1 Find the acute angle between the planes with equations

$$x - 2y + z - 9 = 0 \quad \text{and} \quad x + y - z + 2 = 0.$$ [3]

The planes meet in the line l, and A is the point on l whose position vector is $p\mathbf{i} + q\mathbf{j} + \mathbf{k}$.

(i) Find p and q. [2]

(ii) Find a vector equation for l. [3]

The non-coincident planes Π_1 and Π_2 are both perpendicular to l. The perpendicular distance from A to Π_1 is $\sqrt{14}$ and the perpendicular distance from A to Π_2 is also $\sqrt{14}$. Find equations for Π_1 and Π_2 in the form $ax + by + cz = d$. [5]

Let, $\Pi_3 : x-2y+z=9 \Rightarrow \vec{r}\cdot(\vec{i}-2\vec{j}+\vec{k}) = 9 \therefore \vec{n_3} = \vec{i}-2\vec{j}+\vec{k}$

Let, $\Pi_4 : x+y-z=-2 \Rightarrow \vec{r}\cdot(\vec{i}+\vec{j}-\vec{k}) = -2 \therefore \vec{n_4} = \vec{i}+\vec{j}-\vec{k}$.

$$\vec{n_3}\cdot\vec{n_4} = |\vec{n_1}||\vec{n_2}|\cos\theta$$

$$(\vec{i}-2\vec{j}+\vec{k})\cdot(\vec{i}+\vec{j}-\vec{k}) = \sqrt{1^2+(-2)^2+1^2}\sqrt{1^2+1^2+(-1)^2}\cos\theta$$

$$1-2-1 = \sqrt{6}\sqrt{3}\cos\theta$$

$$\cos\theta = \frac{-2}{\sqrt{18}} \Rightarrow \theta = \cos^{-1}\left(\frac{-2}{\sqrt{18}}\right) = 118.12^\circ$$

$\Rightarrow$ Acute angle between planes $= 180^\circ - \theta = 61.87^\circ \approx 61.9^\circ$.

i) Line of intersection between Π_3 and Π_4 passes through point $A(p,q,1)$. Then point $A(p,q,1)$ lies on both Π_3 and Π_4. Substituting gives:

$$p-2q+1=9 \Rightarrow p = 2q+8 \rightarrow (1)$$

$$p+q-1+2=0 \Rightarrow p=-q-1 \rightarrow (2)$$

Substituting (1) in (2) gives:

$$2q+8=-q-1$$

$$3q=-9 \Rightarrow q=-3.$$

$$\Rightarrow p = -(-3)-1 = 2.$$

$$\therefore p=2, q=-3$$

ii) Direction vector of l would be $\perp$ to both $\vec{n_3}$ and $\vec{n_4}$.

Let, $\vec{b} = \vec{n_1}\times\vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \begin{bmatrix} 2-1 \\ -(-1-1) \\ 1-(-2) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Since line l passes through $A(2,-3,1)$ $\therefore$ equation: $\vec{r} = 2\vec{i}-3\vec{j}+\vec{k}+t(\vec{i}+2\vec{j}+3\vec{k})$.

Since Π_1 and Π_2 are perpendicular to L, the normal vector for both Π_1 and Π_2 would be the direction vector of L. i.e. $\vec{n_1} = \vec{n_2} = \vec{i} + 2\vec{j} + 3\vec{k}$.

Then for both Π_1 and Π_2, cartesian equation is

$$x + 2y + 3z = d.$$

To find a point on Π_1 and Π_2, set $y = z = 0 \Rightarrow x = d$

$\therefore$ $(d, 0, 0)$ is a point on Π_1 and Π_2.

Let, $Q(d, 0, 0)$. $\Rightarrow \vec{AQ} = \vec{OQ} - \vec{OA} = d\vec{i} - (2\vec{i} - 3\vec{j} + \vec{k})$

$$= (d-2)\vec{i} + 3\vec{j} - \vec{k}.$$

$$\vec{AQ} \cdot \vec{n} = \left[(d-2)\vec{i} + 3\vec{j} - \vec{k}\right] \cdot \left[\vec{i} + 2\vec{j} + 3\vec{k}\right]$$

$$\vec{AQ} \cdot \vec{n}. = d - 2 + 6 - 3 = d + 1$$

$$|\vec{n}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}.$$

$\therefore$ Perpendicular distance from Π_1 and Π_2 to A

$$= \left|\frac{\vec{AQ} \cdot \vec{n}}{|\vec{n}|}\right| = \left|\frac{d+1}{\sqrt{14}}\right|$$

Since $\perp$ distance from Π_1 and Π_2 to A is $\sqrt{14}$,

$$\left|\frac{d+1}{\sqrt{14}}\right| = \sqrt{14}$$

$$\frac{d+1}{\sqrt{14}} = \pm\sqrt{14}$$

$$d + 1 = \pm 14$$

$$\Rightarrow d = 13 \text{ or } d = -15$$

$\therefore$ Equation of Π_1: $x + 2y + 3z = 13$

Equation of Π_2: $x + 2y + 3z = -15$.

2 (i) Find the acute angle between the line l whose equation is

$$\mathbf{r} = s(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

and the plane Π_1 whose equation is [3]

$$x - z = 0.$$

(ii) Find, in the form $ax + by + cz = 0$, the equation of the plane Π_2 which contains l and is perpendicular to Π_1. [3]

(iii) Find a vector equation of the line of intersection of the planes Π_1 and Π_2 and hence, or otherwise, show that the vectors $\mathbf{i} - \mathbf{k}$, $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $3\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ are linearly dependent. [3]

(iv) The variable line m passes through the point with position vector $4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$ and is perpendicular to l. The line m meets Π_1 at Q. Find the minimum distance of Q from the origin, as m varies, giving your answer correct to 3 significant figures. [5]

i) $L: \vec{r} = s(2\vec{i}+2\vec{j}+\vec{k}) \Rightarrow$ direction vector, $\vec{b_1} = 2\vec{i}+2\vec{j}+\vec{k}$.

$\Pi_1: x - z = 0 \Rightarrow \vec{r}.(\vec{i}-\vec{k}) = 0 \Rightarrow$ normal vector, $\vec{n_1} = \vec{i}-\vec{k}$.

Let, α be the angle between $\vec{b_1}$ and $\vec{n_1}$.

$\vec{b_1}\cdot\vec{n_1} = |\vec{b_1}||\vec{n_1}|\cos\alpha \Rightarrow (2\vec{i}+2\vec{j}+\vec{k}).(\vec{i}-\vec{k}) = \sqrt{2^2+2^2+1^2}\sqrt{1^2+(-1)^2}\cos\alpha$

$1 = 3\sqrt{2}\cos\alpha$

$\cos\alpha = \dfrac{1}{3\sqrt{2}} \Rightarrow \alpha = 76.36°$

$\therefore$ Acute angle between L and Π_1 : $90° - \alpha = 13.63° \approx 13.6°$.

ii) Π_2 contains L and is $\perp$ to Π_1 $\therefore$ normal vector of Π_2 is $\perp$ to $2\vec{i}+2\vec{j}+\vec{k}$ and $\vec{i}-\vec{k}$.

$$\vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$$

$\therefore$ Equation of Π_2 which contains point $(0,0,0)$:

$\vec{r}.(-2\vec{i}+3\vec{j}-2\vec{k}) = (0\vec{i}+0\vec{j}+0\vec{k}).(-2\vec{i}+3\vec{j}-2\vec{k})$

$-2x + 3y - 2z = 0.$

$\Rightarrow 2x - 3y + 2z = 0.$

iii) Direction vector of line of intersection of Π_1 and Π_2 is $\perp$ to $\vec{n_1}$ and $\vec{n_2}$.

$$\vec{b_2} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ -2 & 3 & -2 \end{vmatrix} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

Observe both planes Π_1 and Π_2 pass through the origin.

$\Rightarrow$ Line of intersection:

$\vec{r} = (0\vec{i}+0\vec{j}+0\vec{k}) + t(3\vec{i}+4\vec{j}+3\vec{k})$

$\Rightarrow \vec{r} = t(3\vec{i}+4\vec{j}+3\vec{k})$.

The vector $2\vec{i}+2\vec{j}+\vec{k}$ is parallel to plane Π_2 since line l is contained in Π_2.
The vector $\vec{i}-\vec{k}$ is perpendicular to plane Π_1 but since Π_2 is perpendicular to Π_1 the vector $\vec{i}-\vec{k}$ is parallel to plane Π_1.
The vector $3\vec{i}+4\vec{j}+3\vec{k}$ is parallel to Π_2 since the line of intersection is contained in Π_2.

Then the three vectors are all parallel to Π_2
$\therefore$ $\vec{i}-\vec{k}$, $2\vec{i}+2\vec{j}+\vec{k}$ and $3\vec{i}+4\vec{j}+3\vec{k}$ are linearly dependent.

iv) Let, Q be a point on Π_1 with coordinates (a,b,c).

Since Q lies on Π_1: $\quad x-z=0$

$$\Rightarrow a-c=0 \Rightarrow a=c.$$

$\therefore$ Q has coordinates (a,b,a).

Since line m passes through $P(4,4,2)$ and $Q(a,b,a)$,

Direction vector of m: $\overrightarrow{PQ}=(a-4)\vec{i}+(b-4)\vec{j}+(a-2)\vec{k}$.

Since $l \perp m$,

$$\overrightarrow{PQ}\cdot(2\vec{i}+2\vec{j}+\vec{k})=0$$

$$2(a-4)+2(b-4)+a-2=0$$

$$2a-8+2b-8+a-2=0$$

$$3a+2b-18=0$$

$$b=\frac{18-3a}{2}.$$

$$\Rightarrow Q=\left(a, \frac{18-3a}{2}, a\right).$$

Distance of Q from origin $=\sqrt{a^2+\left(\frac{18-3a}{2}\right)^2+a^2}=\sqrt{\frac{2a^2+324-108a+9a^2}{4}}$

$$D=\sqrt{\frac{17a^2-108a+324}{4}}$$

For minimum value of d, we set $\frac{d}{da}(D)=0$

$$\frac{1}{2}\left(\frac{17a^2-108a+324}{4}\right)^{-\frac{1}{2}}(34a-108)=0$$

$$\Rightarrow a=\frac{54}{17}.$$

$$\Rightarrow D=\sqrt{\frac{17\left(\frac{54}{17}\right)^2-108\left(\frac{54}{17}\right)+324}{4}}=6.173\approx 6.17$$

3 The points A, B, C have position vectors $a\mathbf{i}$, $b\mathbf{j}$, $c\mathbf{k}$ respectively, where a, b, c are all positive. The plane containing A, B, C is denoted by Π.

(i) Find a vector perpendicular to Π. [3]

(ii) Find the perpendicular distance from the origin to Π, in terms of a, b, c. [3]

i) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = b\vec{j} - a\vec{i} = -a\vec{i} + b\vec{j}$.

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = c\vec{k} - a\vec{i} = -a\vec{i} + c\vec{k}$.

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = \begin{bmatrix} bc \\ ac \\ ab \end{bmatrix}$$

$\therefore$ Vector $\perp$ to $\Pi = (bc)\vec{i} + (ac)\vec{j} + (ab)\vec{k}$.

ii) $\perp$ distance from origin to Π

$$= \left| \frac{\overrightarrow{OA} \cdot \vec{n}}{|\vec{n}|} \right| = \left| \frac{a\vec{i} \cdot \left[(bc)\vec{i} + (ac)\vec{j} + (ab)\vec{k}\right]}{\sqrt{(bc)^2 + (ac)^2 + (ab)^2}} \right| = \frac{abc}{\sqrt{a^2b^2 + a^2c^2 + b^2c^2}}.$$

4 The planes Π_1 and Π_2 have equations

$$x + 2y - 3z + 4 = 0 \qquad \text{and} \qquad 2x + y - 4z - 3 = 0$$

respectively. Show that, for all values of λ, every point which is in both Π_1 and Π_2 is also in the plane

$$x + 2y - 3z + 4 + \lambda(2x + y - 4z - 3) = 0.$$ [2]

The planes Π_1 and Π_2 meet in the line l.

(i) Find the equation of the plane Π_3 which passes through l and the point whose position vector is $a\mathbf{k}$. [3]

(ii) Find the value of a if Π_2 is perpendicular to Π_3. [3]

Let, $P(a,b,c)$ be a point in both plane Π_1 and Π_2.

Then, since P lies in Π_1: $x+2y-3z+4=0 \Rightarrow a+2b-3c+4=0 \rightarrow (1)$

and since P lies in Π_2: $2x+y-4z-3=0 \Rightarrow 2a+b-4c-3=0 \rightarrow (2)$

Let, λ be a scalar. Multiplying (2) by λ gives

$$\lambda(2a+b-4c-3)=0 \rightarrow (3)$$

Adding (1) to (3) gives:

$$a+2b-3c+4+\lambda(2a+b-4c-3)=0$$

Then a point P which lies in Π_1 and Π_2 also lies in the plane:

$$x+2y-3z+4+\lambda(2x+y-4z-3)=0.$$ (Shown).

i) Since Π_1 and Π_2 meet at l, we are looking for a plane which passes through all points contained in both Π_1 and Π_2.

Then equation of Π_3 is of the form $x+2y-3z+4+\lambda(2x+y-4z-3)=0$

Since $(0,0,a)$ lies on Π_3, then

$$0+2(0)-3a+4+\lambda(0+0-4a-3)=0$$

$$4-3a=\lambda(4a+3)$$

$$\lambda=\frac{4-3a}{4a+3}.$$

$\therefore$ Equation of Π_3:

$$x+2y-3z+4+\frac{4-3a}{4a+3}(2x+y-4z-3)=0$$

$$\left(1+\frac{8-6a}{4a+3}\right)x+\left(2+\frac{4-3a}{4a+3}\right)y+\left(-3-4\left(\frac{4-3a}{4a+3}\right)\right)z+4-3\left(\frac{4-3a}{4a+3}\right)=0$$

$$\left(\frac{11-2a}{4a+3}\right)x+\left(\frac{5a+10}{4a+3}\right)y+\left(\frac{-25}{4a+3}\right)z+\left(\frac{25a}{4a+3}\right)=0$$

$$\Rightarrow (11-2a)x+(5a+10)y-25z+25a=0.$$

ii) Since $\Pi_2 \perp \Pi_3$

$$\Rightarrow \quad \overrightarrow{n_2} \cdot \overrightarrow{n_3} = 0$$

$$\left(2\vec{i} + \vec{j} - 4\vec{k}\right) \cdot \left[(11-2a)\vec{i} + (5a+10)\vec{j} - 25\vec{k}\right] = 0$$

$$2(11-2a) + 5a + 10 + 100 = 0$$

$$22 - 4a + 5a + 110 = 0$$

$$a = -132.$$

5 The equation of the plane Π is

$$2x + 3y + 4z = 48.$$

Obtain a vector equation of Π in the form

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c},$$

where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are of the form $p\mathbf{i}$, $q\mathbf{i} + r\mathbf{j}$ and $s\mathbf{i} + t\mathbf{k}$ respectively, and where p, q, r, s, t are integers. [6]

The line l has vector equation $\mathbf{r} = 29\mathbf{i} - 2\mathbf{j} - \mathbf{k} + \theta(5\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})$. Show that l lies in Π. [3]

Find, in the form $ax + by + cz = d$, the equation of the plane which contains l and is perpendicular to Π. [4]

Π: $2x+3y+4z = 48$.

Setting $x=y=0 \Rightarrow 4z = 48 \Rightarrow z = 12. \Rightarrow A(0,0,12)$

Setting $x=z=0 \Rightarrow 3y = 48 \Rightarrow y = 16. \Rightarrow B(0,16,0)$

Setting $y=z=0 \Rightarrow 2x = 48 \Rightarrow x = 24 \Rightarrow C(24,0,0)$.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 16\vec{j} - 12\vec{k} \sim 4\vec{j} - 3\vec{k}$.

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 24\vec{i} - 12\vec{k} \sim 2\vec{i} - \vec{k}$.

$\therefore$ Vector equation of Π:

$$\vec{r} = 12\vec{k} + \lambda(4\vec{j} - 3\vec{k}) + \mu(2\vec{i} - \vec{k}).$$

l: $\vec{r} = 29\vec{i} - 2\vec{j} - \vec{k} + \theta(5\vec{i} - 6\vec{j} + 2\vec{k})$.

$\vec{r} = (29+5\theta)\vec{i} + (-2-6\theta)\vec{j} + (-1+2\theta)\vec{k}$

$\therefore x = 29+5\theta,\ y = -2-6\theta,\ z = -1+2\theta$.

Substituting x, y and z in equation of Π gives:

$$2x + 3y + 4z = 48$$

$$2(29+5\theta) + 3(-2-6\theta) + 4(-1+2\theta) = 48$$

$$58 + 10\theta - 6 - 18\theta - 4 + 8\theta = 48$$

$$48 = 48$$

i.e. equation of Π is satisfied for all values of θ.

Let, Π_2 contains l and is $\perp$ to Π. Then $\vec{n_2}$, the normal vector of Π_2, is $\perp$ to $5\vec{i} - 6\vec{j} + 2\vec{k}$ and $2\vec{i} + 3\vec{j} + 4\vec{k}$.

$$\vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -6 & 2 \\ 2 & 3 & 4 \end{vmatrix} = \begin{bmatrix} -30 \\ -16 \\ 27 \end{bmatrix}$$

Since Π_2 contains the point with position vector $29\vec{i}-2\vec{j}-\vec{k}$,

then
$$\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$$
$$\vec{r}\cdot(-30\vec{i}-16\vec{j}+27\vec{k}) = (29\vec{i}-2\vec{j}-\vec{k})\cdot(-30\vec{i}-16\vec{j}+27\vec{k})$$
$$-30x-16y+27z = -865$$
$$30x+16y-27z = 865.$$

6 The line l_1 passes through the points P and Q whose position vectors are $\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $-2\mathbf{i} + 5\mathbf{j} + 13\mathbf{k}$ respectively. The line l_2 passes through the point S whose position vector is $\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}$ and is parallel to the vector $\mathbf{i} - \mathbf{j} - 3\mathbf{k}$. The point whose position vector is $-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ is on the line l_3, the common perpendicular to l_1 and l_2.

(i) Find, in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, an equation for l_3. [3]

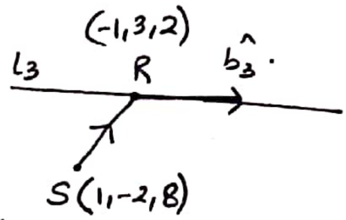
(ii) Find the perpendicular distance from S to l_3. [4]

(iii) Find the perpendicular distance from S to the plane which contains l_3 and passes through P. [4]

i) l_3 is common $\perp$ of l_1 and l_2 $\therefore$ direction vector of l_3, $\vec{b_3}$, is $\perp$ to $\vec{b_1}$ and $\vec{b_2}$, the direction vectors of l_1 and l_2 respectively where $\vec{b_1} = \vec{PQ} = -3\vec{i} + 6\vec{j} + 15\vec{k}$

$$\vec{b_3} = \vec{b_1} \times \vec{b_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 6 & 15 \\ 1 & -1 & -3 \end{vmatrix} = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \sim \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore$ Equation of l_3 : $\vec{r} = -\vec{i} + 3\vec{j} + 2\vec{k} + t(\vec{i} - 2\vec{j} + \vec{k})$.



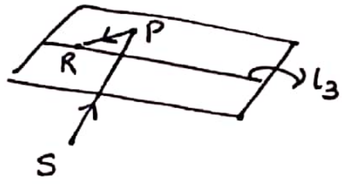
ii) Let, $R = (-1,3,2)$ and $\hat{b_3}$ be unit vector in direction of l_3

$$\hat{b_3} = \frac{1}{\sqrt{1^2+(-2)^2+1^2}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$ and $\vec{SR} = -2\vec{i} + 5\vec{j} - 6\vec{k}$.

$\therefore$ $\perp$ distance from S to l_3

$$= |\vec{SR} \times \hat{b_3}| = \left| \begin{bmatrix} -2 \\ 5 \\ -6 \end{bmatrix} \times \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right| = \frac{1}{\sqrt{6}} \left\| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 5 & -6 \\ 1 & -2 & 1 \end{matrix} \right\|$$

$$= \frac{1}{\sqrt{6}} \left| \begin{bmatrix} -7 \\ -4 \\ -1 \end{bmatrix} \right| = \frac{1}{\sqrt{6}} \sqrt{(-7)^2+(-4)^2+(-1)^2} = \frac{\sqrt{66}}{\sqrt{6}} = \sqrt{11}.$$



iii) We note that vector $\vec{PR}$ is contained in the plane.

$$\vec{PR} = \vec{OR} - \vec{OP} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 4 \end{bmatrix}$$

Then $$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 4 & 4 \\ 1 & -2 & 1 \end{vmatrix} = \begin{bmatrix} 12 \\ 6 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{SP} = \vec{OP} - \vec{OS} = (\vec{i} - \vec{j} - 2\vec{k}) - (\vec{i} - 2\vec{j} + 8\vec{k}) = \vec{j} - 10\vec{k}.$$

$\therefore$ $\perp$ distance from S to plane

$$= \left| \frac{\vec{SP} \cdot \vec{n}}{|\vec{n}|} \right| = \left| \frac{(\vec{j} - 10\vec{k}) \cdot (2\vec{i} + \vec{j})}{\sqrt{2^2+1^2}} \right| = \frac{1}{\sqrt{5}}.$$

7 The position vectors of the points A, B, C, D are

$$7\mathbf{i} + 4\mathbf{j} - \mathbf{k}, \quad 3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}, \quad 2\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}, \quad 2\mathbf{i} + 7\mathbf{j} + \lambda\mathbf{k}$$

respectively. It is given that the shortest distance between the line AB and the line CD is 3.

(i) Show that $\lambda^2 - 5\lambda + 4 = 0$. [7]

(ii) Find the acute angle between the planes through A, B, D corresponding to the values of λ satisfying the equation in part (i). [7]

i) $\overrightarrow{AB} = (3\vec{i}+5\vec{j}-2\vec{k})-(7\vec{i}+4\vec{j}-\vec{k}) = -4\vec{i}+\vec{j}-\vec{k}$.

$\therefore \overrightarrow{AB}: \vec{r} = 7\vec{i}+4\vec{j}-\vec{k} + t(-4\vec{i}+\vec{j}-\vec{k})$.

$\overrightarrow{CD} = (2\vec{i}+7\vec{j}+\lambda\vec{k})-(2\vec{i}+6\vec{j}+3\vec{k}) = \vec{j}+(\lambda-3)\vec{k}$.

$CD: \vec{r} = 2\vec{i}+6\vec{j}+3\vec{k} + s(\vec{j}+(\lambda-3)\vec{k})$.

$\vec{a_2}-\vec{a_1} = \vec{OC}-\vec{OA} = -5\vec{i}+2\vec{j}+4\vec{k}$.

$$\vec{b_1}\times\vec{b_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4 & 1 & -1 \\ 0 & 1 & \lambda-3 \end{vmatrix} = \begin{bmatrix} \lambda-3+1 \\ 4(\lambda-3) \\ -4 \end{bmatrix} = \begin{bmatrix} \lambda-2 \\ 4\lambda-12 \\ -4 \end{bmatrix}.$$

$(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2}) = -5(\lambda-2)+2(4\lambda-12)+4(-4) = -5\lambda+10+8\lambda-24-16$

$= 3\lambda - 30$.

$|\vec{b_1}\times\vec{b_2}| = \sqrt{(\lambda-2)^2+(4\lambda-12)^2+(-4)^2} = \sqrt{\lambda^2-4\lambda+4+16\lambda^2-96\lambda+144+16}$

$= \sqrt{17\lambda^2-100\lambda+164}$.

$\therefore \perp$ distance between line AB and CD $= \left|\dfrac{3\lambda-30}{\sqrt{17\lambda^2-100\lambda+164}}\right|$

$$3^2 = \frac{(3\lambda-30)^2}{17\lambda^2-100\lambda+164}$$

$153\lambda^2-900\lambda+1476 = 9\lambda^2-180\lambda+900$

$144\lambda^2-720\lambda+576 = 0$

$\lambda^2-5\lambda+4 = 0$. (Shown)

ii) $\lambda^2-5\lambda+4=0 \Rightarrow (\lambda-4)(\lambda-1)=0 \Rightarrow \lambda = 4, 1$.

$\lambda = 1 \Rightarrow \vec{OD} = 2\vec{i}+7\vec{j}+\vec{k}$.

Plane ABD: $\overrightarrow{AB} = -4\vec{i}+\vec{j}-\vec{k}$, $\overrightarrow{AD} = (2\vec{i}+7\vec{j}+\vec{k})-(7\vec{i}+4\vec{j}-\vec{k}) = -5\vec{i}+3\vec{j}+2\vec{k}$.

$$\Rightarrow \vec{n}_1 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4 & 1 & -1 \\ -5 & 3 & 2 \end{vmatrix} = \begin{bmatrix} 5 \\ +13 \\ -7 \end{bmatrix}$$

$\lambda = 4 \Rightarrow \overrightarrow{OD} = 2\vec{i} + 7\vec{j} + 4\vec{k}$

$\overrightarrow{AB} = -4\vec{i} + \vec{j} - \vec{k}$, $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = (2\vec{i} + 7\vec{j} + 4\vec{k}) - (7\vec{i} + 4\vec{j} - \vec{k})$

$= -5\vec{i} + 3\vec{j} + 5\vec{k}$.

$\vec{n_2} = \overrightarrow{AB} \times \overrightarrow{AD}$

$$\vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4 & 1 & -1 \\ -5 & 3 & 5 \end{vmatrix} = \begin{bmatrix} 8 \\ +25 \\ -7 \end{bmatrix}$$

Let, angle between $\vec{n_1}$ and $\vec{n_2}$ be α.

$$\vec{n_1} \cdot \vec{n_2} = |\vec{n_1}|\,|\vec{n_2}| \cos\alpha$$

$$(5\vec{i} + 13\vec{j} - 7\vec{k}) \cdot (8\vec{i} + 25\vec{j} - 7\vec{k}) = \sqrt{5^2 + 13^2 + (-7)^2}\,\sqrt{8^2 + 25^2 + (-7)^2} \cos\alpha$$

$$414 = \sqrt{243}\,\sqrt{738} \cos\alpha$$

$$\cos\alpha = \frac{414}{\sqrt{243}\,\sqrt{738}}$$

$$\alpha = 12.14^\circ \simeq 12.1^\circ$$

$\Rightarrow$ Acute Angles between planes $= 12.1^\circ$.

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8 The line l_1 is parallel to the vector $4\mathbf{j} - \mathbf{k}$ and passes through the point A whose position vector is $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$. The variable line l_2 is parallel to the vector $\mathbf{i} - (2\sin t)\mathbf{j}$, where $0 \leqslant t < 2\pi$, and passes through the point B whose position vector is $\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. The points P and Q are on l_1 and l_2, respectively, and PQ is perpendicular to both l_1 and l_2.

(i) Find the length of PQ in terms of t. [5]

(ii) Hence find the values of t for which l_1 and l_2 intersect. [2]

(iii) For the case $t = \frac{1}{4}\pi$, find the perpendicular distance from A to the plane BPQ, giving your answer correct to 3 decimal places. [5]

i) $l_1: \vec{r} = 2\vec{i}+\vec{j}+4\vec{k}+\lambda(4\vec{j}-\vec{k})$

$l_2: \vec{r} = \vec{i}+2\vec{j}+4\vec{k}+\mu\left[\vec{i}-(2\sin t)\vec{j}\right].$

$\vec{a_2}-\vec{a_1} = \vec{OB}-\vec{OA} = (\vec{i}+2\vec{j}+4\vec{k})-(2\vec{i}+\vec{j}+4\vec{k}) = -\vec{i}+\vec{j}$

$$\vec{b_1}\times\vec{b_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 4 & -1 \\ 1 & -2\sin t & 0 \end{vmatrix} = \begin{bmatrix} -2\sin t \\ -1 \\ -4 \end{bmatrix} \sim \begin{bmatrix} 2\sin t \\ 1 \\ 4 \end{bmatrix}.$$

$(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2}) = -(2\sin t)+1 = 1-2\sin t.$

$|\vec{b_1}\times\vec{b_2}| = \sqrt{(2\sin t)^2+1^2+4^2} = \sqrt{4\sin^2 t+17}.$

$$\text{length of } PQ = \left|\frac{1-2\sin t}{\sqrt{4\sin^2 t+17}}\right|$$

ii) For lines to intersect, length of $PQ = 0$

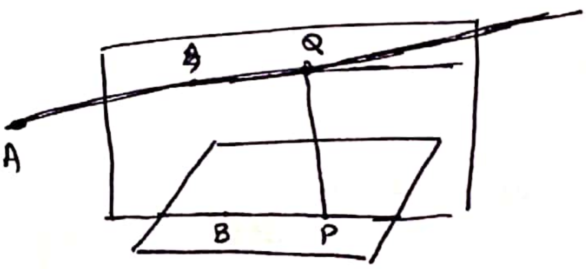
$$\left|\frac{1-2\sin t}{\sqrt{4\sin^2 t+17}}\right| = 0$$

$$1-2\sin t = 0$$

$$\sin t = \frac{1}{2}$$

$$\Rightarrow t = \sin^{-1}\left(\tfrac{1}{2}\right), \pi-\sin^{-1}\left(\tfrac{1}{2}\right)$$

$$t = \frac{\pi}{6}, \frac{5\pi}{6}.$$



iii) normal, $\vec{n}$, of plane BPQ is $\perp$ to $\vec{PQ}$ and direction vector of line l_2.

Note $\vec{PQ}$ is in the same direction as $\vec{b_1}\times\vec{b_2}$

$$= \begin{bmatrix} 2\sin t \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2\sin\left(\frac{\pi}{4}\right) \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 1 \\ 4 \end{bmatrix}$$

$$\vec{b_2} = \begin{bmatrix} 1 \\ -2\sin t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2\sin\left(\frac{\pi}{4}\right) \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 0 \end{bmatrix}.$$

$$\Rightarrow \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \sqrt{2} & 1 & 4 \\ 1 & -\sqrt{2} & 0 \end{vmatrix} = \begin{bmatrix} 4\sqrt{2} \\ 4 \\ -3 \end{bmatrix}.$$

$$\vec{AB} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

$\therefore \perp$ distance from A to plane BPQ

$$= \left| \frac{\vec{AB} \cdot \vec{n}}{|\vec{n}|} \right| = \left| \frac{-1(4\sqrt{2}) + 1(4) + 0(-3)}{\sqrt{(4\sqrt{2})^2 + 4^2 + (-3)^2}} \right| = \left| \frac{4 - 4\sqrt{2}}{\sqrt{57}} \right|$$

$$= 0.2194 \simeq 0.219.$$

9 The lines l_1 and l_2 have vector equations

$$\mathbf{r} = 4\mathbf{i} - 2\mathbf{j} + \lambda(2\mathbf{i} + \mathbf{j} - 4\mathbf{k}) \quad \text{and} \quad \mathbf{r} = 4\mathbf{i} - 5\mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} - \mathbf{j} - \mathbf{k})$$

respectively.

(i) Show that l_1 and l_2 intersect. [3]

(ii) Find the perpendicular distance from the point P whose position vector is $3\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$ to the plane containing l_1 and l_2. [3]

(iii) Find the perpendicular distance from P to l_1. [4]

i) $l_1: \vec{r} = (4+2\lambda)\vec{i} + (-2+\lambda)\vec{j} + (-4\lambda)\vec{k}. \Rightarrow x = 4+2\lambda,\ y = -2+\lambda,\ z = -4\lambda$

$l_2: \vec{r} = (4+\mu)\vec{i} + (-5-\mu)\vec{j} + (2-\mu)\vec{k}. \Rightarrow x = 4+\mu,\ y = -5-\mu,\ z = 2-\mu.$

Setting components equal gives:

$4 + 2\lambda = 4 + \mu \Rightarrow \mu = 2\lambda \rightarrow (1)$

$-2+\lambda = -5-\mu \Rightarrow \mu = -3-\lambda \rightarrow (2)$

$-4\lambda = 2-\mu \Rightarrow \mu = 4\lambda + 2 \rightarrow (3)$

Substitute (1) in (2):

$2\lambda = -3-\lambda \Rightarrow 3\lambda = -3 \Rightarrow \lambda = -1.$

$\therefore \mu = 2(-1) = -2.$

Substituting $\lambda = -1$ and $\mu = -2$ in (3):

$-2 = 4(-1) + 2$

$-2 = -2$ (satisfied).

