

$$x+5 = 1 \quad \text{or} \quad x+5 = -1$$

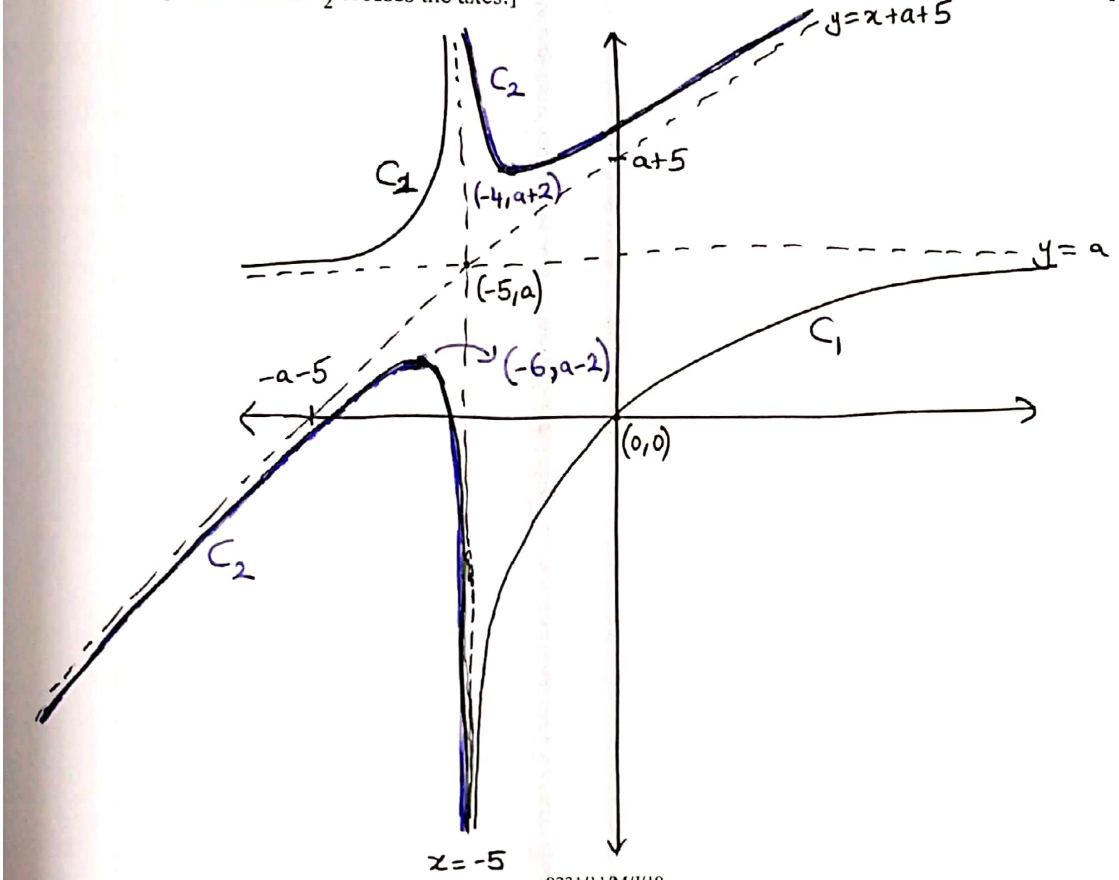
$$x = -4 \quad \text{or} \quad x = -6$$

When $x = -4$, $y = -4 + a + 5 + \frac{1}{-4+5} = a+1+1 = a+2.$

When $x = -6$, $y = -6 + a + 5 + \frac{1}{-6+5} = a-1-1 = a-2.$

$\therefore$ Stationary points: $(-4, a+2)$, $(-6, a-2)$.

(v) Sketch C_1 and C_2 on a single diagram. [You do not need to calculate the coordinates of any points where C_2 crosses the axes.] [3]



24 The curve C has equation

$$y = \frac{x^2}{kx-1},$$

where k is a positive constant.

(i) Obtain the equations of the asymptotes of C. [3]

$$y = \frac{x}{k} + \frac{1}{k^2} + \frac{\frac{1}{k^2}}{kx-1}$$

Oblique asymptote : $y = \frac{x}{k} + \frac{1}{k}$.

Vertical asymptote : $x = \frac{1}{k}$.

$$\frac{x}{k} + \frac{1}{k^2}$$

$$kx-1 \overline{)\, x^2}$$

$$x^2 - \frac{x}{k}$$

$$- \quad +$$

$$\frac{x}{k} + 0$$

$$\frac{x}{k} - \frac{1}{k^2}$$

$$- \quad +$$

$$\frac{1}{k^2}$$

(ii) Find the coordinates of the stationary points of C. [3]

$$\frac{dy}{dx} = \frac{1}{k} - \frac{\frac{1}{k^2}}{(kx-1)^2} \cdot k = \frac{1}{k} - \frac{1}{k(kx-1)^2}$$

For stationary point, $\frac{dy}{dx} = 0 \Rightarrow \frac{1}{k} - \frac{1}{k(kx-1)^2} = 0$

$$\frac{1}{k} = \frac{1}{k(kx-1)^2}$$

$$(kx-1)^2 = 1$$

$$kx-1 = \pm 1$$

$kx-1 = 1$ and $kx-1 = -1$

$kx = 2$ $\quad$ $kx = 0$

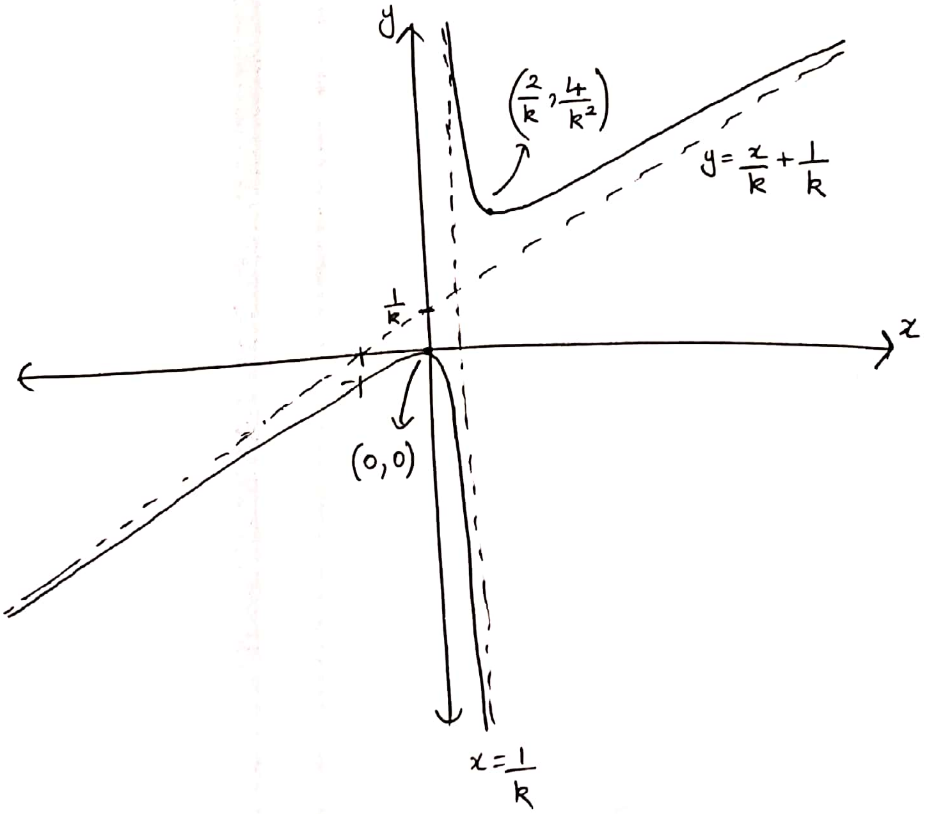
$x = \frac{2}{k}$. $\quad$ $x = 0$

When $x = 0$, $y = \frac{0^2}{k(0)-1} = 0$

When $x = \frac{2}{k}$, $y = \frac{\left(\frac{2}{k}\right)^2}{k \times \frac{2}{k} - 1} = \frac{4}{k^2}$

$\therefore$ Stationary points: $(0,0)$, $\left(\frac{2}{k}, \frac{4}{k^2}\right)$.

(iii) Sketch C. [3]



25 The curve C has equation $y = \frac{5(x-1)(x+2)}{(x-2)(x+3)}$.

(i) Express y in the form $P + \frac{Q}{x-2} + \frac{R}{x+3}$. [3]

(ii) Show that $\frac{dy}{dx} = 0$ for exactly one value of x and find the corresponding value of y. [4]

(iii) Write down the equations of all the asymptotes of C. [3]

(iv) Find the set of values of k for which the line $y = k$ does not intersect C. [4]

i) $y = \frac{5(x^2+x-2)}{x^2+x-6} = \frac{5x^2+5x-10}{x^2+x-6}$

$$\begin{array}{r|l} & 5 \\ \hline x^2+x-6 & 5x^2+5x-10 \\ & 5x^2+5x-30 \\ & - \quad - \quad + \\ \hline & 20 \\ \hline \end{array}$$

$y = 5 + \frac{20}{(x-2)(x+3)}$

$\frac{20}{(x-2)(x+3)} \equiv \frac{Q}{x-2} + \frac{R}{x+3}$

$20 = Q(x+3) + R(x-2)$

$x=2 \Rightarrow 20 = Q(5) \Rightarrow Q = 4.$

$x=-3 \Rightarrow 20 = R(-5) \Rightarrow R = -4$ $\quad \therefore y = 5 + \frac{4}{x-2} - \frac{4}{x+3}.$

ii) $\frac{dy}{dx} = \frac{-4}{(x-2)^2} + \frac{4}{(x+3)^2} = 0 \Rightarrow (x+3)^2 = (x-2)^2$

$x^2+6x+9 = x^2-4x+4$

$10x = -5 \Rightarrow x = \frac{-1}{2}$

$\therefore y = \frac{5(\frac{-1}{2}-1)(\frac{-1}{2}+2)}{(\frac{-1}{2}-2)(\frac{-1}{2}+3)} = \frac{9}{5}.$

$\therefore (\frac{-1}{2}, \frac{9}{5}).$

iii) $y = \frac{5(x-1)(x+2)}{(x-2)(x+3)} = 5 + \frac{4}{x-2} ~~+~~ - \frac{4}{x+3}.$

As $|x| \mapsto \infty, y \mapsto 5$ $\therefore$ Horizontal asymptote $\Rightarrow y = 5.$

For vertical asymptotes, set $x-2=0 \Rightarrow x=2$

and $x+3=0 \Rightarrow x=-3.$

iv) $y = k \Rightarrow k = \frac{5x^2+5x-10}{x^2+x-6}$

$kx^2+kx-6k = 5x^2+5x-10$

$0 = (5-k)x^2 + (5-k)x + 6k - 10$

For no intersection, $b^2 - 4ac < 0$

$$\Rightarrow \quad (5-k)^2 - 4(5-k)(6k-10) < 0$$

$$(5-k)\left[5-k-24k+40\right] < 0$$

$$(5-k)(-25k+45) < 0$$

$$(k-5)(25k-45) < 0$$

$$\Rightarrow \quad \frac{45}{25} < k < 5$$

$$\frac{9}{5} < k < 5.$$

26 The curve C has equation

$$y = \frac{x^2 + 2x - 3}{(\lambda x + 1)(x + 4)},$$

where λ is a constant.

(i) Find the equations of the asymptotes of C for the case where $\lambda = 0$. [4]

(ii) Find the equations of the asymptotes of C for the case where λ is not equal to any of $-1, 0, \frac{1}{4}, \frac{1}{3}$. [3]

(iii) Sketch C for the case where $\lambda = -1$. Show, on your diagram, the equations of the asymptotes and the coordinates of the points of intersection of C with the coordinate axes. [4]

i) $\lambda = 0 \Rightarrow y = \dfrac{x^2+2x-3}{x+4} = x - 2 + \dfrac{5}{x+4}.$

$$\begin{array}{r|l} & x-2 \\ \hline x+4 & x^2+2x-3 \\ & x^2+4x \\ & - \quad - \\ \hline & -2x-3 \\ & -2x-8 \\ & + \quad + \\ \hline & 5 \\ \hline \end{array}$$

Asymptotes : $y = x-2$, $x = -4$.

ii) $y = \dfrac{x^2+2x-3}{\lambda x^2 + x + 4x + 4}$.

As $|x| \mapsto \infty$, $y \mapsto \dfrac{1}{\lambda}$ $\therefore y = \dfrac{1}{\lambda}$ is a vertical asymptote.

For vertical asymptotes : $\lambda x + 1 = 0$ and $x + 4 = 0$

$x = \dfrac{-1}{\lambda}$ $\qquad x = -4$

iii) $\lambda = -1 \Rightarrow y = \dfrac{x^2+2x-3}{(-x+1)(x+4)} = \dfrac{x^2+3x-x-3}{(-x+1)(x+4)} = \dfrac{(x-1)(x+3)}{-(x-1)(x+4)} = -\dfrac{(x+3)}{(x+4)}.$

$\therefore y = \dfrac{-x-3}{x+4} = -1 + \dfrac{1}{x+4}$

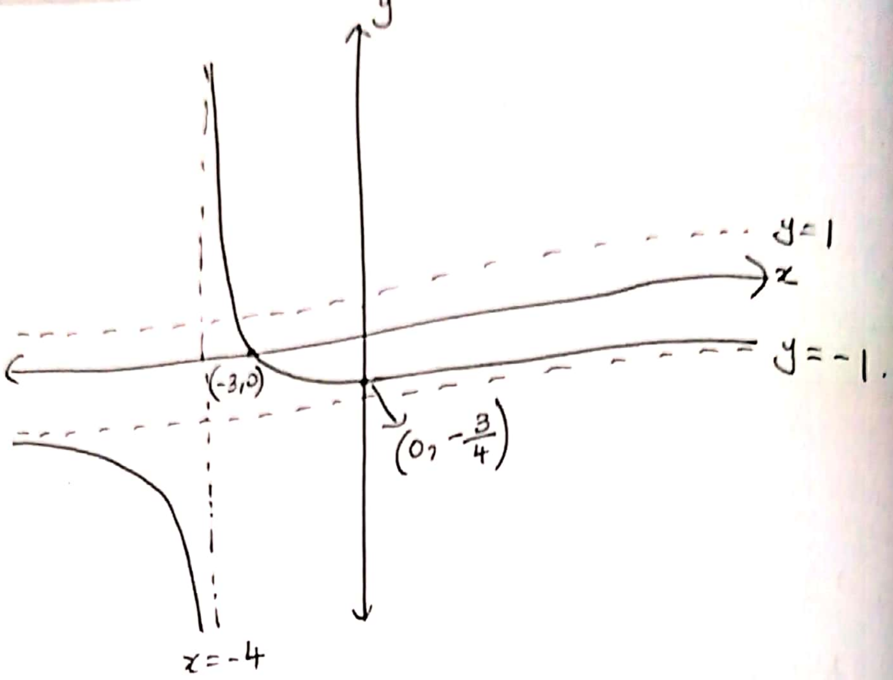
$$\begin{array}{r|l} & -1 \\ \hline x+4 & -x-3 \\ & -x-4 \\ & + \quad + \\ \hline & 1 \\ \hline \end{array}$$

$\therefore$ Asymptotes : $y = -1$, $x = -4$.

y-intercept $\Rightarrow x = 0 \Rightarrow y = \dfrac{-0-3}{0+4} = \dfrac{-3}{4}$ $\therefore \left(0, \dfrac{-3}{4}\right)$.

x-intercept $\Rightarrow y = 0 \Rightarrow 0 = \dfrac{-x-3}{x+4} \Rightarrow x = -3 \therefore (-3, 0)$.

