Roots of the polynomial

1.1 Roots of Polynomials

THEORY: Sum of roots method

(i) Quadratic

$\boxed{ax^2 + bx + c}$ $\boxed{\text{roots} : \alpha, \beta}$

$$x^2 + \frac{b}{a}x + \frac{c}{a} \equiv (x-\alpha)(x-\beta)$$
$$= x^2 - (\alpha+\beta)x + \alpha\beta$$
$$= 0$$

$$\therefore \alpha + \beta = -\frac{b}{a} \qquad \alpha\beta = \frac{c}{a}$$

(ii) Cubic

$\boxed{ax^3 + bx^2 + cx + d}$ $\boxed{\text{roots} : \alpha, \beta, \gamma}$

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} \equiv (x-\alpha)(x-\beta)(x-\gamma)$$
$$= x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\alpha\gamma+\beta\gamma)x - \alpha\beta\gamma$$
$$= 0$$

$$\therefore \alpha+\beta+\gamma = -\frac{b}{a} \qquad \alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a} \qquad \alpha\beta\gamma = -\frac{d}{a}$$

(iii) Quartic

$\boxed{ax^4 + bx^3 + cx^2 + dx + e}$ $\boxed{\text{roots} ; \alpha, \beta, \gamma, \delta}$

$$x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a} \equiv (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$$
$$= x^4 - (\alpha+\beta+\gamma+\delta)x^3 +$$
$$(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)x^2$$
$$+ (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)x + \alpha\beta\gamma\delta$$
$$= 0$$

$$\therefore \Sigma\alpha = -\frac{b}{a} \qquad \Sigma\alpha\beta = \frac{c}{a} \qquad \Sigma\alpha\beta\gamma = -\frac{d}{a} \qquad \alpha\beta\gamma\delta = \frac{e}{a}$$

Q1: The equation $3x^2+2x-7=0$ has roots α and β.
Find the equation whose roots are α^2 and β^2.

(a) Substitution method:

Let $y=x^2$

$x=\sqrt{y}$

$\therefore 3y+2\sqrt{y}-7=0$

$2\sqrt{y}=7-3y$

$4y=(7-3y)^2$

$=49-42y+9y^2$

$\therefore 9y^2-46y+49=0$

(b) Sum of roots method:

Let $\alpha'=\alpha^2 \qquad \beta'=\beta^2$

$\alpha'+\beta'=\alpha^2+\beta^2$

$=(\alpha+\beta)^2-2\alpha\beta$

$=\left(-\frac{2}{3}\right)^2-2\left(-\frac{7}{3}\right)$

$=\frac{46}{9}$

$\alpha'\beta'=\alpha^2\beta^2$

$=(\alpha\beta)^2$

$=\left(-\frac{7}{3}\right)^2$

$=\frac{49}{9}$

Q2: The equation $x^2-7x+3=0$ has roots α, β.
Find the equation which has roots $\alpha+2\beta$, $2\alpha+\beta$.

(a) Substitution method:

$\alpha+\beta=7$

$\rightarrow \alpha+2\beta=(\alpha+\beta)+\beta$

$=7+\beta$

$\rightarrow 2\alpha+\beta=(\alpha+\beta)+\alpha$

$=7+\alpha$

$\alpha+7,\ \beta+7$

sub $y=x+7$ into $x^2-7x+3=0$

$(y-7)^2-7(y-7)+3=0$

$\Rightarrow y^2-21y+101=0$

(b) Sum of roots method:

Let $\alpha'=\alpha+2\beta \qquad \beta'=2\alpha+\beta$

$\alpha'+\beta'=(\alpha+2\beta)+(2\alpha+\beta)$

$=3(\alpha+\beta)$

$=3\times 7$

$=21$

$\alpha'\beta'=(\alpha+2\beta)(2\alpha+\beta)$

$=2\alpha^2+4\alpha\beta+\alpha\beta+2\beta^2$

$=2(\alpha^2+\beta^2)+5\alpha\beta$

$=2[(\alpha+\beta)^2-2\alpha\beta]+5\alpha\beta$

$=2(\alpha+\beta)^2+\alpha\beta$

$=2(7)^2+3$

$=101$

In solving higher order polynomial functions, remember the following notations:

CUBIC:

$$\Sigma\alpha = \alpha + \beta + \gamma$$

$$\Sigma\alpha\beta = \alpha\beta + \alpha\gamma + \beta\gamma$$

$$\Sigma\alpha^2 = \alpha^2 + \beta^2 + \gamma^2$$
$$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$$
$$= (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$$

$$\Sigma\alpha^2\beta^2 = (\alpha\beta + \alpha\gamma + \beta\gamma)^2 - 2(\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2)$$
$$= (\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)$$
$$= (\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma(\Sigma\alpha$$

QUARTIC:

$$\Sigma\alpha\beta(\alpha+\beta) = \alpha\beta(\alpha+\beta) + \alpha\gamma(\alpha+\gamma) + \beta\gamma(\beta+\gamma)$$
$$= \alpha\beta(\alpha+\beta+\gamma) + \alpha\gamma(\alpha+\beta+\gamma) + \beta\gamma(\alpha+\beta+\gamma) - 3\alpha\beta\gamma$$
$$= (\alpha\beta + \alpha\gamma + \beta\gamma)(\alpha+\beta+\gamma) - 3\alpha\beta\gamma$$
$$= (\Sigma\alpha\beta)(\Sigma\alpha) - 3\alpha\beta\gamma$$

CUBIC:

$$\alpha = \alpha + \beta + \gamma$$
$$\alpha\beta = \alpha\beta + \alpha\gamma + \beta\gamma$$
$$\Sigma\alpha^2 = \alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma) = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$$
$$\alpha^2\beta^2 = (\alpha\beta + \alpha\gamma + \beta\gamma)^2 - 2(\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2) = (\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma) = (\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma(\Sigma\alpha)$$
$$\alpha\beta(\alpha+\beta) = \alpha\beta(\alpha+\beta) + \alpha\gamma(\alpha+\gamma) + \beta\gamma(\beta+\gamma) = \alpha\beta(\alpha+\beta+\gamma) + \alpha\gamma(\alpha+\beta+\gamma) + \beta\gamma(\alpha+\beta+\gamma) - 3\alpha\beta\gamma = (\alpha\beta + \alpha\gamma + \beta\gamma)(\alpha+\beta+\gamma) - 3\alpha\beta\gamma$$

QUARTIC:

$$\Sigma\alpha = \alpha + \beta + \gamma + \delta$$

$$\Sigma\alpha\beta = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$$

$$\Sigma\alpha\beta\gamma = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$$

$$\Sigma\alpha^2 \equiv (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$$

$$\Sigma\alpha^2\beta^2 \equiv (\Sigma\alpha\beta)^2 - 2(\Sigma\alpha\beta\gamma)(\Sigma\alpha) - \alpha\beta\gamma\delta$$

$$\Sigma\alpha\beta(\alpha+\beta) \equiv (\Sigma\alpha\beta)(\Sigma\alpha) - 3\Sigma\alpha\beta\gamma$$

THEORY: Evaluation of $\alpha^3+\beta^3+\gamma^3$ and $\alpha^4+\beta^4+\gamma^4$

Suppose $ax^3+bx^2+cx+d=0$

roots: α, β, γ

Then : $\alpha+\beta+\gamma = -\frac{b}{a}$

$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a}$

$\alpha\beta\gamma = -\frac{d}{a}$

$\therefore \alpha^2+\beta^2+\gamma^2 \equiv (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

To find $\alpha^3+\beta^3+\gamma^3$?

THEORY: Evaluation of $\alpha^3+\beta^3+\gamma^3$ and $\alpha^4+\beta^4+\gamma^4$

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To find $\alpha^3+\beta^3+\gamma^3$ & $\alpha^4+\beta^4+\gamma^4$:

Since $ax^3+bx^2+cx+d=0$ has roots α, β, γ, then:

$a\alpha^3+b\alpha^2+c\alpha+d=0 \longrightarrow (1)$

$a\beta^3+b\beta^2+c\beta+d=0 \longrightarrow (2)$

$a\gamma^3+b\gamma^2+c\gamma+d=0 \longrightarrow (3)$

$a(\alpha^3+\beta^3+\gamma^3)+b(\alpha^2+\beta^2+\gamma^2)+c(\alpha+\beta+\gamma)+3d=0$

we can then deduce $\alpha^3+\beta^3+\gamma^3$

Example:

Q1: The equation $x^3 - 2x^2 - 5x + 6 = 0$ has roots α, β, γ.
Find the values of S_2, S_3, S_4, S_5, where $\boxed{S_n = \alpha^n + \beta^n + \gamma^n}$

$$\left.\begin{aligned} a\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha &= 0 \\ a\beta^4 + b\beta^3 + c\beta^2 + d\beta &= 0 \\ a\gamma^4 + b\gamma^3 + c\gamma^3 + d\gamma &= 0 \end{aligned}\right\}$$

$$a(\alpha^4 + \beta^4 + \gamma^4) + b(\alpha^3 + \beta^3 + \gamma^3) + c(\alpha^2 + \beta^2 + \gamma^2) + d(\alpha + \beta + \gamma) = 0$$

we can now deduce $\alpha^4 + \beta^4 + \gamma^4$.

$\therefore\ aS_{n+3} + bS_{n+2} + cS_{n+1} + dS_n = 0$ where

$S_n = \alpha^n + \beta^n + \gamma^n$

$S_n = \alpha$

Example:

Q1: The equation $x^3 - 2x^2 - 5x + 6 = 0$ has roots α, β, γ.
Find the values of S_2, S_3, S_4, S_5, where $\boxed{S_n = \alpha^n + \beta^n + \gamma^n}$

$$\begin{aligned} S_2 &= (\Sigma\alpha)^2 - 2\Sigma\alpha\beta \\ &= 2^2 - 2(-5) \\ &= 14 \end{aligned}$$

Since it has roots α, β, γ,

$\alpha^3 - 2\alpha^2 - 5\alpha + 6 = 0$

$$\left.\begin{aligned} \alpha^3 - 2\alpha^2 - 5\alpha + 6 &= 0 \\ \beta^3 - 2\beta^2 - 5\beta + 6 &= 0 \\ \gamma^3 - 2\gamma^2 - 5\gamma + 6 &= 0 \end{aligned}\right\}$$

multiply by $\alpha^n, \beta^n, \gamma^n$ gives:

$$\alpha^{n+3} - 2\alpha^{n+2} - 5\alpha^{n+1} + 6\alpha^n = 0$$

$$\beta^{n+3} - 2\beta^{n+2} - 5\beta^{n+1} + 6\beta^n = 0$$

$$\gamma^{n+3} - 2\gamma^{n+1} - 5\gamma^{n+1} + 6\gamma^n = 0$$

multiply by a^n, β^n, γ^n gives:

$$\alpha^{n+3} - 2\alpha^{n+2} - 5\alpha^{n+1} + 6\alpha^n = 0$$

$$\beta^{n+3} - 2\beta^{n+2} - 5\beta^{n+1} + 6\beta^n = 0$$

$$\gamma^{n+3} - 2\gamma^{n+1} - 5\gamma^{n+1} + 6\gamma^n = 0$$

$$S_{n+3} - 2S_{n+2} - 5S_{n+1} + 6S_0 = 0$$

$$\therefore \quad S_3 - 2S_2 - 5S_1 + 6S_0 = 0$$

$$S_3 = 2(14) + 5(2) - 6(3) = 20$$

$$\therefore \quad S_4 - 2S_3 - 5S_2 + 6S_1 = 0$$

$$S_4 = 2(20) + 5(14) - 6(2)$$

$$= 9$$

$$S_{n+3} - 2S_{n+2} - 5S_{n+1} + 6S_0 = 0$$

$$\therefore \quad S_3 - 2S_2 - 5S_1 + 6S_0 = 0$$

$$S_3 = 2(14) + 5(2) - 6(3) = 20$$

$$\therefore \quad S_4 - 2S_3 - 5S_2 + 6S_1 = 0$$

$$S_4 = 2(20) + 5(14) - 6(2)$$

$$= 98$$

$$\therefore \quad S_5 - 2S_4 - 5S_3 + 6S_2 = 0$$

$$S_5 = 2(98) + 5(20) - 6(14)$$

$$= 212$$