$\therefore l_1$ and l_2 intersect.

ii) Let, Π be the plane containing l_1 and l_2. Normal vector for Π is $\perp$ to $2\vec{i} + \vec{j} - 4\vec{k}$ and $\vec{i} - \vec{j} - \vec{k}$.

$$\Rightarrow \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -4 \\ 1 & -1 & -1 \end{vmatrix} = \begin{bmatrix} -5 \\ -2 \\ -3 \end{bmatrix} \sim \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}.$$

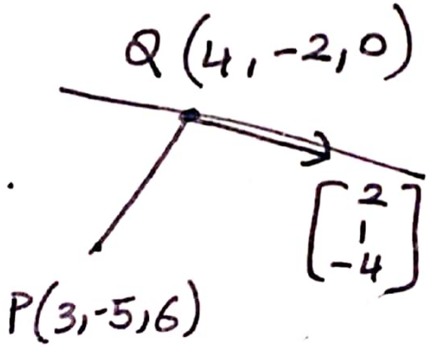
Π contains the point Q with position vector $4\vec{i} - 2\vec{j}$.

$\vec{PQ} = \vec{OQ} - \vec{OP} = (4\vec{i} - 2\vec{j}) - (3\vec{i} - 5\vec{j} + 6\vec{k}) = \vec{i} + 3\vec{j} - 6\vec{k}.$

$\therefore \perp$ distance from P to Π

$$= \left| \frac{\vec{PQ} \cdot \vec{n}}{|\vec{n}|} \right| = \left| \frac{(\vec{i} + 3\vec{j} - 6\vec{k}) \cdot (5\vec{i} + 2\vec{j} + 3\vec{k})}{\sqrt{5^2 + 2^2 + 3^2}} \right| = \left| \frac{-7}{\sqrt{38}} \right| = \frac{7}{\sqrt{38}} = 1.135 \approx 1.14.$$

III) $\overrightarrow{PQ} = \vec{i} + 3\vec{j} - 6\vec{k}$.



Let, $\vec{u}$ be the unit vector in the direction of l_1.

$$\vec{u} = \frac{1}{\sqrt{2^2+1^2+4^2}} \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$$

$$\vec{u} = \frac{1}{\sqrt{21}} \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$$

$\therefore$ $\perp$ distance from P to line l_1

$$= \left| \overrightarrow{PQ} \times \vec{u} \right| = \left| \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} \times \frac{1}{\sqrt{21}} \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} \right|$$

$$= \frac{1}{\sqrt{21}} \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & -6 \\ 2 & 1 & -4 \end{vmatrix} \right\|$$

$$= \frac{1}{\sqrt{21}} \left\| \begin{bmatrix} -6 \\ -8 \\ -5 \end{bmatrix} \right\| = \frac{1}{\sqrt{21}} \sqrt{36+64+25} = \frac{\sqrt{125}}{\sqrt{21}} = 2.439 \simeq 2.44.$$

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10 The line l_1 passes through the point A whose position vector is $3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and is parallel to the vector $\mathbf{i} + \mathbf{j}$. The line l_2 passes through the point B whose position vector is $-\mathbf{i} - \mathbf{k}$ and is parallel to the vector $\mathbf{j} + 2\mathbf{k}$. The point P is on l_1 and the point Q is on l_2 and PQ is perpendicular to both l_1 and l_2.

(i) Find the length of PQ. [4]

(ii) Find the position vector of Q. [5]

(iii) Show that the perpendicular distance from Q to the plane containing AB and the line l_1 is $\sqrt{3}$. [4]

i) $l_1 : \vec{r} = 3\vec{i} + \vec{j} + 2\vec{k} + \lambda(\vec{i} + \vec{j})$

$l_2 : \vec{r} = -\vec{i} - \vec{k} + \mu(\vec{j} + 2\vec{k})$.

$\vec{a_2} - \vec{a_1} = \vec{OB} - \vec{OA} = (-\vec{i} - \vec{k}) - (3\vec{i} + \vec{j} + 2\vec{k}) = -4\vec{i} - \vec{j} - 3\vec{k}$.

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = 2\vec{i} - 2\vec{j} + \vec{k}.$$

$(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = (2\vec{i} - 2\vec{j} + \vec{k}) \cdot (-4\vec{i} - \vec{j} - 3\vec{k}) = -8 + 2 - 3 = -9$.

$|\vec{b_1} \times \vec{b_2}| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$.

$\therefore PQ = \left| \frac{-9}{3} \right| = 3$. i.e. length of $PQ = 3$.

ii) P is on $l_1 \Rightarrow \vec{OP} = \begin{pmatrix} 3+\lambda \\ 1+\lambda \\ 2 \end{pmatrix}$, Q is on $l_2 \Rightarrow \vec{OQ} = \begin{pmatrix} -1 \\ \mu \\ -1+2\mu \end{pmatrix}$.

$$\vec{PQ} = \vec{OQ} - \vec{OP} = \begin{bmatrix} -1 \\ \mu \\ -1+2\mu \end{bmatrix} - \begin{bmatrix} 3+\lambda \\ 1+\lambda \\ 2 \end{bmatrix} = \begin{bmatrix} -4-\lambda \\ -1+\mu-\lambda \\ -3+2\mu \end{bmatrix}.$$

Since $\vec{PQ} \perp l_1 \Rightarrow \vec{PQ} \cdot \vec{i} + \vec{j} = 0$

$-4 - \lambda - 1 + \mu - \lambda = 0$

$\mu = 2\lambda + 5 \rightarrow (1)$

Since $PQ \perp l_2 \Rightarrow \vec{PQ} \cdot \vec{j} + 2\vec{k} = 0$

$-1 + \mu - \lambda + 2(-3 + 2\mu) = 0$

$-1 + \mu - \lambda - 6 + 4\mu = 0$

$\lambda = 5\mu - 7 \rightarrow (2)$

Substituting (1) in (2)

$\Rightarrow \lambda = 5(2\lambda + 5) - 7$

$\lambda = 10\lambda + 18$

$\lambda = -2$

$\Rightarrow \mu = 1$.

$\Rightarrow \vec{OQ} = -\vec{i} + \vec{j} + \vec{k}$.

iii) Plane containing AB and L_1 would have a normal vector $\perp$ to $\overrightarrow{AB}$ and direction vector of L_1.

$$\overrightarrow{AB} = -4\vec{i} - \vec{j} - 3\vec{k}, \quad \vec{b}_1 = \vec{i} + \vec{j}.$$

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4 & -1 & -3 \\ 1 & 1 & 0 \end{vmatrix} = \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix} \sim \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

$$\overrightarrow{QA} = \overrightarrow{OA} - \overrightarrow{OQ} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}.$$

$\therefore \perp$ distance from Q to plane

$$= \left| \frac{\overrightarrow{QA} \cdot \vec{n}}{|\vec{n}|} \right| = \left| \frac{(4\vec{i} + \vec{k}) \cdot (\vec{i} - \vec{j} - \vec{k})}{\sqrt{1^2 + (-1)^2 + (-1)^2}} \right| = \left| \frac{4 - 1}{\sqrt{3}} \right| = \frac{3}{\sqrt{3}} = \sqrt{3}.$$

11 The line l_1 passes through the point with position vector $8\mathbf{i} + 8\mathbf{j} - 7\mathbf{k}$ and is parallel to the vector $4\mathbf{i} + 3\mathbf{j}$. The line l_2 passes through the point with position vector $7\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and is parallel to the vector $4\mathbf{i} - \mathbf{k}$. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. In either order,

(i) show that $PQ = 13$,

(ii) find the position vectors of P and Q. [9]

$l_1: \vec{r} = 8\vec{i} + 8\vec{j} - 7\vec{k} + s(4\vec{i} + 3\vec{j}) = \begin{bmatrix} 8+4s \\ 8+3s \\ -7 \end{bmatrix}.$

Since P lies on l_1, $\vec{OP} = \begin{bmatrix} 8+4s \\ 8+3s \\ -7 \end{bmatrix}.$

$l_2: \vec{r} = 7\vec{i} - 2\vec{j} + 4\vec{k} + t(4\vec{i} - \vec{k}) = \begin{bmatrix} 7+4t \\ -2 \\ 4-t \end{bmatrix}$

Since Q lies on l_2, $\vec{OQ} = \begin{bmatrix} 7+4t \\ -2 \\ 4-t \end{bmatrix}.$

$\vec{PQ} = \vec{OQ} - \vec{OP} = \begin{bmatrix} 7+4t \\ -2 \\ 4-t \end{bmatrix} - \begin{bmatrix} 8+4s \\ 8+3s \\ -7 \end{bmatrix} = \begin{bmatrix} -1+4t-4s \\ -10-3s \\ 11-t \end{bmatrix}$

Since $\vec{PQ} \perp l_1 \therefore \vec{PQ} \cdot (4\vec{i} + 3\vec{j}) = 0$

$4(-1+4t-4s) + 3(-10-3s) = 0$

$-4 + 16t - 16s - 30 - 9s = 0$

$16t - 25s - 34 = 0 \rightarrow (1)$

Since $\vec{PQ} \perp l_2 \therefore \vec{PQ} \cdot (4\vec{i} - \vec{k}) = 0$

$4(-1+4t-4s) - 1(11-t) = 0$

$-4 + 16t - 16s - 11 + t = 0$

$17t - 16s - 15 = 0 \rightarrow (2)$

From (1), $t = \dfrac{25s+34}{16} \rightarrow (3)$

Substituting (3) in (2)

$17\left[\dfrac{25s+34}{16}\right] - 16s - 15 = 0$

$425s + 578 - 256s - 240 = 0$

$169s + 338 = 0$

$\Rightarrow s = -2.$

$\therefore t = \dfrac{25(-2)+34}{16} = -1.$

$$\overrightarrow{PQ} = \begin{bmatrix} -1+4t-4s \\ -10-3s \\ 11-t \end{bmatrix} = \begin{bmatrix} -1-4+8 \\ -10+6 \\ 11+1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 12 \end{bmatrix}$$

$$PQ = |\overrightarrow{PQ}| = \sqrt{3^2+(-4)^2+12^2} = \sqrt{169} = 13 \text{ (Shown)}.$$

$\longleftrightarrow$

ii) $$\overrightarrow{OP} = \begin{bmatrix} 8+4s \\ 8+3s \\ -7 \end{bmatrix} = \begin{bmatrix} 8+4(-2) \\ 8+3(-2) \\ -7 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -7 \end{bmatrix}.$$

$$\overrightarrow{OQ} = \begin{bmatrix} 7+4t \\ -2 \\ 4-t \end{bmatrix} = \begin{bmatrix} 7+4(-1) \\ -2 \\ 4-(-1) \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}.$$

12 The lines l_1 and l_2 have equations

$$l_1:\ \mathbf{r} = 6\mathbf{i} + 5\mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) \quad \text{and} \quad l_2:\ \mathbf{r} = 6\mathbf{i} + 5\mathbf{j} + 4\mathbf{k} + \mu(4\mathbf{i} + 6\mathbf{j} + \mathbf{k}).$$

Find a cartesian equation of the plane Π containing l_1 and l_2. [4]

Find the position vector of the foot of the perpendicular from the point with position vector $\mathbf{i} + 10\mathbf{j} + 3\mathbf{k}$ to Π. [4]

The line l_3 has equation $\mathbf{r} = \mathbf{i} + 10\mathbf{j} + 3\mathbf{k} + \nu(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$. Find the shortest distance between l_1 and l_3. [5]

Π contains l_1 and l_2 $\therefore$ normal vector $\vec{n}$ is $\perp$ to $\vec{i}+\vec{j}+\vec{k}$ and $4\vec{i}+6\vec{j}+\vec{k}$.

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 4 & 6 & 1 \end{vmatrix} = \begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix}.$$

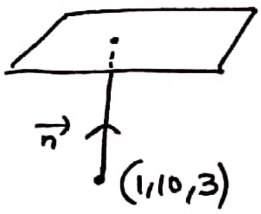
Π contains a point with position vector $6\vec{i}+5\vec{j}+4\vec{k}$.

$$\Rightarrow \vec{r}\cdot(-5\vec{i}+3\vec{j}+2\vec{k}) = (6\vec{i}+5\vec{j}+4\vec{k})\cdot(-5\vec{i}+3\vec{j}+2\vec{k})$$

$$-5x+3y+2z = -7$$

$$\Rightarrow 5x-3y-2z = 7$$

The perpendicular from the point $\vec{i}+10\vec{j}+3\vec{k}$ has the same direction as the normal vector $\vec{n}$ for Π.



$\therefore$ Equation of $\perp$: $\vec{r} = \vec{i}+10\vec{j}+3\vec{k}+t(-5\vec{i}+3\vec{j}+2\vec{k})$

$$\Rightarrow x = 1-5t,\ y = 10+3t,\ z = 3+2t.$$

At foot of perpendicular, equation of plane is satisfied

$$\Rightarrow 5x-3y-2z = 7$$

$$5(1-5t)-3(10+3t)-2(3+2t) = 7$$

$$5-25t-30-9t-6-4t = 7$$

$$-38t = 38$$

$$t = -1.$$

$$\therefore x = 1-5(-1),\ y = 10+3(-1),\ z = 3+2(-1)$$

$$\Rightarrow x = 6,\ y = 7,\ z = 1.$$

$\therefore$ Position vector of foot of perpendicular $= 6\vec{i}+7\vec{j}+\vec{k}$.

$l_1 : \vec{r} = 6\vec{i}+5\vec{j}+4\vec{k}+\lambda(\vec{i}+\vec{j}+\vec{k})$

$l_2 : \vec{r} = \vec{i}+10\vec{j}+3\vec{k}+\nu(2\vec{i}-3\vec{j}+\vec{k})$

$$\vec{a_2}-\vec{a_1} = (\vec{i}+10\vec{j}+3\vec{k}) - (6\vec{i}+5\vec{j}+4\vec{k}) = -5\vec{i}+5\vec{j}-\vec{k}.$$

$$\vec{b_1}\times\vec{b_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & -3 & 1 \end{vmatrix} = \begin{bmatrix} 4 \\ 1 \\ -5 \end{bmatrix} = 4\vec{i}+\vec{j}-5\vec{k}.$$

$$(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2}) = \begin{bmatrix}-5\\5\\-1\end{bmatrix}\cdot\begin{bmatrix}4\\1\\-5\end{bmatrix} = -20+5+5=-10$$

$$|\vec{b_1}\times\vec{b_2}| = \sqrt{4^2+1^2+(-5)^2} = \sqrt{42}.$$

$\Rightarrow$ Shortest distance between l_1 and l_2

$$= \left|\frac{(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})}{|\vec{b_1}\times\vec{b_2}|}\right| = \left|\frac{-10}{\sqrt{42}}\right| = \frac{10}{\sqrt{42}} = 1.543 \approx 1.54.$$

13 The position vectors of the points A, B, C, D are

$$2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}, \quad -2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}, \quad \mathbf{i} + 4\mathbf{j} + \mathbf{k}, \quad \mathbf{i} + 5\mathbf{j} + m\mathbf{k},$$

respectively, where m is an integer. It is given that the shortest distance between the line through A and B and the line through C and D is 3. Show that the only possible value of m is 2. [7]

Find the shortest distance of D from the line through A and C. [3]

Show that the acute angle between the planes ACD and BCD is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$. [4]

$\vec{AB} = \vec{OB} - \vec{OA} = (-2\vec{i} + 5\vec{j} - 4\vec{k}) - (2\vec{i} + 4\vec{j} - 3\vec{k}) = -4\vec{i} + \vec{j} - \vec{k}$.

$AB: \vec{r} = 2\vec{i} + 4\vec{j} - 3\vec{k} + t(-4\vec{i} + \vec{j} - \vec{k})$.

$\vec{CD} = \vec{OD} - \vec{OC} = (\vec{i} + 5\vec{j} + m\vec{k}) - (\vec{i} + 4\vec{j} + \vec{k}) = \vec{j} + (m-1)\vec{k}$.

$CD: \vec{r} = \vec{i} + 4\vec{j} + \vec{k} + s[\vec{j} + (m-1)\vec{k}]$.

$\vec{a_2} - \vec{a_1} = (\vec{i} + 4\vec{j} + \vec{k}) - (2\vec{i} + 4\vec{j} - 3\vec{k}) = -\vec{i} + 4\vec{k}$.

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4 & 1 & -1 \\ 0 & 1 & m-1 \end{vmatrix} = \begin{bmatrix} m-1+1 \\ (-4m+4)\times -1 \\ -4 \end{bmatrix} = \begin{bmatrix} m \\ 4m-4 \\ -4 \end{bmatrix}.$$

$(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = -1(m) + 0(4m-4) + 4(-4) = -m - 16$.

$$|\vec{b_1} \times \vec{b_2}| = \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4 & 1 & -1 \\ 0 & 1 & m-1 \end{vmatrix} \right\| = \sqrt{m^2 + (4m-4)^2 + (-4)^2} = \sqrt{m^2 + 16m^2 - 32m + 16 + 16}$$

$|\vec{b_1} \times \vec{b_2}| = \sqrt{17m^2 - 32m + 32}$

$\therefore$ Shortest distance between lies $= \left| \dfrac{-m-16}{\sqrt{17m^2 - 32m + 32}} \right| = 3$

$$(m+16)^2 = 3^2(17m^2 - 32m + 32)$$

$$\cancel{m^2 + 32m + 256 = 68m^2 - 128m}$$

$$m^2 + 32m + 256 = 9(17m^2 - 32m + 32)$$

$$m^2 + 32m + 256 = 153m^2 - 288m + 288$$

$$0 = 152m^2 - 320m + 32$$

$$0 = 19m^2 - 40m + 4$$

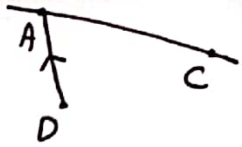
$$0 = 19m^2 - 38m - 2m + 4$$

$$0 = (19m - 2)(m - 2)$$

$$\Rightarrow m = \frac{2}{19} \quad \text{or} \quad m = 2$$

Since m is an integer, ignore $m = \frac{2}{19}$. $\therefore \boxed{m = 2}$

$\overrightarrow{OD} = \vec{i} + 5\vec{j} + 2\vec{k}$.



$\overrightarrow{DA} = \overrightarrow{OA} - \overrightarrow{OD} = (2\vec{i} + 4\vec{j} - 3\vec{k}) - (\vec{i} + 5\vec{j} + 2\vec{k}) = \vec{i} - \vec{j} - 5\vec{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (\vec{i} + 4\vec{j} + \vec{k}) - (2\vec{i} + 4\vec{j} - 3\vec{k}) = -\vec{i} + 4\vec{k}$.

Let, $\vec{u}$ be unit vector in direction of AC $\therefore \vec{u} = \dfrac{1}{\sqrt{(-1)^2 + 4^2}} \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} = \dfrac{1}{\sqrt{17}} \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$.

$\therefore$ Shortest distance from D to AC:

$$= |\overrightarrow{DA} \times \vec{u}| = \left| \begin{bmatrix} 1 \\ -1 \\ -5 \end{bmatrix} \times \frac{1}{\sqrt{17}} \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} \right| = \frac{1}{\sqrt{17}} \left\| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & -5 \\ -1 & 0 & 4 \end{matrix} \right\| = \frac{1}{\sqrt{17}} \left\| \begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix} \right\|$$

$$= \frac{1}{\sqrt{17}} \sqrt{(-4)^2 + 1^2 + (-1)^2} = \frac{\sqrt{18}}{\sqrt{17}} = \sqrt{\frac{18}{17}} = 1.028 \approx 1.03$$

$\longleftrightarrow$

Plane ACD:

$\overrightarrow{OA} = 2\vec{i} + 4\vec{j} - 3\vec{k}$, $\overrightarrow{OC} = \vec{i} + 4\vec{j} + \vec{k}$, $\overrightarrow{OD} = \vec{i} + 5\vec{j} + 2\vec{k}$.

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -\vec{i} + 4\vec{k}$; $\overrightarrow{AD} = -\vec{i} + \vec{j} + 5\vec{k}$.

$$\vec{n_1} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 4 \\ -1 & 1 & 5 \end{vmatrix} = \begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix}$$

Plane BCD:

$\overrightarrow{OB} = -2\vec{i} + 5\vec{j} - 4\vec{k}$, $\overrightarrow{OC} = \vec{i} + 4\vec{j} + \vec{k}$, $\overrightarrow{OD} = \vec{i} + 5\vec{j} + 2\vec{k}$.

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 \\ 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$, $\overrightarrow{BD} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$

$$\vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 5 \\ 3 & 0 & 6 \end{vmatrix} = \begin{bmatrix} -6 \\ -3 \\ 3 \end{bmatrix}$$

Let, α be the angle between $\vec{n_1}$ and $\vec{n_2}$:

$$\vec{n_1} \cdot \vec{n_2} = |\vec{n_1}||\vec{n_2}| \cos\alpha$$

$$\cancel{(3\vec{i} + 6\vec{k}) \cdot (-6\vec{i} - 3\vec{j} + 3\vec{k}) = \sqrt{3^2 + 6^2} \sqrt{(-6)^2 + (-3)^2 + 3^2} \cos\alpha}$$

$$(-4\vec{i} + \vec{j} - \vec{k}) \cdot (-6\vec{i} - 3\vec{j} + 3\vec{k}) = \sqrt{(-4)^2 + 1^2 + (-1)^2} \sqrt{(-6)^2 + (-3)^2 + (3)^2} \cos\alpha$$

$$18 = \sqrt{18}\sqrt{54} \cos\alpha$$

$$\alpha = \cos^{-1}\left(\frac{18}{\sqrt{18}\sqrt{54}}\right)$$

$$\alpha = \cos^{-1}\left(\frac{18}{3\sqrt{2} \times 3\sqrt{2} \times \sqrt{3}}\right)$$

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right). \text{ (Shown)}.$$

14 The plane Π_1 has parametric equation

$$\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} - \mathbf{k}) + \mu(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}).$$

Find a cartesian equation of Π_1. [4]

The plane Π_2 has cartesian equation $3x - 2y - 3z = 4$. Find the acute angle between Π_1 and Π_2. [3]

Find a vector equation of the line of intersection of Π_1 and Π_2. [4]

$$\vec{n_1} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & -1 \\ 1 & 2 & -2 \end{vmatrix} = \begin{bmatrix} 6 \\ 1 \\ 4 \end{bmatrix}$$

Π_1 contains the point with position vector $2\vec{i} - 3\vec{j} + \vec{k}$

$\therefore$ Equation of Π_1: $\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$

$$\vec{r}\cdot(6\vec{i}+\vec{j}+4\vec{k}) = (2\vec{i}-3\vec{j}+\vec{k})\cdot(6\vec{i}+\vec{j}+4\vec{k})$$

$$6x+y+4z = 13.$$

Π_2: $3x - 2y - 3z = 4 \Rightarrow \vec{n_2} = 3\vec{i} - 2\vec{j} - 3\vec{k}$.

Let, α be the acute angle between $\vec{n_1}$ and $\vec{n_2}$.

$$\vec{n_1}\cdot\vec{n_2} = |\vec{n_1}||\vec{n_2}|\cos\alpha$$

$$(6\vec{i}+\vec{j}+4\vec{k})\cdot(3\vec{i}-2\vec{j}-3\vec{k}) = \sqrt{6^2+1^2+4^2}\sqrt{3^2+(-2)^2+(-3)^2}\cos\alpha$$

$$4 = \sqrt{53}\sqrt{22}\cos\alpha$$

$$\alpha = \cos^{-1}\left(\frac{4}{\sqrt{53}\sqrt{22}}\right)$$

$$\alpha = 83.27^\circ \approx 83.3^\circ$$

$\Rightarrow$ Acute angle between Π_1 and $\Pi_2 = 83.3^\circ$.

Π_1: $6x+y+4z = 13 \Rightarrow y = 13 - 6x - 4z \rightarrow (1)$

Π_2: $3x - 2y - 3z = 4 \rightarrow (2)$

Substitute (1) in (2):

$$3x - 2(13 - 6x - 4z) - 3z = 4$$

$$3x - 26 + 12x + 8z - 3z = 4$$

$$15x + 5z = 30$$

$$3x + z = 6$$

$$z = 6 - 3x.$$

Let, $x = t \Rightarrow z = 6 - 3t$.

Substituting in (1):

$$y = 13 - 6(t) - 4(6-3t) = 13 - 6t - 24 + 12t = -11 + 6t.$$

$\Rightarrow$ ~~D~~ Equation of line of intersection of Π_1 and Π_2 :

$$\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ -11+6t \\ 6-3t \end{bmatrix} = \begin{bmatrix} 0 \\ -11 \\ 6 \end{bmatrix} + t \begin{bmatrix} 1 \\ 6 \\ -3 \end{bmatrix}.$$

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15 The points A, B, C and D have coordinates as follows:

$$A(2, 1, -2), \quad B(4, 1, -1), \quad C(3, -2, -1) \quad \text{and} \quad D(3, 6, 2).$$

The plane Π_1 passes through the points A, B and C. Find a cartesian equation of Π_1. [4]

Find the area of triangle ABC and hence, or otherwise, find the volume of the tetrahedron $ABCD$.

[The volume of a tetrahedron is $\frac{1}{3}$ × area of base × perpendicular height.] [6]

The plane Π_2 passes through the points A, B and D. Find the acute angle between Π_1 and Π_2. [4]

Plane ABC (Π_1): $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 2\vec{i} + \vec{k}$.

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \vec{i} - 3\vec{j} + \vec{k}$.

$$\vec{n_1} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 1 \\ 1 & -3 & 1 \end{vmatrix} = \begin{bmatrix} 3 \\ -1 \\ -6 \end{bmatrix}$$

Π_1 passes through A with position vector $2\vec{i} + \vec{j} - 2\vec{k}$.

∴ Equation of Π_1: $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

$$\vec{r} \cdot (3\vec{i} - \vec{j} - 6\vec{k}) = (2\vec{i} + \vec{j} - 2\vec{k}) \cdot (3\vec{i} - \vec{j} - 6\vec{k})$$

$$3x - y - 6z = 17$$

Area of $\Delta ABC = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\sqrt{3^2 + (-1)^2 + (-6)^2} = \frac{\sqrt{46}}{2} = 3.391 \approx 3.39$.

Volume of tetrahedron ABCD where $\overrightarrow{AD} = (3\vec{i} + 6\vec{j} + 2\vec{k}) - (2\vec{i} + \vec{j} - 2\vec{k}) = \vec{i} + 5\vec{j} + 4\vec{k}$.

$$= \frac{1}{6} |(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD}| = \frac{1}{6} \left| \begin{bmatrix} 3 \\ -1 \\ -6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} \right| = \frac{1}{6} |3 - 5 - 24| = \frac{26}{6} = \frac{13}{3}.$$

Π_2 (Plane through A, B and D): $\overrightarrow{AB} = 2\vec{i} + \vec{k}$, $\overrightarrow{AD} = \vec{i} + 5\vec{j} + 4\vec{k}$.

$$\vec{n_2} = \overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 1 \\ 1 & 5 & 4 \end{vmatrix} = \begin{bmatrix} -5 \\ -7 \\ 10 \end{bmatrix}.$$

Let, α be the angle between $\vec{n_1}$ and $\vec{n_2}$:

$$\vec{n_1} \cdot \vec{n_2} = |\vec{n_1}||\vec{n_2}| \cos\alpha$$

$$(3\vec{i} - \vec{j} - 6\vec{k}) \cdot (-5\vec{i} - 7\vec{j} + 10\vec{k}) = \sqrt{3^2 + (-1)^2 + (-6)^2}\sqrt{(-5)^2 + (-7)^2 + 10^2} \cos\alpha$$

$$-68 = \sqrt{46}\sqrt{174} \cos\alpha$$

$$\alpha = \cos^{-1}\left(\frac{-68}{\sqrt{46}\sqrt{174}}\right) = 139.47°$$

⇒ Acute angle between Π_1 and $\Pi_2 = 180° - \alpha = 40.52° \approx 40.5°$.

16 The line l_1 passes through the point A whose position vector is $4\mathbf{i} + 7\mathbf{j} - \mathbf{k}$ and is parallel to the vector $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. The line l_2 passes through the point B whose position vector is $\mathbf{i} + 7\mathbf{j} + 11\mathbf{k}$ and is parallel to the vector $\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$. The points P on l_1 and Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Find the position vectors of P and Q. [8]

Find the shortest distance between the line through A and B and the line through P and Q, giving your answer correct to 3 significant figures. [6]

$l_1: \vec{r} = 4\vec{i}+7\vec{j}-\vec{k}+t(3\vec{i}+2\vec{j}-\vec{k})$.

$l_2: \vec{r} = \vec{i}+7\vec{j}+11\vec{k}+s(\vec{i}-6\vec{j}-2\vec{k})$.

Since P lies on l_1: $\overrightarrow{OP} = \begin{bmatrix} 4+3t \\ 7+2t \\ -1-t \end{bmatrix}$, since Q lies on l_2: $\overrightarrow{OQ} = \begin{bmatrix} 1+s \\ 7-6s \\ 11-2s \end{bmatrix}$.

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{bmatrix} 1+s \\ 7-6s \\ 11-2s \end{bmatrix} - \begin{bmatrix} 4+3t \\ 7+2t \\ -1-t \end{bmatrix} = \begin{bmatrix} -3+s-3t \\ -6s-2t \\ 12-2s+t \end{bmatrix}$.

Since $PQ \perp l_1 \Rightarrow \overrightarrow{PQ} \cdot (3\vec{i}+2\vec{j}-\vec{k}) = 0$

$3(-3+s-3t)+2(-6s-2t)-1(12-2s+t) = 0$

$-9+3s-9t-12s-4t-12+2s-t = 0$

$-21-7s-14t = 0$

$3+s+2t = 0$

$s = -2t-3 \rightarrow (1)$

Since $PQ \perp l_2 \Rightarrow \overrightarrow{PQ} \cdot (\vec{i}-6\vec{j}-2\vec{k}) = 0$

$1(-3+s-3t)-6(-6s-2t)-2(12-2s+t) = 0$

$-3+s-3t+36s+12t-24+4s-2t = 0$

$-27+41s+7t = 0 \rightarrow (2)$

Substitute (1) in (2):

$-27+41(-2t-3)+7t = 0$

$-27-82t-123+7t = 0$

$-75t-150 = 0$

$t = -2$

$\Rightarrow s = -2(-2)-3 = 1$.

$\therefore \overrightarrow{OP} = \begin{bmatrix} 4+3(-2) \\ 7+2(-2) \\ -1-(-2) \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$, $\overrightarrow{OQ} = \begin{bmatrix} 1+1 \\ 7-6 \\ 11-2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 9 \end{bmatrix}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix} - \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 12 \end{bmatrix}$$

$$AB : \vec{r} = \vec{i} + 7\vec{j} + 11\vec{k} + t(-3\vec{i} + 12\vec{k}).$$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{bmatrix} 2 \\ 1 \\ 9 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 8 \end{bmatrix}.$$

$$PQ : \vec{r} = 2\vec{i} + \vec{j} + 9\vec{k} + s(4\vec{i} - 2\vec{j} + 8\vec{k}).$$

$$\vec{a_2} - \vec{a_1} = \begin{bmatrix} 2 \\ 1 \\ 9 \end{bmatrix} - \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ -2 \end{bmatrix}.$$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 0 & 12 \\ 4 & -2 & 8 \end{vmatrix} = \begin{bmatrix} 24 \\ 72 \\ 6 \end{bmatrix} \sim \begin{bmatrix} 4 \\ 12 \\ 1 \end{bmatrix}.$$

$$(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = (\vec{i} - 6\vec{j} - 2\vec{k}) \cdot (4\vec{i} + 12\vec{j} + \vec{k}) = 4 - 72 - 2 = -70.$$

$$|\vec{b_1} \times \vec{b_2}| = \sqrt{4^2 + 12^2 + 1^2} = \sqrt{161}.$$

$\therefore$ Shortest distance between lines AB and PQ

$$= \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right| = \left| \frac{-70}{\sqrt{161}} \right| = \frac{70}{\sqrt{161}} = 5.516 \approx 5.52.$$

17 The line l_1 passes through the points $A(2, 3, -5)$ and $B(8, 7, -13)$. The line l_2 passes through the points $C(-2, 1, 8)$ and $D(3, -1, 4)$. Find the shortest distance between the lines l_1 and l_2. [5]

The plane Π_1 passes through the points A, B and D. The plane Π_2 passes though the points A, C and D. Find the acute angle between Π_1 and Π_2, giving your answer in degrees. [6]

l_1: $\vec{AB} = \vec{OB} - \vec{OA} = (8\vec{i}+7\vec{j}-13\vec{k}) - (2\vec{i}+3\vec{j}-5\vec{k}) = 6\vec{i}+4\vec{j}-8\vec{k}$.