$\dfrac{dy}{dx} = \dfrac{-1}{(x+4)^2} \overset{<}{\cancel{\geq}} 0$ for all x $\therefore$ No turning points exist.


y
y=1
x
y=-1.
(-3,0)
(0, -3/4)
x=-4

27 The curve C has equation

$$y = \frac{x^2}{x + \lambda},$$

where λ is a non-zero constant. Obtain the equations of the asymptotes of C. [3]

In separate diagrams, sketch C for the cases where

(i) $\lambda > 0$,

(ii) $\lambda < 0$. [4]

$$\Rightarrow y = x - \lambda + \frac{\lambda^2}{x+\lambda}$$

$$x+\lambda \overline{\smash{)}\,x^2+0x+0}$$

quotient: $x - \lambda$

$x^2 + \lambda x$

$- \quad -$

$-\lambda x + 0$

$-\lambda x - \lambda^2$

$+ \quad +$

λ^2

$\therefore$ Asymptotes : $y = x - \lambda$, $x = -\lambda$.

i) $\lambda > 0$

$$\frac{dy}{dx} = 1 - \frac{\lambda^2}{(x+\lambda)^2} = 0$$

$$(x+\lambda)^2 = \lambda^2$$

$$x + \lambda = \pm \lambda$$

$$\Rightarrow x+\lambda = \lambda \text{ and } x + \lambda = -\lambda$$

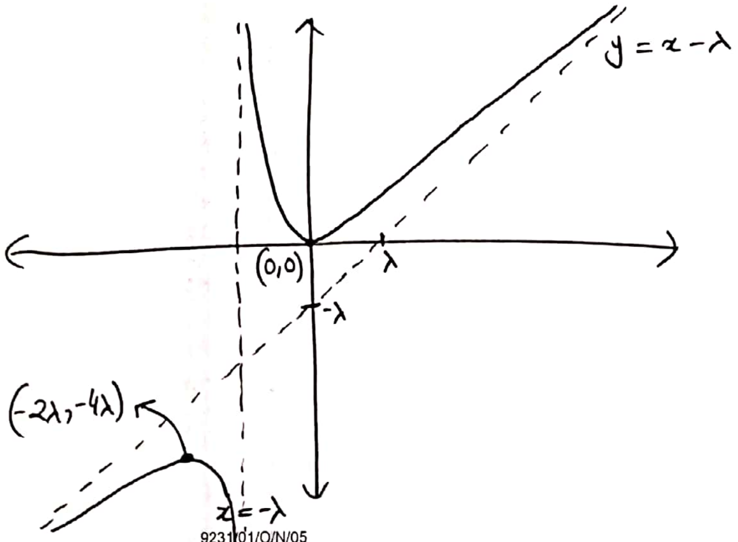
$$x = 0 \text{ and } x = -2\lambda.$$

$$\therefore y = \frac{0^2}{0+\lambda} = 0 \quad \text{and} \quad y = \frac{(-2\lambda)^2}{-2\lambda+\lambda} = \frac{4\lambda^2}{-\lambda} = -4\lambda.$$

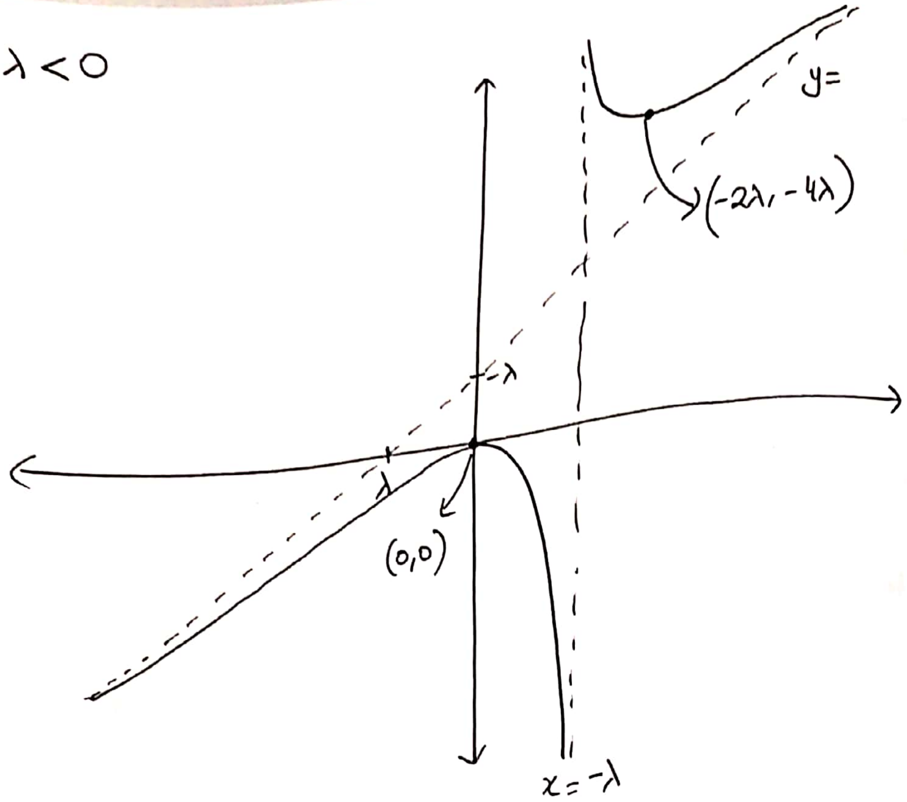
$\therefore$ Turning points $(0,0)$ and $(-2\lambda, -4\lambda)$.

Intercepts : $x = 0 \Rightarrow y = \frac{0^2}{0+\lambda} = 0$

$y = 0 \Rightarrow \frac{x^2}{x+\lambda} = 0 \Rightarrow x = 0$ $\Big\}$ $(0,0)$.



ii) $\lambda < 0$



28 The curve C has equation

$$y = \frac{x^2 + qx + 1}{2x + 3},$$

where q is a positive constant.

(i) Obtain the equations of the asymptotes of C. [3]

(ii) Find the value of q for which the x-axis is a tangent to C, and sketch C in this case. [4]

(iii) Sketch C for the case $q = 3$, giving the exact coordinates of the points of intersection of C with the x-axis. [3]

(iv) It is given that, for all values of the constant λ, the line

$$y = \lambda x + \tfrac{3}{2}\lambda + \tfrac{1}{2}(q - 3)$$

passes through the point of intersection of the asymptotes of C. Use this result, with the diagrams you have drawn, to show that if $\lambda < \frac{1}{2}$ then the equation

$$\frac{x^2 + qx + 1}{2x + 3} = \lambda x + \tfrac{3}{2}\lambda + \tfrac{1}{2}(q - 3)$$

has no real solution if q has the value found in part (ii), but has 2 real distinct solutions if $q = 3$. [4]

i) $y = \frac{x}{2} + \frac{q}{2} - \frac{3}{4} + \frac{\frac{5}{4} - \frac{3q}{2}}{2x+3}$

$\therefore$ Asymptotes: $y = \frac{x}{2} + \frac{q}{2} - \frac{3}{4}$, $x = \frac{-3}{2}$.

$$\begin{array}{r|l} & \frac{x}{2} + \frac{q}{2} - \frac{3}{4} \\ \hline 2x+3 & x^2 + qx + 1 \\ & x^2 + \frac{3x}{2} \\ & - \quad - \\ \hline & \left(q - \frac{3}{2}\right)x + 1 \\ & \left(q - \frac{3}{2}\right)x + \frac{3q}{2} - \frac{1}{4} \\ & - \quad - \quad + \\ \hline & \frac{5}{4} - \frac{3q}{2} \end{array}$$

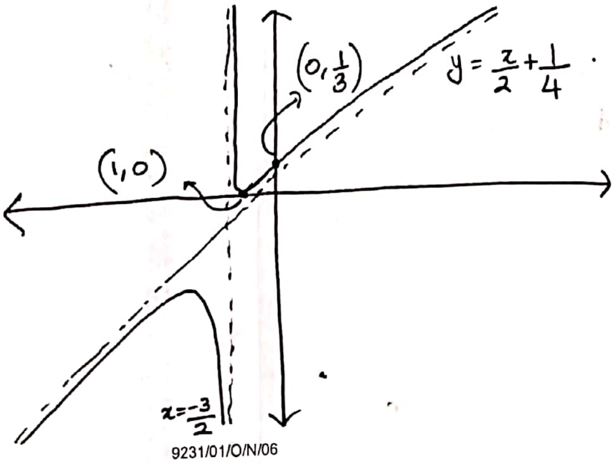
ii) Set $y = 0 \Rightarrow \frac{x^2+qx+1}{2x+3} = 0 \Rightarrow x^2+qx+1=0$

For x-axis to be a tangent, $b^2 - 4ac = 0$

$q^2 - 4(1)(1) = 0$

$q = 2$ since $q > 0$.

$\therefore y = \frac{x^2+2x+1}{2x+3} = \frac{x}{2} + \frac{1}{4} - \frac{\frac{7}{4}}{2x+3}$ $\therefore$ asymptotes: $y = \frac{x}{2} + \frac{1}{4}$, $x = \frac{-3}{2}$.



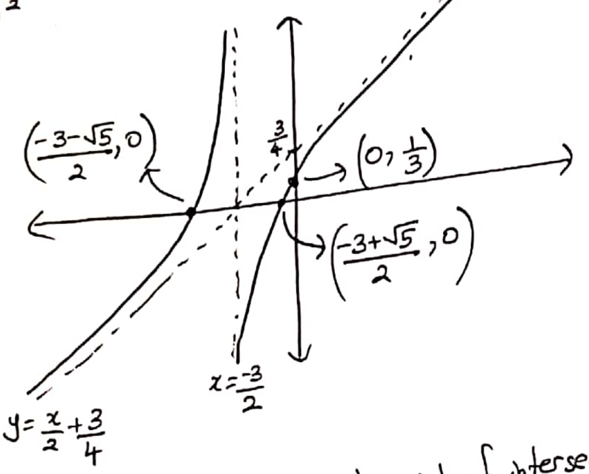
iii) $q = 3 \Rightarrow y = \frac{x^2+3x+1}{2x+3} = \frac{x}{2} + \frac{3}{4} - \frac{\frac{13}{4}}{2x+3}$ ∴ asym

x-intercept, $y = 0 \Rightarrow \frac{x^2+3x+1}{2x+3} = 0 \Rightarrow x = \frac{-3\pm\sqrt{5}}{2}$

$\therefore x = \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2} \Rightarrow \left(\frac{-3+\sqrt{5}}{2}, 0\right), \left(\frac{-3-\sqrt{5}}{2}, 0\right).$

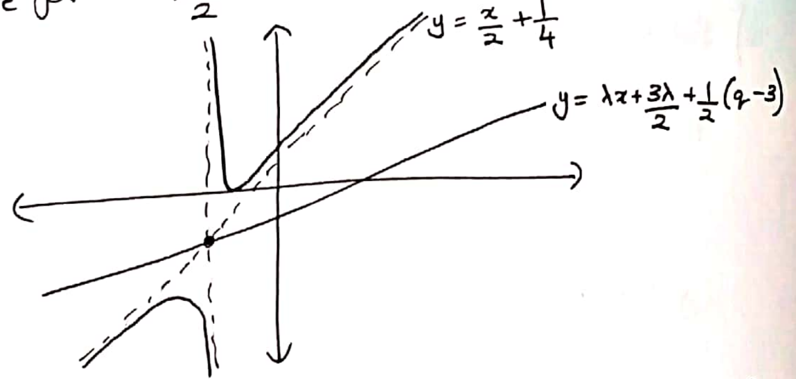
y-intercept, $x = 0 \Rightarrow y = \frac{0^2+3\times0+1}{2(0)+3} = \frac{1}{3} \Rightarrow \left(0, \frac{1}{3}\right).$

$y' = \frac{1}{2} + \frac{\frac{13}{4}}{(2x+3)^2} = 0$ ⇒ No real roots ⇒ No turning points.

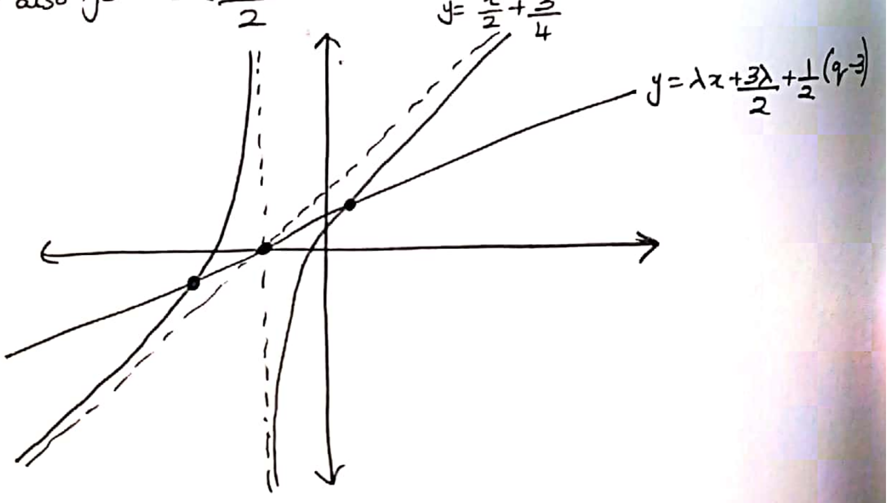


iv) $y = \lambda x + \frac{3}{2}\lambda + \frac{1}{2}(q-3)$ passes through point of intersection of asymptotes.

If $q = 2$, the line with gradient $= \lambda < \frac{1}{2}$ passes below the curve for $x > \frac{-3}{2}$ and above the curve for $x < \frac{-3}{2}$. So there is no point of intersection.



If $q = 3$, the line with gradient $\lambda < \frac{1}{2}$ cuts, the curve twice once for $x > \frac{-3}{2}$ and ~~&~~ also for $x < \frac{-3}{2}$.



29 The curve C has equation

$$y = \frac{ax^2 + bx + c}{x + 4},$$

where a, b and c are constants. It is given that $y = 2x - 5$ is an asymptote of C.

