Vector Geometry Notes

1. Representing Vectors by Directed Line Segments

- A vector is typically represented by a directed line segment.
- The **magnitude** of the vector is the length of the line segment, and the **direction** is the way the segment is pointing.
- A vector $\mathbf{a}$ can be represented as a directed line segment starting at point **A** and ending at point **B**, written as $\mathbf{AB}$.

2. Using Position Vectors

- A **position vector** represents the position of a point relative to an origin. If **O** is the origin and **P** is a point, then the position vector of **P** is denoted as $\mathbf{OP}$.
- The coordinates of a point **P** in 2D or 3D can be written as the vector from the origin to the point. For example:
  - In 2D, if **P** has coordinates (x, y), the position vector is $\mathbf{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$
  - In 3D, if **P** has coordinates (x, y, z), the position vector is $\mathbf{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

3. Sum and Difference of Vectors

- **Vector Addition**: If you have two vectors $\mathbf{a}$ and $\mathbf{b}$, their sum is the vector $\mathbf{a} + \mathbf{b}$, which is found by connecting the tail of $\mathbf{b}$ to the head of $\mathbf{a}$.
- **Vector Subtraction**: To subtract a vector $\mathbf{b}$ from $\mathbf{a}$, you reverse the direction of $\mathbf{b}$ and then add it to $\mathbf{a}$, i.e., $\mathbf{a} - \mathbf{b}$.
- **Expressing Vectors in Terms of Two Coplanar Vectors**: If you have two non-parallel vectors $\mathbf{u}$ and $\mathbf{v}$, any vector in the same plane can be written as a linear combination of these two. For example, $\mathbf{w} = k\mathbf{u} + l\mathbf{v}$, where k and l are scalars.

4. Using Vectors to Solve Geometric Problems

Vectors are powerful tools for solving geometric problems. Here are some common tasks:

- **Show that vectors are parallel**:
  - Two vectors are parallel if one is a scalar multiple of the other. For example, if $\mathbf{a} = k\mathbf{b}$ for some scalar k, then $\mathbf{a}$ and $\mathbf{b}$ are parallel.
- **Show that three points are collinear**:
  - Three points **A**, **B**, and **C** are collinear if the vector $\mathbf{AB}$ is parallel to $\mathbf{BC}$, i.e., if $\mathbf{AB} = k\mathbf{BC}$ for some scalar k. This means that $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ lie on the same straight line.

- **Solve vector problems involving ratio and similarity:**
  - If a point **P** divides the line segment joining points **A** and **B** in the ratio $m : n$, the position vector of **P** is given by the weighted average:

$$\mathbf{P} = \frac{n}{m+n}\mathbf{A} + \frac{m}{m+n}\mathbf{B}$$

  - For similarity problems, vectors are used to establish proportional relationships. For example, when comparing triangles or other geometric shapes, vectors can be used to prove that two figures are similar if their corresponding sides are proportional and their corresponding angles are equal.

Example Problems

Example 1: Show that Two Vectors are Parallel

- **Given**: Vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$.
- **Solution**: To show that the vectors are parallel, check if one is a scalar multiple of the other:

$$\mathbf{b} = 2\mathbf{a}$$

Since $\mathbf{b} = 2\mathbf{a}$, the vectors are parallel.

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Example 2: Show that Three Points are Collinear

- **Given**: Points **A(1, 2)**, **B(3, 6)**, and **C(5, 10)**.
- **Solution**: Calculate the vectors $\mathbf{AB}$ and $\mathbf{BC}$:

$$\mathbf{AB} = \begin{pmatrix} 3-1 \\ 6-2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$\mathbf{BC} = \begin{pmatrix} 5-3 \\ 10-6 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Since $\mathbf{AB} = \mathbf{BC}$, the points are collinear.

Example 3: Find the Point Dividing a Line Segment in a Given Ratio

- **Given**: Points **A(1, 2)** and **B(5, 6)**, and the point **P** divides **AB** in the ratio 2:3.
- **Solution**: Use the section formula to find the position vector of point **P**:

$$\mathbf{P} = \frac{3}{2+3}\mathbf{A} + \frac{2}{2+3}\mathbf{B} = \frac{3}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{2}{5}\begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

$$\mathbf{P} = \begin{pmatrix} \frac{3}{5} \\ \frac{6}{5} \end{pmatrix} + \begin{pmatrix} 2 \\ \frac{12}{5} \end{pmatrix} = \begin{pmatrix} 2.6 \\ 3.6 \end{pmatrix}$$

Therefore, point **P** has coordinates $(2.6, 3.6)$.