$\vec{r} = 2\vec{i}+3\vec{j}-5\vec{k} + t(6\vec{i}+4\vec{j}-8\vec{k})$.

l_2: $\vec{CD} = \vec{OD} - \vec{OC} = (3\vec{i}-\vec{j}+4\vec{k}) - (-2\vec{i}+\vec{j}+8\vec{k}) = 5\vec{i}-2\vec{j}-4\vec{k}$.

$\vec{r} = -2\vec{i}+\vec{j}+8\vec{k} + s(5\vec{i}-2\vec{j}-4\vec{k})$.

$$\vec{a_2} - \vec{a_1} = \begin{bmatrix}-2\\1\\8\end{bmatrix} - \begin{bmatrix}2\\3\\-5\end{bmatrix} = \begin{bmatrix}-4\\-2\\13\end{bmatrix}.$$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 6 & 4 & -8\\ 5 & -2 & -4\end{vmatrix} = \begin{bmatrix}-32\\-16\\-32\end{bmatrix} \sim \begin{bmatrix}2\\1\\2\end{bmatrix}.$$

$$(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2}) = (-4\vec{i}-2\vec{j}+13\vec{k})\cdot(2\vec{i}+\vec{j}+2\vec{k}) = -8-2+26 = 16.$$

$$|\vec{b_1}\times\vec{b_2}| = \sqrt{2^2+1^2+2^2} = 3.$$

$\therefore$ Shortest distance between lines l_1 and l_2

$$= \left|\frac{(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})}{|\vec{b_1}\times\vec{b_2}|}\right| = \left|\frac{16}{3}\right| = \frac{16}{3} \approx 5.33.$$

Plane ABD: $\vec{AB} = 6\vec{i}+4\vec{j}-8\vec{k}$.
(Π_1) $\vec{AD} = \vec{OD} - \vec{OA} = (3\vec{i}-\vec{j}+4\vec{k}) - (2\vec{i}+3\vec{j}-5\vec{k}) = \vec{i}-4\vec{j}+9\vec{k}$.

$$\vec{n_1} = \vec{AB}\times\vec{AD} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 6 & 4 & -8\\ 1 & -4 & 9\end{vmatrix} = \begin{bmatrix}4\\-62\\-28\end{bmatrix} \sim \begin{bmatrix}-2\\31\\14\end{bmatrix}$$

Plane ACD : $\vec{AC} = \vec{OC} - \vec{OA} = (-2\vec{i}+\vec{j}+8\vec{k}) - (2\vec{i}+3\vec{j}-5\vec{k}) = -4\vec{i}-2\vec{j}+13\vec{k}$.
(Π_2) $\vec{AD} = \vec{OD} - \vec{OA} = (3\vec{i}-\vec{j}+4\vec{k}) - (2\vec{i}+3\vec{j}-5\vec{k}) = \vec{i}-4\vec{j}+9\vec{k}$.

$$\vec{n_2} = \vec{AC}\times\vec{AD} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -4 & -2 & 13\\ 1 & -4 & 9\end{vmatrix} = \begin{bmatrix}34\\49\\18\end{bmatrix}$$

Let, angle between $\vec{n_1}$ and $\vec{n_2}$ be α.

$$\vec{n_1}\cdot\vec{n_2} = |\vec{n_1}||\vec{n_2}|\cos\alpha$$

$$(-2\vec{i}+31\vec{j}+14\vec{k})\cdot(34\vec{i}+49\vec{j}+18\vec{k}) = \sqrt{(-2)^2+31^2+14^2}\sqrt{34^2+49^2+18^2}\cos\alpha$$

$$1703 = \sqrt{1161}\,\sqrt{3881}\cos\alpha$$

$$\alpha = \cos^{-1}\left(\frac{1703}{\sqrt{1161}\,\sqrt{3881}}\right)$$

$$\alpha = 36.65^\circ \approx 36.7^\circ$$

$\therefore$ Acute angle between planes $= 36.7^\circ$

18 With respect to an origin O, the point A has position vector $4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and the plane Π_1 has equation

$$\mathbf{r} = (4 + \lambda + 3\mu)\mathbf{i} + (-2 + 7\lambda + \mu)\mathbf{j} + (2 + \lambda - \mu)\mathbf{k},$$

where λ and μ are real. The point L is such that $\overrightarrow{OL} = 3\overrightarrow{OA}$ and Π_2 is the plane through L which is parallel to Π_1. The point M is such that $\overrightarrow{AM} = 3\overrightarrow{ML}$.

(i) Show that A is in Π_1. [1]

(ii) Find a vector perpendicular to Π_2. [2]

(iii) Find the position vector of the point N in Π_2 such that ON is perpendicular to Π_2. [5]

(iv) Show that the position vector of M is $10\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$ and find the perpendicular distance of M from the line through O and N, giving your answer correct to 3 significant figures. [6]

i) $\Pi_1 : \vec{r} = (4\vec{i} - 2\vec{j} + 2\vec{k}) + \lambda(\vec{i} + 7\vec{j} + \vec{k}) + \mu(3\vec{i} + \vec{j} - \vec{k})$

$\therefore$ Point with position vector $4\vec{i} - 2\vec{j} + 2\vec{k}$ lies in Π_1 $\therefore$ A is in Π_1.

ii) Since $\Pi_1 \parallel \Pi_2$, normal vectors of Π_1 and Π_2 are in the same direction.

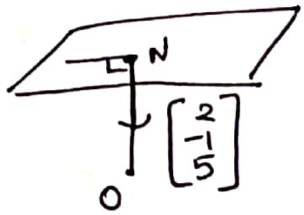
For Π_1, $\vec{n} \perp (\vec{i} + 7\vec{j} + \vec{k})$ and $\vec{n} \perp (3\vec{i} + \vec{j} - \vec{k})$.

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 7 & 1 \\ 3 & 1 & -1 \end{vmatrix} = \begin{bmatrix} -8 \\ 4 \\ -20 \end{bmatrix} \sim \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$$

i.e. vector perpendicular to $\Pi_2 = 2\vec{i} - \vec{j} + 5\vec{k}$.

iii) Equation of Π_2: Note L with position vector $\overrightarrow{OL}$ lies on Π_2.

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$
$$\vec{r} \cdot (2\vec{i} - \vec{j} + 5\vec{k}) = \overrightarrow{OL} \cdot (2\vec{i} - \vec{j} + 5\vec{k})$$
$$2x - y + 5z = 3\,\overrightarrow{OA} \cdot (2\vec{i} - \vec{j} + 5\vec{k})$$
$$2x - y + 5z = 3(4\vec{i} - 2\vec{j} + 2\vec{k}) \cdot (2\vec{i} - \vec{j} + 5\vec{k})$$
$$2x - y + 5z = 3(20)$$
$$2x - y + 5z = 60.$$



Since $\overrightarrow{ON} \perp \Pi_2 \Rightarrow \overrightarrow{ON} = k(\vec{n}) = k(2\vec{i} - \vec{j} + 5\vec{k})$

$\therefore x = 2k, y = -k, z = 5k$

Substituting in equation of plane Π_2 gives:

$$2(2k) - (-k) + 5(5k) = 60$$
$$30k = 60 \Rightarrow k = 2$$

$$\therefore \overrightarrow{ON} = 2(2\vec{i} - \vec{j} + 5\vec{k})$$
$$\overrightarrow{ON} = 4\vec{i} - 2\vec{j} + 10\vec{k}.$$

iv)

Note $\overrightarrow{OA} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$, $\overrightarrow{OL} = 3\begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ -6 \\ 6 \end{bmatrix}$.

$$\overrightarrow{AM} = 3\,\overrightarrow{ML}$$

$$\overrightarrow{OM} - \overrightarrow{OA} = 3\left(\overrightarrow{OL} - \overrightarrow{OM}\right)$$

$$\overrightarrow{OM} - \overrightarrow{OA} = 3\overrightarrow{OL} - 3\overrightarrow{OM}$$

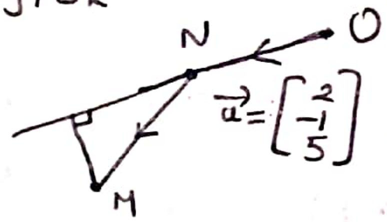
$$4\overrightarrow{OM} = \overrightarrow{OA} + 3\overrightarrow{OL}$$

$$4\overrightarrow{OM} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 12 \\ -6 \\ 6 \end{bmatrix} = \begin{bmatrix} 40 \\ -20 \\ 20 \end{bmatrix}$$

$$\Rightarrow \overrightarrow{OM} = \begin{bmatrix} 10 \\ -5 \\ 5 \end{bmatrix} = 10\vec{i} - 5\vec{j} + 5\vec{k}.$$

line $\overrightarrow{ON} \Rightarrow \vec{r} = t\left(2\vec{i} - \vec{j} + 5\vec{k}\right)$ where $\vec{u} = 2\vec{i} - \vec{j} + 5\vec{k}$.

$$\overrightarrow{NM} = \overrightarrow{OM} - \overrightarrow{ON} = \begin{bmatrix} 10 \\ -5 \\ 5 \end{bmatrix} - \begin{bmatrix} 4 \\ -2 \\ 10 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ -5 \end{bmatrix}$$



$\therefore \perp$ distance from M to line ON

$$= \left| \overrightarrow{NM} \times \hat{u} \right| = \left| \begin{bmatrix} 6 \\ -3 \\ -5 \end{bmatrix} \times \frac{1}{\sqrt{2^2+1^2+5^2}} \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \right|$$

$$= \frac{1}{\sqrt{30}} \left\| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & -3 & -5 \\ 2 & -1 & 5 \end{matrix} \right\| = \frac{1}{\sqrt{30}} \left| \begin{bmatrix} -20 \\ -40 \\ 0 \end{bmatrix} \right|$$

$$= \frac{1}{\sqrt{30}} \sqrt{(-20)^2 + (-40)^2} = \frac{\sqrt{2000}}{\sqrt{30}} = 8.164 \approx 8.16.$$

19 The lines l_1 and l_2 have equations $\mathbf{r} = 8\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j})$ and $\mathbf{r} = 5\mathbf{i} + 3\mathbf{j} - 14\mathbf{k} + \mu(2\mathbf{j} - 3\mathbf{k})$ respectively. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Find the position vector of the point P and the position vector of the point Q. [8]

The points with position vectors $8\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $5\mathbf{i} + 3\mathbf{j} - 14\mathbf{k}$ are denoted by A and B respectively. Find

(i) $\overrightarrow{AP} \times \overrightarrow{AQ}$ and hence the area of the triangle APQ,

(ii) the volume of the tetrahedron $APQB$. (You are given that the volume of a tetrahedron is $\frac{1}{3} \times$ area of base $\times$ perpendicular height.) [6]

l_1: $\vec{r} = (8+\lambda)\vec{i} + (2-2\lambda)\vec{j} + 3\vec{k}$

Since P lies on l_1: $\vec{OP} = \begin{bmatrix} 8+\lambda \\ 2-2\lambda \\ 3 \end{bmatrix}$.

l_2: $\vec{r} = 5\vec{i} + (3+2\mu)\vec{j} + (-14-3\mu)\vec{k}$.

Since Q lies on l_2: $\vec{OQ} = \begin{bmatrix} 5 \\ 3+2\mu \\ -14-3\mu \end{bmatrix}$.

$\vec{PQ} = \vec{OQ} - \vec{OP} = \begin{bmatrix} 5 \\ 3+2\mu \\ -14-3\mu \end{bmatrix} - \begin{bmatrix} 8+\lambda \\ 2-2\lambda \\ 3 \end{bmatrix} = \begin{bmatrix} -3-\lambda \\ 1+2\mu+2\lambda \\ -17-3\mu \end{bmatrix}$.

Since $\vec{PQ} \perp l_1 \Rightarrow \vec{PQ} \cdot (\vec{i} - 2\vec{j}) = 0 \Rightarrow 1(-3-\lambda) - 2(1+2\mu+2\lambda) = 0$

$\Rightarrow -3-\lambda-2-4\mu-4\lambda = 0$

$-5-5\lambda-4\mu = 0$

$5\lambda + 4\mu = -5 \rightarrow (1)$

Since $\vec{PQ} \perp l_2 \Rightarrow \vec{PQ} \cdot (2\vec{j} - 3\vec{k}) = 0 \Rightarrow 2(1+2\mu+2\lambda) - 3(-17-3\mu) = 0$

$2+4\mu+4\lambda+51+9\mu = 0$

$4\lambda + 13\mu = -53 \rightarrow (2)$

From (1), $\lambda = \dfrac{-4\mu-5}{5} \rightarrow (3)$. Substituting (3) in (2) gives

$4\left[\dfrac{-4\mu-5}{5}\right] + 13\mu = -53$

$-16\mu - 20 + 65\mu = -265$

$49\mu = -245 \Rightarrow \mu = -5$

$\therefore \lambda = \dfrac{-4(-5)-5}{5} = 3$.

$\therefore \vec{OP} = \begin{bmatrix} 8+\lambda \\ 2-2\lambda \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \\ 3 \end{bmatrix}$ and $\vec{OQ} = \begin{bmatrix} 5 \\ 3+2\mu \\ -14-3\mu \end{bmatrix} = \begin{bmatrix} 5 \\ 3+2(-5) \\ -14-3(-5) \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \\ 1 \end{bmatrix}$

i) $\vec{AP} = \vec{OP} - \vec{OA} = \begin{bmatrix} 11 \\ -4 \\ 3 \end{bmatrix} - \begin{bmatrix} 8 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 0 \end{bmatrix}$, $\vec{AQ} = \vec{OQ} - \vec{OA} = \begin{bmatrix} 5 \\ -7 \\ 1 \end{bmatrix} - \begin{bmatrix} 8 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -9 \\ -2 \end{bmatrix}$

$$\overrightarrow{AP} \times \overrightarrow{AQ} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -6 & 0 \\ -3 & -9 & -2 \end{vmatrix} = \begin{bmatrix} 12-0 \\ -(-6) \\ -45 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \\ -45 \end{bmatrix}$$

Area of ΔAPQ

$$= \frac{1}{2} \left| \overrightarrow{AP} \times \overrightarrow{AQ} \right| = \frac{1}{2} \sqrt{12^2+6^2+(-45)^2} = \frac{\sqrt{2205}}{2} = 23.47 \approx 23.5$$

ii) $$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{bmatrix} 5 \\ 3 \\ -14 \end{bmatrix} - \begin{bmatrix} 8 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -17 \end{bmatrix}.$$

Volume of tetrahedron

$$= \frac{1}{6} \left| \left(\overrightarrow{AP} \times \overrightarrow{AQ} \right) \cdot \overrightarrow{AB} \right| = \frac{1}{6} \left| \begin{bmatrix} 12 \\ 6 \\ -45 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 1 \\ -17 \end{bmatrix} \right| = \frac{1}{6} \left| 735 \right|$$

$= 122.5$ cubic units.

20 A line, passing through the point A (3, 0, 2), has vector equation $\mathbf{r} = 3\mathbf{i} + 2\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$. It meets the plane Π, which has equation $\mathbf{r}.(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 3$, at the point P. Find the coordinates of P. [3]

Write down a vector $\mathbf{n}$ which is perpendicular to Π, and calculate the vector $\mathbf{w}$, where

$$\mathbf{w} = \mathbf{n} \times (2\mathbf{i} + \mathbf{j} - 2\mathbf{k}).$$ [3]

The point Q lies in Π and is the foot of the perpendicular from A to Π. Use the vector $\mathbf{w}$ to determine an equation of the line PQ in the form $\mathbf{r} = \mathbf{u} + \mu\mathbf{v}$. [4]

Let, line l: $\vec{r} = 3\vec{i}+2\vec{k}+\lambda(2\vec{i}+\vec{j}-2\vec{k}) = \begin{bmatrix} 3+2\lambda \\ \lambda \\ 2-2\lambda \end{bmatrix}$.

$x = 3+2\lambda,\ y = \lambda,\ z = 2-2\lambda$.

plane Π: $\vec{r}.(\vec{i}+2\vec{j}+\vec{k}) = 3$

$\Rightarrow x+2y+z = 3$

Substituting x, y and z from line l into cartesian equation for Π:

$3+2\lambda+2(\lambda)+2-2\lambda = 3$

$5+2\lambda = 3$

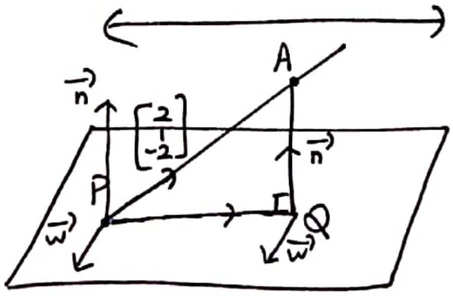
$2\lambda = -2 \Rightarrow \lambda = -1$.

$\therefore\ x = 3+2(-1) = 1,\ y = -1,\ z = 2-2(-1) = 4$

$\therefore P(1, -1, 4)$.

Vector $\vec{n} \perp$ to $\Pi = \vec{i}+2\vec{j}+\vec{k}$.

$\vec{w} = \vec{n} \times (2\vec{i}+\vec{j}-2\vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ 2 & 1 & -2 \end{vmatrix} = \begin{bmatrix} -5 \\ 4 \\ -3 \end{bmatrix} = -5\vec{i}+4\vec{j}-3\vec{k}$.



$\vec{PQ}$ is $\perp$ to both $\vec{n}$ and $\vec{w}$.

$\therefore\ \vec{PQ} = \vec{n} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ -5 & 4 & -3 \end{vmatrix} = \begin{bmatrix} -10 \\ -2 \\ 14 \end{bmatrix} \sim \begin{bmatrix} 5 \\ 1 \\ -7 \end{bmatrix}$.

$\therefore$ Equation of $\vec{PQ}$: $\vec{r} = \vec{OP} + \mu\,\vec{PQ} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + \mu \begin{bmatrix} 5 \\ 1 \\ -7 \end{bmatrix}$.

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21 Find a cartesian equation of the plane Π_1 passing through the points with coordinates (2, −1, 3), (4, 2, −5) and (−1, 3, −2). [4]

The plane Π_2 has cartesian equation $3x - y + 2z = 5$. Find the acute angle between Π_1 and Π_2. [3]

Find a vector equation of the line of intersection of the planes Π_1 and Π_2. [4]

Let, $A(2,-1,3)$, $B(4,2,-5)$ and $C(-1,3,-2)$.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (4\vec{i}+2\vec{j}-5\vec{k}) - (2\vec{i}-\vec{j}+3\vec{k}) = 2\vec{i}+3\vec{j}-8\vec{k}$.

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (-\vec{i}+3\vec{j}-2\vec{k}) - (2\vec{i}-\vec{j}+3\vec{k}) = -3\vec{i}+4\vec{j}-5\vec{k}$.

$$\vec{n_1} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -8 \\ -3 & 4 & -5 \end{vmatrix} = \begin{bmatrix} 17 \\ 34 \\ 17 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$\therefore$ Equation of Π_1 : $\vec{r}\cdot\vec{n_1} = \vec{a}\cdot\vec{n_1}$

$\vec{r}\cdot(\vec{i}+2\vec{j}+\vec{k}) = (2\vec{i}-\vec{j}+3\vec{k})\cdot(\vec{i}+2\vec{j}+\vec{k})$

$x+2y+z=3$.

Π_2: $3x-y+2z=5 \Rightarrow \vec{n_2} = 3\vec{i}-\vec{j}+2\vec{k}$.

Let, θ be the angle between $\vec{n_1}$ and $\vec{n_2}$.

$$\vec{n_1}\cdot\vec{n_2} = |\vec{n_1}||\vec{n_2}|\cos\theta$$

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \sqrt{1^2+2^2+1^2}\sqrt{3^2+(-1)^2+2^2}\cos\theta$$

$$3 = \sqrt{6}\sqrt{14}\cos\theta$$

$$\theta = \cos^{-1}\left(\frac{3}{\sqrt{6}\sqrt{14}}\right) = 70.89^\circ \simeq 70.9^\circ$$

$\therefore$ Acute angle between Π_1 and $\Pi_2 = 70.9^\circ$

Let, $\vec{b}$ be direction vector of line of intersection.

$$\vec{b} = \vec{n_1}\times\vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ 3 & -1 & 2 \end{vmatrix} = \begin{bmatrix} 5 \\ 1 \\ -7 \end{bmatrix}$$

Π_1: $x+2y+z=3$

Π_2: $3x-y+2z=5$

Let, $y=0 \Rightarrow x+z=3 \Rightarrow x=3-z \rightarrow (1)$

and $3x+2z=5 \rightarrow (2)$

Substituting (1) in (2) gives:

$3(3-z)+2z=5 \Rightarrow 9-3z+2z=5$

$-z=-4 \Rightarrow z=4$

$\therefore x=3-4=-1$

i.e. Both planes pass through the point $(-1, 0, 4)$.

$\therefore$ Line of intersection has equation :

$$\vec{r} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} + t \begin{bmatrix} 5 \\ 1 \\ -7 \end{bmatrix} = -\vec{i} + 4\vec{k} + t\left(5\vec{i} + \vec{j} - 7\vec{k}\right).$$

22 The position vectors of the points A, B, C, D are

$$\mathbf{a} = 2\mathbf{i} + \lambda\mathbf{j} - 3\mathbf{k},\ \mathbf{b} = 6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k},\ \mathbf{c} = \mathbf{i} + 2\mathbf{j} - \mathbf{k},\ \mathbf{d} = \mathbf{i} + 7\mathbf{j} + 4\mathbf{k} \text{ respectively.}$$

It is given that the shortest distance between the lines AB and CD is 3.

(i) Show that $\lambda^2 + \lambda - 20 = 0$. [7]

(ii) The planes p_1 and p_2 are the planes through A, B and D corresponding to the two values of λ satisfying the equation in part (i). Find the acute angle between p_1 and p_2. [7]

i) AB: $\vec{AB} = (6\vec{i}+3\vec{j}-2\vec{k}) - (2\vec{i}+\lambda\vec{j}-3\vec{k}) = 4\vec{i}+(3-\lambda)\vec{j}+\vec{k}$.

$\therefore$ Equation of AB: $\vec{r} = 6\vec{i}+3\vec{j}-2\vec{k} + t[4\vec{i}+(3-\lambda)\vec{j}+\vec{k}]$.

CD: $\vec{CD} = (\vec{i}+7\vec{j}+4\vec{k}) - (\vec{i}+2\vec{j}-\vec{k}) = 5\vec{j}+5\vec{k}$.

Equation of CD: $\vec{r} = \vec{i}+2\vec{j}-\vec{k} + s(5\vec{j}+5\vec{k})$

$= \vec{i}+2\vec{j}-\vec{k}+s(\vec{j}+\vec{k})$.

$$\vec{a_2}-\vec{a_1} = \vec{c}-\vec{b} = \begin{bmatrix}1\\2\\-1\end{bmatrix} - \begin{bmatrix}6\\3\\-2\end{bmatrix} = \begin{bmatrix}-5\\-1\\1\end{bmatrix}.$$

$$\vec{b_1}\times\vec{b_2} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 4 & 3-\lambda & 1\\ 0 & 1 & 1\end{vmatrix} = \begin{bmatrix}3-\lambda-1\\-(4-0)\\4-0\end{bmatrix} = \begin{bmatrix}2-\lambda\\-4\\4\end{bmatrix}$$

$$(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2}) = \begin{bmatrix}-5\\-1\\1\end{bmatrix}\cdot\begin{bmatrix}2-\lambda\\-4\\4\end{bmatrix} = -10+5\lambda+4+4 = 5\lambda-2.$$

$$|\vec{b_1}\times\vec{b_2}| = \sqrt{(2-\lambda)^2+(-4)^2+4^2} = \sqrt{4-4\lambda+\lambda^2+16+16} = \sqrt{\lambda^2-4\lambda+36}.$$

$\perp$ distance between lines AB and CD $= \left|\dfrac{(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})}{|\vec{b_1}\times\vec{b_2}|}\right|$

$$3 = \left|\frac{5\lambda-2}{\sqrt{\lambda^2-4\lambda+36}}\right|$$

$$9 = \frac{(5\lambda-2)^2}{\lambda^2-4\lambda+36}$$

$$9\lambda^2-36\lambda+324 = 25\lambda^2-20\lambda+4$$

$$0 = 16\lambda^2+16\lambda-320$$

$$0 = \lambda^2+\lambda-20$$

(Shown)

ii) $\lambda^2+\lambda-20=0 \Rightarrow (\lambda+5)(\lambda-4)=0 \Rightarrow \lambda=-5,4$.

Case 1: $\lambda=-5$

$\Rightarrow \vec{a} = 2\vec{i}-5\vec{j}-3\vec{k},\ \vec{b} = 6\vec{i}+3\vec{j}-2\vec{k},\ \vec{d} = \vec{i}+7\vec{j}+4\vec{k}$.

$\vec{AB} = 4\vec{i}+8\vec{j}+\vec{k},\ \vec{AD} = -\vec{i}+12\vec{j}+7\vec{k}$.

$$\vec{n_1} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 8 & 1 \\ -1 & 12 & 7 \end{vmatrix} = \begin{bmatrix} 44 \\ -29 \\ 56 \end{bmatrix}$$ i.e. normal to plane $P_1 = 44\vec{i} - 29\vec{j} + 56\vec{k}$.

Case 2:

$\lambda = 4 \Rightarrow \vec{a} = 2\vec{i} + 4\vec{j} - 3\vec{k}$, $\vec{b} = 6\vec{i} + 3\vec{j} - 2\vec{k}$, $\vec{d} = \vec{i} + 7\vec{j} + 4\vec{k}$.

$\overrightarrow{AB} = \vec{b} - \vec{a} = 4\vec{i} - \vec{j} + \vec{k}$, $\overrightarrow{AD} = \vec{d} - \vec{a} = -\vec{i} + 3\vec{j} + 7\vec{k}$.

$$\therefore \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -1 & 1 \\ -1 & 3 & 7 \end{vmatrix} = \begin{bmatrix} -10 \\ -29 \\ 11 \end{bmatrix}$$

Let, the angle between $\vec{n_1}$ and $\vec{n_2}$ be θ.

$$\vec{n_1} \cdot \vec{n_2} = |\vec{n_1}||\vec{n_2}| \cos\theta$$

$$\begin{bmatrix} 44 \\ -29 \\ 56 \end{bmatrix} \cdot \begin{bmatrix} -10 \\ -29 \\ 11 \end{bmatrix} = \sqrt{44^2 + (-29)^2 + 56^2}\sqrt{(-10)^2 + (-29)^2 + 11^2} \cos\theta$$

$$1017 = (9\sqrt{73})(3\sqrt{118}) \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{1017}{27\sqrt{73}\sqrt{118}}\right)$$

$$\theta = 66.05^\circ \approx 66.1^\circ$$

$\therefore$ acute Angle between P_1 and $P_2 = 66.1^\circ$

23 The position vectors of the points A, B, C, D are

$$\mathbf{i}+\mathbf{j}+3\mathbf{k}, \quad 3\mathbf{i}-\mathbf{j}+5\mathbf{k}, \quad 3\mathbf{i}-\mathbf{j}+\mathbf{k}, \quad 5\mathbf{i}-5\mathbf{j}+\alpha\mathbf{k},$$

respectively, where α is a positive integer. It is given that the shortest distance between the line AB and the line CD is equal to $2\sqrt{2}$.

(i) Show that the possible values of α are 3 and 5. [7]

$$\vec{AB} = \vec{OB} - \vec{OA} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$$

$$\therefore \text{line } AB: \vec{r} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + s\begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} \text{ where } \vec{a_1} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \vec{b_1} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}.$$

$$\vec{CD} = \vec{OD} - \vec{OC} = \begin{bmatrix} 5 \\ -5 \\ \alpha \end{bmatrix} - \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ \alpha-1 \end{bmatrix}.$$

$$\therefore \text{line } CD: \vec{r} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 2 \\ -4 \\ \alpha-1 \end{bmatrix} \text{ where } \vec{a_2} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \vec{b_2} = \begin{bmatrix} 2 \\ -4 \\ \alpha-1 \end{bmatrix}.$$

$$\vec{a_2} - \vec{a_1} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}.$$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 2 \\ 2 & -4 & \alpha-1 \end{vmatrix} = \begin{bmatrix} -2\alpha+2+8 \\ -[(2\alpha-2)-4] \\ -8+4 \end{bmatrix} = \begin{bmatrix} 10-2\alpha \\ 6-2\alpha \\ -4 \end{bmatrix} \sim \begin{bmatrix} \alpha-5 \\ \alpha-3 \\ 2 \end{bmatrix}.$$

$$(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b}) = \begin{bmatrix} \alpha-5 \\ \alpha-3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} = 2\alpha - 10 - 2\alpha + 6 - 4 = -8.$$

$$\perp \text{ distance between } AB \text{ and } CD = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

$$2\sqrt{2} = \left| \frac{-8}{\sqrt{(\alpha-5)^2 + (\alpha-3)^2 + 2^2}} \right|$$

$$2\sqrt{2} = \left| \frac{\cancel{8} \; 8}{\sqrt{\alpha^2 - 10\alpha + 25 + \alpha^2 - 6\alpha + 9 + 4}} \right|$$

$$2\sqrt{2} = \left| \frac{8}{\sqrt{2\alpha^2 - 16\alpha + 38}} \right|$$

$$(2\sqrt{2})^2 = \frac{64}{2\alpha^2 - 16\alpha + 38}$$

$$8(2\alpha^2 - 16\alpha + 38) = 64$$

$$\alpha^2 - 8\alpha + 19 = 4$$

$$\alpha^2 - 8\alpha + 15 = 0$$

$$(\alpha - 3)(\alpha - 5) = 0$$

$$\Rightarrow \alpha = 3 \text{ or } \alpha = 5. \text{ (shown)}.$$

(ii) Using $\alpha = 3$, find the shortest distance of the point D from the line AC, giving your answer correct to 3 significant figures. [3]

line AC: $\vec{AC} = \vec{OC} - \vec{OA} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}$.

$$\hat{AC} = \frac{1}{\sqrt{2^2+(-2)^2+(-2)^2}}(2\vec{i}-2\vec{j}-2\vec{k}) = \frac{1}{2\sqrt{3}}(2\vec{i}-2\vec{j}-2\vec{k}) = \frac{\vec{i}-\vec{j}-\vec{k}}{\sqrt{3}}.$$

$$\vec{AD} = \vec{OD} - \vec{OA} = \begin{bmatrix} 5 \\ -5 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 0 \end{bmatrix}.$$

Shortest distance from D to line AC

$$= |\hat{AC} \times \vec{AD}| = \left| \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 4 \\ -6 \\ 0 \end{bmatrix} \right| = \frac{1}{\sqrt{3}} \left\| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & -1 \\ 4 & -6 & 0 \end{matrix} \right\|$$

$$= \frac{1}{\sqrt{3}} \left\| \begin{bmatrix} -6 \\ -4 \\ -2 \end{bmatrix} \right\| = \frac{1}{\sqrt{3}} \sqrt{(-6)^2+(-4)^2+(-2)^2} = \frac{\sqrt{56}}{\sqrt{3}} = 4.320 \approx 4.32$$

(iii) Using $\alpha = 3$, find the acute angle between the planes ABC and ABD, giving your answer in degrees. [4]

ABC: $\overrightarrow{AB} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$, $\overrightarrow{AC} = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}$.

$$\therefore \vec{n_1} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 2 \\ 2 & -2 & -2 \end{vmatrix} = \begin{bmatrix} 8 \\ 8 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

ABD: $\overrightarrow{AB} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$, $\overrightarrow{AD} = \begin{bmatrix} 4 \\ -6 \\ 0 \end{bmatrix}$.

$$\overrightarrow{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 2 \\ 4 & -6 & 0 \end{vmatrix} = \begin{bmatrix} 12 \\ 8 \\ -4 \end{bmatrix} \sim \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

Let, θ be the angle between $\vec{n_1}$ and $\overrightarrow{n_2}$.

$$\overrightarrow{n_1} \cdot \overrightarrow{n_2} = |\vec{n_1}||\vec{n_2}| \cos\theta$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \sqrt{1^2+1^2+0^2} \cdot \sqrt{3^2+2^2+(-1)^2} \cos\theta$$

$$5 = \sqrt{2}\sqrt{14} \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{5}{\sqrt{2}\sqrt{14}}\right)$$

$$\theta = 19.10^\circ \simeq 19.1^\circ$$

i.e. acute angle between planes ABC and ABD $= 19.1^\circ$

24 The plane Π_1 passes through the points $(1, 2, 1)$ and $(5, -2, 9)$ and is parallel to the vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

(i) Find the cartesian equation of Π_1. [4]

Let, $A(1,2,1)$ and $B(5,-2,9)$.