(i) Find the values of a and b. [3]

(ii) Given also that C has a turning point at $x = -1$, find the value of c. [3]

(iii) Find the set of values of y for which there are no points on C. [4]

(iv) Draw a sketch of the curve with equation

$$y = \frac{2(x-7)^2 + 3(x-7) - 2}{x - 3}.$$ [3]

[You should state the equations of the asymptotes and the coordinates of the turning points.]

i) $y = ax + b - 4a + \dfrac{c - 4b + 16a}{x + 4}$

$$\begin{array}{r} ax + b - 4a \\ x+4 \,\overline{\big)\, ax^2 + bx + c} \\ -\ \ ax^2 + 4ax \\ \hline (b-4a)x + c \\ -\ \ (b-4a)x + 4b - 16a \\ \hline c - 4b + 16a \\ \hline \end{array}$$

$\Rightarrow$ Oblique asymptote: $y = ax + b - 4a$

$y = ax + b - 4a = 2x - 5$

$\therefore a = 2 \qquad b - 4a = -5$

$b - 8 = -5$

$b = 3$

$\therefore a = 2, b = 3.$

ii) $y = \dfrac{2x^2 + 3x + c}{x + 4} = 2x - 5 + \dfrac{c + 20}{x + 4}.$

$\dfrac{dy}{dx} = 2 - \dfrac{c + 20}{(x + 4)^2} = 0$ when $x = -1 \Rightarrow 2 - \dfrac{c + 20}{(-1 + 4)^2} = 0$

$18 = c + 20$

$-2 = c$

$c = -2.$

iii) $y = \dfrac{2x^2 + 3x - 2}{x + 4} \Rightarrow yx + 4y = 2x^2 + 3x - 2$

$0 = 2x^2 + (3 - y)x - 2 - 4y$

For no points, $b^2 - 4ac < 0 \Rightarrow (3 - y)^2 - 4(2)(-2 - 4y) < 0$

$9 - 6y + y^2 + 16 + 32y < 0$

$y^2 + 26y + 25 < 0$

$(y + 1)(y + 25) < 0$

$\Rightarrow -25 < y < -1.$

For $y = \frac{2x^2+3x-2}{x+4} = 2x-5+\frac{18}{x+4}$

Asymptotes: $y = 2x-5,\ x=-4.$

For turning points, $\frac{dy}{dx} = 0 \Rightarrow 2 - \frac{18}{(x+4)^2} = 0 \Rightarrow 2 = \frac{18}{(x+4)^2}$

$$(x+4)^2 = 9$$
$$x+4 = \pm 3$$
$$x = -1, -7.$$

$\Rightarrow y = -1,\ y = -25.$

$\therefore$ Turning points: $(-1,-1), (-7,-25).$

x - intercept $\Rightarrow y=0 \Rightarrow \frac{2x^2+3x-2}{x+4} = 0 \Rightarrow x = \frac{1}{2}, -2 \therefore (\frac{1}{2},0), (-2,0).$

y-intercept $\Rightarrow x=0 \Rightarrow y = \frac{-2}{4} = \frac{-1}{2} \Rightarrow (0, \frac{-1}{2}).$

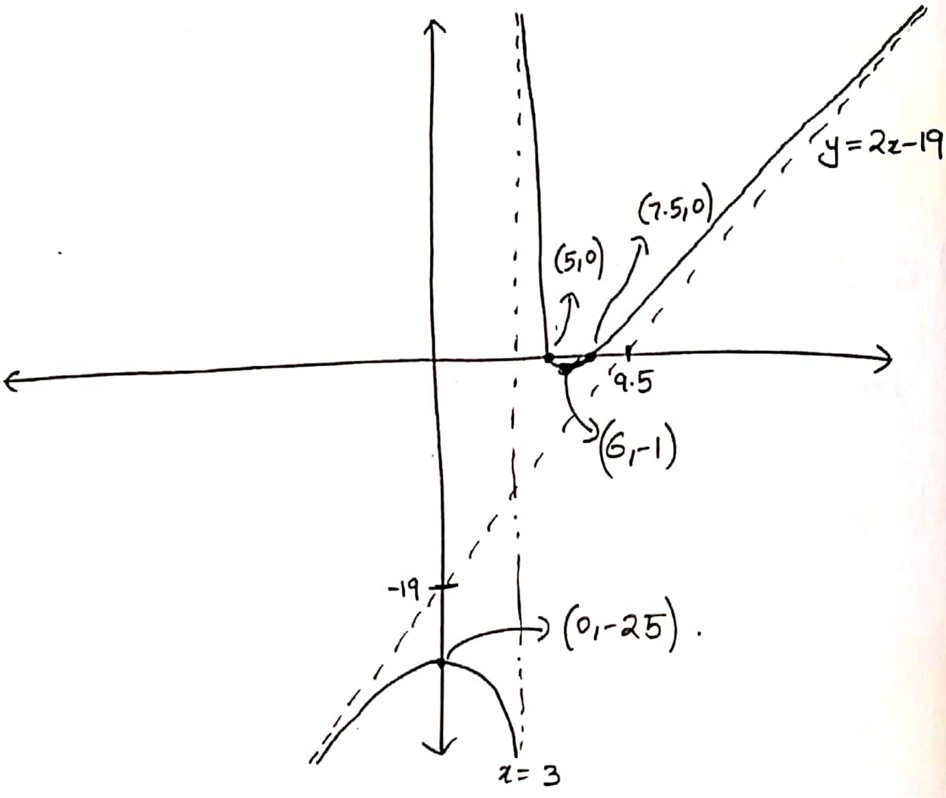
We want to plot $y = \frac{2(x-7)^2+3(x-7)-2}{x-3}$

Note that this involves translating the curve $y = \frac{2x^2+3x-2}{x+4}$ seven units to the right.

Then new asymptotes: $y = 2(x-7)-5 = 2x-19$
$x = -4+7 = 3$

New turning points: $(6,-1)$ and $(0,-25)$.

New x-intercepts: $(7.5,0)$ and $(5,0)$.



10 The curve C has equation

$$y = \frac{(x-2)(x-a)}{(x-1)(x-3)},$$

where a is a constant not equal to 1, 2 or 3.

(i) Write down the equations of the asymptotes of C. [2]

(ii) Show that C meets the asymptote parallel to the x-axis at the point where $x = \frac{2a-3}{a-2}$. [2]

(iii) Show that the x-coordinates of any stationary points on C satisfy

$$(a-2)x^2 + (6-4a)x + (5a-6) = 0,$$

and hence find the set of values of a for which C has stationary points. [6]

(iv) Sketch the graph of C for

(a) $a > 3$,

(b) $2 < a < 3$.

[4]

i) Asymptotes: $y = 1, x = 1, x = 3$.

ii) When $y = 1 \Rightarrow 1 = \frac{(x-2)(x-a)}{(x-1)(x-3)} \Rightarrow x^2 - 4x + 3 = x^2 - 2x - ax + 2a$

$-2x + ax = 2a - 3$

$x = \frac{2a-3}{a-2}$ (Shown).

ii) $y = \frac{x^2 - (2+a)x + 2a}{x^2 - 4x + 3}$.

$$\frac{dy}{dx} = \frac{(x^2-4x+3)[2x-(2+a)] - [x^2-(2+a)x+2a][2x-4]}{(x^2-4x+3)^2}$$

At turning point, $\frac{dy}{dx} = 0 \Rightarrow (x^2-4x+3)(2x-2-a) - (x^2-2x-ax+2a)(2x-4) = 0$

$2x^3 - 8x^2 + 6x - 2x^2 + 8x - 6 - ax^2 + 4ax - 3a$
$- [2x^3 - 4x^2 - 2ax^2 + 4ax - 4x^2 + 8x + 4ax - 8a] = 0$

$2x^3 - 10x^2 - ax^2 + 14x + 4ax - 6 - 3a$
$- [2x^3 - 8x^2 - 2ax^2 + 4ax + 4ax + 8x - 8a] = 0$

$\Rightarrow -2x^2 + ax^2 + 6x - 4ax + 5a - 6 = 0$

$(a-2)x^2 + (6-4a)x + (5a-6) = 0$

(Shown).

For stationary point to exist, $b^2-4ac>0$

$$\Rightarrow (6-4a)^2-4(a-2)(5a-6)>0$$

$$36+16a^2-48a-4[5a^2-10a-6a+12]>0$$

$$-4a^2+16a-12>0$$

~~a^2+4a+~~

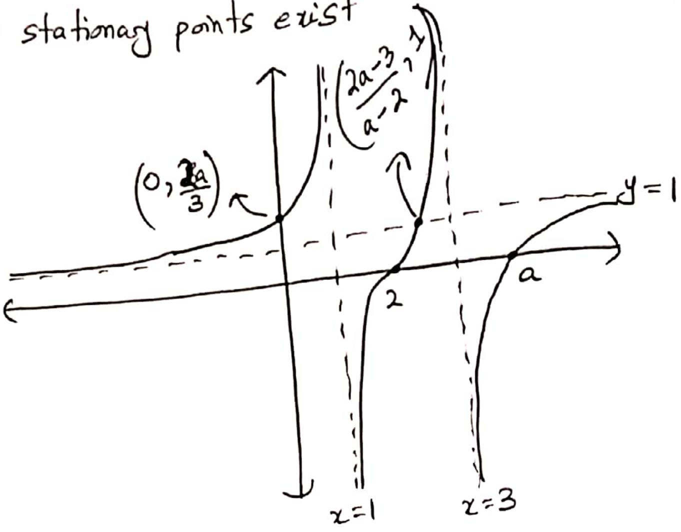
$$a^2-4a+3<0$$

$$(a-3)(a-1)<0$$

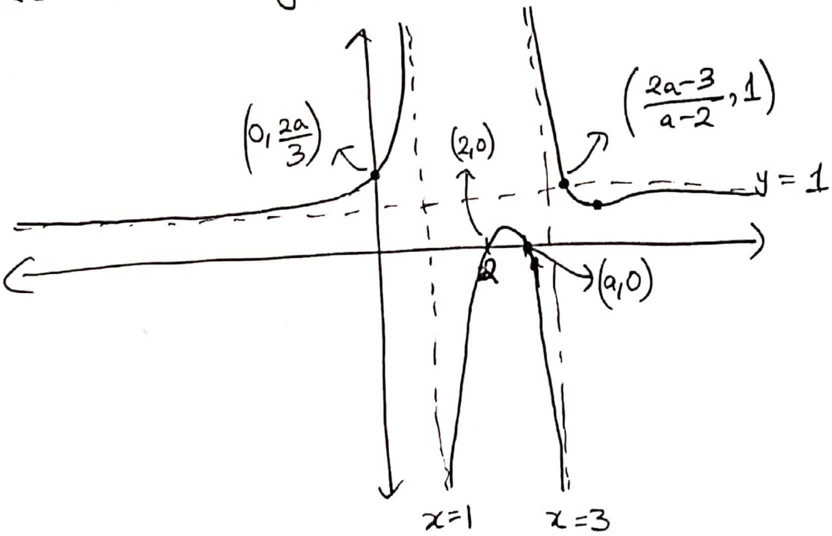
$$\Rightarrow 1<a<3$$

i.e. Stationary points exist for $1<a<3$.

iv) a) $a>3$ ∴ No stationary points exist



b) $2<a<3$ ∴ Two stationary points exist.



Note: x-intercepts $\Rightarrow y=0 \Rightarrow \dfrac{(x-2)(x-a)}{(x-1)(x-3)}=0 \Rightarrow x=2,a.$

∴ $(2,0)$ and $(a,0)$.

y-intercept $\Rightarrow x=0 \Rightarrow y=\dfrac{(0-2)(0-a)}{(0-1)(0-3)}=\dfrac{2a}{3}$ ∴ $\left(0,\frac{2a}{3}\right)$.

31 The curve C has equation

$$y = \frac{x^2 - 5x + 4}{x + 1}.$$

(i) Obtain the coordinates of the points of intersection of C with the axes. [2]

(ii) Obtain the equation of each of the asymptotes of C. [3]

(iii) Draw a sketch of C. [3]

i) At y-axis, $x=0 \Rightarrow y = \frac{0^2-5(0)+4}{0+1} = 4 \therefore (0,4)$.

A x-axis, $y=0 \Rightarrow \frac{x^2-5x+4}{x+1} = 0 \Rightarrow x^2-4x-x+4=0 \Rightarrow (x-1)(x-4)=0$

$\therefore x = 1, 4$

$\therefore (1,0)$ and $(4,0)$.

ii) $y = x-6 + \frac{10}{x+1}$.

$$\begin{array}{r|l} & x-6 \\ x+1 & x^2-5x+4 \\ & x^2+x \\ & - \quad - \\ \hline & -6x+4 \\ & -6x-6 \\ & + \quad + \\ \hline & 10 \\ \hline \end{array}$$

$\therefore$ Asymptotes: $y = x-6$

$x = -1$.

iii) $\frac{dy}{dx} = 1 - \frac{10}{(x+1)^2}$

At turning point, $\frac{dy}{dx} = 0 \Rightarrow 1 - \frac{10}{(x+1)^2} = 0$

$(x+1)^2 = 10 \Rightarrow x+1 = \pm\sqrt{10}$

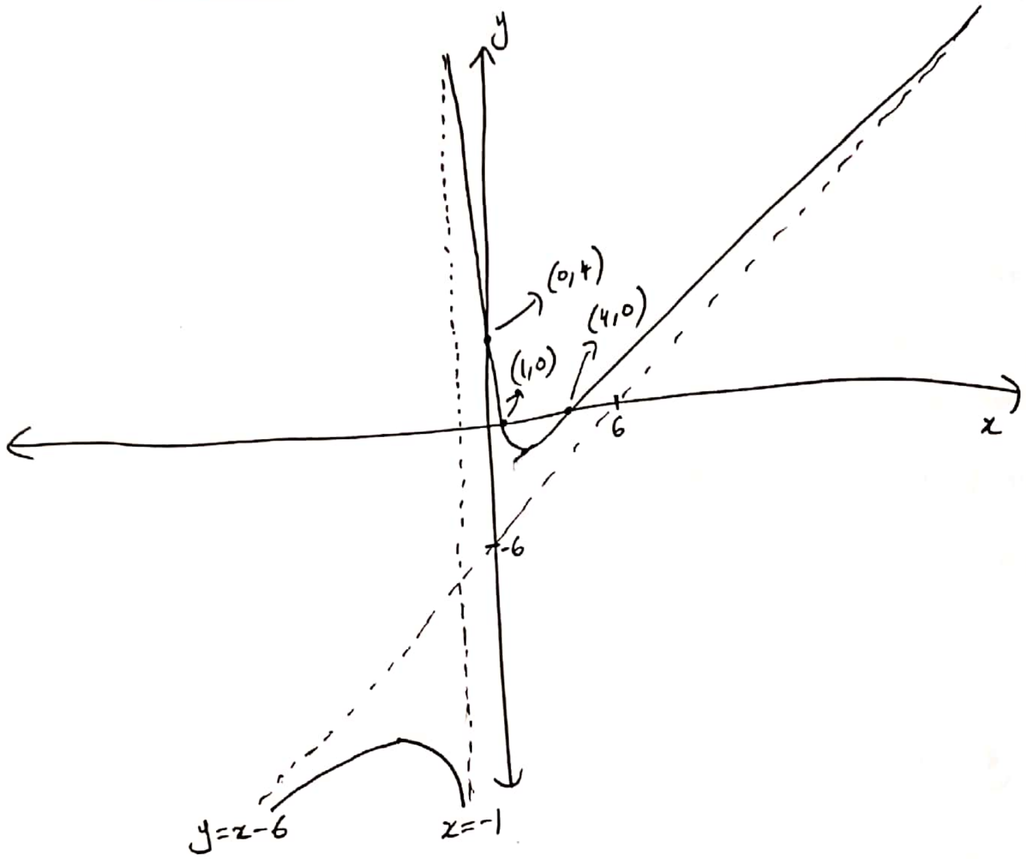
$x = \pm\sqrt{10} - 1$

$x = \sqrt{10}-1 \Rightarrow y = -7+2\sqrt{10}$

$x = -\sqrt{10}-1 \Rightarrow y = -7-2\sqrt{10}$

$\therefore$ Turning points: $(\sqrt{10}-1, -7+2\sqrt{10}), (\sqrt{10}-1, -7-2\sqrt{10})$.