$\Rightarrow \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (5\vec{i}-2\vec{j}+9\vec{k}) - (\vec{i}+2\vec{j}+\vec{k}) = 4\vec{i}-4\vec{j}+8\vec{k}$.

Π_1 contains $\overrightarrow{AB}$ and is // to $\vec{i}+2\vec{j}+3\vec{k}$

$$\therefore \vec{n}_1 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -4 & 8 \\ 1 & 2 & 3 \end{vmatrix} = \begin{bmatrix} -28 \\ -4 \\ 12 \end{bmatrix} \sim \begin{bmatrix} 7 \\ 1 \\ -3 \end{bmatrix} = 7\vec{i}+\vec{j}-3\vec{k}.$$

Π_1 contains the point with position vector $\vec{i}+2\vec{j}+\vec{k}$.

$\therefore$ Equation of Π_1: $\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$

$\vec{r}\cdot(7\vec{i}+\vec{j}-3\vec{k}) = (\vec{i}+2\vec{j}+\vec{k})\cdot(7\vec{i}+\vec{j}-3\vec{k})$

$7x+y-3z = 6$.

The plane Π_2 contains the lines

$$\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \quad \text{and} \quad \mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} + \mu(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}).$$

(ii) Find the cartesian equation of Π_2. [4]

Π_2 contains vectors $\vec{i}-2\vec{j}-\vec{k}$ and $2\vec{i}+3\vec{j}-\vec{k}$.

$$\Rightarrow \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & -1 \\ 2 & 3 & -1 \end{vmatrix} = \begin{bmatrix} 5 \\ -1 \\ 7 \end{bmatrix}$$

Π_2 contains the point with position vector $2\vec{i}-3\vec{j}+\vec{k}$.

$\therefore$ Equation of Π_2: $\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$

$\Rightarrow \vec{r}\cdot(5\vec{i}-\vec{j}+7\vec{k}) = (2\vec{i}-3\vec{j}+\vec{k})\cdot(5\vec{i}-\vec{j}+7\vec{k})$

$5x-y+7z = 20$.

(iii) Find the acute angle between Π_1 and Π_2. [3]

$\Pi_1 : 7x + y - 3z = 6 \Rightarrow \vec{n_1} = 7\vec{i} + \vec{j} - 3\vec{k}$

$\Pi_2 : 5x - y + 7z = 20 \Rightarrow \vec{n_2} = 5\vec{i} - \vec{j} + 7\vec{k}$.

Let, θ be the angle between $\vec{n_1}$ and $\vec{n_2}$.

$$\vec{n_1} \cdot \vec{n_2} = |\vec{n_1}||\vec{n_2}| \cos\theta$$

$$\begin{bmatrix} 7 \\ 1 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -1 \\ 7 \end{bmatrix} = \sqrt{7^2+1^2+(-3)^2}\,\sqrt{5^2+(-1)^2+7^2} \cos\theta$$

$$13 = \sqrt{59}\sqrt{75} \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{13}{\sqrt{59}\sqrt{75}}\right)$$

$$\theta = 78.73^\circ \approx 78.3^\circ$$

$\therefore$ Angle between Π_1 and $\Pi_2 = 78.3^\circ$.

25 The line l_1 is parallel to the vector $a\mathbf{i} - \mathbf{j} + \mathbf{k}$, where a is a constant, and passes through the point whose position vector is $9\mathbf{j} + 2\mathbf{k}$. The line l_2 is parallel to the vector $-a\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and passes through the point whose position vector is $-6\mathbf{i} - 5\mathbf{j} + 10\mathbf{k}$.

(i) It is given that l_1 and l_2 intersect.

(a) Show that $a = -\frac{6}{13}$. [3]

$$l_1: \vec{r} = 9\vec{j} + 2\vec{k} + s(a\vec{i} - \vec{j} + \vec{k}) = \begin{bmatrix} sa \\ 9-s \\ 2+s \end{bmatrix}$$

$$l_2: \vec{r} = -6\vec{i} - 5\vec{j} + 10\vec{k} + t(-a\vec{i} + 2\vec{j} + 4\vec{k}) = \begin{bmatrix} -6-at \\ -5+2t \\ 10+4t \end{bmatrix}.$$

When lines l_1 and l_2 intersect,

$$sa = -6 - at \rightarrow (1)$$

$$9 - s = -5 + 2t \Rightarrow s = 14 - 2t \rightarrow (2)$$

$$2 + s = 10 + 4t \Rightarrow s = 4t + 8 \rightarrow (3)$$

Substituting (3) in (2):

$$4t + 8 = 14 - 2t$$

$$6t = 6$$

$$t = 1$$

$$\Rightarrow s = 4(1) + 8 = 12.$$

Substituting $s = 12, t = 1$ in (1) gives

$$12a = -6 - a(1)$$

$$13a = -6$$

$$a = \frac{-6}{13}.$$

(Shown)

(b) Find a cartesian equation of the plane containing l_1 and l_2. [4]

Plane containing l_1 and l_2 is $\perp$ to vectors $\frac{-6}{13}\vec{i}-\vec{j}+\vec{k}$ and $\frac{6}{13}\vec{i}+2\vec{j}+4\vec{k}$.

$\sim -6\vec{i}-13\vec{j}+13\vec{k}$ $\quad \sim 6\vec{i}+26\vec{j}+52\vec{k}$.

$$\therefore \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -6 & -13 & 13 \\ 6 & 26 & 52 \end{vmatrix} = \begin{bmatrix} -1014 \\ 390 \\ -78 \end{bmatrix} \sim \begin{bmatrix} 13 \\ -5 \\ 1 \end{bmatrix}$$

Plane contains the point with position vector $9\vec{j}+2\vec{k}$.

$\therefore$ Equation of plane: $\quad \vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$

$$\vec{r}\cdot(\cancel{19\vec{j}}\ 13\vec{i}-5\vec{j}+\vec{k}) = (9\vec{j}+2\vec{k})\cdot(13\vec{i}-5\vec{j}+\vec{k})$$

$$13x-5y+z=-43$$

(ii) Given instead that the perpendicular distance between l_1 and l_2 is $3\sqrt{(30)}$, find the value of a. [5]

$l_1 : \vec{r} = \vec{a_1} + s\vec{b_1} = 9\vec{j} + 2\vec{k} + s(a\vec{i} - \vec{j} + \vec{k})$.

$l_2 : \vec{r} = \vec{a_2} + t\vec{b_2} = -6\vec{i} - 5\vec{j} + 10\vec{k} + t(-a\vec{i} + 2\vec{j} + 4\vec{k})$.

$$\vec{a_2} - \vec{a_1} = \begin{bmatrix} -6 \\ -5 \\ 10 \end{bmatrix} - \begin{bmatrix} 0 \\ 9 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ -14 \\ 8 \end{bmatrix}.$$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & -1 & 1 \\ -a & 2 & 4 \end{vmatrix} = \begin{bmatrix} -6 \\ -5a \\ a \end{bmatrix}.$$

$$(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = \begin{bmatrix} -6 \\ -14 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} -6 \\ -5a \\ a \end{bmatrix} = 36 + 70a + 8a = 36 + 78a.$$

$$|\vec{b_1} \times \vec{b_2}| = \sqrt{(-6)^2 + (-5a)^2 + a^2} = \sqrt{36 + 26a^2}.$$

$$\perp \text{ distance between } l_1 \text{ and } l_2 = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

$$\Rightarrow \quad 3\sqrt{30} = \left| \frac{36 + 78a}{\sqrt{36 + 26a^2}} \right|$$

$$(3\sqrt{30})^2 = \frac{(36 + 78a)^2}{36 + 26a^2}$$

$$(36 + 26a^2)(270) = 1296 + 5616a + 6084a^2$$

$$9720 + 7020a^2 = 6084a^2 + 5616a + 1296$$

$$936a^2 - 5616a + 8424 = 0$$

$$a^2 - 6a + 9 = 0$$

$$(a - 3)^2 = 0$$

$$\Rightarrow \quad a = 3.$$

26 The lines l_1 and l_2 have vector equations

$$\mathbf{r} = a\mathbf{i} + 9\mathbf{j} + 13\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \quad \text{and} \quad \mathbf{r} = -3\mathbf{i} + 7\mathbf{j} - 2\mathbf{k} + \mu(-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$$

respectively. It is given that l_1 and l_2 intersect.

(i) Find the value of the constant a. [3]

$$l_1 : \vec{r} = \begin{bmatrix} a+\lambda \\ 9+2\lambda \\ 13+3\lambda \end{bmatrix}, \quad l_2 : \vec{r} = \begin{bmatrix} -3-\mu \\ 7+2\mu \\ -2-3\mu \end{bmatrix}$$

Comparing components at the point of intersection,

$$a + \lambda = -3 - \mu \rightarrow (1)$$

$$9 + 2\lambda = 7 + 2\mu \Rightarrow 2\lambda = 2\mu - 2 \Rightarrow \lambda = \mu - 1 \rightarrow (2)$$

$$13 + 3\lambda = -2 - 3\mu \Rightarrow 3\lambda = -3\mu - 15 \Rightarrow \lambda = -\mu - 5 \rightarrow (3)$$

Substituting (2) in (3):

$$\mu - 1 = -\mu - 5$$

$$2\mu = -4 \Rightarrow \mu = -2$$

$$\therefore \lambda = -2 - 1 = -3.$$

Substituting in (1) gives:

$$a + (-3) = -3 - (-2)$$

$$a = 2.$$

The point P has position vector $3\mathbf{i} + \mathbf{j} + 6\mathbf{k}$.

(ii) Find the perpendicular distance from P to the plane containing l_1 and l_2. [4]

Plane containing l_1 and l_2 is $\perp$ to $\vec{i} + 2\vec{j} + 3\vec{k}$ and $-\vec{i} + 2\vec{j} - 3\vec{k}$.

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ -1 & 2 & -3 \end{vmatrix} = \begin{bmatrix} -12 \\ 0 \\ 4 \end{bmatrix} = -12\vec{i} + 4\vec{k}.$$

Let, $Q = (-3, 7, -2)$ be a point on the plane.

Then $$\vec{PQ} = \vec{OQ} - \vec{OP} = \begin{bmatrix} -3 \\ 7 \\ -2 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \\ -8 \end{bmatrix}.$$

$$\vec{PQ} \cdot \vec{n} = (-6\vec{i} + 6\vec{j} - 8\vec{k}) \cdot (-12\vec{i} + 4\vec{k}) = 72 - 32 = 40.$$

$$|\vec{n}| = \sqrt{(-12)^2 + 4^2} = \sqrt{160}.$$

$\perp$ distance from P to plane containing l_1 and l_2

$$= \left| \frac{\vec{PQ} \cdot \vec{n}}{|\vec{n}|} \right| = \left| \frac{40}{\sqrt{160}} \right| = \sqrt{10} \approx 3.16$$

(iii) Find the perpendicular distance from P to l_2. [4]

Let, $\overrightarrow{OA} = -3\vec{i} + 7\vec{j} - 2\vec{k}$ where A is a point on l_2.

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (3\vec{i} + \vec{j} + 6\vec{k}) - (-3\vec{i} + 7\vec{j} - 2\vec{k}) = 6\vec{i} - 6\vec{j} + 8\vec{k}.$$

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{(-\vec{i} + 2\vec{j} - 3\vec{k})}{\sqrt{(-1)^2 + 2^2 + (-3)^2}} = \frac{-\vec{i} + 2\vec{j} - 3\vec{k}}{\sqrt{14}}.$$

$\perp$ distance from P to l_2

$$= |\overrightarrow{AP} \times \hat{u}| = \left| \begin{bmatrix} 6 \\ -6 \\ 8 \end{bmatrix} \times \frac{1}{\sqrt{14}} \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} \right| = \frac{1}{\sqrt{14}} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & -6 & 8 \\ -1 & 2 & -3 \end{vmatrix}$$

$$= \frac{1}{\sqrt{14}} \left| \begin{bmatrix} 2 \\ 10 \\ 6 \end{bmatrix} \right| = \frac{1}{\sqrt{14}} \sqrt{2^2 + 10^2 + 6^2} = \frac{\sqrt{140}}{\sqrt{14}} = \sqrt{10} \approx 3.16$$

27 The lines l_1 and l_2 have equations $\mathbf{r} = 6\mathbf{i} + 2\mathbf{j} + 7\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j})$ and $\mathbf{r} = 4\mathbf{i} + 4\mathbf{j} + \mu(-6\mathbf{j} + \mathbf{k})$ respectively. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Find the position vectors of P and Q. [8]

$l_1 : \vec{r} = (6+\lambda)\vec{i} + (2+\lambda)\vec{j} + 7\vec{k}.$

Since P lies on l_1, $\overrightarrow{OP} = \begin{bmatrix} 6+\lambda \\ 2+\lambda \\ 7 \end{bmatrix}.$

$l_2 : \vec{r} = (4-6\mu)\vec{j} + (4)\vec{i} + (\mu)\vec{k}.$

Since Q lies on l_2, $\overrightarrow{OQ} = \begin{bmatrix} 4 \cancel{-6\mu} \\ 4-6\mu \\ \mu \end{bmatrix}.$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{bmatrix} 4 \cancel{-6\mu} \\ 4-6\mu \\ \mu \end{bmatrix} - \begin{bmatrix} 6+\lambda \\ 2+\lambda \\ 7 \end{bmatrix} = \begin{bmatrix} -2 \cancel{-6\mu} - \lambda \\ -6\mu+2-\lambda \\ \mu - 7 \end{bmatrix}.$$

Since $\overrightarrow{PQ} \perp l_1 \therefore \overrightarrow{PQ} \cdot (\vec{i}+\vec{j}) = 0$

$\Rightarrow -2-6\mu-\lambda+2-\lambda = 0$

$-6\mu - 2\lambda = 0 \Rightarrow 3\mu + \lambda = 0 \Rightarrow \lambda = -3\mu \rightarrow (1)$

Since $\overrightarrow{PQ} \perp l_2 \therefore \overrightarrow{PQ} \cdot (-6\vec{j}+\vec{k}) = 0$

-6μ

$-6(2-\lambda) + 1(\mu-7) = 0$

$36\mu - 12 + 6\lambda + \mu - 7 = 0$

$37\mu = 19 - 6\lambda \rightarrow (2)$

Substituting (1) in (2) gives:

$37\mu = 19 - 6(-3\mu)$

$37\mu = 19 + 18\mu$

$19\mu = 19$

$\mu = 1$

$\Rightarrow \lambda = -3(1) = -3.$

$$\therefore \overrightarrow{OP} = \begin{bmatrix} 6+\lambda \\ 2+\lambda \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 7 \end{bmatrix} \text{ and } \overrightarrow{OQ} = \begin{bmatrix} 4 \\ 4-6\mu \\ \mu \end{bmatrix} = \begin{bmatrix} 4 \\ 4-6(1) \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

28 The line l_1 passes through the points $A(-3, 1, 4)$ and $B(-1, 5, 9)$. The line l_2 passes through the points $C(-2, 6, 5)$ and $D(-1, 7, 5)$.

(i) Find the shortest distance between the lines l_1 and l_2. [5]

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (-\vec{i}+5\vec{j}+9\vec{k}) - (-3\vec{i}+\vec{j}+4\vec{k}) = 2\vec{i}+4\vec{j}+5\vec{k}$.

$l_1 : \vec{r} = \overrightarrow{OA} + t\,\overrightarrow{AB} = (-3\vec{i}+\vec{j}+4\vec{k}) + t(2\vec{i}+4\vec{j}+5\vec{k})$.

$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = (-\vec{i}+7\vec{j}+5\vec{k}) - (-2\vec{i}+6\vec{j}+5\vec{k}) = \vec{i}+\vec{j}$

$l_2 : \vec{r} = \overrightarrow{OC} + s\,\overrightarrow{CD} = (-2\vec{i}+6\vec{j}+5\vec{k}) + t(\vec{i}+\vec{j})$.

$$\vec{a_2} - \vec{a_1} = \begin{bmatrix} -2 \\ 6 \\ 5 \end{bmatrix} - \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}.$$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 4 & 5 \\ 1 & 1 & 0 \end{vmatrix} = \begin{bmatrix} -5 \\ 5 \\ -2 \end{bmatrix}$$

$\therefore$ Shortest distance between l_1 and l_2

$$= \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right| = \left| \frac{(\vec{i}+5\vec{j}+\vec{k}) \cdot (-5\vec{i}+5\vec{j}-2\vec{k})}{\sqrt{(-5)^2+(5)^2+(-2)^2}} \right|$$

$$= \left| \frac{18}{\sqrt{54}} \right| = \sqrt{6} = 2.449 \approx 2.45.$$

(ii) Find the acute angle between the line l_2 and the plane containing A, B and D. [5]

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 2\vec{i} + 4\vec{j} + 5\vec{k}.$$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = (-\vec{i} + 7\vec{j} + 5\vec{k}) - (-3\vec{i} + \vec{j} + 4\vec{k}) = 2\vec{i} + 6\vec{j} + \vec{k}.$$

$$\vec{n_2} = \overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 4 & 5 \\ 2 & 6 & 1 \end{vmatrix} = \begin{bmatrix} -26 \\ 8 \\ 4 \end{bmatrix} \sim \begin{bmatrix} -13 \\ 4 \\ 2 \end{bmatrix}$$

i.e. normal for plane containing A, B and D $= -13\vec{i} + 4\vec{j} + 2\vec{k}$.

For line l_2,

$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \begin{bmatrix} -1 \\ 7 \\ 5 \end{bmatrix} - \begin{bmatrix} -2 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

i.e. direction vector for l_2 : $\vec{i} + \vec{j}$.

Let, θ be the angle between $\vec{b}$ and $\vec{n_2}$.

$$\vec{b} \cdot \vec{n_2} = |\vec{b}||\vec{n_2}| \cos\theta$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -13 \\ 4 \\ 2 \end{bmatrix} = \sqrt{1^2 + 1^2} \sqrt{(-13)^2 + 4^2 + 2^2} \cos\theta$$

$$-9 = \sqrt{2}\sqrt{189} \cos\theta$$

$$\cos\theta = \frac{-9}{\sqrt{2}\sqrt{189}} \Rightarrow \theta = 117.57^\circ$$

$\therefore$ Acute angle between line and plane $= 117.57^\circ - 90^\circ = 27.57^\circ \approx 27.6^\circ$

29 The line l_1 passes through the point A with position vector $\mathbf{i}-\mathbf{j}-2\mathbf{k}$ and is parallel to the vector $3\mathbf{i}-4\mathbf{j}-2\mathbf{k}$. The variable line l_2 passes through the point $(1+5\cos t)\mathbf{i}-(1+5\sin t)\mathbf{j}-14\mathbf{k}$, where $0 \leqslant t < 2\pi$, and is parallel to the vector $15\mathbf{i}+8\mathbf{j}-3\mathbf{k}$. The points P and Q are on l_1 and l_2 respectively, and PQ is perpendicular to both l_1 and l_2.

(i) Find the length of PQ in terms of t. [4]

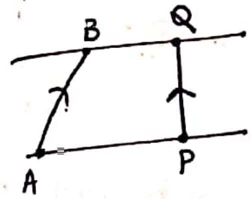
(ii) Hence show that the lines l_1 and l_2 do not intersect, and find the maximum length of PQ as t varies. [3]

(iii) The plane Π_1 contains l_1 and PQ; the plane Π_2 contains l_2 and PQ. Find the angle between the planes Π_1 and Π_2, correct to the nearest tenth of a degree. [4]

i) Since $\overrightarrow{PQ}$ is $\perp$ to both l_1 and l_2,

$$\overrightarrow{PQ} = (3\vec{i}-4\vec{j}-2\vec{k}) \times (15\vec{i}+8\vec{j}-3\vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -4 & -2 \\ 15 & 8 & -3 \end{vmatrix} = \begin{bmatrix} 28 \\ -21 \\ 84 \end{bmatrix} \sim \begin{bmatrix} 4 \\ -3 \\ 12 \end{bmatrix}$$

Let, $\overrightarrow{OB} = \begin{bmatrix} 1+5\cos t \\ -(1+5\sin t) \\ -14 \end{bmatrix}$ where B is a point on line l_2.



$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{bmatrix} 1+5\cos t \\ -1-5\sin t \\ -14 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 5\cos t \\ -5\sin t \\ -12 \end{bmatrix}.$$

$$\overrightarrow{AB} \cdot \overrightarrow{PQ} = 4(5\cos t) - 3(-5\sin t) - 12(+12) = 20\cos t + 15\sin t - 144.$$

$$|\overrightarrow{PQ}| = \sqrt{4^2+(-3)^2+12^2} = \sqrt{169} = 13.$$

$$PQ = \left| \frac{\overrightarrow{AB} \cdot \overrightarrow{PQ}}{|\overrightarrow{PQ}|} \right| = \left| \frac{20\cos t + 15\sin t - 144}{13} \right|.$$

ii)
$$20\cos t + 15\sin t = R\cos(t-\alpha)$$
$$20\cos t + 15\sin t = R\cos t\cos\alpha + R\sin t\sin\alpha$$
$$20\cos t + 15\sin t = (R\cos\alpha)\cos t + (R\sin\alpha)\sin t$$

$\Rightarrow R\cos\alpha = 20 \Rightarrow \cos\alpha = \frac{20}{R}$ and $R\sin\alpha = 15 \Rightarrow \sin\alpha = \frac{15}{R}$.

$$\sin^2\alpha + \cos^2\alpha = 1 \Rightarrow \left(\frac{20}{R}\right)^2 + \left(\frac{15}{R}\right)^2 = 1 \Rightarrow 625 = R^2 \Rightarrow R = 25.$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{15}{R}}{\frac{20}{R}} = \frac{3}{4} \Rightarrow \alpha = \tan^{-1}\left(\frac{3}{4}\right) = 36.86^\circ$$

$$\therefore 20\cos t + 15\sin t = 25\cos(t - 36.86^\circ)$$

$\Rightarrow -25 \leq 20\cos t + 15\sin t \leq 25$ since $-25 \leq 25\cos(t-36.86^\circ) \leq 25$

$$\Rightarrow -169 \leq 20\cos t + 15\sin t - 144 \leq -119$$

$$-13 \leq \frac{20\cos t + 15\sin t - 144}{13} \leq \frac{-119}{13}$$

$$\Rightarrow \quad \frac{119}{13} \leq \left| \frac{20\cos t + 15\sin t - 144}{13} \right| \leq 13$$

i.e. $\quad \frac{119}{13} \leq PQ \leq 13$

$\therefore$ Since minimum length of PQ is $\frac{119}{13}$, then l_1 and l_2 do not intersect.

Maximum length of $PQ = 13$.

iii) Π_1 ~~PQ~~ contains vector $\overrightarrow{PQ} = 4\vec{i} - 3\vec{j} + 12\vec{k}$ and direction vector of l_1 : $3\vec{i} - 4\vec{j} - 2\vec{k}$.

$$\therefore \overrightarrow{n_1} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -3 & 12 \\ 3 & -4 & -2 \end{vmatrix} = \begin{bmatrix} 54 \\ 44 \\ -7 \end{bmatrix}$$

Π_2 contains vectors $\overrightarrow{PQ} = 4\vec{i} - 3\vec{j} + 12\vec{k}$ and direction vector of l_2: $15\vec{i} + 8\vec{j} - 3\vec{k}$.

$$\therefore \overrightarrow{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -3 & 12 \\ 15 & 8 & -3 \end{vmatrix} = \begin{bmatrix} -87 \\ 192 \\ 77 \end{bmatrix}$$

Let, angle between Π_1 and Π_2 be θ.

Then $\quad \overrightarrow{n_1} \cdot \overrightarrow{n_2} = |\overrightarrow{n_1}||\overrightarrow{n_2}| \cos\theta$

$$\begin{bmatrix} 54 \\ 44 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} -87 \\ 192 \\ 77 \end{bmatrix} = \sqrt{54^2 + 44^2 + (-7)^2} \sqrt{(-87)^2 + (192)^2 + (77)^2} \cos\theta$$

$$\cos\theta = \frac{3211}{\sqrt{4901}\sqrt{50362}}$$

$$\theta = 78.20° \approx 78.2°$$

30 The line l_1 passes through the point A, whose position vector is $3\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$, and is parallel to the vector $3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$. The line l_2 passes through the point B, whose position vector is $2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, and is parallel to the vector $\mathbf{i} - \mathbf{j} - 4\mathbf{k}$. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. The plane Π_1 contains PQ and l_1, and the plane Π_2 contains PQ and l_2.

(i) Find the length of PQ. [4]

(ii) Find a vector perpendicular to Π_1. [2]

(iii) Find the perpendicular distance from B to Π_1. [3]

(iv) Find the angle between Π_1 and Π_2. [3]

i) $l_1: \vec{r} = 3\vec{i} - 5\vec{j} - 4\vec{k} + t(3\vec{i} + 4\vec{j} + 2\vec{k}) = \begin{bmatrix} 3+3t \\ -5+4t \\ -4+2t \end{bmatrix}$

Since P lies on $l_1 \Rightarrow \vec{OP} = \begin{bmatrix} 3t+3 \\ 4t-5 \\ 2t-4 \end{bmatrix}$.

$l_2: \vec{r} = 2\vec{i} + 3\vec{j} + 5\vec{k} + s(\vec{i} - \vec{j} - 4\vec{k}) = \begin{bmatrix} 2+s \\ 3-s \\ 5-4s \end{bmatrix}$.

Since Q lies on $l_2 \Rightarrow \vec{OQ} = \begin{bmatrix} 2+s \\ 3-s \\ 5-4s \end{bmatrix}$.

$\vec{PQ} = \vec{OQ} - \vec{OP} = \begin{bmatrix} 2+s \\ 3-s \\ 5-4s \end{bmatrix} - \begin{bmatrix} 3+3t \\ -5+4t \\ -4+2t \end{bmatrix} = \begin{bmatrix} -1+s-3t \\ 8-s-4t \\ 9-4s-2t \end{bmatrix}$

Since $\vec{PQ} \perp l_1 \therefore \vec{PQ} \cdot (3\vec{i} + 4\vec{j} + 2\vec{k}) = 0 \Rightarrow -3+3s-9t+32-4s-16t+18-8s-4t = 0$

$-9s - 29t + 47 = 0 \rightarrow (1)$

Since $\vec{PQ} \perp l_2 \therefore \vec{PQ} \cdot (\vec{i} - \vec{j} - 4\vec{k}) = 0 \Rightarrow -1+s-3t-8+s+4t-36+16s+8t = 0$

$18s + 9t - 45 = 0 \rightarrow (2)$

Solving simultaneously $\Rightarrow s = 2, t = 1$.

$\therefore \vec{PQ} = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix} \Rightarrow |\vec{PQ}| = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = 3$.

ii) Since Π_1 contains PQ and l_1, $\vec{n_1} = \vec{PQ} \times 3\vec{i} + 4\vec{j} + 2\vec{k}$.

$\vec{n_1} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 2 & -1 \\ 3 & 4 & 2 \end{vmatrix} = \begin{bmatrix} 8 \\ +1 \\ -14 \end{bmatrix} = 8\vec{i} + \vec{j} - 14\vec{k}$.

i.e. vector perpendicular to $\Pi_1 = 8\vec{i} + \vec{j} - 14\vec{k}$.

iii) The point A $(3, -5, -4)$ lies on l_1 and hence lies on Π_1.

$\vec{AB} = \vec{OB} - \vec{OA} = (2\vec{i} + 3\vec{j} + 5\vec{k}) - (3\vec{i} - 5\vec{j} - 4\vec{k}) = -\vec{i} + 8\vec{j} + 9\vec{k}$.

$\vec{AB} \cdot \vec{n_1} = \begin{bmatrix} -1 \\ 8 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ +1 \\ -14 \end{bmatrix} = -8 + 8 - 126 = -126$

$|\vec{n_1}| = \sqrt{8^2 + 1^2 + (-14)^2} = \sqrt{261}$.

$\perp$ distance from B to $\Pi_1 = \left| \dfrac{\overrightarrow{AB} \cdot \overrightarrow{n_1}}{|\overrightarrow{n_1}|} \right| = \left| \dfrac{-126}{\sqrt{261}} \right| = \dfrac{126}{\sqrt{261}} = 7.799 \approx 7.80.$

iv) Π_2 contains vectors $\overrightarrow{PQ}$ and l_2's direction vector $\vec{i} - \vec{j} - 4\vec{k}$.

$$\therefore \overrightarrow{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 2 & -1 \\ 1 & -1 & -4 \end{vmatrix} = \begin{bmatrix} -9 \\ -9 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$$

Let, angle between $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ be θ.

$$\therefore \overrightarrow{n_1} \cdot \overrightarrow{n_2} = |\overrightarrow{n_1}| \, |\overrightarrow{n_2}| \cos\theta$$

$$\begin{bmatrix} 8 \\ 1 \\ -14 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = \sqrt{8^2 + 1^2 + (-14)^2} \sqrt{3^2 + 3^2 + 0^2} \cos\theta$$

$$27 = \sqrt{261} \sqrt{18} \cos\theta$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{27}{\sqrt{261}\sqrt{18}} \right)$$

$$\theta = 66.80^\circ \approx 66.8^\circ$$

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31 The planes Π_1 and Π_2 have vector equations

$$\mathbf{r} = \lambda_1(\mathbf{i}+\mathbf{j}-\mathbf{k}) + \mu_1(2\mathbf{i}-\mathbf{j}+\mathbf{k}) \quad \text{and} \quad \mathbf{r} = \lambda_2(\mathbf{i}+2\mathbf{j}+\mathbf{k}) + \mu_2(3\mathbf{i}+\mathbf{j}-\mathbf{k})$$

respectively. The line l passes through the point with position vector $4\mathbf{i}+5\mathbf{j}+6\mathbf{k}$ and is parallel to both Π_1 and Π_2. Find a vector equation for l. [6]

Find also the shortest distance between l and the line of intersection of Π_1 and Π_2. [4]

For Π_1 : $\vec{n_1} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 2 & -1 & 1 \end{vmatrix} = \begin{bmatrix} 0 \\ -3 \\ -3 \end{bmatrix} \sim \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$

For Π_2 : $\vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ 3 & 1 & -1 \end{vmatrix} = \begin{bmatrix} -3 \\ 4 \\ -5 \end{bmatrix}$

Line l is // to Π_1 and Π_2, then its direction vector $\vec{b}$ is $\perp$ to $\vec{n_1}$ and $\vec{n_2}$.

$\vec{b} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ -3 & 4 & -5 \end{vmatrix} = \begin{bmatrix} -9 \\ -3 \\ 3 \end{bmatrix} \sim \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

$\therefore$ vector Equation of l : $\vec{r} = (4\vec{i}+5\vec{j}+6\vec{k}) + t(3\vec{i}+\vec{j}-\vec{k})$

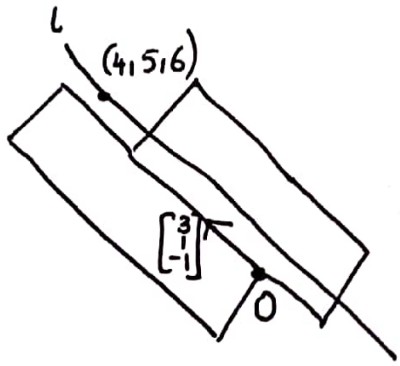
Note that both Π_1 and Π_2 pass through the origin.

Point P with position vector $4\vec{i}+5\vec{j}+6\vec{k}$ lies on line l.

The line of intersection of Π_1 and Π_2 has direction vector $3\vec{i}+\vec{j}-\vec{k}$.

Then distance from l to ~~point~~ line of intersection

$= $ distance from P to line of intersection



$= \left|\vec{OP} \times \hat{u}\right| = \left|(4\vec{i}+5\vec{j}+6\vec{k}) \times \frac{(3\vec{i}+\vec{j}-\vec{k})}{\sqrt{9+1+1}}\right|$

$= \frac{1}{\sqrt{11}} \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 5 & 6 \\ 3 & 1 & -1 \end{vmatrix} \right\| = \frac{1}{\sqrt{11}} \left\| \begin{bmatrix} -11 \\ 22 \\ -11 \end{bmatrix} \right\|$

$= \frac{1}{\sqrt{11}} \sqrt{(-11)^2+22^2+(-11)^2} = \frac{1}{\sqrt{11}} \sqrt{726} = \sqrt{66} \approx 8.12$

32 With O as origin, the points A, B, C have position vectors

$$\mathbf{i}, \quad \mathbf{i}+\mathbf{j}, \quad \mathbf{i}+\mathbf{j}+2\mathbf{k}$$

respectively. Find a vector equation of the common perpendicular of the lines AB and OC. [6]

Show that the shortest distance between the lines AB and OC is $\frac{2}{5}\sqrt{5}$. [2]

Find, in the form $ax + by + cz = d$, an equation for the plane containing AB and the common perpendicular of the lines AB and OC. [3]

Line AB: $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{i}+\vec{j}-\vec{i} = \vec{j}$.