$(2.16, -0.67), (-4.16, -13.32)$.


y
(0,4)
(1,0)
(4,0)
6
x
-6
y=x-6
x=-1

The curve C has equation

$$y = \frac{x^2 + 2\lambda x}{x^2 - 2x + \lambda},$$

where λ is a constant and $\lambda \neq -1$.

(i) Show that C has at most two stationary points. [3]

(ii) Show that if C has **exactly** two stationary points then $\lambda > -\frac{5}{4}$. [2]

(iii) Find the set of values of λ such that C has two vertical asymptotes. [2]

(iv) Find the x-coordinates of the points of intersection of C with

(a) the x-axis,

(b) the horizontal asymptote.

[3]

(v) Sketch C in each of the cases

(a) $\lambda < -2$,

(b) $\lambda > 2$.

[4]

i) $$\frac{dy}{dx} = \frac{(x^2-2x+\lambda)(2x+2\lambda) - (x^2+2\lambda x)(2x-2)}{(x^2-2x+\lambda)^2}$$

$$= \frac{2x^3 - 4x^2 + 2\lambda x + 2\lambda x^2 - 4\lambda x + 2\lambda^2 - (2x^3 + 4\lambda x^2 - 2x^2 - 4\lambda x)}{(x^2-2x+\lambda)^2}$$

$$= \frac{2x^3 - 4x^2 + 2\lambda x^2 - 2\lambda x + 2\lambda^2 - 2x^3 - 4\lambda x^2 + 2x^2 + 4\lambda x}{(x^2-2x+\lambda)^2}$$

$$\frac{dy}{dx} = \frac{-2\lambda x^2 - 2x^2 + 2\lambda x + 2\lambda^2}{(x^2-2x+\lambda)^2}$$

For stationary points, set $\frac{dy}{dx} = 0 \Rightarrow \frac{-2\lambda x^2 - 2x^2 + 2\lambda x + 2\lambda^2}{(x^2-2x+\lambda)^2} = 0$

$\Rightarrow -2\lambda x^2 - 2x^2 + 2\lambda x + 2\lambda^2 = 0$

$\lambda x^2 + x^2 - \lambda x - \lambda^2 = 0$

$(\lambda+1)x^2 - \lambda x - \lambda^2 = 0.$

The quadratic equation has at most two solutions ∴ C has at most two stationary points.

ii) For exactly two stationary points, we require

$b^2 - 4ac > 0 \Rightarrow (-\lambda)^2 - 4(\lambda+1)(-\lambda^2) > 0$

$\lambda^2 + 4\lambda^2(\lambda+1) > 0$

$\Rightarrow \lambda^2(1 + 4\lambda + 4) > 0$

$4\lambda + 5 > \frac{0}{\lambda^2}$

$\lambda > \frac{-5}{4}$. (Shown)

iii) For vertical asymptotes: $x^2-2x+\lambda=0$

For two vertical asymptotes, $b^2-4ac>0$

$$(-2)^2-4(1)(\lambda)>0$$
$$4-4\lambda>0$$
$$4>4\lambda$$
$$1>\lambda$$

$\therefore \lambda<1 \Rightarrow$ Two vertical asymptotes.

iv) a) At x-axis, $y=0 \Rightarrow \frac{x^2+2\lambda x}{x^2-2x+\lambda}=0 \Rightarrow x(x+2\lambda)=0 \therefore x=0,-2\lambda$.

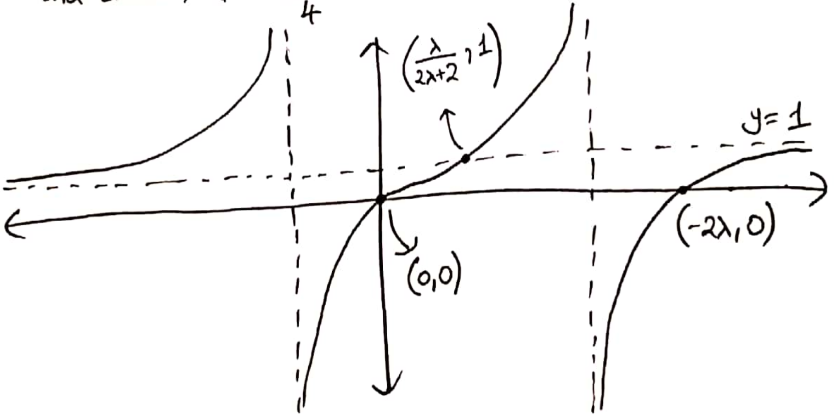
b) $y=\frac{x^2+2\lambda x}{x^2-2x+\lambda}$: As $|x|\mapsto\infty, y\mapsto 1 \therefore$ horizontal asymptote : $y=1$

At $y=1 \Rightarrow 1=\frac{x^2+2\lambda x}{x^2-2x+\lambda} \therefore$ $x^2-2x+\lambda=x^2+2\lambda x$

$$\lambda=(2\lambda+2)x$$

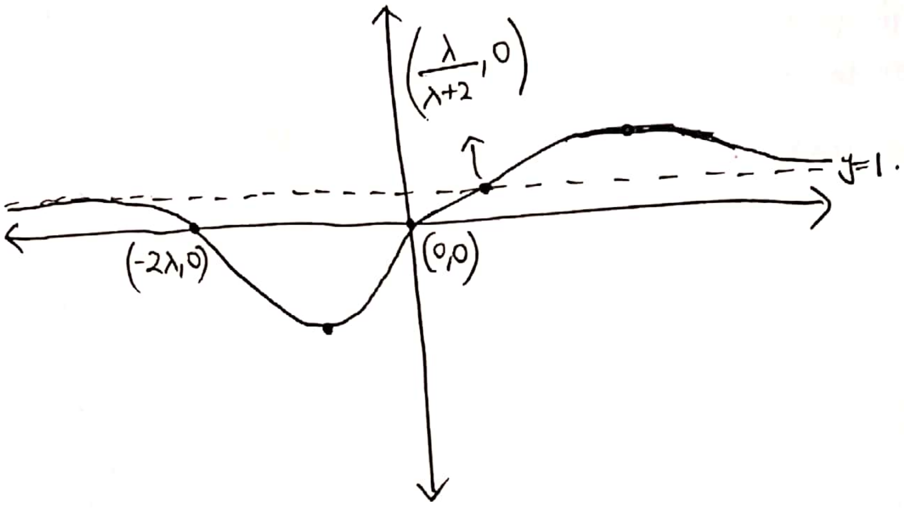
$$\therefore x=\frac{\lambda}{2\lambda+2}.$$

v) a) $\lambda<-2$. Since $\lambda<1 \therefore$ there are two vertical asymptotes and since $\lambda \cancel{>} <\frac{-5}{4} \therefore$ there are no stationary points.



b) $\lambda>2$ Since $\lambda>1 \therefore$ there are no vertical asymptotes.

Since $\lambda>\frac{-5}{4} \therefore$ there are two stationary points.



33 The curve C has equation $y = \frac{x^2 + px + 1}{x - 2}$, where p is a constant. Given that C has two asymptotes, find the equation of each asymptote. [3]

Find the set of values of p for which C has two distinct turning points. [5]

Sketch C in the case $p = -1$. Your sketch should indicate the coordinates of any intersections with the axes, but need not show the coordinates of any turning points. [3]

$$y = x + p + 2 + \frac{2p+5}{x-2}.$$

Asymptotes: $y = x + p + 2.$

$x = 2.$

$$\begin{array}{r} x + p+2 \\ x-2 \overline{\big) x^2+px+1} \\ x^2-2x \\ - \quad + \\ \hline (p+2)x+1. \\ (p+2)x-2p-4 \\ - \quad + \quad + \\ \hline 2p+5 \\ \hline \end{array}$$

$$\frac{dy}{dx} = 1 - \frac{2p+5}{(x-2)^2}.$$

At turning point, $\frac{dy}{dx} = 0$

$$\Rightarrow 1 - \frac{2p+5}{(x-2)^2} = 0$$

$$1 = \frac{2p+5}{(x-2)^2}$$

$$x^2 - 4x + 4 = 2p + 5$$

$$x^2 - 4x - 2p - 1 = 0.$$

For two distinct roots, $b^2 - 4ac > 0$

$$\Rightarrow (-4)^2 - 4(1)(-2p-1) > 0$$

$$16 + 8p + 4 > 0$$

$$8p > -20$$

$$p > \frac{-5}{2}.$$

For two distinct turning points, $p > \frac{-5}{2}.$

$$p = -1 \Rightarrow y = \frac{x^2 - x + 1}{x - 2} = x + 1 + \frac{3}{x-2}.$$

Asymptotes: $y = x + 1$, $x = 2.$

For y-intercept, $x=0 \Rightarrow y = \frac{0^2-0+1}{0-2} = \frac{-1}{2}$ $\therefore (0, \frac{\;}{2})$

For x-intercept, $y=0 \Rightarrow 0 = \frac{x^2-x+1}{x-2} \Rightarrow x^2-x+1=0$

$x = \frac{1 \pm \sqrt{-3}}{2}$ (No real roots).

$\Rightarrow$ No x-intercepts.

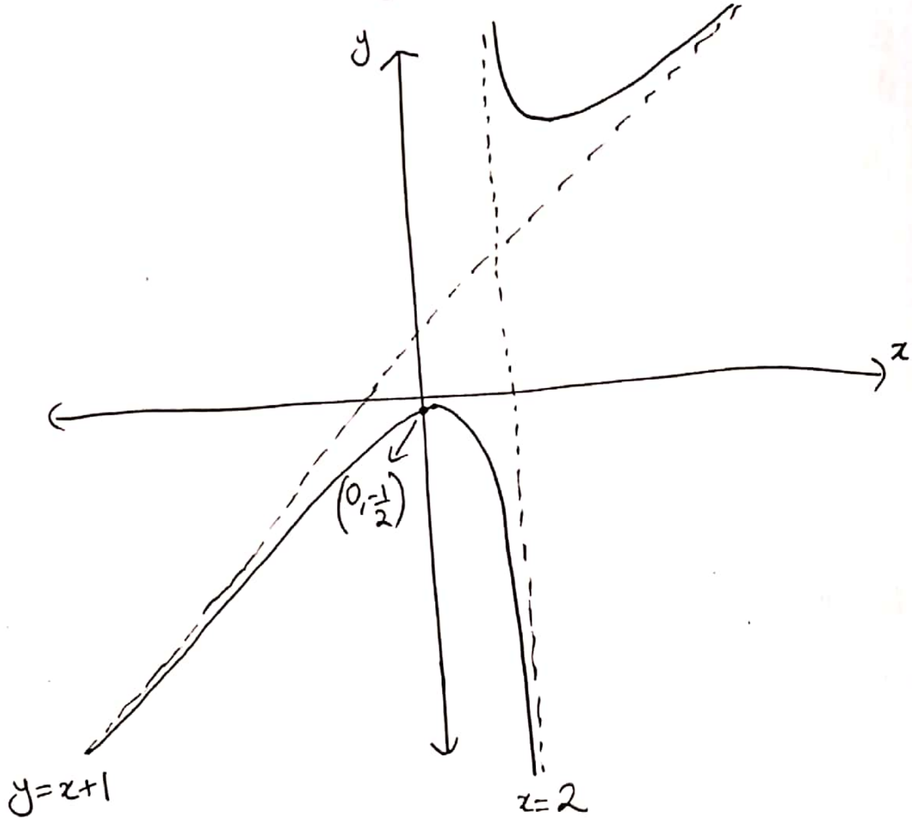
For turning point, set $\frac{dy}{dx}=0 \Rightarrow 1 - \frac{3}{(x-2)^2} = 0$

$\Rightarrow x^2-4x+1=0$

$x = 2+\sqrt{3}, x = 2-\sqrt{3}$

$y = 3+2\sqrt{3}, y = 3-2\sqrt{3}$

$\therefore$ Turning points: $(3.73, 6.46), (0.267, -0.464)$.



34 A curve C has equation

$$y = \frac{5(x^2 - x - 2)}{x^2 + 5x + 10}.$$

Find the coordinates of the points of intersection of C with the axes. [2]

Show that, for all real values of x, $-1 \leqslant y \leqslant 15$. [4]

Sketch C, stating the coordinates of any turning points and the equation of the horizontal asymptote. [7]

For y-intercept, set $x=0 \Rightarrow y = \frac{5(0^2-0-2)}{0^2+5(0)+10} = \frac{-10}{10} = -1 \therefore (0,-1).$

For x-intercept, set $y=0 \Rightarrow \frac{5(x^2-x-2)}{x^2+5x+10} = 0 \Rightarrow x^2-x-2=0$

$(x-2)(x+1)=0$

$x=-1, 2$

$\therefore (-1,0)$ and $(2,0)$.

$$y = \frac{5x^2-5x-10}{x^2+5x+10}$$

$$yx^2+5yx+10y = 5x^2-5x-10$$

$$0 = (5-y)x^2+(-5-5y)x-10-10y$$

For real values of x, $b^2-4ac \geqslant 0$

$$(-5-5y)^2-4(5-y)(-10-10y) \geqslant 0$$

$$25+25y^2+50y - 4[-50+10y-50y+10y^2] \geqslant 0$$

$$25+25y^2+50y+200+160y-40y^2 \geqslant 0$$

$$-15y^2+210y+225 \geqslant 0$$

$$y^2-14y-15 \leq 0$$

$$(y-15)(y+1) \leq 0$$

$\Rightarrow \quad -1 \leq y \leq 15.$ (shown).