$\therefore$ Equation of AB: $\vec{r} = \vec{i} + s(\vec{j})$.

Line OC: Equation: $\vec{r} = t(\vec{i}+\vec{j}+2\vec{k})$.

Let, P be a point on line AB and Q a point on OC such that PQ is a common perpendicular to both AB and OC.

Since P is on AB $\therefore$ $\overrightarrow{OP} = \vec{i}+s\vec{j}$ and since Q is on OC, $\overrightarrow{OQ} = t\vec{i}+t\vec{j}+2t\vec{k}$.

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (t\vec{i}+t\vec{j}+2t\vec{k}) - (\vec{i}+s\vec{j}) = (t-1)\vec{i} + (t-s)\vec{j} + (2t)\vec{k}$.

Since $\overrightarrow{PQ} \perp AB$, then $\overrightarrow{PQ}\cdot(\vec{j}) = 0 \Rightarrow (t-s)(1) = 0 \Rightarrow t = s \rightarrow (1)$.

Since $\overrightarrow{PQ} \perp OC$, then $\overrightarrow{PQ}\cdot(\vec{i}+\vec{j}+2\vec{k}) = 0 \Rightarrow (t-1)(1)+(t-s)(1)+2(2t) = 0$

$t-1+t-s+4t = 0$

$6t-s-1 = 0 \rightarrow (2)$

Substitute (1) in (2)

$\Rightarrow 6t - t - 1 = 0 \Rightarrow 5t = 1 \Rightarrow t = \frac{1}{5} \therefore s = \frac{1}{5}$.

$\therefore \overrightarrow{OP} = \vec{i} + \frac{1}{5}\vec{j}$, $\overrightarrow{OQ} = \frac{1}{5}\vec{i} + \frac{1}{5}\vec{j} + \frac{2}{5}\vec{k} \Rightarrow \overrightarrow{PQ} = \frac{-4}{5}\vec{i} + 0\vec{j} + \frac{2}{5}\vec{k}$.

Vector equation of PQ:

$\vec{r} = \overrightarrow{OP} + \lambda\overrightarrow{PQ} = \vec{i} + \frac{1}{5}\vec{j} + \lambda\left(\frac{-4}{5}\vec{i} + \frac{2}{5}\vec{k}\right)$

$\vec{r} = \vec{i} + \frac{1}{5}\vec{j} + \lambda(2\vec{i} - \vec{k})$.

Shortest distance between AB and OC

$= |\overrightarrow{PQ}| = \sqrt{\left(\frac{-4}{5}\right)^2 + \left(\frac{2}{5}\right)^2} = \sqrt{\frac{4}{5}} = \frac{2}{5}\sqrt{5}$. (Shown).

Plane contains vectors $\vec{j}$ (direction vector of AB) and $\overrightarrow{PQ} = \frac{-4}{5}\vec{i} + \frac{2}{5}\vec{k}$

$\sim 2\vec{i} - \vec{k}$.

$\therefore \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{vmatrix} = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Plane contains the point with position vector $\vec{i}$.

$\therefore$ Equation of plane: $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

$$\vec{r} \cdot (\vec{i} + 2\vec{k}) = \vec{i} \cdot (\vec{i} + 2\vec{k})$$

$$\underline{x + 2z = 1}.$$

33 The points A, B and C have position vectors $2\mathbf{i}$, $3\mathbf{j}$ and $4\mathbf{k}$ respectively. Find a vector which is perpendicular to the plane Π_1 containing A, B and C. [3]

The plane Π_2 has equation

$$\mathbf{r} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j}) + \mu(\mathbf{j} - \mathbf{k}).$$

Find the acute angle between the planes Π_1 and Π_2. [5]

$\vec{OA} = 2\vec{i}$, $\vec{OB} = 3\vec{j}$, $\vec{OC} = 4\vec{k}$.

$\vec{AB} = \vec{OB} - \vec{OA} = 3\vec{j} - 2\vec{i} = -2\vec{i} + 3\vec{j}$

$\vec{AC} = \vec{OC} - \vec{OA} = 4\vec{k} - 2\vec{i} = -2\vec{i} + 4\vec{k}$.

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 3 & 0 \\ -2 & 0 & 4 \end{vmatrix} = \begin{bmatrix} 12 \\ 8 \\ 6 \end{bmatrix} \sim \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix} = 6\vec{i} + 4\vec{j} + 3\vec{k}.$$

$\therefore$ Vector perpendicular to $\Pi_1 = 6\vec{i} + 4\vec{j} + 3\vec{k}$.

$\Pi_2 : \vec{r} = \vec{i} + 4\vec{j} + 2\vec{k} + \lambda(\vec{i} - \vec{j}) + \mu(\vec{j} - \vec{k})$.

$$\vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let, angle between $\vec{n_1}$ and $\vec{n_2}$ be θ.

$$\vec{n_1} \cdot \vec{n_2} = |\vec{n_1}||\vec{n_2}| \cos\theta$$

$$\begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \sqrt{6^2+4^2+3^2}\sqrt{1^2+1^2+1^2} \cos\theta$$

$$13 = \sqrt{61}\sqrt{3} \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{13}{\sqrt{61}\sqrt{3}}\right) = 16.05° \approx 16.1°$$

i.e. acute angles between Π_1 and $\Pi_2 = 16.1°$.

34 The plane Π_1 has equation

$$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \theta(2\mathbf{j} - \mathbf{k}) + \phi(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}).$$

Find a vector normal to Π_1 and hence show that the equation of Π_1 can be written as $2x + 3y + 6z = 14$. [4]

The line l has equation

$$\mathbf{r} = 3\mathbf{i} + 8\mathbf{j} + 2\mathbf{k} + t(4\mathbf{i} + 6\mathbf{j} + 5\mathbf{k}).$$

The point on l where $t = \lambda$ is denoted by P. Find the set of values of λ for which the perpendicular distance of P from Π_1 is not greater than 4. [4]

The plane Π_2 contains l and the point with position vector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Find the acute angle between Π_1 and Π_2. [4]

Π_1 contains vectors $2\vec{j} - \vec{k}$ and $3\vec{i} + 2\vec{j} - 2\vec{k}$.

$$\therefore \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & -1 \\ 3 & 2 & -2 \end{vmatrix} = \begin{bmatrix} -2 \\ -3 \\ -6 \end{bmatrix} \sim \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$$

Π_1 contains the point with position vector $\vec{i} + 2\vec{j} + \vec{k}$.

$\therefore$ Equation of Π_1: $\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$

$$\vec{r}\cdot(2\vec{i} + 3\vec{j} + 6\vec{k}) = (\vec{i} + 2\vec{j} + \vec{k})\cdot(2\vec{i} + 3\vec{j} + 6\vec{k})$$

$$2x + 3y + 6z = 2 + 6 + 6$$

$$2x + 3y + 6z = 14 \quad \text{(Shown)}.$$

Line l: $\vec{r} = 3\vec{i} + 8\vec{j} + 2\vec{k} + t(4\vec{i} + 6\vec{j} + 5\vec{k})$.

At P, $t = \lambda \Rightarrow \vec{r} = (3 + 4\lambda)\vec{i} + (8 + 6\lambda)\vec{j} + (2 + 5\lambda)\vec{k}$

$$\therefore \vec{OP} = \begin{bmatrix} 3 + 4\lambda \\ 8 + 6\lambda \\ 2 + 5\lambda \end{bmatrix}.$$

Let, $Q(1, 2, 1)$ be a point on Π_1 $\therefore \vec{OQ} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

$$\vec{PQ} = \vec{OQ} - \vec{OP} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 + 4\lambda \\ 8 + 6\lambda \\ 2 + 5\lambda \end{bmatrix} = \begin{bmatrix} -2 - 4\lambda \\ -6 - 6\lambda \\ -1 - 5\lambda \end{bmatrix}.$$

$$\vec{PQ}\cdot\vec{n} = \begin{bmatrix} -2 - 4\lambda \\ -6 - 6\lambda \\ -1 - 5\lambda \end{bmatrix}\cdot\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} = -4 - 8\lambda - 18 - 18\lambda - 6 - 30\lambda = -56\lambda - 28.$$

$$|\vec{n}| = \sqrt{2^2 + 3^2 + 6^2} = 7.$$

$$\therefore \perp \text{ distance from } P \text{ to } \Pi_1 = \left|\frac{\vec{PQ}\cdot\vec{n}}{|\vec{n}|}\right| = \left|\frac{-56\lambda - 28}{7}\right| = |-8\lambda - 4|$$

Since $\perp$ distance ≤ 4

$$\Rightarrow |-8\lambda - 4| \leq 4 \Rightarrow |8\lambda + 4| \leq 4 \Rightarrow -4 \leq 8\lambda + 4 \leq 4$$

$$-8 \leq 8\lambda \leq 0$$

$$-1 \leq \lambda \leq 0$$

Π_2: Since Π_2 contains the points $3\vec{i}+8\vec{j}+2\vec{k}$ and $\vec{i}+2\vec{j}+\vec{k}$, then Π_2 contains the vector $(3\vec{i}+8\vec{j}+2\vec{k})-(\vec{i}+2\vec{j}+\vec{k}) = 2\vec{i}+6\vec{j}+\vec{k}$.

Then normal vector for Π_2 is $\vec{n_2} = (2\vec{i}+6\vec{j}+\vec{k}) \times (4\vec{i}+6\vec{j}+5\vec{k})$.

$$\therefore \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 6 & 1 \\ 4 & 6 & 5 \end{vmatrix} = \begin{bmatrix} 24 \\ -6 \\ -12 \end{bmatrix} \sim \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}.$$

Π_1 : $\vec{n_1} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$.

For angle θ between Π_1 and Π_2,

$$\vec{n_1} \cdot \vec{n_2} = |\vec{n_1}|\,|\vec{n_2}| \cos\theta$$

$$\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix} = \sqrt{2^2+3^2+6^2}\,\sqrt{4^2+(-1)^2+(-2)^2}\,\cos\theta$$

$$-7 = 7\sqrt{21}\cos\theta$$

$$\theta = \cos^{-1}\left(\frac{-7}{7\sqrt{21}}\right) = 102.60^\circ$$

$\therefore$ Acute angle between Π_1 and Π_2 : $180^\circ - \theta = 77.39^\circ \approx 77.4^\circ$

35 Relative to an origin O, the points A, B, C have position vectors

$$\mathbf{i}, \quad \mathbf{j}+\mathbf{k}, \quad \mathbf{i}+\mathbf{j}+\theta\mathbf{k},$$

respectively. The shortest distance between the lines AB and OC is $\frac{1}{\sqrt{2}}$. Find the value of θ. [6]

Line $\overleftrightarrow{AB}$: $\vec{r} = \overrightarrow{OA} + t\,\overrightarrow{AB} = \vec{i} + t(\vec{j}+\vec{k}-\vec{i})$

$\vec{r} = \vec{i} + t(-\vec{i}+\vec{j}+\vec{k})$ where $\vec{a_1} = \vec{i}$, $\vec{b_1} = -\vec{i}+\vec{j}+\vec{k}$.

line OC : $\vec{r} = 0\vec{i}+0\vec{j}+0\vec{k} + s(\vec{i}+\vec{j}+\theta\vec{k})$

$\vec{r} = s(\vec{i}+\vec{j}+\theta\vec{k})$ where $\vec{a_2} = 0\vec{i}+0\vec{j}+0\vec{k}$, $\vec{b_2} = \vec{i}+\vec{j}+\theta\vec{k}$.

$$\vec{a_2} - \vec{a_1} = (0\vec{i}+0\vec{j}+0\vec{k}) - (\vec{i}) = -\vec{i}.$$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 1 \\ 1 & 1 & \theta \end{vmatrix} = \begin{bmatrix} \theta-1 \\ -(-\theta-1) \\ -1-1 \end{bmatrix} = \begin{bmatrix} \theta-1 \\ \theta+1 \\ -2 \end{bmatrix}.$$

$$(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = (-\vec{i}) \cdot ((\theta-1)\vec{i} + (\theta+1)\vec{j} - 2\vec{k})$$
$$= -(\theta-1) = 1-\theta.$$

$$|\vec{b_1} \times \vec{b_2}| = \sqrt{(\theta-1)^2 + (\theta+1)^2 + (-2)^2} = \sqrt{\theta^2 - 2\theta + 1 + \theta^2 + 2\theta + 1 + 4}$$
$$= \sqrt{2\theta^2 + 6}$$

$\therefore$ Distance between OC and AB $= \left| \dfrac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{(\vec{b_1} \times \vec{b_2})} \right|$

$$\frac{1}{\sqrt{2}} = \left| \frac{1-\theta}{\sqrt{2\theta^2+6}} \right|$$

$$\frac{1}{2} = \frac{(1-\theta)^2}{2\theta^2+6}$$

$$2\theta^2 + 6 = 2(1 - 2\theta + \theta^2)$$

$$\cancel{2\theta^2} + 6 = 2 - 4\theta + \cancel{2\theta^2}$$

$$4\theta = -4$$

$$\theta = -1.$$

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36 The plane Π_1 has equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) + \mu(-\mathbf{i} + \mathbf{k})$. Obtain a cartesian equation of Π_1 in the form $px + qy + rz = d$. [4]

The plane Π_2 has equation $\mathbf{r}.(\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}) = 12$. Find a vector equation of the line of intersection of Π_1 and Π_2. [3]

The line l passes through the point A with position vector $a\mathbf{i} + (2a + 1)\mathbf{j} - 3\mathbf{k}$ and is parallel to $3c\mathbf{i} - 3\mathbf{j} + c\mathbf{k}$, where a and c are positive constants. Given that the perpendicular distance from A to Π_1 is $\frac{15}{\sqrt{6}}$ and that the acute angle between l and Π_1 is $\sin^{-1}\left(\frac{2}{\sqrt{6}}\right)$, find the values of a and c. [7]

Π_1 contains vectors $2\vec{i}+3\vec{j}+4\vec{k}$ and $-\vec{i}+\vec{k}$.

$$\vec{n_1} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 4 \\ -1 & 0 & 1 \end{vmatrix} = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} = 3\vec{i} - 6\vec{j} + 3\vec{k}.$$

Π_1 passes through $2\vec{i}+\vec{j}+4\vec{k}$ $\therefore$ Equation of Π_1:

$$\vec{r}\cdot\vec{n_1} = \vec{a}\cdot\vec{n_1}$$
$$\vec{r}\cdot(3\vec{i}-6\vec{j}+3\vec{k}) = (2\vec{i}+\vec{j}+4\vec{k})\cdot(3\vec{i}-6\vec{j}+3\vec{k})$$
$$3x-6y+3z = 12$$
$$x-2y+z = 4 \rightarrow (1)$$

Π_2:
$$\vec{r}\cdot(\vec{i}-4\vec{j}+5\vec{k}) = 12$$
$$\Rightarrow x - 4y + 5z = 12 \rightarrow (2)$$

From Π_1: $x = 4+2y-z \rightarrow (3)$

Substitute (3) in (2):
$$4+2y-z-4y+5z = 12.$$
$$-2y+4z = 8$$
$$-y+2z = 4$$
$$y = 2z-4.$$

Let, $z = t \Rightarrow y = 2t-4$

$$\therefore x = 4+2(2t-4)-t = 4+4t-8-t = 3t-4$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3t-4 \\ 2t-4 \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -4 \\ -4 \\ 0 \end{bmatrix}$$

i.e. vector equation of line of intersection: $\vec{r} = -4\vec{i}-4\vec{j}+t(3\vec{i}+2\vec{j}+\vec{k})$.

Line L: $\vec{r} = \begin{bmatrix} a \\ 2a+1 \\ -3 \end{bmatrix} + s\begin{bmatrix} 3c \\ -3 \\ c \end{bmatrix}$ where $\overrightarrow{OA} = \begin{bmatrix} a \\ 2a+1 \\ -3 \end{bmatrix}$

Plane Π_1: $x - 2y + z = 4$. contains point P with position vector $\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$.

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} - \begin{bmatrix} a \\ 2a+1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2-a \\ -2a \\ 7 \end{bmatrix}$$

$$\overrightarrow{AP} \cdot \vec{n} = \begin{bmatrix} 2-a \\ -2a \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 2-a+4a+7 = 9+3a.$$

$$|\vec{n}| = \sqrt{1^2+(-2)^2+1^2} = \sqrt{6}.$$

$\therefore$ Distance from A to $\Pi_1 = \left|\frac{\overrightarrow{PQ} \cdot \vec{n}}{|\vec{n}|}\right| = \left|\frac{9+3a}{\sqrt{6}}\right| = \frac{15}{\sqrt{6}}$

$\Rightarrow \frac{9+3a}{\sqrt{6}} = \frac{15}{\sqrt{6}}$ ignore negative since $a>0$.

$\Rightarrow 3a = 6 \Rightarrow a = 2.$

Let, angle between direction vector of L and $\vec{n}$ of Π_1 be θ.

$\Rightarrow \begin{bmatrix} 3c \\ \dots \end{bmatrix} \begin{bmatrix} 1 \\ \dots \end{bmatrix} = \sqrt{(3c)^2+(-3)^2+c^2}\sqrt{1^2+(-2)^2+1^2}\cos\theta$

$$\Rightarrow \quad 3a = 6 \Rightarrow a = 2.$$

Let, angle between direction vector of L and $\vec{n}$ of Π_1 be θ.

$$\Rightarrow \begin{bmatrix} 3c \\ -3 \\ c \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \sqrt{(3c)^2+(-3)^2+c^2}\,\sqrt{1^2+(-2)^2+1^2}\,\cos\theta$$

$$3c + 6 + c = \sqrt{10c^2+9}\,\sqrt{6}\,\cos\theta$$

$$\frac{4c+6}{\sqrt{10c^2+9}\,\sqrt{6}} = \cos\theta.$$

$$\Rightarrow \cos\theta = \frac{4c+6}{\sqrt{60c^2+54}} \Rightarrow \theta = \cos^{-1}\left(\frac{4c+6}{\sqrt{60c^2+54}}\right)$$

Acute angle between line L and $\Pi_1 = 90° - \theta$

$$\sin^{-1}\left(\frac{2}{\sqrt{6}}\right) = 90° - \cos^{-1}\left(\frac{4c+6}{\sqrt{60c^2+54}}\right)$$

$$\cos^{-1}\left(\frac{4c+6}{\sqrt{60c^2+54}}\right) = 90° - \sin^{-1}\left(\frac{2}{\sqrt{6}}\right)$$

$$\frac{4c+6}{\sqrt{60c^2+54}} = 0.8164$$

$$\frac{(4c+6)^2}{60c^2+54} = \frac{2}{3}$$

$$\frac{16c^2+48c+36}{60c^2+54} = \frac{2}{3} \Rightarrow 48c^2+144c+108 = 120c^2+108$$

$$72c^2 - 144c \cancel{-84} = 0$$

$$72c\,(c-2) = 0$$

$c = 0$ (Ignore) or $\boxed{c = 2}$

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37 The position vectors of points A, B, C, relative to the origin O, are **a**, **b**, **c**, where

$$\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \qquad \mathbf{b} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}, \qquad \mathbf{c} = 3\mathbf{i} - \mathbf{j} - \mathbf{k}.$$

Find $\mathbf{a} \times \mathbf{b}$ and deduce the area of the triangle OAB. [3]

Hence find the volume of the tetrahedron $OABC$, given that the volume of a tetrahedron is $\frac{1}{3} \times$ area of base $\times$ perpendicular height. [2]

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & -1 \\ 4 & -3 & 2 \end{vmatrix} = \begin{bmatrix} 1 \\ -10 \\ -17 \end{bmatrix} = \vec{i} - 10\vec{j} - 17\vec{k}.$$

$$\text{Area of } \Delta OAB = \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}\sqrt{1^2 + (-10)^2 + (-17)^2} = \frac{\sqrt{390}}{2} = 9.874 \approx 9.87$$

Volume of tetrahedron OABC

$$= \frac{1}{6}\left|\left(\vec{a} \times \vec{b}\right) \cdot \vec{c}\right| = \frac{1}{6}\left|\left(\vec{i} - 10\vec{j} - 17\vec{k}\right) \cdot \left(3\vec{i} - \vec{j} - \vec{k}\right)\right|$$

$$= \frac{1}{6}\left|30\right| = 5 \text{ cubic units.}$$

38 Find a cartesian equation of the plane Π containing the lines

$$\mathbf{r} = 3\mathbf{i} + \mathbf{k} + s(2\mathbf{i} + \mathbf{j} - \mathbf{k}) \quad \text{and} \quad \mathbf{r} = 3\mathbf{i} - 7\mathbf{j} + 10\mathbf{k} + t(\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}).$$ [4]

The line l passes through the point P with position vector $6\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and is parallel to the vector $2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$. Find

(i) the position vector of the point where l meets Π, [3]

(ii) the perpendicular distance from P to Π, [3]

(iii) the acute angle between l and Π. [3]

Π has a normal $\perp$ to $2\vec{i}+\vec{j}-\vec{k}$ and $\vec{i}-3\vec{j}+4\vec{k}$.

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 1 & -3 & 4 \end{vmatrix} = \begin{bmatrix} 1 \\ -9 \\ -7 \end{bmatrix}.$$

Π contains the point $3\vec{i}+\vec{k}$ $\therefore$ Equation of Π:

$$\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$$

$$\vec{r}\cdot(\vec{i}-9\vec{j}-7\vec{k}) = (3\vec{i}+\vec{k})\cdot(\vec{i}-9\vec{j}-7\vec{k})$$

$$x - 9y - 7z = -4$$

Equation of line

i) l: $\vec{r} = 6\vec{i}-2\vec{j}+\vec{k}+\lambda(2\vec{i}+\vec{j}-4\vec{k}) = \begin{bmatrix} 6+2\lambda \\ -2+\lambda \\ 1-4\lambda \end{bmatrix}$ $\therefore$ $x = 6+2\lambda$, $y = -2+\lambda$, $z = 1-4\lambda$

when line meets Π: $(6+2\lambda)-9(-2+\lambda)-7(1-4\lambda) = -4$

$$6+2\lambda+18-9\lambda-7+28\lambda+4 = 0$$

$$21\lambda = -21$$

$$\lambda = -1$$

$$\therefore \vec{r} = \begin{bmatrix} 6+2(-1) \\ -2+(-1) \\ 1-4(-1) \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix} = 4\vec{i}-3\vec{j}+5\vec{k}.$$

i.e. position vector of point of intersection $= 4\vec{i}-3\vec{j}+5\vec{k}$.

ii) Π: $x-9y-7z=4$ has $\vec{n} = \vec{i}-9\vec{j}-7\vec{k}$ and passes through point Q with position vector $3\vec{i}+\vec{k}$.

$$\vec{PQ} = \vec{OQ}-\vec{OP} = (3\vec{i}+\vec{k})-(6\vec{i}-2\vec{j}+\vec{k}) = -3\vec{i}+2\vec{j}$$

$$\vec{PQ}\cdot\vec{n} = (-3\vec{i}+2\vec{j})\cdot(\vec{i}-9\vec{j}-7\vec{k}) = -3+2(-9) = -21.$$

$$|\vec{n}| = \sqrt{(1)^2+(-9)^2+(-7)^2} = \sqrt{1+81+49} = \sqrt{131}$$

$$\therefore \perp \text{ distance between } P \text{ and } \Pi = \left|\frac{\vec{PQ}\cdot\vec{n}}{|\vec{n}|}\right| = \left|\frac{-21}{\sqrt{131}}\right| = \frac{21}{\sqrt{131}} = 1.834 \approx 1.83.$$

iii) Let, angle between direction vector of line and normal vector of plane be α.

$$\vec{b} \cdot \vec{n} = |\vec{b}|\,|\vec{n}| \cos\alpha$$

$$\begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -9 \\ -7 \end{bmatrix} = \sqrt{2^2+1^2+(-4)^2}\,\sqrt{1^2+(-9)^2+(-7)^2}\,\cos\alpha$$

$$21 = \sqrt{21}\,\sqrt{131}\,\cos\alpha$$

$$\cos\alpha = \frac{21}{\sqrt{21}\,\sqrt{131}}$$

$$\alpha = \cos^{-1}\left(\frac{21}{\sqrt{21}\,\sqrt{131}}\right) = 66.39^\circ \approx 66.4^\circ$$

$\therefore$ Acute angle between l and Π : $90^\circ - \alpha = 23.6^\circ$.

39 The plane Π has equation

$$\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) + \mu(3\mathbf{i} + \mathbf{j} - 2\mathbf{k}).$$

The line l, which does not lie in Π, has equation

$$\mathbf{r} = 3\mathbf{i} + 6\mathbf{j} + 12\mathbf{k} + t(8\mathbf{i} + 5\mathbf{j} - 8\mathbf{k}).$$

Show that l is parallel to Π. [4]

Find the position vector of the point at which the line with equation $\mathbf{r} = 5\mathbf{i} - 4\mathbf{j} + 7\mathbf{k} + s(2\mathbf{i} - \mathbf{j} + \mathbf{k})$ meets Π. [4]

Find the perpendicular distance from the point with position vector $9\mathbf{i} + 11\mathbf{j} + 2\mathbf{k}$ to l. [4]

Π contains vectors $\begin{bmatrix}1\\-2\\2\end{bmatrix}$ and $\begin{bmatrix}3\\1\\-2\end{bmatrix}$ $\therefore$ its normal vector $\vec{n} = \begin{bmatrix}1\\-2\\2\end{bmatrix} \times \begin{bmatrix}3\\1\\-2\end{bmatrix}$.

$$\vec{n} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -2 & 2\\ 3 & 1 & -2\end{vmatrix} = \begin{bmatrix}2\\8\\7\end{bmatrix} = 2\vec{i} + 8\vec{j} + 7\vec{k}.$$

For line l, direction vector $= 8\vec{i} + 5\vec{j} - 8\vec{k}$.

$$\begin{bmatrix}2\\8\\7\end{bmatrix} \cdot \begin{bmatrix}8\\5\\-8\end{bmatrix} = 16 + 40 - 56 = 0.$$

Since normal of Π $\perp$ to direction vector of L, then L is parallel to Π.

$$\vec{r} = 5\vec{i} - 4\vec{j} + 7\vec{k} + s(2\vec{i} - \vec{j} + \vec{k}) = \begin{bmatrix}5+2s\\-4-s\\7+s\end{bmatrix}$$

$$\therefore x = 5 + 2s,\ y = -4 - s,\ z = 7 + s.$$

At point of intersection with Π, parametric expression satisfy cartesian equation of Π.

Cartesian equation of Π:

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$$\vec{r} \cdot (2\vec{i} + 8\vec{j} + 7\vec{k}) = (2\vec{i} + 3\vec{j} - \vec{k}) \cdot (2\vec{i} + 8\vec{j} + 7\vec{k})$$

$$2x + 8y + 7z = 21.$$

Substituting gives $2(5+2s) + 8(-4-s) + 7(7+s) = 21$

$$10 + 4s - 32 - 8s + 49 + 7s = 21$$

$$3s = -6$$

$$s = -2.$$

$\therefore$ Point of intersection: $x = 5 + 2(-2) = 1$

$$y = -4 - (-2) = -2.$$

$$z = 7 + (-2) = 5$$

$\therefore$ Position vector of point of intersection $= \vec{i} - 2\vec{j} + 5\vec{k}$.

Let, $P(9,11,2)$ and $A(3,6,12)$

$$\vec{AP} = \vec{OP} - \vec{OA} = \begin{bmatrix} 9 \\ 11 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \\ 12 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ -10 \end{bmatrix} = 6\vec{i} + 5\vec{j} - 10\vec{k}.$$

$$\hat{u} = \frac{8\vec{i} + 5\vec{j} - 8\vec{k}}{\sqrt{8^2+5^2+(-8)^2}} = \frac{8\vec{i} + 5\vec{j} - 8\vec{k}}{\sqrt{153}}.$$

$\therefore \perp$ distance from P to L

$$= |\vec{AP} \times \hat{u}| = \left| \begin{bmatrix} 6 \\ 5 \\ -10 \end{bmatrix} \times \frac{1}{\sqrt{153}} \begin{bmatrix} 8 \\ 5 \\ -8 \end{bmatrix} \right|$$

$$= \frac{1}{\sqrt{153}} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & 5 & -10 \\ 8 & 5 & -8 \end{vmatrix} = \frac{1}{\sqrt{153}} \left| \begin{bmatrix} 10 \\ -32 \\ -10 \end{bmatrix} \right| = \frac{\sqrt{10^2+(-32)^2+(-10)^2}}{(\sqrt{153})}$$

$$= \frac{6\sqrt{34}}{\sqrt{153}} = 2\sqrt{2} \approx 2.828 \approx 2.83.$$

40 The points A, B and C have position vectors $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $2\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ respectively. Find $\overrightarrow{AB} \times \overrightarrow{AC}$. [3]

Deduce, in either order, the exact value of

(i) the area of the triangle ABC,

(ii) the perpendicular distance from C to AB. [3]

$$\vec{AB} = \vec{OB} - \vec{OA} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{AC} = \vec{OC} - \vec{OA} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \vec{i} + \vec{j} - \vec{k}.$$

i) Area of ΔABC

$$= \frac{1}{2} \left| \vec{AB} \times \vec{AC} \right| = \frac{1}{2} \sqrt{1^2 + 1^2 + (-1)^2} = \frac{1}{2} \sqrt{3} \text{ sq. units.}$$

ii) Area of $\Delta ABC = \frac{1}{2} \times AB \times$ (Perpendicular distance from C to AB)

$$\frac{1}{2}\sqrt{3} = \frac{1}{2} \times |\vec{AB}| \times d$$

$$\sqrt{3} = \sqrt{1^2 + 2^2 + 3^2} \times d$$

$$\sqrt{3} = \sqrt{14} \times d$$

$$\Rightarrow d = \frac{\sqrt{3}}{\sqrt{14}} = \sqrt{\frac{3}{14}} = 0.4629 \approx 0.463.$$

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41 The plane Π_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$. Find a cartesian equation of Π_1. [3]

The plane Π_2 has equation $2x - y + z = 10$. Find the acute angle between Π_1 and Π_2. [2]

Find an equation of the line of intersection of Π_1 and Π_2, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. [5]

Π_1 : Plane has a normal vector $\perp$ to $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$.

$$\vec{n_1} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 1 & -1 & -2 \end{vmatrix} = \begin{bmatrix} 0-(-1) \\ -(-2-1) \\ -1-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}.$$

Cartesian equation of Π_1: $\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$

$$\vec{r}\cdot(\vec{i}+3\vec{j}-\vec{k}) = (2\vec{i}+3\vec{j}-\vec{k})\cdot(\vec{i}+3\vec{j}-\vec{k})$$

$$x+3y-z = 12$$

Π_2 : $2x-y+z = 10 \Rightarrow \vec{r}\cdot(2\vec{i}-\vec{j}+\vec{k}) = 10 \quad \therefore \vec{n_2} = 2\vec{i}-\vec{j}+\vec{k}$.

Let, ~~acute~~ angle between Π_1 and Π_2 be θ.

$$\vec{n_1}\cdot\vec{n_2} = |\vec{n_1}||\vec{n_2}|\cos\theta$$

$$(\vec{i}+3\vec{j}-\vec{k})\cdot(2\vec{i}-\vec{j}+\vec{k}) = \sqrt{1^2+3^2+(-1)^2}\sqrt{2^2+(-1)^2+1^2}\cos\theta$$

$$-2 = \sqrt{11}\sqrt{6}\cos\theta$$

$$\cos\theta = \frac{-2}{\sqrt{11}\sqrt{6}} \Rightarrow \theta = \cos^{-1}\left(\frac{-2}{\sqrt{11}\sqrt{6}}\right) = 104.25^\circ$$

$\therefore$ Acute angle between Π_1 and Π_2 $= 180^\circ - 104.25^\circ = 75.74^\circ \approx 75.7^\circ$.

Π_1: $x+3y-z=12 \Rightarrow x = 12-3y+z \rightarrow (3)$

Π_2: $2x-y+z = 10 \rightarrow (2)$

Substitute (3) in (2)

$$2(12-3y+z) - y + z = 10$$

$$24 - 6y + 2z - y + z - 10 = 0$$

$$-7y + 3z + 14 = 0$$

$$y = \frac{3z+14}{7}$$

Let, $z = t \Rightarrow y = \frac{3t+14}{7}$

$$\therefore x = 12 - 3\left(\frac{3t+14}{7}\right) + t = 12 - \frac{9t}{7} - 6 + t = 6 - \frac{2t}{7}.$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 - 2t/7 \\ 2 + 3t/7 \\ t \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2/7 \\ 3/7 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ 7 \end{bmatrix}$$

i.e. Line of intersection, $\vec{r} = \begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ 7 \end{bmatrix}$

42 The points A, B, C have position vectors

$$4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}, \qquad 5\mathbf{i} + 7\mathbf{j} + 8\mathbf{k}, \qquad 2\mathbf{i} + 6\mathbf{j} + 4\mathbf{k},$$

respectively, relative to the origin O. Find a cartesian equation of the plane ABC. [4]

The point D has position vector $6\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$. Find the coordinates of E, the point of intersection of the line OD with the plane ABC. [4]

Find the acute angle between the line ED and the plane ABC. [3]

$\vec{OA} = 4\vec{i}+5\vec{j}+6\vec{k}$, $\vec{OB} = 5\vec{i}+7\vec{j}+8\vec{k}$, $\vec{OC} = 2\vec{i}+6\vec{j}+4\vec{k}$.

$$\vec{AB} = \vec{OB} - \vec{OA} = \begin{bmatrix}5\\7\\8\end{bmatrix} - \begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}1\\2\\2\end{bmatrix}, \quad \vec{AC} = \vec{OC} - \vec{OA} = \begin{bmatrix}2\\6\\4\end{bmatrix} - \begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}-2\\1\\-2\end{bmatrix}$$

$$\therefore \vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 2 & 2\\ -2 & 1 & -2\end{vmatrix} = \begin{bmatrix}-6\\-2\\5\end{bmatrix} = -6\vec{i}-2\vec{j}+5\vec{k}.$$

$\therefore$ Equation of plane ABC : $\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$

$$\vec{r}\cdot(-6\vec{i}-2\vec{j}+5\vec{k}) = (4\vec{i}+5\vec{j}+6\vec{k})\cdot(-6\vec{i}-2\vec{j}+5\vec{k})$$

$$-6x-2y+5z = -4$$

$$\Rightarrow \quad 6x+2y-5z = 4.$$

Line $\vec{OD}$: $\vec{r} = (0\vec{i}+0\vec{j}+0\vec{k}) + t(6\vec{i}+3\vec{j}+6\vec{k}) = (6t)\vec{i}+(3t)\vec{j}+(6t)\vec{k}$.

$\therefore \; x = 6t, \; y = 3t, \; z = 6t.$

At point of intersection with plane ABC,

$$6x+2y-5z = 4$$

$$6(6t)+2(3t)-5(6t) = 4$$

$$12t = 4 \Rightarrow t = \frac{1}{3}.$$

$$\therefore \vec{r} = \left(6\times\frac{1}{3}\right)\vec{i}+\left(3\times\frac{1}{3}\right)\vec{j}+\left(6\times\frac{1}{3}\right)\vec{k} = 2\vec{i}+\vec{j}+2\vec{k}.$$

i.e. point of intersection $E = (2,1,2)$.

$$\vec{ED} = \vec{OD} - \vec{OE} = (6\vec{i}+3\vec{j}+6\vec{k}) - (2\vec{i}+\vec{j}+2\vec{k}) = 4\vec{i}+2\vec{j}+4\vec{k}.$$

Let, the angle between $\vec{ED}$ (direction vector of line ED) and $\vec{n}$ be α.

$$\vec{ED}\cdot\vec{n} = |\vec{ED}||\vec{n}|\cos\alpha$$

$$\begin{bmatrix}4\\2\\4\end{bmatrix}\cdot\begin{bmatrix}-6\\-2\\5\end{bmatrix} = \sqrt{4^2+2^2+4^2}\,\sqrt{(-6)^2+(-2)^2+5^2}\,\cos\alpha$$

$$-8 = 6\sqrt{65}\cos\alpha \Rightarrow \alpha = \cos^{-1}\left[\frac{-8}{6\sqrt{65}}\right] = 99.51^\circ$$

$\Rightarrow$ Angle between line ED and plane ABC

$= \alpha - 90^\circ = 99.519^\circ - 90^\circ = 9.519^\circ \approx 9.52^\circ$

43 The line l_1 is parallel to the vector $\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$ and passes through the point A, whose position vector is $3\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$. The line l_2 is parallel to the vector $-2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and passes through the point B, whose position vector is $-3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Find

(i) the length PQ, [5]

(ii) the cartesian equation of the plane Π containing PQ and l_2, [4]

(iii) the perpendicular distance of A from Π. [3]

i) $l_1 : \vec{r} = 3\vec{i} + 3\vec{j} - 4\vec{k} + t(\vec{i} - 2\vec{j} - 3\vec{k})$, where $\vec{a_1} = 3\vec{i} + 3\vec{j} - 4\vec{k}$, $\vec{b_1} = \vec{i} - 2\vec{j} - 3\vec{k}$.

$l_2 : \vec{r} = -3\vec{i} - \vec{j} + 2\vec{k} + s(-2\vec{i} + \vec{j} + 3\vec{k})$ and $\vec{a_2} = -3\vec{i} - \vec{j} + 2\vec{k}$, $\vec{b_2} = -2\vec{i} + \vec{j} + 3\vec{k}$.

$$\vec{a_2} - \vec{a_1} = \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} -6 \\ -4 \\ 6 \end{bmatrix}.$$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & -3 \\ -2 & 1 & 3 \end{vmatrix} = \begin{bmatrix} -6+3 \\ -(3-6) \\ 1-4 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ -3 \end{bmatrix} \sim \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = \begin{bmatrix} -6 \\ -4 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = -6 + 4 + 6 = 4.$$

$$|\vec{b_1} \times \vec{b_2}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}.$$

$\therefore$ Length $PQ = \perp$ distance between l_1 and $l_2 = \left| \dfrac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right| = \dfrac{4}{\sqrt{3}}$.

ii) If plane Π ~~PQ~~ contains PQ, then it contains the vector $\vec{i} - \vec{j} + \vec{k}$.

Similarly, if plane Π contains l_2, then it contains vector $-2\vec{i} + \vec{j} + 3\vec{k}$.

Then, normal to Π, $\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ -2 & 1 & 3 \end{vmatrix} = \begin{bmatrix} -4 \\ -5 \\ -1 \end{bmatrix} \sim \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$.

Π contains the point B (a point on line l_2) with position vector $\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}$

$\therefore$ Equation of Π: $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

$$\vec{r} \cdot (4\vec{i} + 5\vec{j} + \vec{k}) = (-3\vec{i} - \vec{j} + 2\vec{k}) \cdot (4\vec{i} + 5\vec{j} + \vec{k})$$

$$4x + 5y + z = -12 - 5 + 2$$

$$4x + 5y + z = -15$$

iii) B is a point on Π. $\vec{AB} = \vec{OB} - \vec{OA} = \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} -6 \\ -4 \\ 6 \end{bmatrix}$

$$\vec{AB} \cdot \vec{n} = \begin{bmatrix} -6 \\ -4 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} = -24 - 20 + 6 = -38.$$

$$|\vec{n}| = \sqrt{4^2+5^2+1^2} = \sqrt{42}.$$

$$\therefore \perp \text{ distance from } A \text{ to } \Pi = \left| \frac{\vec{AB}\cdot\vec{n}}{|\vec{n}|} \right|$$

$$= \left| \frac{-38}{\sqrt{42}} \right| = \frac{38}{\sqrt{42}} = 5.863 \approx 5.86.$$

44 The points A, B and C have position vectors $\mathbf{i}$, $2\mathbf{j}$ and $4\mathbf{k}$ respectively, relative to an origin O. The point N is the foot of the perpendicular from O to the plane ABC. The point P on the line-segment ON is such that $OP = \frac{3}{4}ON$. The line AP meets the plane OBC at Q. Find a vector perpendicular to the plane ABC and show that the length of ON is $\frac{4}{\sqrt{(21)}}$. [4]

Find the position vector of the point Q. [5]

Show that the acute angle between the planes ABC and ABQ is $\cos^{-1}(\frac{2}{3})$. [5]

$\vec{OA} = \vec{i}$, $\vec{OB} = 2\vec{j}$, $\vec{OC} = 4\vec{k}$.

$\vec{AB} = \vec{OB} - \vec{OA} = 2\vec{j} - \vec{i} = -\vec{i} + 2\vec{j}$

$\vec{AC} = \vec{OC} - \vec{OA} = 4\vec{k} - \vec{i} = -\vec{i} + 4\vec{k}$

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 0 \\ -1 & 0 & 4 \end{vmatrix} = \begin{bmatrix} 8 \\ 4 \\ 2 \end{bmatrix} \sim \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

i.e. Vector perpendicular to plane ABC $= 4\vec{i} + 2\vec{j} + \vec{k}$.

Shortest distance from O to plane ABC

$$= \left| \frac{\vec{OA} \cdot \vec{n}}{|\vec{n}|} \right| = \left| \frac{\vec{i} \cdot (4\vec{i} + 2\vec{j} + \vec{k})}{\sqrt{4^2 + 2^2 + 1^2}} \right| = \frac{4}{\sqrt{21}} \text{ (Shown)}$$

Since $\vec{ON} \perp$ Plane ABC $\therefore \vec{ON}$ lies in the direction of $4\vec{i} + 2\vec{j} + \vec{k} = \vec{n}$.

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{4\vec{i} + 2\vec{j} + \vec{k}}{\sqrt{4^2 + 2^2 + 1^2}} = \frac{4\vec{i} + 2\vec{j} + \vec{k}}{\sqrt{21}}$$

$$\therefore \vec{ON} = \cancel{|\vec{n}|} \, |\vec{ON}| \times \hat{n} = \frac{4}{\sqrt{21}} \times \frac{4\vec{i} + 2\vec{j} + \vec{k}}{\sqrt{21}} = \frac{4(4\vec{i} + 2\vec{j} + \vec{k})}{21}.$$

Since $OP = \frac{3}{4}ON \therefore \vec{OP} = \frac{3}{4} \times \vec{ON} = \frac{3}{4} \times \frac{4(4\vec{i} + 2\vec{j} + \vec{k})}{21} = \frac{4\vec{i} + 2\vec{j} + \vec{k}}{7}$.

$$\vec{AP} = \vec{OP} - \vec{OA} = \frac{4\vec{i} + 2\vec{j} + \vec{k}}{7} - \vec{i} = \frac{-3\vec{i} + 2\vec{j} + \vec{k}}{7}$$

$\therefore$ Equation of AP: $\vec{r} = \vec{OA} + t\,\vec{AP} = \vec{i} + t\left[\frac{-3\vec{i} + 2\vec{j} + \vec{k}}{7}\right]$

$$\Rightarrow \vec{r} = \vec{i} + t(-3\vec{i} + 2\vec{j} + \vec{k}).$$

$$\vec{r} = \begin{bmatrix} 1 - 3t \\ 2t \\ t \end{bmatrix}$$

Plane OBC : $\vec{r} = \lambda(2\vec{j}) + \mu(4\vec{k})$.

$\therefore$ normal vector $= \vec{i}$ $\therefore$ equation is of the form $x = 0$.

Then when line AP meets plane OBC, $x=0$

$$\Rightarrow 1-3t=0 \Rightarrow t=\frac{1}{3}.$$

$$\therefore \text{Point of intersection, } \vec{r} = \begin{bmatrix} 1-3\left(\frac{1}{3}\right) \\ 2\left(\frac{1}{3}\right) \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$\therefore$ Q has position vector $= \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}$.

$\longleftrightarrow$

For plane ABQ,

$$\vec{AB} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad \vec{AQ} = \vec{OQ} - \vec{OA} = \begin{bmatrix} 0 \\ 2/3 \\ 1/3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$\vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 0 \\ -1 & 2/3 & 1/3 \end{vmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 4/3 \end{bmatrix} \sim \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

For plane ABC, $\vec{n_1} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$

Let, acute angle between ABQ and ABC be θ.

$$\vec{n_1} \cdot \vec{n_2} = |\vec{n_1}|\,|\vec{n_2}| \cos\theta$$

$$\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \sqrt{4^2+2^2+1^2}\sqrt{2^2+1^2+4^2}\cos\theta$$

$$14 = \sqrt{21}\sqrt{21}\cos\theta$$

$$\theta = \cos^{-1}\left(\frac{14}{\sqrt{21}\sqrt{21}}\right) = \cos^{-1}\left(\frac{14}{21}\right)$$

$$\theta = \cos^{-1}\left(\frac{2}{3}\right). \text{ (Shown)}.$$

45 The lines l_1 and l_2 have equations

$$\mathbf{r} = 6\mathbf{i} - 3\mathbf{j} + s(3\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}) \quad \text{and} \quad \mathbf{r} = 2\mathbf{i} - \mathbf{j} - 4\mathbf{k} + t(\mathbf{i} - 3\mathbf{j} - \mathbf{k})$$

respectively. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Show that the position vector of P is $3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and find the position vector of Q. [7]

Find, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, an equation of the plane Π which passes through P and is perpendicular to l_1. [3]

The plane Π meets the plane $\mathbf{r} = p\mathbf{i} + q\mathbf{j}$ in the line l_3. Find a vector equation of l_3. [4]

$l_1: \vec{r} = \begin{pmatrix} 6+3s \\ -3-4s \\ -2s \end{pmatrix}$. Since P lies on l_1 $\therefore \overrightarrow{OP} = \begin{bmatrix} 6+3s \\ -3-4s \\ -2s \end{bmatrix}$.

$l_2: \vec{r} = \begin{pmatrix} 2+t \\ -1-3t \\ -4-t \end{pmatrix}$. Since Q lies on l_2 $\therefore \overrightarrow{OQ} = \begin{bmatrix} 2+t \\ -1-3t \\ -4-t \end{bmatrix}$.

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{bmatrix} 2+t \\ -1-3t \\ -4-t \end{bmatrix} - \begin{bmatrix} 6+3s \\ -3-4s \\ -2s \end{bmatrix} = \begin{bmatrix} -4+t-3s \\ 2-3t+4s \\ -4-t+2s \end{bmatrix}$.

Since $\overrightarrow{PQ} \perp l_1$ $\therefore \overrightarrow{PQ} \cdot (3\vec{i}-4\vec{j}-2\vec{k}) = 0 \Rightarrow 3(-4+t-3s)-4(2-3t+4s)-2(-4-t+2s)=0$

$-12+3t-9s-8+12t-16s+8+2t-4s=0$

$-12+17t-29s=0 \rightarrow (1)$

Since $\overrightarrow{PQ} \perp l_2$, $\overrightarrow{PQ} \cdot (\vec{i}-3\vec{j}-\vec{k}) = 0 \Rightarrow 1(-4+t-3s)-3(2-3t+4s)-1(-4-t+2s)=0$

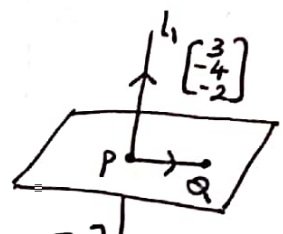
$-4+t-3s-6+9t-12s+4+t-2s=0$

$-6+11t-17s=0 \rightarrow (2)$

Solving simultaneously gives $s = t = -1$.

$\therefore \overrightarrow{OP} = \begin{bmatrix} 6+3(-1) \\ -3-4(-1) \\ -2(-1) \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = 3\vec{i}+\vec{j}+2\vec{k}$.

$\overrightarrow{OQ} = \begin{bmatrix} 2+(-1) \\ -1-3(-1) \\ -4-(-1) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \vec{i}+2\vec{j}-3\vec{k}$.



For Π, the vector $3\vec{i}-4\vec{j}-2\vec{k}$ is a normal vector. $\overrightarrow{PQ}$ is $\perp$ to l_1 and so lies in Π. We can find a vector parallel to the plane by finding $\overrightarrow{PQ} \times 3\vec{i}-4\vec{j}-2\vec{k}$. where

$\overrightarrow{PQ} = \begin{bmatrix} -4-1+3 \\ 2+3-4 \\ -4+1-2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix}$

$\overrightarrow{PQ} \times (3\vec{i}-4\vec{j}-2\vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 1 & -5 \\ 3 & -4 & -2 \end{vmatrix}$

$= \begin{bmatrix} -2-(20) \\ -(4-(-15)) \\ 8-3 \end{bmatrix} = \begin{bmatrix} -22 \\ -19 \\ 5 \end{bmatrix}$

Then plane passes through point P with position vector $\begin{bmatrix}3\\1\\2\end{bmatrix}$ and has direction vectors $\begin{bmatrix}-22\\-19\\5\end{bmatrix}$ and $\begin{bmatrix}2\\-1\\5\end{bmatrix}$.

$\Rightarrow$ Equation of plane Π: $\vec{r} = \begin{bmatrix}3\\1\\2\end{bmatrix} + \lambda\begin{bmatrix}-22\\-19\\5\end{bmatrix} + \mu\begin{bmatrix}2\\-1\\5\end{bmatrix}$.

Cartesian equation of Π:

$$\vec{n} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -22 & -19 & 5\\ 2 & -1 & 5\end{vmatrix} = \begin{bmatrix}-90\\120\\60\end{bmatrix} \sim \begin{bmatrix}-3\\4\\2\end{bmatrix}$$

$$\therefore \vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$$

$$\vec{r}\cdot(-3\vec{i}+4\vec{j}+2\vec{k}) = (3\vec{i}+\vec{j}+2\vec{k})\cdot(-3\vec{i}+4\vec{j}+2\vec{k})$$

$$-3x+4y+2z=-1 \rightarrow (3)$$

For plane $\vec{r} = p\vec{i}+q\vec{j}$,

$$\vec{n} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & 0\\ 0 & 1 & 0\end{vmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\Rightarrow \vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$$

$$\vec{r}\cdot(\vec{k}) = (0\vec{i}+0\vec{j}+0\vec{k})\cdot(\vec{k})$$

$$z = 0 \rightarrow (4)$$

Substituting (4) in (3):

$$-3x+4y+2(0) = -1.$$

$$-3x+4y=-1$$

$$4y = 3x-1$$

$x = t \Rightarrow 4y = 3t-1 \Rightarrow y = \frac{1}{4}\left[3t-1\right] = \frac{3}{4}t - \frac{1}{4}$.

$$\therefore \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}t\\ \frac{3}{4}t-\frac{1}{4}\\ 0\end{bmatrix} = \begin{bmatrix}0\\ \frac{-1}{4}\\ 0\end{bmatrix} + t\begin{bmatrix}1\\ \frac{3}{4}\\ 0\end{bmatrix} = \begin{bmatrix}0\\ \frac{-1}{4}\\ 0\end{bmatrix} + t\begin{bmatrix}4\\3\\0\end{bmatrix}$$

$\therefore$ Vector equation of line L_3:

$$\vec{r} = \begin{bmatrix}0\\-1/4\\0\end{bmatrix} + t\begin{bmatrix}4\\3\\0\end{bmatrix}$$

46 The plane Π_1 has equation

$$\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

(i) Find a cartesian equation of Π_1. [3]

Let, $\vec{n_1}$ be a normal vector of Π_1.

$$\vec{n_1} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4 & 1 & 3 \\ 0 & 1 & 2 \end{vmatrix} = \begin{bmatrix} -1 \\ -(-8) \\ -4 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ -4 \end{bmatrix} = -\vec{i}+8\vec{j}-4\vec{k}.$$

$\therefore$ Equation of Π_1: $\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$

$$\vec{r}\cdot(-\vec{i}+8\vec{j}-4\vec{k}) = (5\vec{i}+\vec{j})\cdot(-\vec{i}+8\vec{j}-4\vec{k})$$

$$-x+8y-4z = -5+8$$

$$-x+8y-4z = 3.$$

The plane Π_2 has equation $3x + y - z = 3$.

(ii) Find the acute angle between Π_1 and Π_2, giving your answer in degrees. [2]

Π_1: $-x+8y-4z=3 \Rightarrow \vec{n_1} = -\vec{i}+8\vec{j}-4\vec{k}$.

Π_2: $3x+y-z=3 \Rightarrow \vec{n_2} = 3\vec{i}+\vec{j}-\vec{k}$.

Let, acute angle between Π_1 and Π_2 be θ.

$$\vec{n_1}\cdot\vec{n_2} = |\vec{n_1}||\vec{n_2}|\cos\theta$$

$$\begin{bmatrix} -1 \\ 8 \\ -4 \end{bmatrix}\cdot\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \sqrt{(-1)^2+8^2+(-4)^2}\sqrt{3^2+1^2+(-1)^2}\cos\theta$$

$$-3+8+4 = \sqrt{81}\sqrt{11}\cos\theta$$

$$\theta = \cos^{-1}\left(\frac{9}{\sqrt{81}\sqrt{11}}\right) = 72.45^\circ$$

$$\theta \approx 72.5^\circ$$

(iii) Find an equation of the line of intersection of Π_1 and Π_2, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. [5]

$\Pi_1 : -x + 8y - 4z = 3 \rightarrow (1)$

$\Pi_2 : 3x + y - z = 3 \rightarrow (2)$

From (1), $-x = 3 - 8y + 4z \Rightarrow x = -3 + 8y - 4z \rightarrow (3)$

Substituting (3) in (2):

$$3(-3+8y-4z) + y - z = 3$$

$$-9 + 24y - 12z + y - z = 3$$

$$25y - 13z = 12$$

$$\Rightarrow 13z = 25y - 12$$

$$z = \frac{25}{13}y - \frac{12}{13}$$

Let, $y = t$

$$\Rightarrow z = \frac{25}{13}t - \frac{12}{13}.$$

Substituting in (3)

$$x = -3 + 8t - 4\left(\frac{25}{13}t - \frac{12}{13}\right) = -3 + 8t - \frac{100t}{13} + \frac{48}{13}$$

$$x = \cancel{\frac{33}{5} - 12t} \; \frac{9}{13} + \frac{4t}{13}.$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{9}{13} + \frac{4t}{13} \\ t \\ -\frac{12}{13} + \frac{25t}{13} \end{bmatrix} = \begin{bmatrix} \frac{9}{13} \\ 0 \\ \frac{-12}{13} \end{bmatrix} + t\begin{bmatrix} \frac{4}{13} \\ 1 \\ \frac{25}{13} \end{bmatrix} = \begin{bmatrix} \frac{9}{13} \\ 0 \\ \frac{-12}{13} \end{bmatrix} + t\begin{bmatrix} 4 \\ 13 \\ 25 \end{bmatrix}$$

i.e. equation of line of intersection

$$\Rightarrow \vec{r} = \begin{bmatrix} 9/13 \\ 0 \\ -12/13 \end{bmatrix} + t\begin{bmatrix} 4 \\ 13 \\ 25 \end{bmatrix}$$

Alternatively, direction vector of line of intersection is $\perp$ to $\vec{n_1}$ and $\vec{n_2}$.

$$\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 8 & -4 \\ 3 & 1 & -1 \end{vmatrix} = \begin{bmatrix} -8-(-4) \\ -(1-(-12)) \\ -1-24 \end{bmatrix} = \begin{bmatrix} -4 \\ -13 \\ -25 \end{bmatrix} \sim \begin{bmatrix} 4 \\ 13 \\ 25 \end{bmatrix}$$

$\Pi_1 : -x + 8y - 4z = 3$

$\Pi_2 : 3x + y - z = 3$

Let, $y = 0 \Rightarrow -x - 4z = 3 \Rightarrow x = -3 - 4z \rightarrow (3)$

$$3x - z = 3 \rightarrow (4)$$

Substitute (3) in (4)

$$3(-3-4z) - z = 3$$

$$-9 - 12z - z = 3$$

$$-13z = 3 + 9$$

$$z = \frac{-12}{13}.$$

$$\therefore x = -3 - 4\left(\frac{-12}{13}\right) = \frac{9}{13}.$$

$\therefore \left(\frac{9}{13}, 0, \frac{-12}{13}\right)$ is a point that lies on both planes.

$\therefore$ Line of intersection of Π_1 and Π_2:

$$\vec{r} = \begin{bmatrix} 9/13 \\ 0 \\ -12/13 \end{bmatrix} + t \begin{bmatrix} 4 \\ 13 \\ 25 \end{bmatrix}.$$

47 The position vectors of the points A, B, C, D are

$$\mathbf{i}+\mathbf{j}+3\mathbf{k}, \quad 3\mathbf{i}+4\mathbf{j}+5\mathbf{k}, \quad -\mathbf{i}+3\mathbf{k}, \quad m\mathbf{j}+4\mathbf{k},$$

respectively, where m is a constant.

(i) Show that the lines AB and CD are parallel when $m = \frac{3}{2}$. [1]

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{bmatrix}3\\4\\5\end{bmatrix} - \begin{bmatrix}1\\1\\3\end{bmatrix} = \begin{bmatrix}2\\3\\2\end{bmatrix}.$$

$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \begin{bmatrix}0\\m\\4\end{bmatrix} - \begin{bmatrix}-1\\0\\3\end{bmatrix} = \begin{bmatrix}1\\m\\1\end{bmatrix}.$$

With $m = \frac{3}{2}$, $\overrightarrow{CD} = \begin{bmatrix}1\\3/2\\1\end{bmatrix}$. But then $\overrightarrow{CD} = \frac{1}{2}\overrightarrow{AB}$ $\therefore$ $CD \parallel AB$.

(ii) Given that $m \neq \frac{3}{2}$, find the shortest distance between the lines AB and CD. [5]

line AB: $\vec{r} = \begin{bmatrix}1\\1\\3\end{bmatrix} + \lambda\begin{bmatrix}2\\3\\2\end{bmatrix}$ where $\vec{a_1} = \begin{bmatrix}1\\1\\3\end{bmatrix}$, $\vec{b_1} = \begin{bmatrix}2\\3\\2\end{bmatrix}$.

line CD: $\vec{r} = \begin{bmatrix}-1\\0\\3\end{bmatrix} + \mu\begin{bmatrix}1\\m\\1\end{bmatrix}$ where $\vec{a_2} = \begin{bmatrix}-1\\0\\3\end{bmatrix}$, $\vec{b_2} = \begin{bmatrix}1\\m\\1\end{bmatrix}$

$$\vec{a_2} - \vec{a_1} = \begin{bmatrix}-1\\0\\3\end{bmatrix} - \begin{bmatrix}1\\1\\3\end{bmatrix} = \begin{bmatrix}-2\\-1\\0\end{bmatrix}$$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 3 & 2\\ 1 & m & 1\end{vmatrix} = \begin{bmatrix}3-2m\\0\\2m-3\end{bmatrix} = \begin{bmatrix}-(2m-3)\\0\\(2m-3)\end{bmatrix} \sim \begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = (-2\vec{i} - \vec{j}) \cdot (-\vec{i} + \vec{k}) = -2(-1) - 1(0) + 0(1) = 2.$$

$$|\vec{b_1} \times \vec{b_2}| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}.$$

$\therefore$ Shortest distance between lines AB and CD

$$= \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_2} \times \vec{b_1}|} \right| = \frac{2}{\sqrt{2}} = \sqrt{2} = 1.414 \approx 1.41$$

Alternatively, if $\vec{b_1} \times \vec{b_2}$ is not simplified $\vec{b_1} \times \vec{b_2} = \begin{bmatrix}-(2m-3)\\0\\2m-3\end{bmatrix}$.

$$(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = (-2\vec{i} - \vec{j}) \cdot [-(2m-3)\vec{i} + (2m-3)\vec{k}]$$
$$= -2(-1)(2m-3) = 2(2m-3).$$

$$|\vec{b_1} \times \vec{b_2}| = \sqrt{[-(2m-3)]^2 + [2m-3]^2} = \sqrt{2}(2m-3).$$

Shortest distance between lines AB and CD

$$= \left| \frac{(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})}{|\vec{b_1}\times\vec{b_2}|} \right| = \left| \frac{2(2m-3)}{\sqrt{2}\,(2m-3)} \right| = \frac{2}{\sqrt{2}} = \sqrt{2} \approx 1.41.$$

(iii) When $m = 2$, find the acute angle between the planes ABC and ABD, giving your answer in degrees. [6]

For plane ABC,

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (3\vec{i}+4\vec{j}+5\vec{k})-(\vec{i}+\vec{j}+3\vec{k}) = 2\vec{i}+3\vec{j}+2\vec{k}$.

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (-\vec{i}+3\vec{k}) - (\vec{i}+\vec{j}+3\vec{k}) = -2\vec{i}-\vec{j}$

$$\therefore \vec{n_1} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 2 \\ -2 & -1 & 0 \end{vmatrix} = \begin{bmatrix} 0-(-2) \\ -(0-(-4)) \\ -2-(-6) \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix} \sim \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

For plane ABD,

$\overrightarrow{AB} = 2\vec{i}+3\vec{j}+2\vec{k}$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = (2\vec{j}+4\vec{k}) - (\vec{i}+\vec{j}+3\vec{k}) = -\vec{i}+\vec{j}+\vec{k}$.

$$\therefore \vec{n_2} = \overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 2 \\ -1 & 1 & 1 \end{vmatrix} = \begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix}.$$

Let, acute angle between planes ABC and ABD be θ.

$$\vec{n_1} \cdot \vec{n_2} = |\vec{n_1}||\vec{n_2}| \cos\theta$$

$$\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix} = \sqrt{1^2+(-2)^2+2^2}\sqrt{1^2+(-4)^2+5^2} \cos\theta$$

$$19 = \sqrt{9}\sqrt{42} \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{19}{\sqrt{9}\sqrt{42}}\right)$$

$$\theta = 12.24° \approx 12.2°$$

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48 With O as the origin, the points A, B, C have position vectors

$$\mathbf{i}-\mathbf{j}, \quad 2\mathbf{i}+\mathbf{j}+7\mathbf{k}, \quad \mathbf{i}-\mathbf{j}+\mathbf{k}$$

respectively.

(i) Find the shortest distance between the lines OC and AB. [5]

Line $\vec{OC}$: $\vec{r} = \begin{bmatrix}0\\0\\0\end{bmatrix} + t\begin{bmatrix}1\\-1\\1\end{bmatrix} = t\begin{bmatrix}1\\-1\\1\end{bmatrix}$

Line AB:

$\vec{OA} = \vec{i}-\vec{j}$, $\vec{OB} = 2\vec{i}+\vec{j}+7\vec{k} \Rightarrow \vec{AB} = \vec{OB}-\vec{OA} = \begin{bmatrix}2\\1\\7\end{bmatrix} - \begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}1\\2\\7\end{bmatrix}$

Equation : $\vec{r} = \begin{bmatrix}1\\-1\\0\end{bmatrix} + s\begin{bmatrix}1\\2\\7\end{bmatrix}$

Let, equation of line OC : $\vec{r} = \vec{a_1} + t\vec{b_1}$ where $\vec{a_1} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, $\vec{b_1} = \begin{bmatrix}1\\-1\\1\end{bmatrix}$

equation of line AB : $\vec{r} = \vec{a_2} + s\vec{b_2}$ where $\vec{a_2} = \begin{bmatrix}1\\-1\\0\end{bmatrix}$, $\vec{b_2} = \begin{bmatrix}1\\2\\7\end{bmatrix}$.

$\vec{a_2} - \vec{a_1} = \begin{bmatrix}1\\-1\\0\end{bmatrix} - \begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\-1\\0\end{bmatrix}$.

$\vec{b_1} \times \vec{b_2} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -1 & 1\\ 1 & 2 & 7\end{vmatrix} = \begin{bmatrix}-7-2\\-(7-1)\\2-(-1)\end{bmatrix} = \begin{bmatrix}-9\\-6\\3\end{bmatrix} \sim \begin{bmatrix}3\\2\\-1\end{bmatrix}$.

$|\vec{b_1} \times \vec{b_2}| = \sqrt{3^2+2^2+(-1)^2} = \sqrt{9+4+1} = \sqrt{14}$.

Shortest distance between lines OC and AB

$= \left|\dfrac{(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})}{|\vec{b_1}\times\vec{b_2}|}\right| = \left|\dfrac{(\vec{i}-\vec{j})\cdot(3\vec{i}+2\vec{j}-\vec{k})}{\sqrt{14}}\right| = \left|\dfrac{3-2}{\sqrt{14}}\right| = \dfrac{1}{\sqrt{14}}$

$= 0.2672 \approx 0.267$

(ii) Find the cartesian equation of the plane containing the line OC and the common perpendicular of the lines OC and AB. [4]

The common perpendicular to lines OC and AB is $\perp$ to $\vec{b_1}$ and $\vec{b_2}$.