$y = \frac{5x^2-5x-10}{x^2+5x+10} \Rightarrow$ As $|x| \mapsto \infty, y \mapsto 5$

$\therefore y=5$ is a horizontal asymptote.

When $y=-1 \Rightarrow -1 = \frac{5x^2-5x-10}{x^2+5x+10} \Rightarrow -x^2-5x-10 = 5x^2-5x-10$

$0 = 6x^2$

$\Rightarrow x=0$

$\therefore (0,-1).$

When $y = 15$, $\quad 15 = \frac{5x^2 - 5x - 10}{x^2 + 5x + 10}$

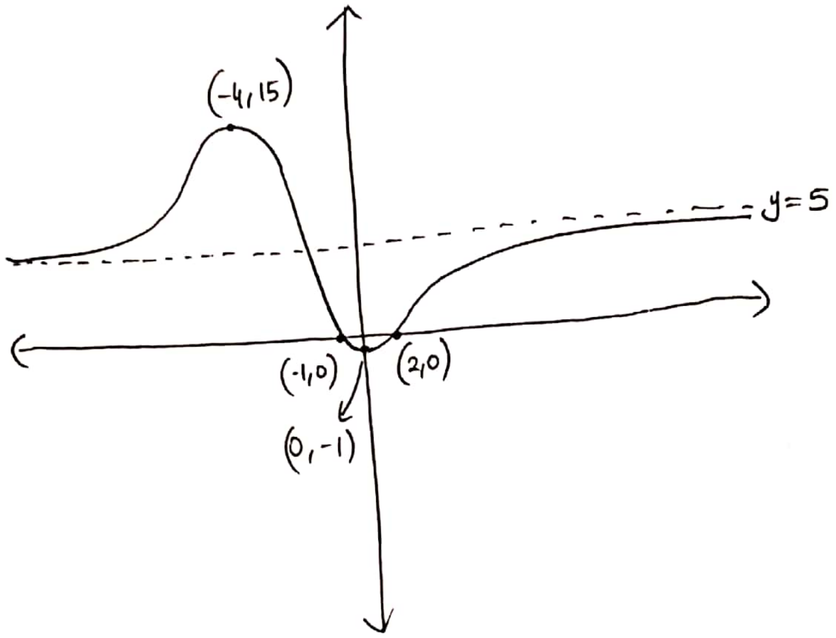
$$15x^2 + 75x + 150 = 5x^2 - 5x - 10$$

$$10x^2 + 80x + 160 = 0$$

$$x^2 + 8x + 16 = 0$$

$(x+4)^2 = 0 \Rightarrow x = -4 \quad \therefore (-4, 15)$.

$\therefore$ Turning points : $(-4, 15)$ and $(0, -1)$.



15 The curve C has equation

$$y = \lambda x + \frac{x}{x-2},$$

where λ is a non-zero constant. Find the equations of the asymptotes of C. [3]

Show that C has no turning points if $\lambda < 0$. [3]

Sketch C in the case $\lambda = -1$, stating the coordinates of the intersections with the axes. [3]

$\therefore y = \lambda x + \frac{x}{x-2} = \lambda x + 1 + \frac{2}{x-2}$.

$$x-2 \overline{\big)\, x} \quad \begin{array}{r} 1 \\ x-2 \\ - \;\; + \\ \hline 2 \\ \hline \end{array}$$

Asymptotes: $y = \lambda x + 1$

$x = 2$.

$\frac{dy}{dx} = \lambda - \frac{2}{(x-2)^2}$

At turning point, $\frac{dy}{dx} = 0$

$\Rightarrow \lambda - \frac{2}{(x-2)^2} = 0$

$\lambda = \frac{2}{(x-2)^2}$

$(x-2)^2 = \frac{2}{\lambda}$

No real roots if $\frac{2}{\lambda} < 0 \Rightarrow \lambda < 0$

Then C has no turning points if $\lambda < 0$.

$\lambda = -1 \Rightarrow y = -x + 1 + \frac{2}{x-2}$.

y intercept $\Rightarrow x = 0 \therefore y = -0 + 1 + \frac{2}{0-2} = 1 - 1 = 0 \therefore (0,0)$.

x-intercept $\Rightarrow y = 0 \Rightarrow 1 - x + \frac{2}{x-2} = 0$

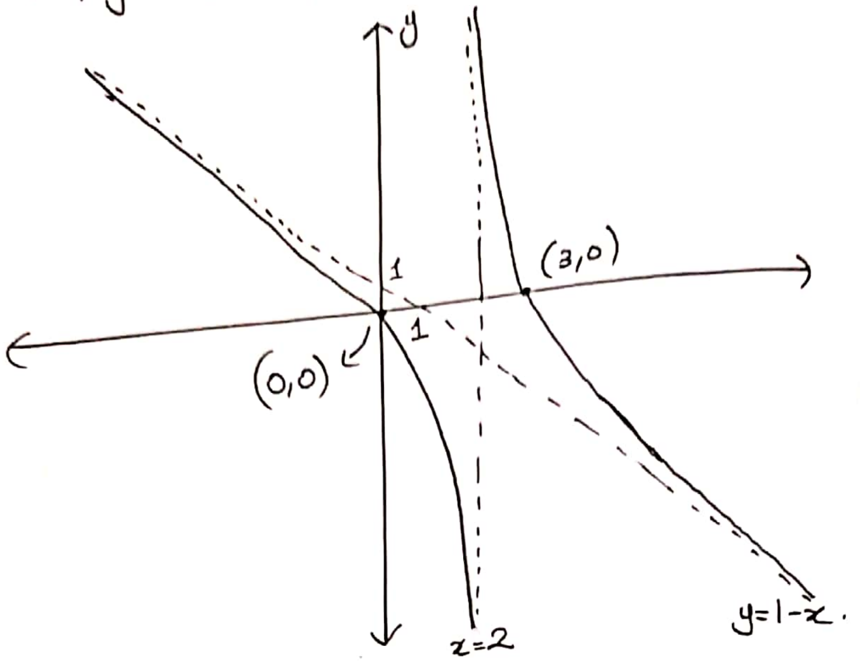
$\frac{2}{x-2} = x - 1$

$2 = x^2 - 3x + 2$

$0 = (x-3)(x)$

$\Rightarrow x = 0, 3 \therefore (0,0)$ and $(3,0)$.

$\lambda = -1 \Rightarrow y = -x+1$ (Oblique asymptote).



36 The curve C has equation $y = \dfrac{x^2 - 3x + 3}{x - 2}$. Find the equations of the asymptotes of C. [3]

Show that there are no points on C for which $-1 < y < 3$. [4]

Find the coordinates of the turning points of C. [3]

Sketch C. [2]

$$y = x - 1 + \frac{1}{x-2}$$

$$\begin{array}{r}
x - 1 \\
x-2 \,\overline{)\, x^2 - 3x + 3} \\
x^2 - 2x \\
- \quad + \\
\hline
-x + 3 \\
-x + 2 \\
+ \quad - \\
\hline
1 \\
\hline
\end{array}$$

Asymptotes: $y = x - 1$, $x = 2$.

For the curve to not exist,

$$\frac{x^2 - 3x + 3}{x - 2} = y$$

$$x^2 - 3x + 3 = yx - 2y$$

$$x^2 - 3x - yx + 3 + 2y = 0$$

$$x^2 + x(-3-y) + 3 + 2y = 0$$

set $b^2 - 4ac < 0$

$$(-3-y)^2 - 4(1)(3+2y) < 0$$

$$9 + y^2 + 6y - 12 - 8y < 0$$

$$y^2 - 2y - 3 < 0$$

$$(y-3)(y+1) < 0$$

$$\Rightarrow \quad -1 < y < 3$$

i.e. curve does not exist for $-1 < y < 3$.

$$y = -1 \Rightarrow \quad -1 = \frac{x^2 - 3x + 3}{x - 2}$$

$$2 - x = x^2 - 3x + 3$$

$$0 = x^2 - 2x + 1$$

$$0 = (x-1)^2$$

$$\Rightarrow x = 1$$

$$\therefore (1, -1)$$

$$y = 3 \Rightarrow \quad 3 = \frac{x^2 - 3x + 3}{x - 2}$$

$$3x - 6 = x^2 - 3x + 3$$

$$0 = x^2 - 6x + 9 = (x-3)^2$$

$x = 3 \quad \therefore (3, 3) \quad \therefore$ Turning points: $(3, 3)$, $(1, -1)$

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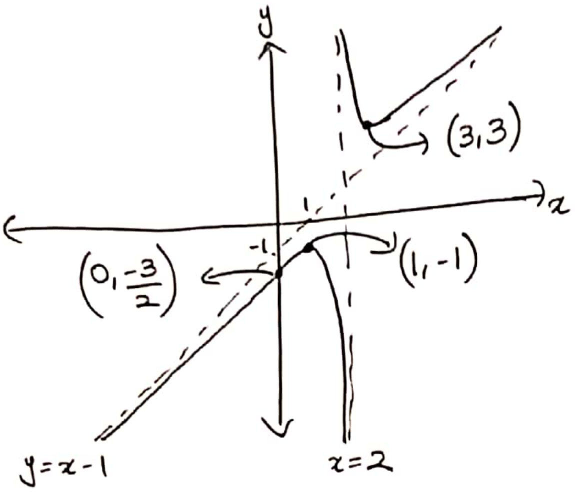
x-intercept $\Rightarrow$ $y=0$ $\therefore$ $\dfrac{x^2-3x+3}{x-2}=0$

$$x^2-3x+3=0$$

$$x=\frac{-(-3)\pm\sqrt{(-3)^2-4(1)(3)}}{2(1)}=\frac{3\pm\sqrt{-3}}{2}$$

No x-intercepts.

y-intercept $\Rightarrow$ $x=0$ $\therefore$ $y=\dfrac{0^2-3(0)+3}{0-2}=\dfrac{-3}{2}$ $\therefore$ $\left(0,\dfrac{-3}{2}\right)$.



37 The curve C has equation

$$y = \frac{px^2 + 4x + 1}{x + 1},$$

where p is a positive constant and $p \neq 3$.

(i) Obtain the equations of the asymptotes of C. [3]

(ii) Find the value of p for which the x-axis is a tangent to C, and sketch C in this case. [4]

(iii) For the case $p = 1$, show that C has no turning points, and sketch C, giving the exact coordinates of the points of intersection of C with the x-axis. [5]

i) $y = px + 4 - p + \frac{p-3}{x+1}$.

$\therefore$ Asymptotes: $y = px + 4 - p$.

$x = -1$

$$\begin{array}{r|l} & px + 4 - p \\ \hline x+1 & px^2 + 4x + 1 \\ & px^2 + px \\ & - \quad - \\ \hline & (4-p)x + 1 \\ & (4-p)x + 4 - p \\ & - \quad - \; + \\ \hline & p - 3 \\ \hline \end{array}$$

ii) Solve for $y = 0 \Rightarrow \frac{px^2+4x+1}{x+1} = 0$

$px^2 + 4x + 1 = 0$

We want $y = 0$ to be a tangent

Then $4^2 - 4(p)(1) = 0$

$16 = 4p$

$\Rightarrow p = 4$.

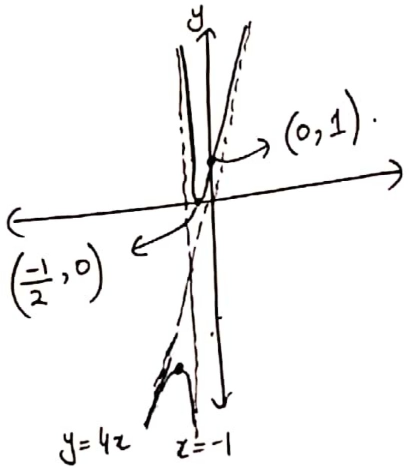
$p = 4 \Rightarrow y = 4x + \frac{1}{x+1}$ $\therefore$ Asymptotes: $y = 4x, x = -1$.

$y = 0 \Rightarrow 4x^2 + 4x + 1 = 0 \Rightarrow (2x+1)^2 = 0$

$x = \frac{-1}{2}$.

$\Rightarrow$ Point of tangency: $\left(\frac{-1}{2}, 0\right)$.

~~y~~ Set $x = 0 \Rightarrow y = 4(0) + \frac{1}{0+1} = 1 \therefore (0, 1)$.



$p=1 \Rightarrow y = \frac{x^2+4x+1}{x+1} = x+3-\frac{2}{x+1}$

$\frac{dy}{dx} = 1 + \frac{2}{(x+1)^2}$.

At turning point, $\frac{dy}{dx} = 0 \Rightarrow 1+\frac{2}{(x+1)^2} = 0 \Rightarrow \frac{2}{(x+1)^2} = -1 \Rightarrow (x+1)^2 = -2$

No real roots.

Then the curve has no turning points.

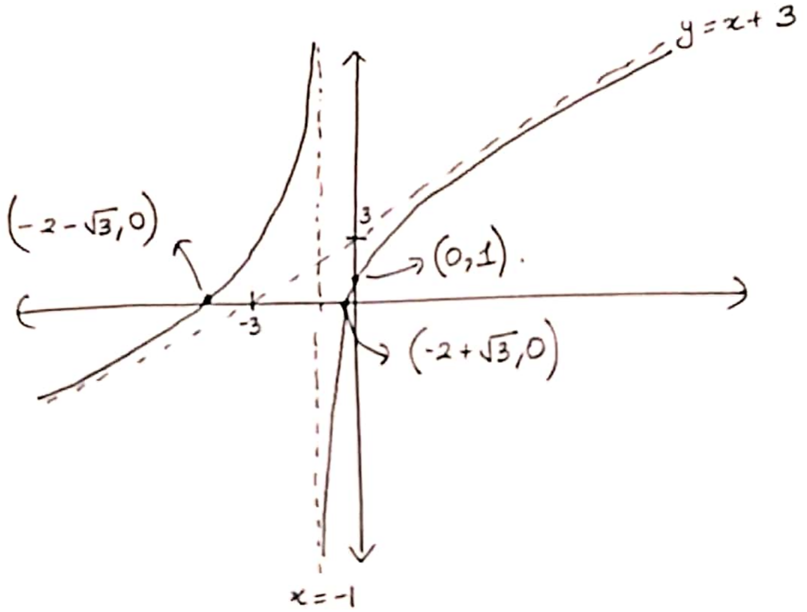
For x-intercept, $y=0 \Rightarrow \frac{x^2+4x+1}{x+1} = 0 \Rightarrow x = \frac{-4 \pm \sqrt{4^2-4(1)(1)}}{2(1)}$

$x = \frac{-4\pm\sqrt{12}}{2} = \frac{-4\pm2\sqrt{3}}{2}$

$x = -2\pm\sqrt{3}$

$\therefore$ x-intercepts: $(-2+\sqrt{3},0)$, $(-2-\sqrt{3},0)$.

y-intercept: $x=0 \Rightarrow y = \frac{0^2+4(0)+1}{0+1} = 1 \Rightarrow (0,1)$.