But we know $\vec{n_1} = \vec{b_1} \times \vec{b_2} = 3\vec{i} + 2\vec{j} - \vec{k}$.

Then our plane contains the vector $3\vec{i} + 2\vec{j} - \vec{k}$ and the direction vector of line OC: $\vec{i} - \vec{j} + \vec{k}$.

Then our plane is $\perp$ to $3\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{i} - \vec{j} + \vec{k}$.

$$\therefore \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \begin{bmatrix} 2-1 \\ -(3-(-1)) \\ -3-2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ -5 \end{bmatrix}$$

Our plane ~~considers~~ contains line OC so it contains the origin.

Then equation of plane:

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$$\vec{r} \cdot (\vec{i} - 4\vec{j} - 5\vec{k}) = (0\vec{i} + 0\vec{j} + 0\vec{k}) \cdot (\vec{i} - 4\vec{j} - 5\vec{k})$$

$$x - 4y - 5z = 0 - 0 - 0$$

$$x - 4y - 5z = 0.$$

49 The points A, B, C have position vectors

$$-\mathbf{i}+\mathbf{j}+2\mathbf{k}, \quad -2\mathbf{i}-\mathbf{j}, \quad 2\mathbf{i}+2\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

$\overrightarrow{OA} = -\vec{i}+\vec{j}+2\vec{k}$, $\overrightarrow{OB} = -2\vec{i}-\vec{j}$, $\overrightarrow{OC} = 2\vec{i}+2\vec{k}$.

$\overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA} = \left(-2\vec{i}-\vec{j}\right)-\left(-\vec{i}+\vec{j}+2\vec{k}\right) = -\vec{i}-2\vec{j}-2\vec{k}$.

$\overrightarrow{AC} = \overrightarrow{OC}-\overrightarrow{OA} = \left(2\vec{i}+2\vec{k}\right)-\left(-\vec{i}+\vec{j}+2\vec{k}\right) = 3\vec{i}-\vec{j}$

The normal vector of plane ABC is $\perp$ to $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\therefore \vec{n} = \overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -2 & -2 \\ 3 & -1 & 0 \end{vmatrix} = \begin{bmatrix} 0-2 \\ -(0-(-6)) \\ 1-(-6) \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \\ 7 \end{bmatrix}.$$

$\therefore$ Equation of ABC:

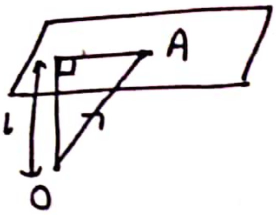
$$\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$$

$$\vec{r}\cdot\left(-2\vec{i}-6\vec{j}+7\vec{k}\right) = \left(2\vec{i}+2\vec{k}\right)\cdot\left(-2\vec{i}-6\vec{j}+7\vec{k}\right)$$

$$-2x-6y+7z = -4+14$$

$$-2x-6y+7z = 10.$$

(b) Find the perpendicular distance from O to the plane ABC. [2]



$\overrightarrow{OA} = -\vec{i}+\vec{j}+2\vec{k}$.

$\overrightarrow{OA}\cdot\vec{n} = (-\vec{i}+\vec{j}+2\vec{k})\cdot(-2\vec{i}-6\vec{j}+7\vec{k}) = 10$.

$\therefore \perp$ distance from O to plane ABC

$$= \left|\frac{\overrightarrow{OA}\cdot\vec{n}}{|\vec{n}|}\right| = \frac{10}{\sqrt{(-2)^2+(6)^2+(7)^2}} = \frac{10}{\sqrt{4+36+49}} = \frac{10}{\sqrt{89}} = 1.059 \simeq 1.06.$$

(c) Find the acute angle between the planes OAB and ABC. [4]

For plane OAB, the normal vector $\vec{n_2}$ is $\perp$ to $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

$$\vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 2 \\ -2 & -1 & 0 \end{vmatrix} = \begin{bmatrix} 0-(-2) \\ -(0-(-4)) \\ 1-(-2) \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix} = 2\vec{i}-4\vec{j}+3\vec{k}$$

For plane ABC, normal vector is $\vec{n_1} = -2\vec{i}-6\vec{j}+7\vec{k}$.

Let, acute angle between planes OAB and ABC be θ.

$$\vec{n_1}\cdot\vec{n_2} = |\vec{n_1}||\vec{n_2}|\cos\theta$$

$$(2\vec{i}-4\vec{j}+3\vec{k})\cdot(-2\vec{i}-6\vec{j}+7\vec{k}) = \sqrt{2^2+(-4)^2+3^2}\sqrt{(-2)^2+(-6)^2+7^2}\cos\theta$$

$$-4+24+21 = \sqrt{29}\sqrt{89}\cos\theta$$

$$\theta = \cos^{-1}\left[\frac{41}{\sqrt{29}\sqrt{89}}\right]$$

$$\theta = 36.19^\circ \approx 36.2^\circ.$$

50 The points A, B, C have position vectors

$$-2\mathbf{i}+2\mathbf{j}-\mathbf{k}, \quad -2\mathbf{i}+\mathbf{j}+2\mathbf{k}, \quad -2\mathbf{j}+\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

$\vec{OA} = -2\vec{i}+2\vec{j}-\vec{k}$, $\vec{OB} = -2\vec{i}+\vec{j}+2\vec{k}$, $\vec{OC} = -2\vec{j}+\vec{k}$.

$\vec{AB} = \vec{OB} - \vec{OA} = (-2\vec{i}+\vec{j}+2\vec{k}) - (-2\vec{i}+2\vec{j}-\vec{k}) = -\vec{j}+3\vec{k}$.

$\vec{AC} = \vec{OC} - \vec{OA} = (-2\vec{j}+\vec{k}) - (-2\vec{i}+2\vec{j}-\vec{k}) = 2\vec{i}-4\vec{j}+2\vec{k}$.

$$\vec{n_1} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & 3 \\ 2 & -4 & 2 \end{vmatrix} = \begin{bmatrix} -2-(-12) \\ -[0-6] \\ 0-(-2) \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \\ 2 \end{bmatrix} \sim \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

$\therefore$ Equation of ABC: $\vec{r}\cdot\vec{n_1} = \vec{a}\cdot\vec{n_1}$

$$\vec{r}\cdot[5\vec{i}+3\vec{j}+\vec{k}] = [-2\vec{i}+2\vec{j}-\vec{k}]\cdot[5\vec{i}+3\vec{j}+\vec{k}]$$

$$5x+3y+z = -10+6-1$$

$$5x+3y+z = -5.$$

(b) Find the acute angle between the planes OBC and ABC. [4]

For plane OBC, the normal vector $\vec{n_2}$ is $\perp$ to $\vec{OB}$ and $\vec{OC}$.

$$\therefore \vec{n_2} = \vec{OB} \times \vec{OC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 1 & 2 \\ 0 & -2 & 1 \end{vmatrix} = \begin{bmatrix} 1-(-4) \\ -[-2-0] \\ 4-0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$$

Let, the angle between planes OBC and ABC be θ.

$$\vec{n_1}\cdot\vec{n_2} = |\vec{n_1}||\vec{n_2}|\cos\theta$$

$$\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} = \sqrt{5^2+3^2+1^2}\sqrt{5^2+2^2+4^2}\cos\theta$$

$$25+6+4 = \sqrt{35}\sqrt{45}\cos\theta$$

$$\theta = \cos^{-1}\left(\frac{35}{\sqrt{35}\sqrt{45}}\right) = 28.12^\circ \approx 28.1^\circ$$

The point D has position vector $t\mathbf{i}-\mathbf{j}$.

(c) Given that the shortest distance between the lines AB and CD is $\sqrt{10}$, find the value of t. [6]

Line AB : $\vec{r} = \vec{OA} + \lambda \vec{AB} = -2\vec{i}+2\vec{j}-\vec{k} + \lambda\left[\left(-2\vec{i}+\vec{j}+2\vec{k}\right)-\left(-2\vec{i}+2\vec{j}-\vec{k}\right)\right]$

$\vec{r} = -2\vec{i}+2\vec{j}-\vec{k}+\lambda\left[-\vec{j}+3\vec{k}\right]$

Line CD : $\vec{CD} = \vec{OD} - \vec{OC} = \left(t\vec{i}-\vec{j}\right) - \left(-2\vec{j}+\vec{k}\right) = t\vec{i}+\vec{j}-\vec{k}$.

$\therefore \vec{r} = -2\vec{j}+\vec{k}+\mu\left(t\vec{i}+\vec{j}-\vec{k}\right)$.

Let, line AB have equation $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ where $\vec{a_1} = -2\vec{i}+2\vec{j}-\vec{k}$, $\vec{b_1} = -\vec{j}+3\vec{k}$.

line CD have equation $\vec{r}=\vec{a_2}+\lambda\vec{b_2}$ where $\vec{a_2} = -2\vec{j}+\vec{k}$, $\vec{b_2} = t\vec{i}+\vec{j}-\vec{k}$.

$\vec{a_2}-\vec{a_1} = \left(-2\vec{j}+\vec{k}\right) - \left(-2\vec{i}+2\vec{j}-\vec{k}\right) = 2\vec{i}-4\vec{j}+2\vec{k}$.

$$\vec{b_1}\times\vec{b_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & 3 \\ t & 1 & -1 \end{vmatrix} = \begin{bmatrix} -2 \\ -(-3t) \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 3t \\ t \end{bmatrix}.$$

$\left(\vec{a_2}-\vec{a_1}\right)\cdot\left(\vec{b_1}\times\vec{b_2}\right) = 2(-2)-4(3t)+2(t) = -4-12t+2t = -4-10t$.

$\left|\vec{b_1}\times\vec{b_2}\right| = \sqrt{(-2)^2+(3t)^2+t^2} = \sqrt{4+9t^2+t^2} = \sqrt{4+10t^2}$.

$$\text{Distance between AB and CD} = \left|\frac{\left(\vec{a_2}-\vec{a_1}\right)\cdot\left(\vec{b_1}\times\vec{b_2}\right)}{\left|\vec{b_1}\times\vec{b_2}\right|}\right|$$

$$\sqrt{10} = \left|\frac{-4-10t}{\sqrt{4+10t^2}}\right|$$

$$\sqrt{10} = \frac{|-4-10t|}{\sqrt{4+10t^2}}$$

$$\sqrt{10(4+10t^2)} = 4+10t$$

$$10(4+10t^2) = (4+10t)^2$$

$$40+100t^2 = 16+80t+100t^2$$

$$24 = 80t$$

$$\Rightarrow t = \frac{24}{80} = \frac{3}{10}.$$

51 The lines l_1 and l_2 have equations $\mathbf{r} = 3\mathbf{i} + 3\mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} + 4\mathbf{k})$ and $\mathbf{r} = 3\mathbf{i} - 5\mathbf{j} - 6\mathbf{k} + \mu(5\mathbf{j} + 6\mathbf{k})$ respectively.

(a) Find the shortest distance between l_1 and l_2. [5]

Let, $l_1 : \vec{r} = \vec{a_1} + \lambda \vec{b_1}$ where $\vec{a_1} = 3\vec{i} + 3\vec{k}$, $\vec{b_1} = \vec{i} + 4\vec{j} + 4\vec{k}$.

$l_2 : \vec{r} = \vec{a_2} + \lambda \vec{b_2}$ where $\vec{a_2} = 3\vec{i} - 5\vec{j} - 6\vec{k}$, $\vec{b_2} = 5\vec{j} + 6\vec{k}$.

$$\vec{a_2} - \vec{a_1} = (3\vec{i} - 5\vec{j} - 6\vec{k}) - (3\vec{i} + 3\vec{k}) = -5\vec{j} - 9\vec{k}.$$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & 4 \\ 0 & 5 & 6 \end{vmatrix} = \begin{bmatrix} 24-20 \\ -(6-0) \\ 5-0 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 5 \end{bmatrix} = 4\vec{i} - 6\vec{j} + 5\vec{k}.$$

$$(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = (-5\vec{j} - 9\vec{k}) \cdot (4\vec{i} - 6\vec{j} + 5\vec{k})$$
$$= -5(-6) - 9(5) = 30 - 45 = -15.$$

$$|\vec{b_1} \times \vec{b_2}| = \sqrt{4^2 + (-6)^2 + 5^2} = \sqrt{77}.$$

$\therefore$ Shortest distance between l_1 and l_2

$$= \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right| = \left| \frac{-15}{\sqrt{77}} \right| = \frac{15}{\sqrt{77}} = 1.709 \simeq 1.71$$

The plane Π contains l_1 and is parallel to the vector $\mathbf{i}+\mathbf{k}$.

[4]

(b) Find the equation of Π, giving your answer in the form $ax+by+cz=d$.

Π has a normal vector $\perp$ to $\vec{i}+4\vec{j}+4\vec{k}$ and $\vec{i}+\vec{k}$.

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & 4 \\ 1 & 0 & 1 \end{vmatrix} = \begin{bmatrix} 4-0 \\ -(1-4) \\ 0-4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ -4 \end{bmatrix}$$

Π contains the point with position vector $3\vec{i}+3\vec{k}$.

$\therefore$ Equation of Π: $\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$

$$\vec{r}\cdot(4\vec{i}+3\vec{j}-4\vec{k}) = (3\vec{i}+3\vec{k})\cdot(4\vec{i}+3\vec{j}-4\vec{k})$$

$$4x+3y-4z = 12-12$$

$$4x+3y-4z = 0$$

[3]

(c) Find the acute angle between l_2 and Π.

Direction vector of $l_2 = 5\vec{j}+6\vec{k}$. Let, angle between $\vec{b_2}$ and $\vec{n}$ be α.

$$\Rightarrow \quad \vec{b_2}\cdot\vec{n} = |\vec{b_2}||\vec{n}|\cos\alpha$$

$$(5\vec{j}+6\vec{k})\cdot(4\vec{i}+3\vec{j}-4\vec{k}) = \sqrt{5^2+6^2}\ \sqrt{4^2+3^2+(-4)^2}\ \cos\alpha$$

$$15-24 = \sqrt{61}\ \sqrt{41}\ \cos\alpha$$

$$\alpha = \cos^{-1}\left(\frac{-9}{\sqrt{61}\sqrt{41}}\right) = 100.36^\circ$$

$\therefore$ Acute angle between l_2 and Π: $\overset{(\alpha-90^\circ)}{=}100.36^\circ - 90^\circ = 10.36^\circ \approx 10.4^\circ$.

52 The lines l_1 and l_2 have equations $\mathbf{r} = -5\mathbf{j} + \lambda(5\mathbf{i} + 2\mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} + \mu(\mathbf{j} + \mathbf{k})$ respectively. The plane Π contains l_1 and is parallel to l_2.

(a) Find the equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

Π has a normal vector $\perp$ to $5\vec{i} + 2\vec{k}$ and $\vec{j} + \vec{k}$.

$$\Rightarrow \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = \begin{bmatrix} 0-2 \\ -(5-0) \\ 5-0 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \\ 5 \end{bmatrix}.$$

Π contains the point $-5\vec{j}$.

$\therefore$ Equation of Π: $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

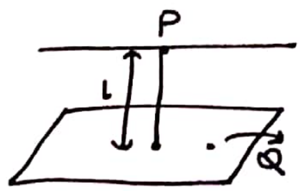
$$\vec{r} \cdot (-2\vec{i} - 5\vec{j} + 5\vec{k}) = -5\vec{j} \cdot (-2\vec{i} - 5\vec{j} + 5\vec{k})$$

$$-2x - 5y + 5z = 25.$$

(b) Find the distance between l_2 and Π. [3]

Let, $Q(0,-5,0)$ $\therefore$ $\vec{OQ} = -5\vec{j}$. and $\vec{OP} = (4\vec{i} + 2\vec{j} - 2\vec{k})$ so $P(4,2,-2)$.

Π contains Q while P is a point on l_2.



$$\vec{PQ} = \vec{OQ} - \vec{OP} = \begin{bmatrix} 0 \\ -5 \\ 0 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \\ 2 \end{bmatrix}.$$

$$\vec{PQ} \cdot \vec{n} = \begin{bmatrix} -4 \\ -7 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -5 \\ 5 \end{bmatrix} = 8 + 35 + 10 = 53.$$

$$|\vec{n}| = \sqrt{(-2)^2 + (-5)^2 + 5^2} = \sqrt{4 + 25 + 25} = \sqrt{54}.$$

Distance between l_2 and Π

$$= \left| \frac{\vec{PQ} \cdot \vec{n}}{|\vec{n}|} \right| = \frac{53}{\sqrt{54}} = 7.212 \approx 7.21$$

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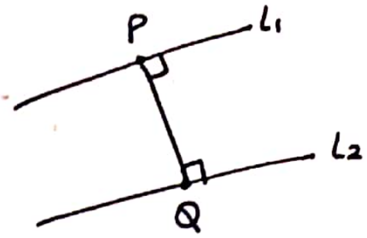
The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2.

(c) Show that P has position vector $\frac{55}{27}\mathbf{i} - 5\mathbf{j} + \frac{22}{27}\mathbf{k}$ and state a vector equation for PQ. [8]



$$l_1 : \vec{r} = \begin{bmatrix} 0 \\ -5 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5\lambda \\ -5 \\ 2\lambda \end{bmatrix}.$$

Since P lies on l_1 $\therefore$ $\vec{OP} = \begin{bmatrix} 5\lambda \\ -5 \\ 2\lambda \end{bmatrix}$.

$$l_2 : \vec{r} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2+\mu \\ -2+\mu \end{bmatrix}.$$

Since Q lies on l_2 $\therefore$ $\vec{OQ} = \begin{bmatrix} 4 \\ 2+\mu \\ -2+\mu \end{bmatrix}$

$$\therefore \vec{PQ} = \vec{OQ} - \vec{OP} = \begin{bmatrix} 4 \\ 2+\mu \\ -2+\mu \end{bmatrix} - \begin{bmatrix} 5\lambda \\ -5 \\ 2\lambda \end{bmatrix} = \begin{bmatrix} 4-5\lambda \\ 7+\mu \\ -2+\mu-2\lambda \end{bmatrix}.$$

Since $\vec{PQ} \perp l_1$ $\therefore$ $\vec{PQ} \cdot [5\vec{i} + 2\vec{k}] = 0$

$$\Rightarrow 5(4-5\lambda) + 2(-2+\mu-2\lambda) = 0$$
$$20 - 25\lambda - 4 + 2\mu - 4\lambda = 0$$
$$-29\lambda + 2\mu + 16 = 0 \rightarrow (1)$$

Since $\vec{PQ} \perp l_2$ $\therefore$ $\vec{PQ} \cdot [\vec{j} + \vec{k}] = 0$

$$1(7+\mu) + 1(-2+\mu-2\lambda) = 0$$
$$7 + \mu - 2 + \mu - 2\lambda = 0$$
$$2\mu - 2\lambda + 5 = 0 \rightarrow (2)$$

From (1), $2\mu = -16 + 29\lambda \rightarrow (3)$

Substitute (3) in (2)

$$-16 + 29\lambda - 2\lambda + 5 = 0$$
$$27\lambda = 11 \Rightarrow \lambda = \frac{11}{27}.$$

$$\therefore 2\mu = -16 + 29\left(\frac{11}{27}\right) \Rightarrow \mu = \frac{-113}{54}.$$

$$\vec{OP} = \begin{bmatrix} 5\lambda \\ -5 \\ 2\lambda \end{bmatrix} = \begin{bmatrix} 5 \times \frac{11}{27} \\ -5 \\ 2 \times \frac{11}{27} \end{bmatrix} = \frac{55}{27}\vec{i} - 5\vec{j} + \frac{22}{27}\vec{k}. \text{ (Shown)}.$$

$$\vec{PQ} = \begin{bmatrix} 4-5\lambda \\ 7+\mu \\ -2+\mu-2\lambda \end{bmatrix} = \begin{bmatrix} 4-5\left(\frac{11}{27}\right) \\ 7+\left(\frac{-113}{54}\right) \\ -2+\left(-\frac{113}{54}\right)-2\left(\frac{11}{27}\right) \end{bmatrix} = \begin{bmatrix} 53/27 \\ 265/54 \\ -265/54 \end{bmatrix}$$

$\therefore$ Vector equation of PQ :

$$\vec{r} = \frac{55}{27}\vec{i} - 5\vec{j} + \frac{22}{27}\vec{k} + \lambda\left(\frac{53}{27}\vec{i} + \frac{265}{54}\vec{j} - \frac{265}{54}\vec{k}\right)$$

$$= \frac{1}{27}\left[55\vec{i} - 135\vec{j} + 22\vec{k}\right] + \frac{53\lambda}{54}\left(2\vec{i} + 5\vec{j} - 5\vec{k}\right)$$

$$\vec{r} = \frac{1}{27}\begin{bmatrix} 55 \\ -135 \\ 22 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 5 \\ -5 \end{bmatrix}$$

53 Let t be a positive constant.

The line l_1 passes through the point with position vector $t\mathbf{i}+\mathbf{j}$ and is parallel to the vector $-2\mathbf{i}-\mathbf{j}$. The line l_2 passes through the point with position vector $\mathbf{j}+t\mathbf{k}$ and is parallel to the vector $-2\mathbf{j}+\mathbf{k}$.

It is given that the shortest distance between the lines l_1 and l_2 is $\sqrt{21}$.

(a) Find the value of t. [5]

$$l_1 : \vec{r} = \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad l_2 : \vec{r} = \begin{pmatrix} 0 \\ 1 \\ t \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}.$$

Shortest distance between l_1 and l_2 $= \left| \dfrac{(\vec{a_2}-\vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right|$

$$\vec{a_2}-\vec{a_1} = \begin{pmatrix} 0 \\ 1 \\ t \end{pmatrix} - \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix}$$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -1 & 0 \\ 0 & -2 & 1 \end{vmatrix} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$$

$$\Rightarrow (\vec{a_2}-\vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = -t(-1) + 0(2) + t(4) = 5t.$$

$$|\vec{b_1} \times \vec{b_2}| = \sqrt{(-1)^2 + 2^2 + 4^2} = \sqrt{21}.$$

$\Rightarrow$ Shortest distance between l_1 and l_2 $= \left| \dfrac{5t}{\sqrt{21}} \right|$

$$\sqrt{21} = \frac{5t}{\sqrt{21}} \quad \text{since } t > 0$$

$$21 = 5t$$

$$t = \frac{21}{5} = 4.2.$$

The plane Π_1 contains l_1 and is parallel to l_2.

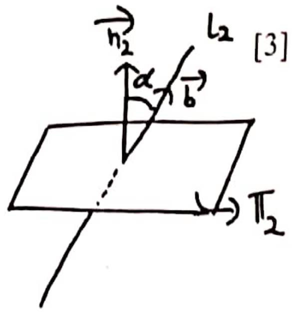
(b) Write down an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. [1]

$$l_1 : \vec{r} = \begin{pmatrix} 4.2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \Pi_1 : \vec{r} = \begin{pmatrix} 4.2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

The plane Π_2 has Cartesian equation $5x - 6y + 7z = 0$.

(c) Find the acute angle between l_2 and Π_2. [3]



$$\vec{n_2} = \begin{bmatrix} 5 \\ -6 \\ 7 \end{bmatrix} \text{ and } \vec{b_2} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

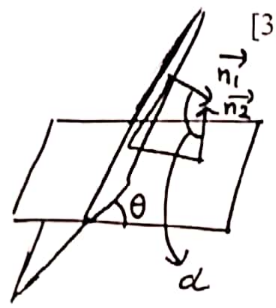
$$\cos\alpha = \frac{\vec{n_2}\cdot\vec{b_2}}{|\vec{n_2}||\vec{b_2}|} = \frac{5(0)-6(-2)+7(1)}{\sqrt{5^2+(-6)^2+7^2}\sqrt{0^2+(-2)^2+1^2}}$$

$$\cos\alpha = \frac{19}{\sqrt{110}\sqrt{5}} \Rightarrow \alpha = 35.88^\circ$$

$\Rightarrow$ Angle between l_2 and $\Pi_2 = 90^\circ - \alpha$

$$= 54.11^\circ \approx 54.1^\circ.$$

(d) Find the acute angle between Π_1 and Π_2. [3]



$$\vec{n_1} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -1 & 0 \\ 0 & -2 & 1 \end{vmatrix} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} \text{ and } \vec{n_2} = \begin{bmatrix} 5 \\ -6 \\ 7 \end{bmatrix}$$

$$\cos\alpha = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \frac{-1(5)+2(-6)+4(7)}{\sqrt{(-1)^2+2^2+4^2}\sqrt{5^2+(-6)^2+7^2}}$$

$$\alpha = \cos^{-1}\left(\frac{11}{\sqrt{21}\sqrt{110}}\right) = 76.76^\circ \approx 76.8^\circ$$

Then, acute angle between Π_1 and Π_2 is 76.8°.

54 The lines l_1 and l_2 have equations $\mathbf{r} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k} + s(2\mathbf{i} - 3\mathbf{j})$ and $\mathbf{r} = 3\mathbf{i} - 2\mathbf{k} + t(3\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ respectively.

The plane Π_1 contains l_1 and the point P with position vector $-2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$.

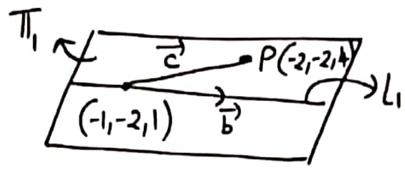
(a) Find an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. [2]

$$\vec{r} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$$

$$\vec{c} = \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$$

or

$$\vec{r} = \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} +2 \\ -3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}.$$



The plane Π_2 contains l_2 and is parallel to l_1.

(b) Find an equation of Π_2, giving your answer in the form $ax + by + cz = d$. [4]

$$\Pi_2 : \vec{r} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$$

$$\vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 3 \\ 2 & -3 & 0 \end{vmatrix} = \begin{bmatrix} 9 \\ 6 \\ -7 \end{bmatrix}$$

$$\vec{r} \cdot \vec{n_2} = \vec{a} \cdot \vec{n_2}$$

$$\vec{r} \cdot \begin{pmatrix} 9 \\ 6 \\ -7 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 6 \\ -7 \end{pmatrix}$$

$$9x + 6y - 7z = 41.$$

(c) Find the acute angle between Π_1 and Π_2. [5]

$$\vec{n_1} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 0 \\ -1 & 0 & 3 \end{vmatrix} = \begin{bmatrix} -9 \\ -6 \\ -3 \end{bmatrix} \sim \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \text{ and } \vec{n_2} = \begin{bmatrix} 9 \\ 6 \\ -7 \end{bmatrix}$$

$$\cos\alpha = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|} \Rightarrow \cos\alpha = \frac{3(9)+2(6)+1(-7)}{\sqrt{3^2+2^2+1^2}\sqrt{9^2+6^2+(-7)^2}}$$

$$\cos\alpha = \frac{32}{\sqrt{14}\sqrt{166}}$$

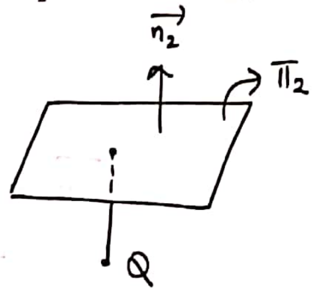
$$\alpha = 48.41^\circ \approx 48.4^\circ$$

$\Rightarrow$ Acute angle between Π_1 and $\Pi_2 = \alpha = 48.4^\circ$.

(d) The point Q is such that $\overrightarrow{OQ} = -5\overrightarrow{OP}$.

Find the position vector of the foot of the perpendicular from the point Q to Π_2. [4]

$$\overrightarrow{OQ} = -5\overrightarrow{OP} = -5\begin{bmatrix}-2\\-2\\4\end{bmatrix} = \begin{bmatrix}10\\10\\-20\end{bmatrix}$$



Line through Q $\perp$ to Π_2: $\vec{r} = \begin{bmatrix}10\\10\\-20\end{bmatrix} + \lambda\begin{bmatrix}9\\6\\-7\end{bmatrix}$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}10+9\lambda\\10+6\lambda\\-20-7\lambda\end{bmatrix}$$

Π_2: $9x+6y-7z = 41 \rightarrow (1)$

Substituting $x = 10+9\lambda$, $y = 10+6\lambda$ and $z = -20-7\lambda$ in (1) gives:

$$9(10+9\lambda)+6(10+6\lambda)-7(-20-7\lambda) = 41$$

$$90+81\lambda+60+36\lambda+140+49\lambda = 41$$

$$166\lambda = -249$$

$$\lambda = \frac{-3}{2}.$$

$$\Rightarrow x = 10+9\left(\frac{-3}{2}\right) = \frac{-7}{2}$$

$$y = 10+6\left(\frac{-3}{2}\right) = 1$$

$$z = -20-7\left(\frac{-3}{2}\right) = \frac{-19}{2}.$$

$\therefore$ Position vector of foot of perpendicular from Q to Π_2 $= \begin{bmatrix}-7/2\\1\\-19/2\end{bmatrix}$.

55 The points A, B, C have position vectors

$$4\mathbf{i}-4\mathbf{j}+\mathbf{k}, \quad -4\mathbf{i}+3\mathbf{j}-4\mathbf{k}, \quad 4\mathbf{i}-\mathbf{j}-2\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

$$\vec{OA}=\begin{bmatrix}4\\-4\\1\end{bmatrix}, \quad \vec{OB}=\begin{bmatrix}-4\\3\\-4\end{bmatrix} \text{ and } \vec{OC}=\begin{bmatrix}4\\-1\\-2\end{bmatrix}$$

$$\vec{AB}=\vec{OB}-\vec{OA}=\begin{bmatrix}-4\\3\\-4\end{bmatrix}-\begin{bmatrix}4\\-4\\1\end{bmatrix}=\begin{bmatrix}-8\\7\\-5\end{bmatrix}.$$

$$\vec{AC}=\vec{OC}-\vec{OA}=\begin{bmatrix}4\\-1\\-2\end{bmatrix}-\begin{bmatrix}4\\-4\\1\end{bmatrix}=\begin{bmatrix}0\\3\\-3\end{bmatrix}$$

Plane ABC: $\vec{r}=\begin{bmatrix}4\\-4\\1\end{bmatrix}+\lambda\begin{bmatrix}-8\\7\\-5\end{bmatrix}+\mu\begin{bmatrix}0\\3\\-3\end{bmatrix}$

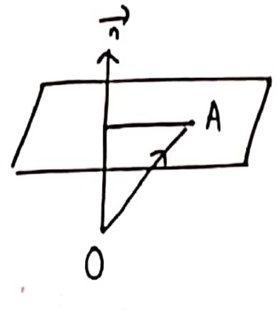
$$\vec{n}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -8 & 7 & -5\\ 0 & 3 & -3\end{vmatrix}=\begin{bmatrix}-6\\-24\\-24\end{bmatrix}\sim\begin{bmatrix}1\\4\\4\end{bmatrix}$$

$$\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$$

$$\vec{r}\cdot\begin{bmatrix}1\\4\\4\end{bmatrix}=\begin{bmatrix}4\\-4\\1\end{bmatrix}\cdot\begin{bmatrix}1\\4\\4\end{bmatrix}$$

$$x+4y+4z=-8.$$

(b) Find the perpendicular distance from O to the plane ABC. [2]



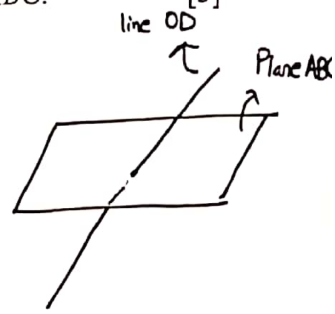
$\perp$ distance from O to plane ABC

$$= \left| \frac{\vec{OA} \cdot \vec{n}}{|\vec{n}|} \right| = \left| \frac{\begin{bmatrix} 4 \\ -4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}}{\sqrt{1^2+4^2+4^2}} \right| = \left| \frac{-8}{\sqrt{33}} \right|$$

$$= \frac{8}{\sqrt{33}} \simeq 1.39 \text{ units.}$$

(c) The point D has position vector $2\mathbf{i}+3\mathbf{j}-3\mathbf{k}$.

Find the coordinates of the point of intersection of the line OD with the plane ABC. [3]



line $\vec{OD}$: $\vec{r} = t\begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 2t \\ 3t \\ -3t \end{pmatrix} \Rightarrow x=2t, y=3t, z=-3t$.

Plane ABC: $x+4y+4z=-8 \rightarrow (1)$

Substituting $x=2t, y=3t, z=-3t$ in (1):

$$2t+4(3t)+4(-3t)=-8$$
$$2t=-8 \Rightarrow t=-4.$$

$$\Rightarrow x=2(-4)=-8$$
$$y=3(-4)=-12$$
$$z=-3(-4)=12.$$

$\therefore$ Coordinates of point of intersection: $(-8,-12,12)$.