38 The curve C has equation

$$y = \frac{2x^2 + 5x - 1}{x + 2}.$$

Find the equations of the asymptotes of C. [3]

Show that $\frac{dy}{dx} > 2$ at all points on C. [3]

Sketch C. [3]

$$y = 2x + 1 - \frac{3}{x+2}$$

$$\begin{array}{r|l} & 2x + 1 \\ \hline x+2 & 2x^2 + 5x - 1 \\ & 2x^2 + 4x \\ & - \quad - \\ \hline & x - 1 \\ & x + 2 \\ & - \quad - \\ \hline & -3 \\ \hline \end{array}$$

Asymptotes: $y = 2x + 1$

$x = -2.$

$$\frac{dy}{dx} = 2 + \frac{3}{(x+2)^2}.$$

Note $\frac{1}{(x+2)^2} > 0$ as $y = 0$ is an asymptote to the curve $y = \frac{1}{(x+2)^2}$.

$\hookrightarrow$ for all values of x.

Then $\frac{3}{(x+2)^2} > 0$

$\Rightarrow 2 + \frac{3}{(x+2)^2} > 0 + 2$

$\frac{dy}{dx} > 2$ at all points on C.

---

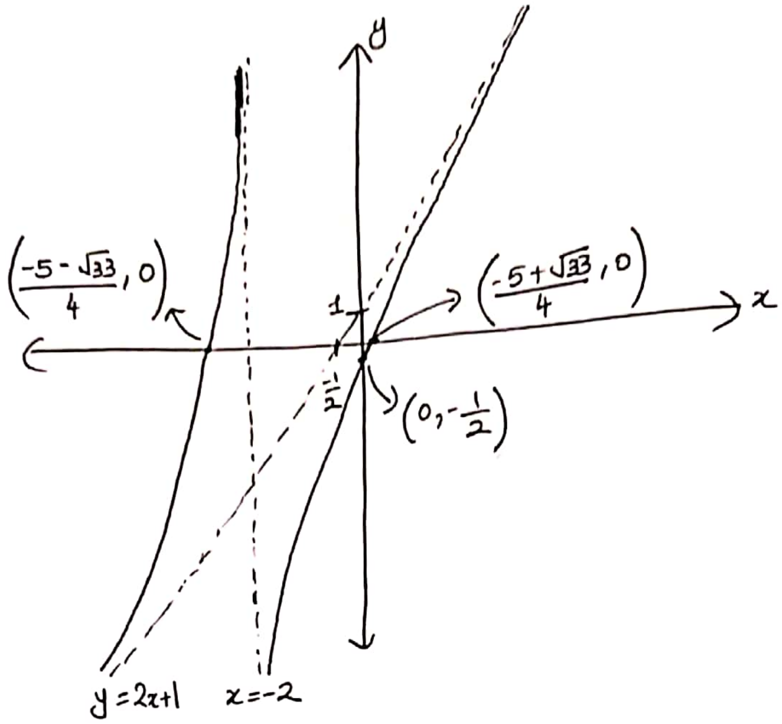
x-intercept : $y = 0 \Rightarrow \frac{2x^2+5x-1}{x+2} = 0$

$$x = \frac{-5 \pm \sqrt{25+8}}{2(2)} = \frac{-5 \pm \sqrt{33}}{4}$$

$$\therefore \left(\frac{-5+\sqrt{33}}{4}, 0\right) \text{ and } \left(\frac{-5-\sqrt{33}}{4}, 0\right).$$

y-intercept : $x = 0 \Rightarrow y = \frac{2(0)^2+5(0)-1}{0+2} = \frac{-1}{2}.$

$$\therefore \left(0, \frac{-1}{2}\right).$$


y
x
$\left(\frac{-5-\sqrt{33}}{4}, 0\right)$
$\left(\frac{-5+\sqrt{33}}{4}, 0\right)$
1
$-\frac{1}{2}$
$\left(0, -\frac{1}{2}\right)$
y = 2x+1
x = -2

39 A curve C has equation $y = \frac{2x^2 + x - 1}{x - 1}$. Find the equations of the asymptotes of C. [3]

Show that there is no point on C for which $1 < y < 9$. [4]

$$y = \frac{2x^2 + x - 1}{x - 1}$$

$$y = 2x + 3 + \frac{2}{x - 1}$$

$$\begin{array}{r} 2x + 3 \\ x - 1 \overline{\smash{)}\, 2x^2 + x - 1} \\ 2x^2 - 2x \\ - \quad + \\ \hline 3x - 1 \\ 3x - 3 \\ - \quad + \\ \hline 2 \\ \hline \end{array}$$

Asymptotes: $y = 2x + 3$

$x = \underline{1}$.

$$y = \frac{2x^2 + x - 1}{x - 1}$$

$$yx - y = 2x^2 + x - 1$$

$$0 = 2x^2 + x - yx + y - 1$$

$$0 = 2x^2 + (1-y)x + y - 1$$

For x to not be real, $b^2 - 4ac < 0$

$$\Rightarrow (1-y)^2 - 4(2)(y-1) < 0$$

$$1 + y^2 - 2y - 8y + 8 < 0$$

$$y^2 - 10y + 9 < 0$$

$$(y-9)(y-1) < 0$$

$$\Rightarrow 1 < y < 9$$

$\therefore$ There is no point on C for which $1 < y < 9$.

10 The curve C has equation $y = \dfrac{2x^2 + kx}{x+1}$, where k is a constant. Find the set of values of k for which C has no stationary points. [5]

For the case $k = 4$, find the equations of the asymptotes of C and sketch C, indicating the coordinates of the points where C intersects the coordinate axes. [6]

$$y = 2x + k - 2 + \frac{2-k}{x+1}.$$

$$\begin{array}{r} 2x + k - 2 \\ x+1 \overline{)\, 2x^2 + kx } \\ 2x^2 + 2x \\ - \quad - \\ \hline (k-2)x \\ (k-2)x + k - 2 \\ - \quad\quad - \quad + \\ \hline -k+2 \\ \hline \end{array}$$

$$\frac{dy}{dx} = 2 + \frac{(2-k)(-1)}{(x+1)^2} = 2 + \frac{k-2}{(x+1)^2}.$$

For stationary point, $\frac{dy}{dx} = 0$

$$\Rightarrow 2 + \frac{k-2}{(x+1)^2} = 0$$

$$2 = \frac{2-k}{(x+1)^2}$$

$$2(x^2 + 2x + 1) = 2 - k$$

$$2x^2 + 4x + 2 - 2 + k = 0$$

$$2x^2 + 4x + k = 0$$

For no real roots, $b^2 - 4ac < 0$

$$4^2 - 4(2)(k) < 0$$

$$16 - 8k < 0$$

$$16 < 8k$$

$$2 < k$$

Then for $k > 2$, there are no stationary points.

$k = 4 \Rightarrow y = 2x + 2 + \dfrac{(2-4)}{x+1} = 2x + 2 - \dfrac{2}{x+1}. = \dfrac{2x^2 + 4x}{x+1}$

Asymptotes : $y = 2x + 2$, $x = -1$.

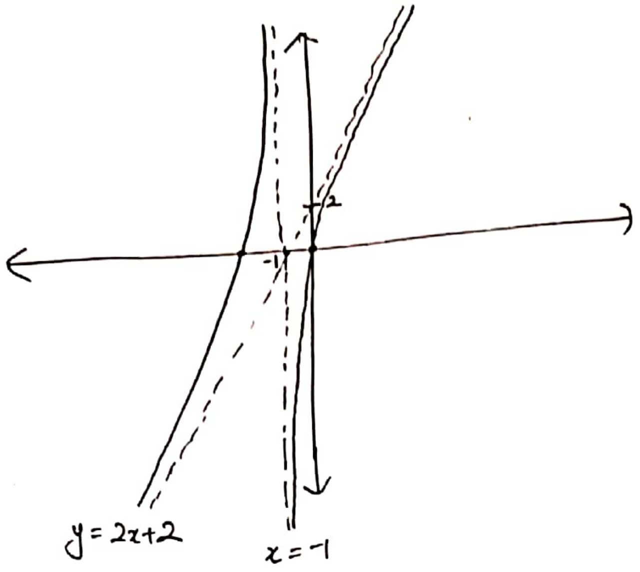
x-intercepts $\Rightarrow y = 0 \Rightarrow \dfrac{2x^2 + 4x}{x+1} = 0$

$$2x(x+2) = 0$$

$$x = 0, -2$$

$\therefore (0,0)$ and $(-2,0)$.

y-intercept $\Rightarrow x = 0 \Rightarrow y = \dfrac{2\times 0^2 + 4\times 0}{0+1} = 0 \Rightarrow (0,0)$.


2
-1
y = 2x+2
x = -1

41 The curve C has equation

$$y = \frac{3x - 9}{(x - 2)(x + 1)}.$$

(i) Find the equations of the asymptotes of C. [2]

As $|x| \mapsto \infty$, $y \mapsto 0$ $\therefore y = 0$ is horizontal asymptote.

$(x-2)(x+1) = 0 \Rightarrow x = 2$ ~~and~~ $x = -1$ are vertical asymptotes.

(ii) Show that there is no point on C for which $\frac{1}{3} < y < 3$. [4]

$$y = \frac{3x-9}{x^2 - x - 2}$$

$$yx^2 - yx - 2y = 3x - 9$$

$$yx^2 - yx - 3x - 2y + 9 = 0$$

$$yx^2 + x(-y-3) + 9 - 2y = 0.$$

When x is complex/imaginary, $b^2 - 4ac < 0$

$$\Rightarrow (-y-3)^2 - 4(y)(9-2y) < 0$$

$$y^2 + 6y + 9 - 36y + 8y^2 < 0$$

$$9y^2 - 30y + 9 < 0$$

$$3y^2 - 10y + 3 < 0$$

$$3y^2 - 9y - y + 3 < 0$$

$$(3y-1)(y-3) < 0$$

$$\Rightarrow \frac{1}{3} < y < 3$$

Then the curve does not exist for $\frac{1}{3} < y < 3$.

(iii) Find the coordinates of the turning points of C. [3]

$y = \frac{1}{3} \Rightarrow \frac{3x-9}{x^2-x-2} = \frac{1}{3}$

$9x - 27 = x^2 - x - 2$

$0 = x^2 - 10x + 25$

$0 = (x-5)^2$

$\Rightarrow x = 5$

$\therefore \left(5, \frac{1}{3}\right)$

$y = 3 \Rightarrow \frac{3x-9}{x^2-x-2} = 3$

$3x - 9 = 3x^2 - 3x - 6$

$0 = 3x^2 - 6x + 3$

$0 = x^2 - 2x + 1$

$0 = (x-1)^2$

$\Rightarrow x = 1$

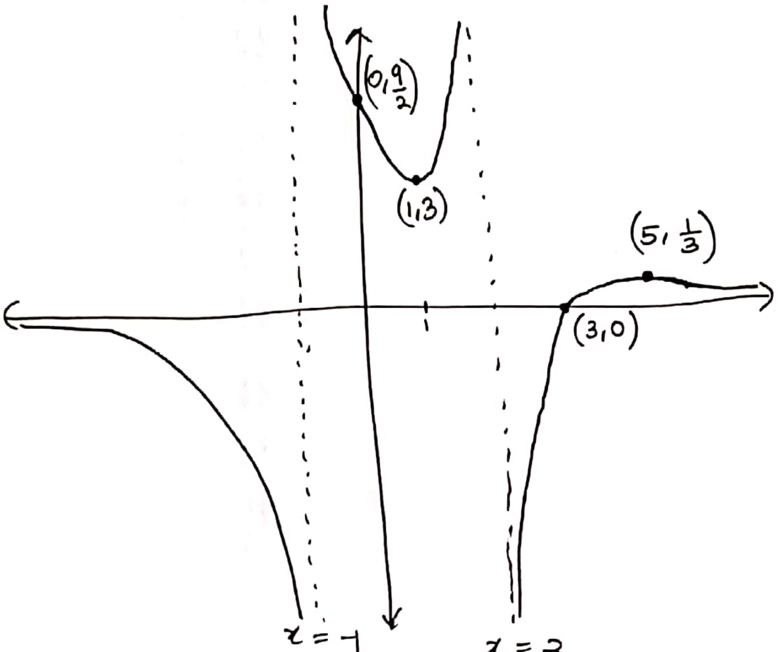
$(1, 3)$

i.e. Coordinates of turning points: $\left(5, \frac{1}{3}\right)$ and $(1, 3)$.

(iv) Sketch C. [3]

x-intercept $\Rightarrow y = 0 \Rightarrow \frac{3x-9}{x^2-x-2} = 0 \Rightarrow x = 3 \therefore (3, 0)$

y-intercept $\Rightarrow x = 0 \Rightarrow y = \frac{3(0)-9}{0^2-0-2} = \frac{9}{2} \therefore \left(0, \frac{9}{2}\right)$



12 The curve C has equation

$$y = \frac{x^2 + ax - 1}{x + 1},$$

where a is constant and $a > 1$.

(i) Find the equations of the asymptotes of C. [3]

$$y = x + a - 1 - \frac{a}{x+1}$$

$x + a - 1$

$x+1 \,\big)\, x^2 + ax - 1$

$x^2 + x$

$-\quad -$

$(a-1)x - 1$

$(a-1)x + a - 1$

$-\quad - \quad +$

$-a$

Asymptotes: $y = x + a - 1$

$x = -1$.

(ii) Show that C intersects the x-axis twice. [1]

At x-axis, $y = 0 \Rightarrow \dfrac{x^2+ax-1}{x+1} = 0$

$\Rightarrow x^2 + ax - 1 = 0$

$b^2 - 4ac = (a)^2 - 4(1)(-1) = a^2 + 4$

Note $a > 1 \Rightarrow a^2 > 1 \Rightarrow a^2 + 4 > 5$

Since discriminant > 0, then the equation has two distinct roots.

Then the curve intersects the x-axis twice.

(iii) Justifying your answer, find the number of stationary points on C. [2]

$$y = x+a-1 - \frac{a}{x+1}$$

$$\frac{dy}{dx} = 1 + \frac{a}{(x+1)^2}$$

$$0 = 1+\frac{a}{(x+1)^2}$$

$$-1 = \frac{a}{(x+1)^2}$$

$$\Rightarrow (x+1)^2 = -a$$

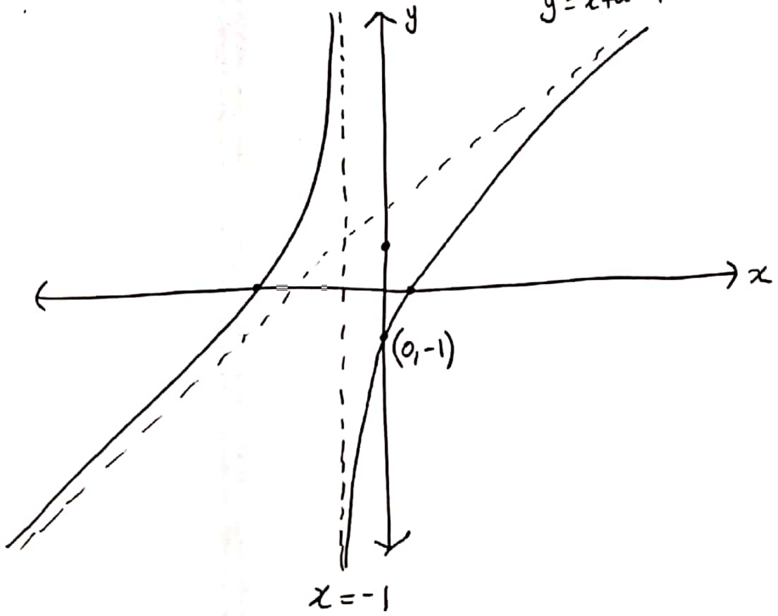
$x+1 = \pm\sqrt{-a}$ (No real roots).