56 The position vectors of the points A, B, C, D are

$$7\mathbf{i}+4\mathbf{j}-\mathbf{k}, \qquad 11\mathbf{i}+3\mathbf{j}, \qquad 2\mathbf{i}+6\mathbf{j}+3\mathbf{k}, \qquad 2\mathbf{i}+7\mathbf{j}+\lambda\mathbf{k}$$

respectively.

(a) Given that the shortest distance between the line AB and the line CD is 3, show that $\lambda^2-5\lambda+4=0$. [7]

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}11\\3\\0\end{pmatrix}-\begin{pmatrix}7\\4\\-1\end{pmatrix}=\begin{pmatrix}4\\-1\\1\end{pmatrix}.$$

$$\text{line } \overrightarrow{AB}: \vec{r}=\begin{pmatrix}7\\4\\-1\end{pmatrix}+\lambda\begin{pmatrix}4\\-1\\1\end{pmatrix}.$$

$$\overrightarrow{CD}=\overrightarrow{OD}-\overrightarrow{OC}=\begin{pmatrix}2\\7\\\lambda\end{pmatrix}-\begin{pmatrix}2\\6\\3\end{pmatrix}=\begin{pmatrix}0\\1\\\lambda-3\end{pmatrix}.$$

$$\text{line } \overrightarrow{CD}: \vec{r}=\begin{pmatrix}2\\6\\3\end{pmatrix}+\mu\begin{pmatrix}0\\1\\\lambda-3\end{pmatrix}.$$

Shortest distance between line AB and CD $=\left|\dfrac{(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})}{|\vec{b_1}\times\vec{b_2}|}\right|$

$$\vec{a_2}-\vec{a_1}=\begin{pmatrix}2\\6\\3\end{pmatrix}-\begin{pmatrix}7\\4\\-1\end{pmatrix}=\begin{pmatrix}-5\\2\\4\end{pmatrix}$$

$$\vec{b_1}\times\vec{b_2}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\4&-1&1\\0&1&\lambda-3\end{vmatrix}=\begin{bmatrix}-\lambda+3-1\\-4\lambda+12\\4\end{bmatrix}=\begin{bmatrix}2-\lambda\\12-4\lambda\\4\end{bmatrix}$$

$$(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})=-5(2-\lambda)+2(12-4\lambda)+4(4)=-10+5\lambda+24-8\lambda+16$$
$$=30-3\lambda.$$

$$|\vec{b_1}\times\vec{b_2}|=\sqrt{(2-\lambda)^2+(12-4\lambda)^2+4^2}=\sqrt{4-4\lambda+\lambda^2+144-96\lambda+16\lambda^2+16}$$
$$=\sqrt{164-100\lambda+17\lambda^2}.$$

$$\therefore \text{Shortest distance}=\frac{30-3\lambda}{\sqrt{164-100\lambda+17\lambda^2}}$$

$$3^2=\frac{(30-3\lambda)^2}{164-100\lambda+17\lambda^2}$$

$$9(164-100\lambda+17\lambda^2)=9(100-20\lambda+\lambda^2)$$
$$16\lambda^2-80\lambda+64=0$$
$$\lambda^2-5\lambda+4=0$$
(shown).

Let Π_1 be the plane ABD when $\lambda = 1$.

Let Π_2 be the plane ABD when $\lambda = 4$.

(b) (i) Write down an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$. [2]

$$\lambda = 1 \Rightarrow \overrightarrow{AB} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}, \overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix} - \begin{bmatrix} 7 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix}.$$

$$\therefore \Pi_1 : \vec{r} = \begin{bmatrix} 7 \\ 4 \\ -1 \end{bmatrix} + s\begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix}$$

(ii) Find an equation of Π_2, giving your answer in the form $ax + by + cz = d$. [4]

$$\lambda = 4 \Rightarrow \overrightarrow{AB} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}, \overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix} - \begin{bmatrix} 7 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 5 \end{bmatrix}.$$

$$\Pi_2 : \vec{r} = \begin{bmatrix} 7 \\ 4 \\ -1 \end{bmatrix} + s\begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -5 \\ 3 \\ 5 \end{bmatrix}.$$

$$\vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -1 & 1 \\ -5 & 3 & 5 \end{vmatrix} = \begin{bmatrix} -8 \\ -25 \\ 7 \end{bmatrix}$$

$$\vec{r} \cdot \vec{n_2} = \vec{a} \cdot \vec{n_2}$$

$$\vec{r} \cdot \begin{bmatrix} -8 \\ -25 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -8 \\ -25 \\ 7 \end{bmatrix}$$

$$-8x - 25y + 7z = -163$$

(c) Find the acute angle between Π_1 and Π_2. [5]

$$\vec{n_1} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -1 & 1 \\ -5 & 3 & 2 \end{vmatrix} = \begin{bmatrix} -5 \\ -13 \\ 7 \end{bmatrix} \quad \text{and} \quad \vec{n_2} = \begin{bmatrix} -8 \\ -25 \\ 7 \end{bmatrix}$$

$$\cos\alpha = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\,|\vec{n_2}|}$$

$$= \frac{-5(-8) - 13(-25) + 7(7)}{\sqrt{(-5)^2+(-13)^2+7^2}\,\sqrt{(-8)^2+(-25)^2+7^2}}$$

$$\cos\alpha = \frac{+414}{\sqrt{243}\,\sqrt{738}} \Rightarrow \alpha = \cos^{-1}\left(\frac{414}{\sqrt{243}\,\sqrt{738}}\right) = 12.1^\circ$$

$\therefore$ Acute angle between Π_1 and Π_2 $= 12.1^\circ$.

57 The plane Π has equation $\mathbf{r} = -2\mathbf{i}+3\mathbf{j}+3\mathbf{k}+\lambda(\mathbf{i}+\mathbf{k})+\mu(2\mathbf{i}+3\mathbf{j})$.

(a) Find a Cartesian equation of Π, giving your answer in the form $ax+by+cz=d$. [4]

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 2 & 3 & 0 \end{vmatrix} = \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}$$

$$\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$$

$$\vec{r}\cdot\begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 3 \end{bmatrix}\cdot\begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}$$

$$-3x+2y+3z = 21$$

The line l passes through the point P with position vector $2\mathbf{i}-3\mathbf{j}+5\mathbf{k}$ and is parallel to the vector $\mathbf{k}$.

(b) Find the position vector of the point where l meets Π. [3]

line l: $\vec{r} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 5+t \end{bmatrix} \Rightarrow x=2, y=-3, z=5+t.$

Π: $-3x+2y+3z = 21 \rightarrow (1)$

Substituting $x=2, y=-3$ and $z=5+t$ in (1):

$$-3(2)+2(-3)+3(5+t) = 21$$

$$-6-6+15+3t = 21$$

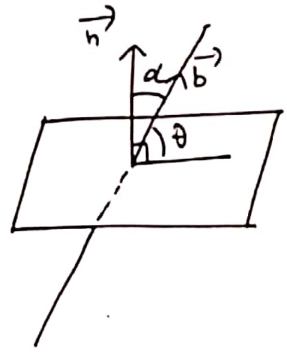
$$3t = 18 \Rightarrow t=6.$$

$t=6 \Rightarrow x=2, y=-3, z=11.$

i.e. position vector of point where l meets Π: $\begin{pmatrix} 2 \\ -3 \\ 11 \end{pmatrix}$.

(c) Find the acute angle between l and Π. [3]

$$L: \vec{r} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \Pi: \vec{r}\cdot\begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix} = 21$$



$$\cos\alpha = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}$$

$$\cos\alpha = \frac{0(-3)+0(2)+1(3)}{\sqrt{0^2+0^2+1^2}\,\sqrt{(-3)^2+2^2+3^2}}$$

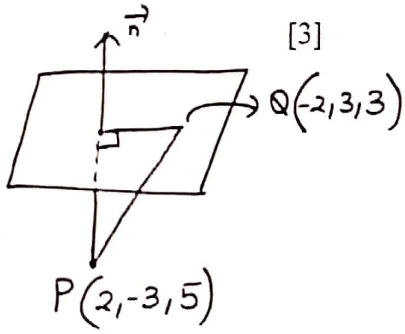
$$\cos\alpha = \frac{3}{1(\sqrt{22})}$$

$$\alpha = 50.23^\circ$$

$\therefore$ Acute angle between L and $\Pi = 90^\circ - \alpha$

$$\theta = 39.76^\circ \simeq 39.8^\circ$$

(d) Find the perpendicular distance from P to Π. [3]



$$\vec{PQ} = \vec{OQ} - \vec{OP} = \begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \\ -2 \end{pmatrix}$$

$\perp$ distance from P to Π:

$$= \left|\frac{\vec{PQ}\cdot\vec{n}}{|\vec{n}|}\right| = \left|\frac{\begin{pmatrix} -4 \\ 6 \\ -2 \end{pmatrix}\cdot\begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix}}{\sqrt{(-3)^2+2^2+3^2}}\right|$$

$$= \frac{18}{\sqrt{22}} = 3.837 \simeq 3.84$$

58 The points A, B, C have position vectors

$$2\mathbf{i}+2\mathbf{j}, \quad -\mathbf{j}+\mathbf{k} \quad \text{and} \quad 2\mathbf{i}+\mathbf{j}-7\mathbf{k}$$

respectively, relative to the origin O.

(a) Find an equation of the plane OAB, giving your answer in the form $\mathbf{r.n} = p$. [3]

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \text{ and } \overrightarrow{OB} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

$$\vec{n} = \overrightarrow{OA} \times \overrightarrow{OB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 0 \\ 0 & -1 & 1 \end{vmatrix} = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} \sim \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$$\vec{r} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$\vec{r} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 0$$

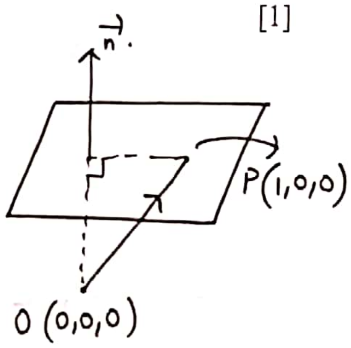
The plane Π has equation $x - 3y - 2z = 1$.

(b) Find the perpendicular distance of Π from the origin. [1]

$\Pi: \ x - 3y - 2z = 1$

$y = z = 0 \Rightarrow x = 1 \Rightarrow P = (1, 0, 0)$

$$\vec{n} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$$



$\perp$ distance of Π from the origin

$$= \left| \frac{\overrightarrow{OP} \cdot \vec{n}}{|\vec{n}|} \right| = \left| \frac{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}}{\sqrt{1^2 + (-3)^2 + (-2)^2}} \right| = \frac{1}{\sqrt{14}} \approx 0.267$$

(c) Find the acute angle between the planes OAB and Π. [3]

For OAB : $\vec{n_1} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ and for Π : $\vec{n_2} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$.

$$\cos\alpha = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \frac{\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}}{\sqrt{3}\sqrt{14}} = \frac{6}{\sqrt{42}}$$

$$\alpha = \cos^{-1}\left(\frac{6}{\sqrt{42}}\right) = 22.20^\circ \simeq 22.2^\circ$$

i.e. Acute angle between planes $= 22.2^\circ$.

(d) Find an equation for the common perpendicular to the lines OC and AB. [10]

line OC : $\vec{r} = t\begin{pmatrix} 2 \\ 1 \\ -7 \end{pmatrix}$

$\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$

$\vec{AB} = \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$

line $\overline{AB}$: $\vec{r} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + s\begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$

Let, P be a point on line OC and Q be a point on AB such that $PQ \perp OC$ and $PQ \perp AB$.

$\vec{OP} = \begin{pmatrix} 2t \\ t \\ -7t \end{pmatrix}$ and $\vec{OQ} = \begin{pmatrix} 2-2s \\ 2-3s \\ s \end{pmatrix}$.

$$\vec{PQ} = \vec{OQ} - \vec{OP} = \begin{pmatrix} 2-2s \\ 2-3s \\ s \end{pmatrix} - \begin{pmatrix} 2t \\ t \\ -7t \end{pmatrix} = \begin{pmatrix} 2-2s-2t \\ 2-3s-t \\ s+7t \end{pmatrix}.$$

Since $\vec{PQ} \perp OC \Rightarrow \vec{PQ}\cdot\begin{pmatrix} 2 \\ 1 \\ -7 \end{pmatrix} = 0$

$$2(2-2s-2t)+1(2-3s-t)-7(s+7t)=0$$
$$4-4s-4t+2-3s-t-7s-49t=0$$
$$6-14s-54t=0.$$
$$6-54t=14s \rightarrow (1)$$

Since $\overrightarrow{PQ} \perp AB \Rightarrow \overrightarrow{PQ} \cdot \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix} = 0$

$$-2(2-2s-2t)-3(2-3s-t)+1(s+7t)=0$$
$$-4+4s+4t-6+9s+3t+s+7t=0$$
$$-10+14s+14t=0$$
$$14s = 10-14t \rightarrow (2)$$

Substitute (1) in (2):

$$6-54t = 10-14t$$
$$-40t = 4$$
$$t = -0.1$$
$$\Rightarrow 14s = 10-14(-0.1)$$
$$s = \frac{57}{70}.$$

$t=-0.1 \Rightarrow \overrightarrow{OP} = \begin{pmatrix} -0.2 \\ -0.1 \\ 0.7 \end{pmatrix}$ and $\overrightarrow{PQ} = \begin{pmatrix} 2-2\left(\frac{57}{70}\right)-2(-0.1) \\ 2-3\left(\frac{57}{70}\right)-(-0.1) \\ \frac{57}{70}+7(-0.1) \end{pmatrix} = \begin{pmatrix} 4/7 \\ -12/35 \\ 4/35 \end{pmatrix}$

$\therefore \vec{b}$ (direction vector of line $\overrightarrow{PQ}$) $\sim \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$.

$\therefore$ Equation of common perpendicular of lines OC and AB:

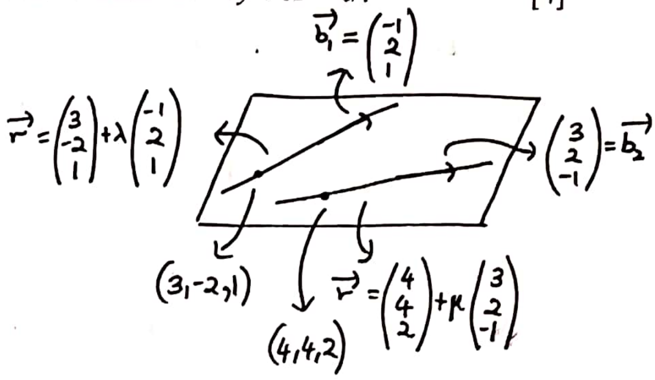
$$\vec{r} = \begin{pmatrix} -0.2 \\ -0.1 \\ 0.7 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}.$$

59 The plane Π contains the lines $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \mu(3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$.

(a) Find a Cartesian equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

$\vec{n} = \vec{b_1} \times \vec{b_2}$

$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 1 \\ 3 & 2 & -1 \end{vmatrix} = \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \sim \begin{bmatrix} -2 \\ 1 \\ -4 \end{bmatrix}$



$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$

$-2x + y - 4z = -12$

$2x - y + 4z = 12.$

The line l passes through the point P with position vector $2\mathbf{i}+3\mathbf{j}+\mathbf{k}$ and is parallel to the vector $\mathbf{j}+\mathbf{k}$.

(b) Find the acute angle between l and Π. [3]

Line L: $\vec{r} = \vec{a} + t\vec{b} = \begin{pmatrix}2\\3\\1\end{pmatrix} + t\begin{pmatrix}0\\1\\1\end{pmatrix} \Rightarrow \vec{b} = \begin{pmatrix}0\\1\\1\end{pmatrix}$

$\vec{b}\cdot\vec{n} = \begin{pmatrix}0\\1\\1\end{pmatrix}\cdot\begin{pmatrix}-2\\1\\-4\end{pmatrix} = \left(0(-2)+1(1)+1(-4)\right) = -3.$

$|\vec{b}| = \sqrt{1^2+1^2} = \sqrt{2}$

$|\vec{n}| = \sqrt{(-2)^2+1^2+(-4)^2} = \sqrt{21}.$

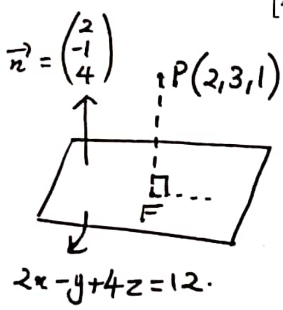
$\cos\theta = \dfrac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|} = \dfrac{-3}{\sqrt{2}\sqrt{21}} \Rightarrow \theta = \cos^{-1}\left(\dfrac{-3}{\sqrt{2}\sqrt{21}}\right) = 117.57°$

$\therefore$ Acute angle between line L and plane $P = \theta - 90° = 117.57° - 90° = 27.57° \approx 27.6°.$

(c) Find the position vector of the foot of the perpendicular from P to Π. [4]

Let, F be the foot of the perpendicular from P to Π.

PF lies in the same direction as $\vec{n}$.



Equation of PF: $\vec{r} = \vec{a} + s\vec{b} = \begin{pmatrix}2\\3\\1\end{pmatrix} + s\begin{pmatrix}2\\-1\\4\end{pmatrix} = \begin{pmatrix}2+2s\\3-s\\1+4s\end{pmatrix}.$

Since F lies of line PF, $F = (2+2s, 3-s, 1+4s)$.

Substituting into equation of plane:

$2(2+2s) - (3-s) + 4(1+4s) = 12$

$4 + 4s - 3 + s + 4 + 16s = 12$

$21s = 7$

$s = \frac{1}{3}.$

$\Rightarrow \vec{OF} = \begin{pmatrix}2+2(1/3)\\3-(1/3)\\1+4(1/3)\end{pmatrix} = \begin{pmatrix}8/3\\8/3\\7/3\end{pmatrix} = \frac{1}{3}\begin{pmatrix}8\\8\\7\end{pmatrix}.$

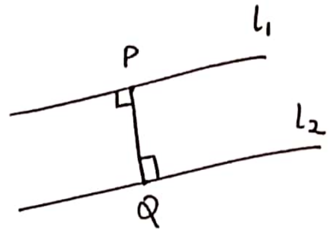
60 The lines l_1 and l_2 have equations $\mathbf{r} = 2\mathbf{i}+\mathbf{k}+\lambda(\mathbf{i}-\mathbf{j}+2\mathbf{k})$ and $\mathbf{r} = 2\mathbf{j}+6\mathbf{k}+\mu(\mathbf{i}+2\mathbf{j}-2\mathbf{k})$ respectively.

The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2.

(a) Find the length PQ. [5]

Line l_1: $\vec{r} = \begin{pmatrix}2\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\2\end{pmatrix} \Rightarrow \vec{a_1} = \begin{pmatrix}2\\0\\1\end{pmatrix}$ and $\vec{b_1} = \begin{pmatrix}1\\-1\\2\end{pmatrix}$.

Line l_2: $\vec{r} = \begin{pmatrix}0\\2\\6\end{pmatrix} + \mu\begin{pmatrix}1\\2\\-2\end{pmatrix} \Rightarrow \vec{a_2} = \begin{pmatrix}0\\2\\6\end{pmatrix}$ and $\vec{b_2} = \begin{pmatrix}1\\2\\-2\end{pmatrix}$.



$$\vec{a_2} - \vec{a_1} = \begin{pmatrix}0\\2\\6\end{pmatrix} - \begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}-2\\2\\5\end{pmatrix}$$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -1 & 2\\ 1 & 2 & -2\end{vmatrix} = \begin{bmatrix}-2\\4\\3\end{bmatrix}.$$

$|\vec{PQ}|$ = shortest distance between lines l_1 and l_2

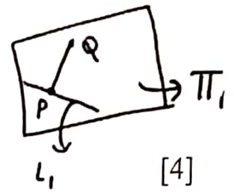
$$= \left|\frac{(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})}{|\vec{b_1}\times\vec{b_2}|}\right| = \left|\frac{\begin{pmatrix}-2\\2\\5\end{pmatrix}\cdot\begin{pmatrix}-2\\4\\3\end{pmatrix}}{\sqrt{(-2)^2+4^2+3^2}}\right| = \left|\frac{27}{\sqrt{29}}\right| = \frac{27}{\sqrt{29}}.$$

The plane Π_1 contains PQ and l_1.

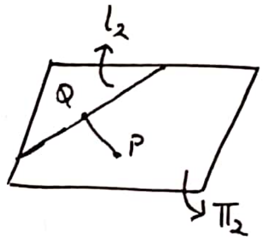
The plane Π_2 contains PQ and l_2.

(b) **(i)** Write down an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$. [1]

$$\Pi_1: \vec{r} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}.$$



(ii) Find an equation of Π_2, giving your answer in the form $ax + by + cz = d$. [4]



$$\Pi_2: \vec{r} = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} + t\begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}.$$

$$\vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -2 \\ -2 & 4 & 3 \end{vmatrix} = \begin{bmatrix} 14 \\ 1 \\ 8 \end{bmatrix}.$$

$$\vec{r} \cdot \vec{n_2} = \vec{a_2} \cdot \vec{n_2}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 14 \\ 1 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 14 \\ 1 \\ 8 \end{pmatrix}$$

$$14x + y + 8z = 50.$$

(c) Find the acute angle between Π_1 and Π_2. [5]

$$\vec{n_1} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 2 \\ -2 & 4 & 3 \end{vmatrix} = \begin{bmatrix} -11 \\ -7 \\ 2 \end{bmatrix} \text{ and } \vec{n_2} = \begin{bmatrix} 14 \\ 1 \\ 8 \end{bmatrix}.$$

$$\vec{n_1} \cdot \vec{n_2} = |\vec{n_1}||\vec{n_2}| \cos\theta$$

$$-11(14) + (-7)(1) + 2(8) = \sqrt{(-11)^2 + (-7)^2 + 2^2}\sqrt{14^2 + 1^2 + 8^2} \cos\theta.$$

$$-145 = \sqrt{174}\sqrt{261} \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{-145}{\sqrt{174}\sqrt{261}}\right) = 132.87^\circ$$

$\Rightarrow$ Acute angle between Π_1 and $\Pi_2 = 47.12^\circ \approx 47.1^\circ$.

61 The plane Π_1 has equation $r = -4\mathbf{j} - 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k})$.

(a) Obtain an equation of Π_1 in the form $px + qy + rz = d$. [4]

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \sim \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$y + z = -7.$$

(b) The plane Π_2 has equation $\mathbf{r}.(-5\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) = 4$.

Find a vector equation of the line of intersection of Π_1 and Π_2. [4]

Π_2: $\vec{r} \cdot (-5\vec{i} + 3\vec{j} + 5\vec{k}) = 4 \Rightarrow -5x + 3y + 5z = 4 \rightarrow (1)$

Π_1: $y + z = -7 \Rightarrow y = -7 - z \rightarrow (2)$

Substitute (2) in (1): $-5x + 3(-7 - z) + 5z = 4$

$$-5x - 21 - 3z + 5z = 4$$
$$-5x + 2z = 25$$
$$2z = 5x + 25$$
$$z = 2.5x + 12.5$$

Let, $x = t \Rightarrow z = 2.5t + 12.5$

$\Rightarrow y = -7 - (2.5t + 12.5) = -7 - 2.5t - 12.5 = -19.5 - 2.5t.$

$$\therefore \vec{r} = \begin{pmatrix} t \\ -2.5t - 19.5 \\ 2.5t + 12.5 \end{pmatrix} = \begin{pmatrix} t \\ -2.5t \\ 2.5t \end{pmatrix} + \begin{pmatrix} 0 \\ -19.5 \\ 12.5 \end{pmatrix} = \begin{pmatrix} 0 \\ -19.5 \\ 12.5 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2.5 \\ 2.5 \end{pmatrix}$$

$$\Rightarrow \vec{r} = \begin{pmatrix} 0 \\ -19.5 \\ 12.5 \end{pmatrix} + t \begin{pmatrix} 2 \\ -5 \\ 5 \end{pmatrix}.$$

The line l passes through the point A with position vector $a\mathbf{i}+a\mathbf{j}+(a-7)\mathbf{k}$ and is parallel to $(1-b)\mathbf{i}+b\mathbf{j}+b\mathbf{k}$, where a and b are positive constants.

(c) Given that the perpendicular distance from A to Π_1 is $\sqrt{2}$, find the value of a. [2]

$$\overrightarrow{OA}=\begin{pmatrix}a\\a\\a-7\end{pmatrix},\ \overrightarrow{OP}=\begin{pmatrix}0\\-4\\-3\end{pmatrix} \Rightarrow \overrightarrow{AP}=\overrightarrow{OP}-\overrightarrow{OA}=\begin{pmatrix}0\\-4\\-3\end{pmatrix}-\begin{pmatrix}a\\a\\a-7\end{pmatrix}=\begin{pmatrix}-a\\-4-a\\4-a\end{pmatrix}$$

$$\perp \text{ distance from } A \text{ to } \Pi_1 = \left|\frac{\overrightarrow{AP}\cdot\overrightarrow{n_1}}{|\overrightarrow{n_1}|}\right| = \left|\frac{\begin{pmatrix}-a\\-4-a\\4-a\end{pmatrix}\cdot\begin{pmatrix}0\\1\\1\end{pmatrix}}{\sqrt{0^2+1^2+1^2}}\right| = \left|\frac{-0-4-a+4-a}{\sqrt{2}}\right|$$

$$\sqrt{2}=\left|\frac{-2a}{\sqrt{2}}\right|$$

$$\pm\sqrt{2}=\frac{-2a}{\sqrt{2}} \Rightarrow \text{Either, } \frac{-2a}{\sqrt{2}}=\sqrt{2} \text{ or } \frac{-2a}{\sqrt{2}}=-2$$

$$-2a=2 \qquad -2a=-2$$

$$a=-1 \qquad a=1$$

(Ignore)

$\therefore$ Value of $a=1$.

(d) Given that the obtuse angle between l and Π_1 is $\frac{3}{4}\pi$, find the exact value of b. [4]

$$\vec{b}=\begin{pmatrix}1-b\\b\\b\end{pmatrix},\ \overrightarrow{n_1}=\begin{pmatrix}0\\1\\1\end{pmatrix}.$$

$$\vec{b}\cdot\overrightarrow{n_1}=0+b+b=2b$$

$$|\vec{b}|=\sqrt{(1-b)^2+b^2+b^2}=\sqrt{3b^2-2b+1}.$$

$$|\overrightarrow{n_1}|=\sqrt{0^2+1^2+1^2}=\sqrt{2}.$$

$$\vec{b}\cdot\overrightarrow{n_1}=|\vec{b}||\overrightarrow{n_1}|\cos\theta$$

$$2b=\sqrt{3b^2-2b+1}\sqrt{2}\cos\left(\frac{3\pi}{4}\right).$$

$$2b=\sqrt{3b^2-2b+1}\cdot\sqrt{2}\cdot\frac{-1}{\sqrt{2}}$$

$$2b=-\sqrt{3b^2-2b+1}$$

$$4b^2=3b^2-2b+1$$

$$b^2+2b-1=0$$

$$b=\frac{-2\pm\sqrt{2^2-4(1)(-1)}}{2(1)}=\frac{-2\pm\sqrt{8}}{2}=\frac{-2\pm2\sqrt{2}}{2}=-1\pm\sqrt{2}$$

Ignore $b=-1-\sqrt{2}$ since $b>0$

$\therefore b=-1+\sqrt{2}$.

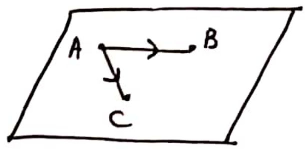
62 The points A, B, C have position vectors

$$\mathbf{i}+\mathbf{j}, \quad -\mathbf{i}+2\mathbf{j}+4\mathbf{k}, \quad -2\mathbf{i}+\mathbf{j}+3\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}-1\\2\\4\end{pmatrix}-\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}-2\\1\\4\end{pmatrix}.$$



$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\begin{pmatrix}-2\\1\\3\end{pmatrix}-\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}-3\\0\\3\end{pmatrix}$$

$$\vec{n}=\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\-2&1&4\\-3&0&3\end{vmatrix}=\begin{bmatrix}3\\-6\\3\end{bmatrix}\sim\begin{bmatrix}1\\-2\\1\end{bmatrix}.$$

$$\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

$$x-2y+z=-1$$

(b) Find the perpendicular distance from O to the plane ABC. [2]

$$\overrightarrow{OA}=\begin{pmatrix}1\\1\\0\end{pmatrix},\ \vec{n}=\begin{pmatrix}1\\-2\\1\end{pmatrix}.$$

$\perp$ distance from O to the plane ABC

$$=\left|\frac{\overrightarrow{OA}\cdot\vec{n}}{|\vec{n}|}\right|=\left|\frac{\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix}}{\sqrt{1^2+(-2)^2+1^2}}\right|=\left|\frac{1(1)+1(-2)+0(1)}{\sqrt{6}}\right|=\left|\frac{-1}{\sqrt{6}}\right|=\frac{1}{\sqrt{6}}.$$

(c) Find a vector equation of the common perpendicular to the lines OC and AB. [8]

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}-1\\2\\4\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}-2\\1\\4\end{pmatrix}.$$

$$\Rightarrow \text{Line } AB: \vec{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} + t\begin{pmatrix}-2\\1\\4\end{pmatrix} = \begin{pmatrix}1-2t\\1+t\\4t\end{pmatrix}.$$

$$\text{Line } OC: \vec{r} = \begin{pmatrix}0\\0\\0\end{pmatrix} + s\begin{pmatrix}-2\\1\\3\end{pmatrix} = \begin{pmatrix}-2s\\s\\3s\end{pmatrix}.$$

Let, P and Q be points on line AB and OC such that PQ is the common perpendicular.

$$\overrightarrow{OP} = \begin{pmatrix}1-2t\\1+t\\4t\end{pmatrix} \text{ and } \overrightarrow{OQ} = \begin{pmatrix}-2s\\s\\3s\end{pmatrix}.$$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix}-2s\\s\\3s\end{pmatrix} - \begin{pmatrix}1-2t\\1+t\\4t\end{pmatrix} = \begin{pmatrix}-1-2s+2t\\-1+s-t\\3s-4t\end{pmatrix}.$$

$$\overrightarrow{PQ} \perp \overrightarrow{AB} \Rightarrow \overrightarrow{PQ}\cdot\overrightarrow{AB} = 0 \Rightarrow -2(-1-2s+2t)+1(-1+s-t)+4(3s-4t) = 0$$

$$2+4s-4t-1+s-t+12s-16t = 0$$

$$17s-21t+1 = 0 \rightarrow (1)$$

$$\overrightarrow{PQ} \perp \overrightarrow{OC} \Rightarrow \overrightarrow{PQ}\cdot\overrightarrow{OC} = 0 \Rightarrow -2(-1-2s+2t)+1(-1+s-t)+3(3s-4t) = 0$$

$$2+4s-4t-1+s-t+9s-12t = 0$$

$$14s-17t+1 = 0 \rightarrow (2)$$

From (1), $17s = 21t - 1 \Rightarrow s = \dfrac{21t-1}{17} \rightarrow (3)$

Substitute (3) in (2): $14\left(\dfrac{21t-1}{17}\right) - 17t + 1 = 0 \Rightarrow 294t - 14 - 289t + 17 = 0$

$$5t = -3$$

$$t = -0.6$$

$$\Rightarrow s = \frac{21(-0.6)-1}{17} = -0.8$$

$$t = -0.6 \Rightarrow \overrightarrow{OP} = \begin{pmatrix}1-2(-0.6)\\1+(-0.6)\\4(-0.6)\end{pmatrix} = \begin{pmatrix}2.2\\0.4\\-2.4\end{pmatrix}$$

$$s = -0.8 \Rightarrow \overrightarrow{OQ} = \begin{pmatrix}-2(-0.8)\\-0.8\\3(-0.8)\end{pmatrix} = \begin{pmatrix}1.6\\-0.8\\-2.4\end{pmatrix}$$

$$\Rightarrow \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix}-0.6\\-1.2\\0\end{pmatrix}$$

$\therefore$ Equation of common perpendicular: $\vec{r} = \begin{pmatrix}2.2\\0.4\\-2.4\end{pmatrix} + \lambda\begin{pmatrix}-0.6\\-1.2\\0\end{pmatrix} \Rightarrow \vec{r} = \begin{pmatrix}2.2\\0.4\\-2.4\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\0\end{pmatrix}.$