Then $\frac{dy}{dx} = 0$ has no real roots.

$\Rightarrow$ There are no stationary points.

(iv) Sketch C, stating the coordinates of its point of intersection with the y-axis. [3]

At y-axis, $x=0 \Rightarrow y = \frac{0^2+a(0)-1}{0+1} = -1$. $\therefore$ $(0,-1)$.



43 The curve C has equation

$$y = \frac{5x^2 + 5x + 1}{x^2 + x + 1}.$$

(i) Find the equation of the asymptote of C. [2]

As $|x| \mapsto \infty$, $y \mapsto \frac{5}{1} = 5$.

$\therefore$ Horizontal asymptote : $y = 1$.

$x^2 + x + 1 = 0$ has no real roots.

$\Rightarrow$ No vertical asymptote exists.

(ii) Show that, for all real values of x, $-\frac{1}{3} \leqslant y < 5$. [4]

$$y = \frac{5x^2 + 5x + 1}{x^2 + x + 1}$$

$$yx^2 + yx + y = 5x^2 + 5x + 1$$

$$yx^2 - 5x^2 + yx - 5x + y - 1 = 0$$

$$(y-5)x^2 + (y-5)x + y - 1 = 0$$

For x to be real, $b^2 - 4ac \geqslant 0$

$$\Rightarrow (y-5)^2 - 4(y-5)(y-1) \geqslant 0$$

$$y^2 - 10y + 25 - 4(y^2 - 5y - y + 5) \geqslant 0$$

$$y^2 - 10y + 25 - 4(y^2 - 6y + 5) \geqslant 0$$

$$-3y^2 + 14y + 5 \geqslant 0$$

$$3y^2 - 14y - 5 \leqslant 0$$

$$3y^2 - 15y + y - 5 \leqslant 0$$

$$(3y+1)(y-5) \leqslant 0$$

$$\Rightarrow \frac{-1}{3} \leqslant y \leqslant 5.$$

Note $y = 5$ is not possible as $y = 5$ is the horizontal asymptote.

$\therefore$ For all real values of x, $\frac{-1}{3} \leqslant y \leqslant 5$.

(iii) Find the coordinates of any stationary points of C. [2]

Set $y = \frac{-1}{3} \Rightarrow \frac{5x^2+5x+1}{x^2+x+1} = \frac{-1}{3}$

$$15x^2+15x+3 = -x^2-x-1$$

$$16x^2+16x+4 = 0$$

$$4(4x^2+4x+1) = 0$$

$$(2x+1)^2 = 0$$

$$\Rightarrow x = \frac{-1}{2}.$$

$\therefore$ Coordinates of stationary point $= \left(\frac{-1}{2}, \frac{-1}{3}\right)$

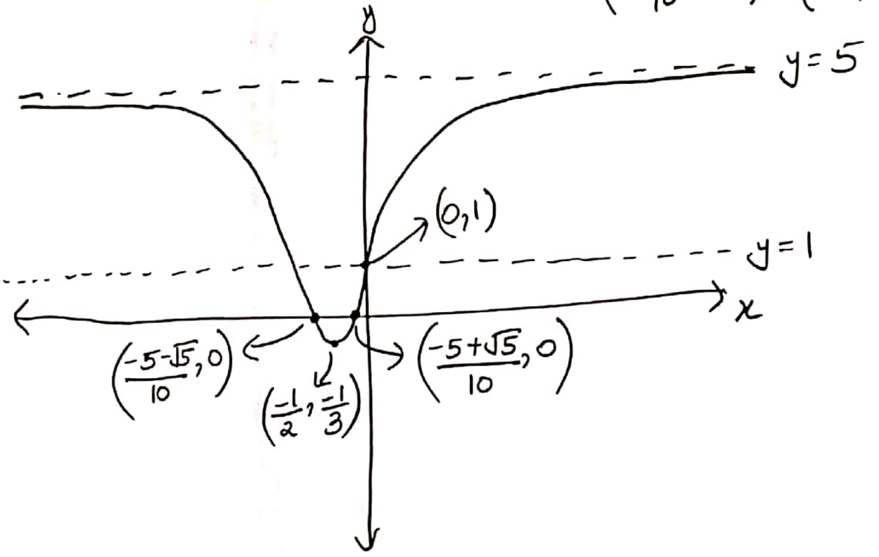
Note $y=5$ is an asymptote so setting $y=5$ gives no solutions.

(iv) Sketch C, stating the coordinates of any intersections with the y-axis. [2]

For y-intercept, set $x=0 \Rightarrow y = \frac{5(0)^2+5(0)+1}{0^2+0+1} = 1 \Rightarrow (0,1)$.

For x-intercept, set $y=0 \Rightarrow \frac{5x^2+5x+1}{x^2+x+1} = 0 \Rightarrow x = \frac{-5+\sqrt{5}}{10}, \frac{-5-\sqrt{5}}{10}$.

$\therefore \left(\frac{-5+\sqrt{5}}{10}, 0\right), \left(\frac{-5-\sqrt{5}}{10}, 0\right)$.



4 The line $y = 2x + 1$ is an asymptote of the curve C with equation

$$y = \frac{x^2 + 1}{ax + b}.$$

(i) Find the values of the constants a and b. [3]

$$\begin{array}{r} \frac{x}{a} - \frac{b}{a^2} \\ ax+b \,\overline{)\, x^2 + 0x + 1} \\ x^2 + \frac{bx}{a} \\ - \qquad - \\ \hline \frac{-bx}{a} + 1 \\ -\frac{bx}{a} - \frac{b^2}{a^2} \\ + \qquad + \\ \hline 1 + \frac{b^2}{a^2} \\ \hline \end{array}$$

$$y = \frac{x}{a} - \frac{b}{a^2} + \frac{1 + \frac{b^2}{a^2}}{ax+b}$$

Oblique asymptote: $y = \frac{x}{a} - \frac{b}{a^2}$.

By comparison to $y = 2x + 1$,

$\frac{1}{a} = 2$ and $\frac{-b}{a^2} = 1$

$\Rightarrow a = \frac{1}{2}$

$-b = a^2$

$\Rightarrow -b = \left(\frac{1}{2}\right)^2$

$b = \frac{-1}{4}$.

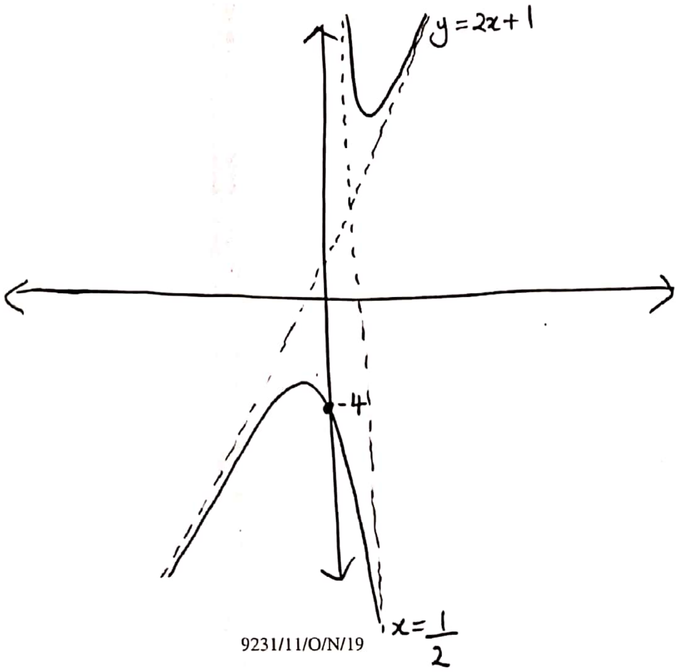
(ii) State the equation of the other asymptote of C. [1]

$$y = \frac{x^2 + 1}{\frac{1}{2}x - \frac{1}{4}}$$

For vertical asymptote: $\frac{1}{2}x - \frac{1}{4} = 0 \Rightarrow \frac{1}{2}x = \frac{1}{4} \Rightarrow x = \frac{1}{2}$.

(iii) Sketch C. [Your sketch should indicate the coordinates of any points of intersection with the y-axis. You do not need to find the coordinates of any stationary points.] [3]

At y-axis, $x = 0 \Rightarrow y = \frac{0^2 + 1}{\frac{1}{2}\times 0 - \frac{1}{4}} = \frac{1}{\frac{-1}{4}} = -4 \quad \therefore (0, -4)$.



15 The curve C has equation $y = \dfrac{x^2+x-1}{x-1}$.

(a) Find the equations of the asymptotes of C. [3]

$$y = x+2+\frac{1}{x-1}$$

$$x-1\,\overline{\smash{)}\,x^2+x-1}^{\;x+2}$$

x^2-x

$-\quad+$

$2x-1$

$2x-2$

$-\quad+$

1

Asymptotes : $y = x+2$

$x = 1$

(b) Show that there is no point on C for which $1 < y < 5$. [4]

$$y = \frac{x^2+x-1}{x-1}$$

$$yx - y = x^2+x-1$$

$$0 = x^2+x-yx+y-1$$

$$0 = x^2+(1-y)x+y-1$$

For real values of x, we require $b^2-4ac \geq 0$

$$\Rightarrow (1-y)^2-4(1)(y-1) \geq 0$$

$$1-2y+y^2-4y+4 \geq 0$$

$$y^2-6y+5 \geq 0$$

$$(y-5)(y-1) \geq 0$$

$$\Rightarrow y \leq 1 \text{ or } y \geq 5$$

Then curve exists for $y \leq 1$ or $y \geq 5$

$\Rightarrow$ There is no point for which $1<y<5$.

(c) Find the coordinates of the intersections of C with the axes, and sketch C. [3]

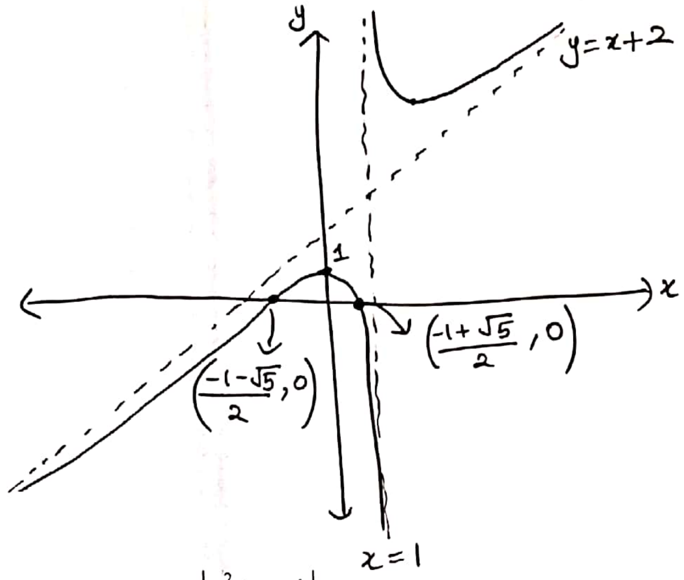
x-intercept: $y = 0 \Rightarrow \frac{x^2+x-1}{x-1} = 0 \Rightarrow x^2+x-1=0$

$$x = \frac{-1 \pm \sqrt{1-4(1)(-1)}}{2}$$

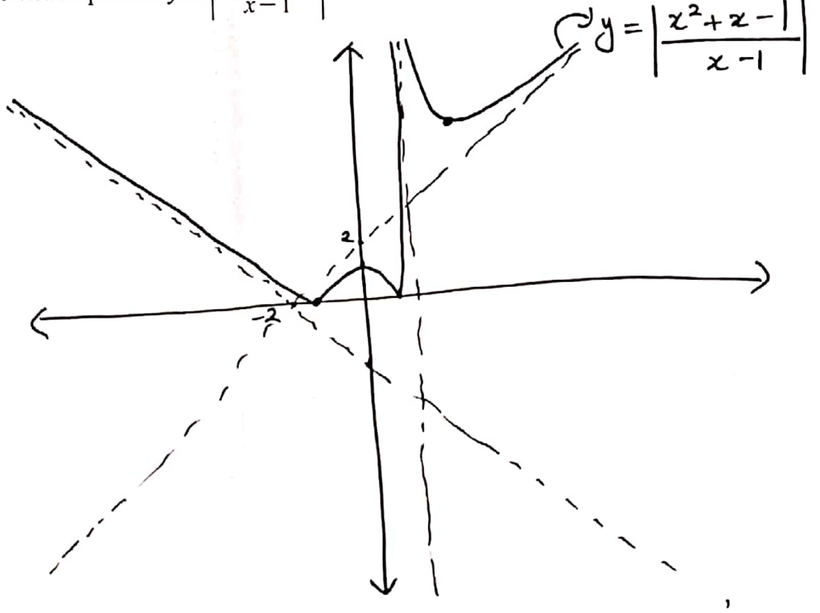
$$x = \frac{-1 \pm \sqrt{5}}{2}$$

$\therefore \left(\frac{-1+\sqrt{5}}{2}, 0\right)$ and $\left(\frac{-1-\sqrt{5}}{2}, 0\right)$.

y-intercept: $x = 0 \Rightarrow y = \frac{0^2+0-1}{0-1} = 1 \therefore (0,1)$.



(d) Sketch the curve with equation $y = \left|\frac{x^2+x-1}{x-1}\right|$. [2]



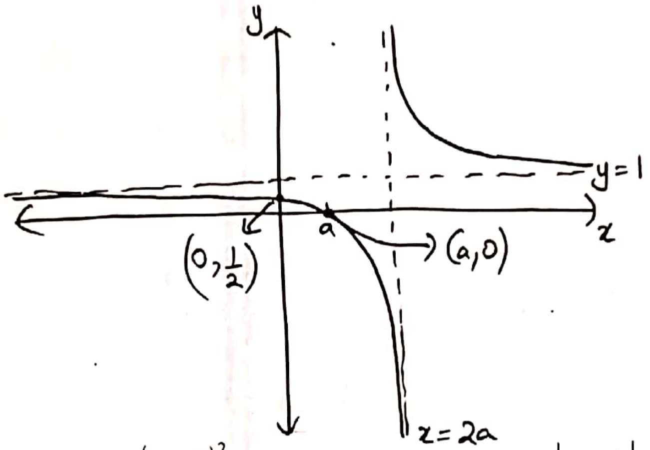
16 Let a be a positive constant.

(a) The curve C_1 has equation $y = \frac{x-a}{x-2a}$.

Sketch C_1. [2]

Asymptotes: $y = 1$, $x = 2a$.

$x = 0 \Rightarrow y = \frac{1}{2}$ ∴ $\left(0, \frac{1}{2}\right)$ and $y = 0 \Rightarrow x = a$ ∴ $(a, 0)$.



The curve C_2 has equation $y = \left(\frac{x-a}{x-2a}\right)^2$. The curve C_3 has equation $y = \left|\frac{x-a}{x-2a}\right|$.

(b) (i) Find the coordinates of any stationary points of C_2. [3]

$$C_2 : y = \left[\frac{x-a}{x-2a}\right]^2.$$

$$\frac{dy}{dx} = 2\left[\frac{x-a}{x-2a}\right] \cdot \left[\frac{(x-2a)(1) - (x-a)(1)}{(x-2a)^2}\right]$$

$$\frac{dy}{dx} = \frac{2(x-a)(-a)}{(x-2a)^3}$$

At turning point, $\frac{dy}{dx} = 0 \Rightarrow \frac{2(x-a)(-a)}{(x-2a)^3} = 0$

$$\Rightarrow x - a = 0$$

$$\therefore x = a$$

$$\Rightarrow y = \left[\frac{a-a}{a-2a}\right]^2 = 0.$$

∴ Stationary points of C_2 : $(a, 0)$.

(ii) Find also the coordinates of any points of intersection of C_2 and C_3. [3]

For C_2 and C_3 to meet, $\left(\frac{x-a}{x-2a}\right)^2 = \left|\frac{x-a}{x-2a}\right|$

Either, $\left(\frac{x-a}{x-2a}\right)^2 = \frac{x-a}{x-2a}$ or $\left(\frac{x-a}{x-2a}\right)^2 = -\left(\frac{x-a}{x-2a}\right)$

$$(x-2a)(x-a)^2 = (x-a)(x-2a)^2$$

$$(x-2a)(x-a)^2 - (x-a)(x-2a)^2 = 0$$

$$(x-a)(x-2a)\left[x-a-x+2a\right] = 0$$

$$(x-a)(x-2a)(a) = 0$$

$x = a, 2a$.

Ignore $x = 2a$ since curves are undefined.

$x = a \Rightarrow y = \left|\frac{a-a}{a-2a}\right| = \left|\frac{0}{-a}\right| = 0$

$$\frac{(x-a)^2}{(x-2a)^2} = \frac{-(x-a)}{(x-2a)}$$

$$(x-a) = -(x-2a)$$

$$x-a = -x+2a$$

$$2x = 3a$$

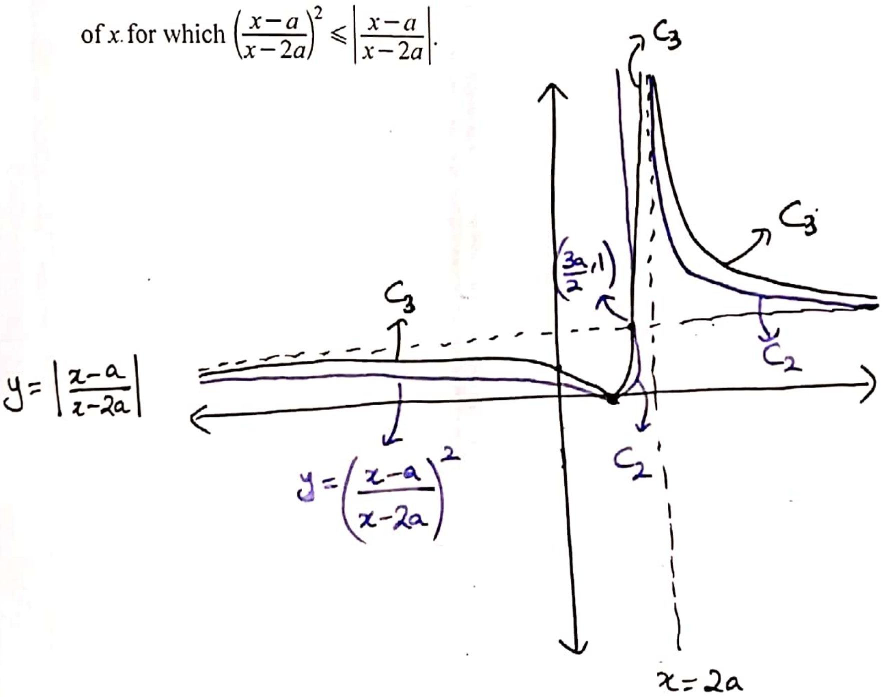
$$x = \frac{3a}{2}.$$

$\Rightarrow y = \left|\frac{\frac{3a}{2}-a}{\frac{3a}{2}-2a}\right| = \left|\frac{\frac{a}{2}}{\frac{-a}{2}}\right| = |-1|$

$y = 1$

$\therefore$ Points of intersection: $(a, 0)$ and $\left(\frac{3a}{2}, 1\right)$.

(c) Sketch C_2 and C_3 on a single diagram, clearly identifying each curve. Hence find the set of values of x for which $\left(\frac{x-a}{x-2a}\right)^2 \leqslant \left|\frac{x-a}{x-2a}\right|$. [5]



$\left(\frac{x-a}{x-2a}\right)^2 \leq \left|\frac{x-a}{x-2a}\right|$ for $x \leq \frac{3a}{2}$.

17 The curve C has equation $y = \frac{x^2}{2x+1}$.

(a) Find the equations of the asymptotes of C. [3]

$$y = \frac{x}{2} - \frac{1}{4} + \frac{\frac{1}{4}}{2x+1}.$$

$$\begin{array}{r} \frac{x}{2} - \frac{1}{4} \\ 2x+1 \,\big)\, x^2 \\ x^2 + \frac{x}{2} \\ - \quad - \\ \hline -\frac{x}{2} \\ -\frac{x}{2} - \frac{1}{4} \\ + \quad + \\ \hline \frac{1}{4} \\ \hline \end{array}$$

Asymptotes : $y = \frac{x}{2} - \frac{1}{4}$.

$x = \frac{-1}{2}$.

(b) Find the coordinates of the stationary points on C. [3]

$$\frac{dy}{dx} = \frac{1}{2} + \frac{-\frac{1}{4}}{(2x+1)^2} \cdot 2 = \frac{1}{2} - \frac{1}{2(2x+1)^2}.$$

At stationary point, $\frac{dy}{dx} = 0 \Rightarrow \frac{1}{2} - \frac{1}{2(2x+1)^2} = 0$

$$\frac{1}{2} = \frac{1}{2(2x+1)^2}$$

$$(2x+1)^2 = 1$$

$$2x+1 = \pm 1$$

Either $2x+1 = 1$ or $2x+1 = -1$

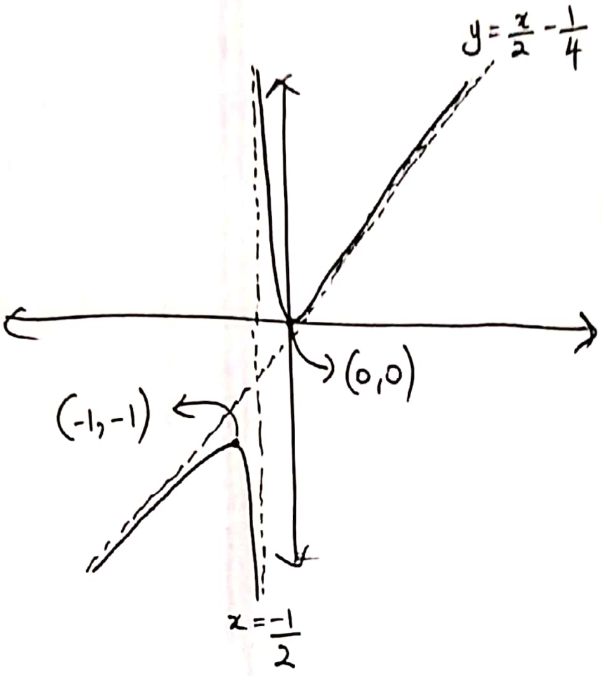
$x = 0$ or $x = -1$.

$\Rightarrow x = 0 \Rightarrow y = \frac{0^2}{2\times 0+1} = 0$

$\Rightarrow x = -1 \Rightarrow y = \frac{(-1)^2}{2\times -1+1} = \frac{1}{-1} = -1$

$\therefore$ Stationary points:
$(0,0)$ and $(-1,-1)$.

(c) Sketch C. [3]



48 The curve C has equation $y = \frac{10 + x - 2x^2}{2x - 3}$.

(a) Find the equations of the asymptotes of C. [3]

$$\therefore y = -x - 1 + \frac{7}{2x-3}$$

$$\begin{array}{r|l} & -x - 1 \\ \hline 2x - 3 & -2x^2 + x + 10 \\ & -2x^2 + 3x \\ & + \quad - \\ \hline & -2x + 10 \\ & -2x + 3 \\ & + \quad - \\ \hline & 7 \\ \hline \end{array}$$

$\therefore$ Asymptotes: $y = -x - 1$ (oblique)

$x = \frac{3}{2}$ (vertical).

(b) Show that C has no turning points. [3]

$$\frac{dy}{dx} = -1 - \frac{7}{(2x-3)^2} \cdot 2 = -1 - \frac{14}{(2x-3)^2}.$$

At turning points, $\frac{dy}{dx} = 0$

$$-1 - \frac{14}{(2x-3)^2} = 0$$

$$\frac{-14}{(2x-3)^2} = 1$$

$$(2x-3)^2 = -14$$

$\Rightarrow$ No real roots exist.

$\therefore$ There are no turning points.

(c) Sketch C_2 stating the coordinates of the intersections with the axes. [3]

y-intercept $\Rightarrow x=0 \therefore y = \frac{10+0-2(0)^2}{2(0)-3} = \frac{10}{-3} \therefore \left(0, -\frac{10}{3}\right)$

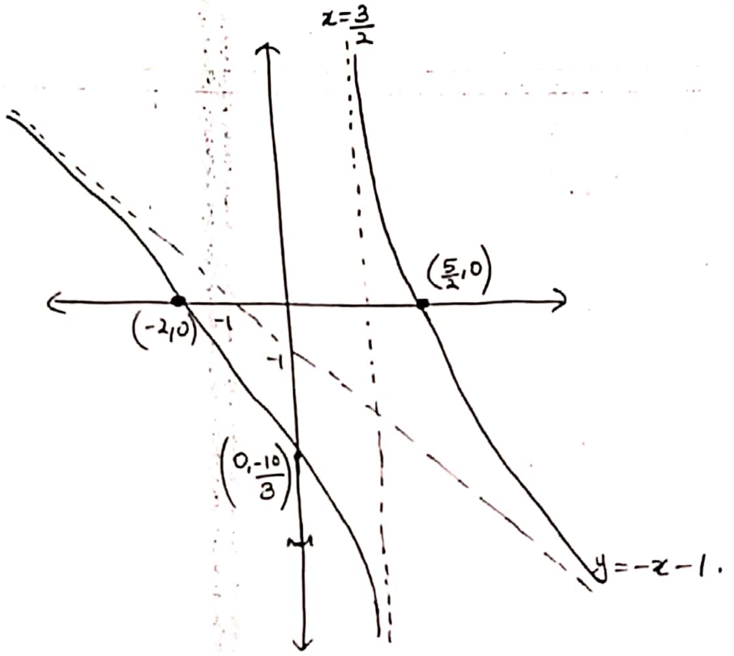
x-intercept $\Rightarrow y=0 \therefore \frac{10+x-2x^2}{2x-3} = 0$

$$2x^2-x-10=0$$

$$2x^2-5x+4x-10=0$$

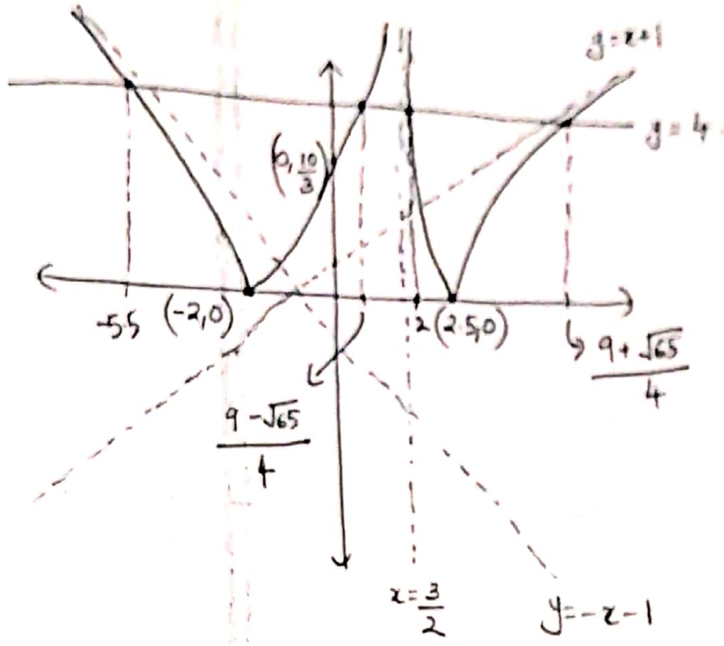
$$(2x-5)(x+2)=0$$

$\therefore x=\frac{5}{2}, -2 \therefore (-2,0), \left(\frac{5}{2},0\right).$



(d) Sketch the curve with equation $y = \left|\frac{10 + x - 2x^2}{2x - 3}\right|$ and find in exact form the set of values of x for which $\left|\frac{10 + x - 2x^2}{2x - 3}\right| < 4$.

[6]



Set $\left|\frac{10+x-2x^2}{2x-3}\right| = 4$

$\Rightarrow \frac{10+x-2x^2}{2x-3} = 4$ or $\frac{10+x-2x^2}{2x-3} = -4$

$-2x^2+x+10 = 8x-12$

$0 = 2x^2+7x-22$

$0 = 2x^2+11x-4x-22$

$0 = (2x+1)(x-2)$

$\Rightarrow x = \frac{-11}{2}, 2$

$-2x^2+x+10 = -8x+12$

$0 = 2x^2-9x+2$

$x = \frac{-(-9) \pm \sqrt{(-9)^2-4(2)(2)}}{2(2)}$

$x = \frac{9 \pm \sqrt{65}}{4}$

$x = \frac{9+\sqrt{65}}{4}, \frac{9-\sqrt{65}}{4}$

$\left|\frac{10+x-2x^2}{2x-3}\right| < 4 \Rightarrow -5.5 < x < \frac{9-\sqrt{65}}{4}$

and $2 < x < \frac{9+\sqrt{65}}{4}$.